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Nonlinear Elliptic Equations Involving the Critical Sobolev Exponent—Survey and Perspectives

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1. INTRODUCTION

Let Ω be a (smooth) bounded domain in $\mathbf{R}^N$ with $N > 3$. Consider the problem of finding a function $u(x)$ which satisfies

$$\left.\begin{aligned} -\Delta u &= u^p + a(x)u \quad \text{on } \Omega, \\ u &> 0 \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \qquad \text{(I)}$$

where $p = (N + 2)/(N - 2)$ and $a(x)$ is a given smooth function. The question is to determine conditions on $a(x)$ and Ω which guarantee that (I) has a solution. For example, an obvious <u>necessary condition</u> is that the linear operator $L = -\Delta - a$ should be <u>coercive</u>, that is,

$$\int (|\nabla\varphi|^2 - a\varphi^2) > \delta\int\varphi^2 \qquad \forall\varphi \in H_0^1 \text{ with } \delta > 0 . \qquad (1)$$

Indeed, let μ_1 denote the first eigenvalue of L (with zero Dirichlet condition) and let $\varphi_1 > 0$ be a corresponding eigenfunction. Multiplying (I) through by φ_1 we obtain

$$\mu_1 \int u\varphi_1 = \int u^p\varphi_1 ,$$

so that μ_1 must be positive if a solution of (I) exists. It is tempting to assert that (1) is also a sufficient condition; and this is indeed the case when $1 < p < (N + 2)/(N - 2)$.

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However, when $p = (N + 2)/(N - 2)$ the situation is completely different. A celebrated result of Pohozaev [34] asserts that if Ω is starshaped and $a(x) \equiv 0$ there is no solution of (I) (despite the fact that (1) holds). Therefore, extra assumptions, in addition to (1), are needed. Two different kinds of conditions have been found so far:

a) The function $a(x)$ is positive somewhere on Ω when $N \geq 4$ (and no assumption on Ω); a more delicate (and global) assumption when $N = 3$.

b) The domain Ω has nontrivial topology - for example Ω has a "hole" - (and no assumption on the function $a(x)$).

Case a) has been investigated in [17] following a technique introduced by Th. Aubin [8]; it is discussed in Sections 2 and 3. Case b) is related to the splendid work of Bahri-Coron [10-11] which is discussed in Section 5.

A closely related problem is to find a function $u(x)$ which satisfies

$$\left.\begin{array}{lll} -\Delta u = |u|^{p-1}u + a(x)u & \text{on} & \Omega\,, \\ u \not\equiv 0 & \text{on} & \Omega\,, \\ u = 0 & \text{on} & \partial\Omega\,, \end{array}\right\} \qquad \text{(II)}$$

where, again, $p = (N + 2)/(N - 2)$ and $a(x)$ is a given (smooth) function. Clearly, any solution of (I) is a solution of (II). Nevertheless, problem (II) is of independent interest for the following reasons:

a) The conditions to impose on $a(x)$ or Ω may be weaker; for example (1) is not anymore a necessary condition.

b) Problem (II) may have many solutions - possibly infinitely many - in cases where (I) has just one solution.

Problems such as (I) or (II) have attracted much attention in recent years. Despite their simple form they have a very rich structure. The main reason is that they admit a <u>variational formulation which lacks the Palais-Smale (PS) condition</u>. For example, solutions of (II) correspond to nonzero critical points of the functional

$$F(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1} - \frac{1}{2}\int au^2 .$$

The functional F is well defined on the Sobolev space H_0^1 (because of the Sobolev imbedding $H^1 \subset L^{p+1} = L^{2N/(N-2)}$. However, F does not satisfy the (PS) compactness condition (because this imbedding is not compact). So that, all the standard variational techniques (minimization, Morse theory, Ljusternik-Schnirelman theory etc. ...) do not apply directly to (I) or (II). These kinds of problems are fascinating experimental laboratories for discovering new tools to overcome this lack of compactness. Related questions - such as the Yamabe problem - occur in geometry (see e.g. Kazdan [28]) and in physics (see e.g. the work of Taubes [40] dealing with the Yang-Mills equations in dimension four).

The plan is the following:

1. Introduction.
2. Problem (I) when $N > 4$.
3. Problem (I) when $N = 3$.
4. Problem (II).
5. The effect of topology: the work of Bahri-Coron.
6. Various related problems.

2. PROBLEM (I) WHEN $N > 4$

Throughout Sections 2 and 3 we shall assume that (1) holds. The first - and simplest - approach in order to find a solution of (I) is minimization with constraint. More precisely, set

$$J = \operatorname*{Inf}_{\varphi \in H_0^1} \left\{ \frac{|\nabla\varphi|^2 - a\varphi^2}{\|\varphi\|_{p+1}^2} \right\} = \operatorname*{Inf}_{\substack{\varphi \in H_0^1 \\ \|\varphi\|_{p+1}=1}} \left\{ \int |\nabla\varphi|^2 - a\varphi^2 \right\} . \qquad (2)$$

Note that $J > 0$ (because (1) holds). It is not clear whether this infimum is achieved - the reason is that the constraint $\|\varphi\|_{p+1} = 1$ is not preserved under weak H^1 limits. However, if we assume that J is achieved, then we obtain a solution of (I). Indeed let φ be a minimizer; we may always suppose that $\varphi > 0$ - otherwise replace φ by $|\varphi|$. The Euler equation for (2) is

$$-\Delta\varphi - a\varphi = J\varphi^p \quad \text{on} \quad \Omega .$$

It follows from the strong maximum principle that $\varphi > 0$ on Ω and finally we obtain a solution of (I) by scaling out the constant J.

As pointed out by Th. Aubin [7-8] the best Sobolev constant S plays an important role in such problems. S is defined by

$$S = \operatorname*{Inf}_{\varphi \in H_0^1} \left\{ \frac{\int |\nabla\varphi|^2}{\|\varphi\|^2_{p+1}} \right\} = \operatorname*{Inf}_{\substack{\varphi \in H_0^1 \\ \|\varphi\|_{p+1}=1}} \left\{ \int |\nabla\varphi|^2 \right\} . \tag{3}$$

It is easy to see that S is independent of Ω (it depends only on N) and that the infimum in (3) is <u>not</u> achieved in any bounded domain.

The success (or the failure) of this minimization approach is <u>completely settled</u> by the following:

<u>Theorem 1</u> (Brezis-Nirenberg): Assume Ω is any bounded domain in $\mathbb{R}^N$ with $N \geq 4$. The following properties are equivalent:

$$a(x) > 0 \quad \text{somewhere on} \quad \Omega , \tag{4}$$

$$J < S , \tag{5}$$

$$J \quad \text{is achieved (and therefore (I) has a solution)} . \tag{6}$$

The proof of Theorem 1 is essentially contained in [17] (see also [14]). The most technical part is the proof of (4) ==> (5), which is achieved by constructing explicitly a function φ such that

$$Q(\varphi) = \frac{\int |\nabla\varphi|^2 - a\varphi^2}{\|\varphi\|^2_{p+1}} < S .$$

More precisely, φ is chosen of the form

$$\varphi(x) = \zeta(x)/(\varepsilon + |x|^2)^{(N-2)/2}$$

with $\varepsilon > 0$ small enough and $\zeta \in C_0^\infty(\Omega)$ is such that $\zeta \equiv 1$ near a point x_0 where $a(x_0) > 0$. It is in this part of the argument that the assumption $N \geq 4$ comes in.

In fact, the implication (4) ==> (5) fails when $N = 3$. The other implications (5) ==> (6) and (6) ==> (4) are rather easy and hold even when $N > 3$. This leads us to:

Question 1: Is there a direct proof of the implication (6) ==> (5) which does not make use of (4)? Is it true that (6) ==> (5) when $N = 3$?

There are related - "dual" - maximization problems which are likely to have solutions; for example:

Question 2: Assume $a(x) > 0$ somewhere on Ω and consider

$$\sup_{\substack{\varphi \in H_0^1 \\ \int |\nabla\varphi|^2 < 1}} \left\{ \frac{1}{p+1} \int |\varphi|^{p+1} + \frac{1}{2} \int a\varphi^2 \right\} .$$

Is this supremum achieved when $N > 4$? Same question when $N = 3$.

Note that the minimization technique (2) may not be the right approach for finding a solution of Problem (I). In particular, if $a(x) < 0$ everywhere on Ω, Problem (I) may still have a solution. Of course, Theorem 1 prevents that solution from being reached by minimization and it is a challenge to find the appropriate tool. Here are two examples:

Example 1: Assume (for simplicity) that $a(x) \equiv 0$ and that Ω has nontrivial topology. Then, Problem (I) has a solution (see Section 5).

Example 2: Let Ω be any domain (for instance a ball) and fix any function $f \in C_0^\infty(\Omega)$ with $f > 0$, $f \not\equiv 0$. Let v be the solution of the problem

$$\begin{cases} -\Delta v = f & \text{on } \Omega , \\ v = 0 & \text{on } \partial\Omega , \end{cases}$$

so that $v > 0$ on Ω. Set

$$a = \frac{f}{v} - \mu^{p-1} v^{p-1}$$

where μ is a constant large enough so that $a < 0$ on Ω. It is clear that $u = \mu v$ is a solution of Problem (I).

This leads us to the following:

Question 3: Assume $a(x) < 0$ everywhere on Ω. Find conditions on $a(x)$ which guarantee that Problem (I) has a solution. The answer is unknown even if Ω is a ball and $a(x)$ is a radial function.

Remark 1: The Pohozaev identity applied to a solution u of Problem (I) says that

$$\int_\Omega \left(a + \frac{1}{2}\Sigma x_i \frac{\partial a}{\partial x_i}\right)u^2 = \frac{1}{2}\int_{\partial\Omega} (x \cdot n)\left(\frac{\partial u}{\partial n}\right)^2 . \tag{7}$$

If Ω is starshaped with respect to the origin, an obvious necessary condition for the existence of a solution is that $(a + \frac{1}{2}\Sigma x_i \frac{\partial a}{\partial x_i})$ should be positive somewhere on Ω. In particular, if $a(x) \equiv \lambda$ is a constant and Ω is starshaped we find that Problem (I) has a solution for every $\lambda \in (0,\lambda_1)$ and no solution otherwise, where λ_1 denotes the first eigenvalue of $-\Delta$ with zero Dirichlet condition.

3. PROBLEM (I) WHEN $N = 3$

For some strange reason this case is much more delicate than the case $N \geq 4$. The most recent contribution is due, independently to B. McLeod [31] and R. Schoen [37] (actually R. Schoen was working on the Yamabe problem, but his idea for solving the Yamabe problem in low dimension applies in our situation as well). In order to describe their result one has to introduce the Green's function $G(x,y)$ of the operator $L = -\Delta - a$, that is, $G(\cdot,y)$ is the solution of

$$\begin{cases} -\Delta G - aG = \delta(\cdot - y) & \text{on } \Omega \\ G = 0 & \text{on } \partial\Omega . \end{cases}$$

Note that G is well defined because of assumption (1). Write

$$G(x,y) = \frac{1}{4\pi|x-y|} + g(x,y) ;$$

the function $g(x,y)$ is called the regular part of the Green's function and g is continuous on $\Omega \times \Omega$ (including the diagonal (x,x)).

Theorem 2 (McLeod, Schoen): Assume Ω is any bounded domain in R^3. The following implications hold:

$g(x,x) > 0$ somewhere on Ω , (8)

$J < S$, (9)

J is achieved (and therefore (I) has a solution) . (10)

For the proof of Theorem 2 we refer to [31], [37] and also [14]. As in Theorem 1, the most technical part is the proof of (8) ==> (9) which is achieved by constructing a function φ such that $Q(\varphi) < S$. As pointed out by Bahri and Coron [11], it is convenient to choose $\varphi = \varphi_\varepsilon$ to be the solution of the problem:

$$\begin{cases} -\Delta\varphi_\varepsilon - a\varphi_\varepsilon = \dfrac{1}{(\varepsilon + |x - x_0|^2)^{5/2}} & \text{on } \Omega , \\ \varphi_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

where x_0 is any point in Ω such that $g(x_0,x_0) > 0$. An easy expansion as $\varepsilon \to 0$ (see [14]) shows that

$$Q(\varphi_\varepsilon) = S - Cg(x_0,x_0)\varepsilon^{1/2} + o(\varepsilon^{1/2})$$

where C is some positive (universal) constant.

It is tempting to compare Theorems 1 and 2 and to raise the following:

<u>Question 4</u>: Are the properties (8), (9), (10) equivalent?

The answer to Question 4 is not known, even in the case where $a(x) \equiv \lambda$ is a constant and Ω is, say, convex. However, the case where $a(x) \equiv \lambda$ is a constant and Ω is a ball is completely settled (see [17]): Problem (I) has a solution if $\lambda \in (\frac{1}{4}\lambda_1, \lambda_1)$ and no solution otherwise. In that case the answer to Question 4 is positive.

As in Section 2, I must emphasize that the minimization technique (2) may not be the right approach for finding a solution of Problem (I). In particular, if $a(x) < 0$ everywhere on Ω, it follows (from the maximum principle) that $g(x,x) < 0$ everywhere on Ω - and Problem (I) may still have a solution (see Example 2 in Section 2). Again, it is a challenge to find other tools besides minimization.

4. <u>PROBLEM (II)</u>

Here, assumption (1) plays no role and is not required. When p is <u>subcritical</u>, i.e. $1 < p < (N + 2)/(N - 2)$, it

is well-known (see [1]) that Problem (II) possesses infinitely many solutions without any additional assumption (on Ω or $a(x)$). However, in the critical case, $p = (N + 2)/(N - 2)$ further assumptions are needed. Indeed, the Pohozaev identity (7) still holds for any solution u of Problem (II). In particular, if Ω is starshaped and $a(x) \equiv \lambda$ is a constant, we find that λ must be positive if a solution of (II) exists. In view of this obstruction one can anticipate two kinds of assumptions which may force solutions to exist:

a) the function $a(x)$ is positive somewhere on Ω (and no assumption about Ω);

b) the domain Ω has nontrivial topology (and no assumption about $a(x)$).

So far, little is known about Problem (II), but some interesting progresses have been achieved during the last two years for the case $a(x) \equiv \lambda$ is a constant:

Theorem 3 (Capozzi-Fortunato-Palmieri): Assume Ω is any bounded domain in $\mathbb{R}^N$ with $N \geq 4$. Then, for every $\lambda > 0$, Problem (II) has a solution.

For the proof we refer to [18]. An elegant alternative proof has been given by Ambrosetti-Struwe [3], using a dual formulation proposed in [2] and which is similar to the dual formulation of Clarke-Ekeland [23] for Hamiltonian systems (see also [13]).

Theorem 4 (Cerami-Solimini-Struwe, D. Zhang): Assume Ω is any bounded domain in $\mathbb{R}^N$ with $N \geq 6$. Then, for every $\lambda \in (0,\lambda_1)$, Problem (II) has at least two pairs of solutions.

Theorem 5 (Solimini): Assume Ω is a ball in $\mathbb{R}^N$ with $N \geq 7$. Then for every $\lambda \in (0,\infty)$, Problem (II) has infinitely many radial solutions.

For the proof we refer to [38] (see also [21], [27] and [42] for related results). The idea is to use a min-max argument as in Ambrosetti-Rabinowitz [1] - except that the failure of the (PS) condition requires additional devices inspired by [17] and [39]. A result of a more "local" nature (for $\lambda < \lambda_j$ and $\lambda_j - \lambda$ small enough) had been

obtained previously by Cerami-Fortunato-Struwe [20]. Naturally, we are led to the following:

Question 5: Assume Ω is any bounded domain in R^N with $N \geq 4$ and assume $a(x) > 0$ somewhere in Ω. Does Problem (II) have (at least) one solution? infinitely many solutions? What happens when $a(x) \leq 0$ everywhere on Ω? How about $N = 3$?

5. THE EFFECT OF TOPOLOGY: THE WORK OF BAHRI-CORON.

A remarkable result of Bahri-Coron [11] shows that the topology of Ω plays an important role which may cancel the "Pohozaev obstruction":

Theorem 6: Assume Ω is a domain in R^N, $N \geq 3$, with nontrivial topology and assume $a(x) \equiv 0$. Then, there exists a solution of Problem (I).

The precise meaning of the assumption "Ω has nontrivial topology" is expressed in terms of homology groups: there exists an integer $k \geq 1$ such that either $H_{2k-1}(\Omega;\mathbb{Q}) \neq 0$ or $H_k(\Omega;\mathbb{Z}/2\mathbb{Z}) \neq 0$. When $N = 3$, Ω has nontrivial topology iff Ω is not contractible. When $N \geq 4$, the assumption "Ω has nontrivial topology" covers a large variety of domains. For example, a domain which has the topology of a solid torus or any domain with a hole is OK. (However, when $N \geq 4$ it is not known whether the conclusion of Theorem 6 holds under the only assumption that Ω is not contractible.)

The result of Bahri-Coron prompts many questions:

Question 6: Assume Ω has nontrivial topology and $a(x)$ is any function such that (1) holds. Is there always a solution of Problem (I)? Likewise, if Ω is replaced by a manifold M of dimension N, without boundary.

Question 7 (Bott): Does the topology of Ω affect the number of solutions of Problem (I)? For example, if Ω has two holes, are there at least two solutions of Problem (I)?

Question 8: Assume Ω has nontrivial topology and $a(x)$ is any function (without assuming (1)). Is there at least one solution of Problem (II)? Are there infinitely many solutions of Problem (II)? Likewise, if Ω is replaced by a manifold M without boundary.

The proof of Theorem 6 is rather difficult and it involves many new ideas. I will sketch briefly the basic structure of the argument. First, recall that the minimization approach used in Sections 2 and 3 definitely fails. Indeed, if we consider

$$\inf_{\substack{\varphi \in H_0^1 \\ \|\varphi\|_{p+1}=1}} \{\int |\nabla\varphi|^2\} ,$$

this is precisely the best Sobolev constant S and it is never achieved in any domain. Therefore, if Problem (I) has a solution it corresponds, in the variational formulation, to a critical point which is not a minimum. That is why it is tempting to use Morse theory. Unfortunately, the classical Morse theory involves a compactness condition - namely the (PS) condition which fails in our problem. What Bahri and Coron have done is to analyze very precisely how the (PS) condition fails and they have overcome this lack of compactness by a kind of "compactification at infinity".
The abstract setting is the following:
Let H be a Hilbert space. Let $F : H \to \mathbf{R}$ be a function of class C^2. Given $a \in \mathbf{R}$ we set, as usual

$$F_a = \{u \in H;\ F(u) < a\} .$$

A critical point of F is an element $u \in H$ such that $F'(u) = 0$. A critical value c is a real number such that $c = F(u)$ for some critical point u. The standard (PS) condition says that:

$$\left.\begin{array}{l}\text{every sequence } (u_n) \text{ in } H \text{ such that } |F(u_n)| \\ \text{is bounded and } \|F'(u_n)\| \to 0 \text{ is relatively} \\ \text{compact in } H .\end{array}\right\} \quad (PS)$$

For our purpose, it is convenient to use a weaker, "localized", version of the (PS) condition which has been introduced in [16]. Given $c \in \mathbf{R}$ we say that $(PS)_c$ holds if

$$\left.\begin{array}{l}\text{every sequence } (u_n) \text{ in } H \text{ such that} \\ F(u_n) \to c \text{ and } \|F'(u_n)\| \to 0 \text{ is} \\ \text{relatively compact in } H\,.\end{array}\right\} \qquad (PS)_c$$

The (PS) condition prevents critical points from "leaking at infinity". More precisely, if the $(PS)_c$ conditions fails at some level c, we may say that c is a critical value which corresponds to a "critical point at infinity" - a concept introduced by A. Bahri in [9].

A basic principle of Morse theory - based on a standard deformation argument along the gradient flow - asserts that $F_a \simeq F_b$ (homotopy equivalence) provided:

i) F has no critical value in the interval $[a,b]$,

ii) F satisfies $(PS)_c$ for every $c \in [a,b]$.

For the study of Problem (I), we take $H = H_0^1(\Omega)$ and

$$F(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u^+|^{p+1}$$

where $u^+ = \max(u,0)$. Clearly $u = 0$ is a critical point of F and the nonzero critical points of F correspond precisely to the solutions of Problem (I). The proof of Theorem 6 is by contradiction and henceforth we shall assume that

$$u = 0 \text{ is the only critical point of } F\,. \qquad (11)$$

It is rather easy to see that the $(PS)_c$ condition fails at the levels $c = k\Sigma$ where $k = 1,2,3,\ldots$ and $\Sigma = (1/N)S^{N/2}$. This is done by constructing explicitly a sequence (u_n) such that $u_n \rightharpoonup 0$ weakly in H_0^1 and $F(u_n) \to k\Sigma$ (see e.g. [14]). It is a striking fact - which is much more difficult to prove - that the $(PS)_c$ condition fails only at the levels $k\Sigma$; the argument relies on a blowup technique originally due to J. Sacks-Uhlenbeck [36] and subsequently developed by many authors (e.g. P. L. Lions [30], M. Struwe [39], H. Brezis-J. M. Coron [15]); for details, see [11] and [14].

In view of assumption (11) and the basic principle of Morse theory we see that changes in the topology of the sets F_c may occur only across the levels $c = k\Sigma$, $k = 0,1,2,\ldots$. Bahri and Coron have obtained an explicit

representation of the jump in topology at each of the levels $k\Sigma$. For example, the pair $(F_{\Sigma+\varepsilon}, F_{\Sigma-\varepsilon})$ has the same topology (homotopy equivalence) as the pair $(\Omega \times B^1, \Omega \times S^0)$ where $B^1 = \{x \in R;\ |x| < 1\}$ and $S^0 = \partial B^1 = \{-1,+1\}$. This is achieved by a deformation method, i.e. pushing down with the gradient flow. But it requires a delicate Morse analysis of the Hessian matrix D^2F near the critical points infinity - which correspond to sequences (u_n) such that $F(u_n) \to \Sigma$ and $F'(u_n) \to 0$.

At the level $k\Sigma$, $k > 2$, there is still an <u>explicit</u> formula describing the topology of the pair $(F_{k\Sigma+\varepsilon}, F_{k\Sigma-\varepsilon})$ in terms of Ω and the Green's function $G(x,y)$ of Δ (with zero Dirichlet condition). However, the formula is complicated and I shall not write it down; see e.g. [11] and [14].

Using that formula, Bahri and Coron are able to see that:

1) For any domain Ω, there is an integer k_0 (depending on Ω) such that:

   $(F_{k\Sigma+\varepsilon}, F_{k\Sigma-\varepsilon})$ is trivial for $k > k_0$;

   in other words $F_{k\Sigma+\varepsilon} \simeq F_{k\Sigma-\varepsilon}$ for $k > k_0$, i.e. there is no change in topology in F_c for $c > k_0\Sigma$.

2) Assuming that Ω has nontrivial topology, then for <u>each</u> $k > 1$ the pair $(F_{k\Sigma+\varepsilon}, F_{k\Sigma-\varepsilon})$ is nontrivial.

This last claim is by induction on k (it is obvious when $k = 1$) and relies heavily on tools from algebraic topology. Therefore, we are led to a contradiction and, thus assumption (11) is absurd.

6. VARIOUS RELATED PROBLEMS

6.1. The equation $-\Delta u = u^p + \mu u^q$.

Let Ω be a (smooth) bounded domain in R^N with $N > 3$. Consider the problem of finding a function $u(x)$ which satisfies

$$\left.\begin{array}{lll} -\Delta u = u^p + \mu u^q & \text{on} & \Omega \\ u > 0 & \text{on} & \Omega \\ u = 0 & \text{on} & \partial\Omega \end{array}\right\} \qquad (12)$$

where $p = (N + 2)/(N - 2)$, $1 < q < p$, and μ is a constant.

Theorem 7 (Brezis-Nirenberg): Let Ω be any domain in R^N with $N > 4$. Then, for every $\mu > 0$ there exists a solution of (12).

For the proof we refer to [17]. Again, the argument consists of finding a nonzero critical point of the functional

$$F(u) = \frac{1}{2} \int |\nabla u|^2 - \frac{1}{p+1} \int (u^+)^{p+1} - \frac{\mu}{q+1} \int (u^+)^{q+1} .$$

The mountain pass theorem of Ambrosetti-Rabinowitz [1] applies, except for the (PS) condition and therefore an additional argument is needed. Roughly speaking, we are saved by the fact that F satisfies the $(PS)_c$ condition provided $c < \Sigma$ (the same Σ as in Section 5).

If Ω is starshaped, Pohozaev's identity shows that there is no solution of (12) when $\mu < 0$. However, the results of Section 5 suggest the following:

Question 9. Assume $\Omega \subset R^N$, with $N > 3$, has nontrivial topology and $\mu \in R$. Is there a solution of Problem (12)? Likewise on a manifold M without boundary.

When $N = 3$, so that $p = 5$, the situation is rather peculiar. The following results have been established:

Theorem 8 (Brezis-Nirenberg): Assume Ω is any domain in R^3 and $3 < q < 5$. Then, for every $\mu > 0$ there exists a solution of (12). However, if $1 < q < 3$ one has to assume that μ is large enough in order to have a solution; if $1 < q < 3$ and Ω is starshaped no solution exists for $\mu > 0$ small.

For the proof, see [17]. There is a more precise result in the case of a ball:

Theorem 9 (Atkinson-Peletier): Assume Ω is a ball in R^3 and $1 < q < 3$. Then, there exists a constant $\mu_0 > 0$ such that:

(i) if $\mu > \mu_0$, Problem (12) has at least two solutions,
(ii) if $\mu < \mu_0$, Problem (12) has no solution.

For the proof see [4]. Naturally, we may ask the following:

Question 10: Assume Ω is any domain in $\mathbf{R}^3$ and $1 < q < 3$. Does Problem (12) have at least two solutions for μ large enough?

Of course, we may also replace (12) by

$$\left.\begin{array}{lll} -\Delta u = |u|^{p-1}u + \mu|u|^{q-1}u & \text{on} & \Omega \\ u \not\equiv 0 & \text{on} & \Omega \\ u = 0 & \text{on} & \partial\Omega \end{array}\right\} \qquad (13)$$

and ask:

Question 11: Are there infinitely many solutions of Problem (13) under the natural restrictions? By the natural restrictions we mean one of the following cases:

a) $N \geq 4$, Ω is any domain and $\mu > 0$,

b) $N = 3$, Ω is any domain and $3 < q < 5$ or $1 < q < 3$ and μ is large enough,

c) $N \geq 3$, Ω is a domain with topology and no condition on $\mu \in \mathbf{R}$.

Along these lines, I should also mention a work by F. Atkinson-L. Peletier-J. Serrin (in preparation) dealing with the problem:

$$\left.\begin{array}{lll} -\mathrm{div}(|\nabla u|^{m-2}\nabla u) = u^p + \mu u^q & \text{on} & \Omega\,, \\ u > 0 & \text{on} & \Omega\,, \\ u = 0 & \text{on} & \partial\Omega\,, \end{array}\right\} \qquad (14)$$

in the critical exponent case $\left(\frac{1}{p+1} = \frac{1}{m} - \frac{1}{N}\right)$ and $1 < q < p$ (see also [35]).

6.2. The equation $-\Delta u - \frac{1}{2}x \cdot \nabla u = u^p + \lambda u$ on $\mathbf{R}^N$.

Consider the problem of finding a function $u(x)$ which satisfies

$$\left.\begin{array}{lll} -\Delta u - \frac{1}{2}x \cdot \nabla u = u^p + \lambda u & \text{on} & \mathbf{R}^N\,, \\ u > 0 & \text{on} & \mathbf{R}^N\,, \\ \int (|u|^2 + |\nabla u|^2)\exp(|x|^2/4) < \infty\,. & & \end{array}\right\} \qquad (15)$$

Such a problem arises in the study of self-similar solutions of the evolution equation $u_t - \Delta u = u^p$. Again, the critical exponent $p = (N+2)/(N-2)$ plays a special role and one has the following:

Theorem 10 (Escobedo-Kavian, Atkinson-Peletier): Assume $p = (N + 2)/(N - 2)$. Then

(i) if $N \geq 4$ and $\lambda \in (\frac{N}{4}, \frac{N}{2})$ there is at least one solution of (14); no solution when $\lambda \notin (\frac{N}{4}, \frac{N}{2})$,

(ii) if $N = 3$ and $\lambda \in (1, \frac{3}{2})$ there is at least one solution of (14); no solution when $\lambda \notin (1, \frac{3}{2})$.

For the proofs, see [26] and [5]. There is a striking similarity with the results of Section 2 (and 3). For example, when $N \geq 4$, the restriction $\lambda > N/4$ corresponds to a "Pohozaev type obstruction" while the restriction $\lambda < N/2$ corresponds to the assumption that the linear operator $L = -\Delta - \frac{1}{2} x \cdot \nabla - \lambda$ is coercive on an appropriate weighted Sobolev space.

6.3. The equation $-\Delta u = K(x)u^p + a(x)u$.

Consider the problem of finding a function $u(x)$ which satisfies

$$\left.\begin{array}{lll} Lu \equiv -\Delta u - a(x)u = K(x)u^p & \text{on} & \Omega \\ u > 0 & \text{on} & \Omega \\ u = 0 & \text{on} & \partial\Omega \end{array}\right\} \qquad (16)$$

where $p = (N + 2)/(N - 2)$, $a(x)$ and $K(x)$ are given (smooth) functions. Likewise, one may consider the same problem on a manifold M without boundary - which is of interest in differential geometry see e.g. [28].

Little is known about the solvability (or nonsolvability) of (15). Obstructions to existence (related to a Pohozaev-type identity) have been found by Kazdan-Warner [29] and also Bourguignon-Ezin [12]. Some positive results have been obtained by Ni [33] (when $\Omega = R^N$), Escobar-Schoen [25] and Escobar [24]. The most interesting positive result is the following:

Theorem 11 (Bahri-Coron): Assume $M = S^3$ and L is the conformal Laplacian on S^3. Let $K(x)$ be a positive function of class C^2 with finitely many nondegenerate critical points $y_1, y_2, \ldots, y_n$, such that $(LK)(y_i) \neq 0$. Let k_i be the Morse index of $K(x)$ at y_i and assume that

$$\sum_{(LK)(y_i)<0} (-1)^{k_i} \neq -1 .$$

Then, there exists a solution of (15) on S^3.

The proof (see [10]) involves, as in Section 5, a careful Morse analysis of the critical points at infinity together with an Euler-Poincaré characteristic argument. In some sense, it suggests that a combination of topological properties of Ω (or M) and topological properties of the function $K(x)$ may force existence. Much work remains to be done in that direction.

6.4. Neumann boundary condition.

Little is known if one replaces in Problem (I) the Dirichlet boundary condition by a Neumann condition, for example,

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega$$

or by a nonlinear boundary condition, such as

$$-\frac{\partial u}{\partial n} = u^q \quad \text{on} \quad \partial\Omega$$

where $q = (N+1)/(N-1)$ is the critical Sobolev exponent on the boundary; see however Cherrier [22].

6.5. The case $N = 2$.

When $N = 2$ the critical Sobolev exponent should be replaced by the Trudinger-Moser embedding theorem (see [41] and [32]). Consider, for example the problem

$$\left.\begin{array}{lll} -\Delta u = f(u) & \text{on} & \Omega \subset \mathbf{R}^2 \\ u > 0 & \text{on} & \Omega \\ u = 0 & \text{on} & \partial\Omega \end{array}\right\} \tag{17}$$

where f is a positive function on $(0,\infty)$ such that $f(0) = 0$, $0 < f'(0) < \lambda_1$ and $f(u)$ behaves, as $u \to \infty$, like $e^{\alpha u^2}$ (or $ue^{\alpha u^2}$) $(\alpha > 0)$.

Unexpected existence results have been obtained by Carleson-Chang [19] and Atkinson-Peletier [6]. It is not clear, at the moment, where is the border line between existence and nonexistence. It would be interesting to answer the following:

Question 12: Can one find some function f, as above, for which Problem (16) has no solution, say when Ω is a ball?

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