Elliptic Equations with Limiting Sobolev Exponents—The Impact of Topology

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Introduction

In recent years there has been much interest in nonlinear elliptic equations of the form

$$(1)\qquad \begin{aligned} -\Delta u &= u^p + a(x)u && \text{on } \Omega,\\ u &> 0 && \text{on } \Omega,\\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geqq 3$, $a(x)$ is a given (smooth) function and $p = (N+2)/(N-2)$ is the critical exponent for the Sobolev imbedding. Alternatively, one may also consider the same question on a Riemannian manifold M of dimension N, without boundary, that is,

$$(2)\qquad \begin{aligned} -\Delta u &= u^p + a(x)u && \text{on } M,\\ u &> 0 && \text{on } M, \end{aligned}$$

where $a(x)$ is a given function on M.

The original interest in such questions grew out of Yamabe's problem (see [40], [39], [2], [27], [15]) which corresponds to the special case where $a(x) = -((N-2)/4(N-1))R(x)$ and $R(x)$ is the scalar curvature of M. It turns out that, despite its simple form, equation (1) (or (2)) has a very rich structure and provides an amazing source of open problems and new ideas. The main reason is that (1) (or (2)) can be expressed as a variational problem in the Sobolev space $H_0^1(\Omega)$ (or $H^1(M)$); however *it lacks compactness*—in other words, the Palais-Smale condition (PS) fails—because the exponent $p = (N+2)/(N-2)$ is critical and the Sobolev imbedding $H^1 \subset L^{2N/(N-2)}$ is *not* compact.

The first contribution to problem (1) is a negative result due to Pohozaev. Consider the special case of (1) where $a(x) \equiv 0$, i.e.,

$$(3)\qquad \begin{aligned} -\Delta u &= u^p && \text{on } \Omega,\\ u &> 0 && \text{on } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

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THEOREM 1 (Pohozaev [24]). *Assume Ω is star-shaped, then* (3) *has* <u>*no*</u> *solution.*

In view of Pohozaev's theorem we are left with two choices in order to regain the existence of a solution:

(i) exploit the lower-order term $a(x)u$,
(ii) exploit the topology of the domain Ω (or the manifold M).

This program is supported by the following simple observations:

1. Assume Ω is any domain and let λ_1 denote the first eigenvalue of $-\Delta$ with zero Dirichlet condition. If $a(x) \equiv \lambda$ is a constant with $\lambda < \lambda_1$ and $|\lambda - \lambda_1|$ small enough, then (1) has a solution; this follows from general bifurcation theory (see e.g. [25]).
2. Assume Ω is an annulus, i.e.,

$$\Omega = \left\{ x \in \mathbb{R}^N ;\, r_1 < |x| < r_2 \right\}.$$

If $a(x) \equiv 0$, it is easy to see—as pointed out by Kazdan-Warner [16]—that (3) has a (radial) solution. This leads to a natural question, which I heard originally from Louis Nirenberg: what happens to (3) if Ω is a "perturbed" annulus—is there still a solution?

Accordingly, I shall divide my lecture into two parts:

1. The effect of $a(x)u$.
2. The impact of topology.

In Section 1, no special assumption is made about Ω, but restrictive assumptions are imposed on $a(x)$. The existence of a solution is established by showing that some functional achieves its minimum. The most recent development in that direction is the complete solution of Yanabe's problem by R. Schoen [27] (following earlier contributions of N. Trudinger [39] and Th. Aubin [2]). I shall discuss a related phenomenon for domains Ω.

In Section 2, I shall present a remarkable recent result of A. Bahri and J. M. Coron.

THEOREM 2 (Bahri-Coron [4]). *Assume Ω has nontrivial topology. Then, a solution of* (3) *exists.*

The meaning of the assumption "Ω has nontrivial topology" will be explained in Section 2. For example, if $N = 3$ it corresponds precisely to the assumption that Ω is not contractible to a point. Here, *one cannot obtain a solution by* *minimization*. Instead, Bahri and Coron use a kind of Morse analysis to prove the existence of critical points at "high" levels. Standard Morse theory does not apply because the (PS) condition fails and some very interesting new ideas have been introduced by Bahri and Coron to overcome such a difficulty. It is likely that this breakthrough will lead to similar results for other variational problems with lack of compactness; see e.g. [7] for a detailed list of such problems—some of them motivated by geometry or physics. The works of J. Sacks–K. Uhlenbeck [26], C. Taubes [35], [36], [37], [38] and S. Donaldson [12] provide other examples of beautiful interplays between analysis and topology in a related spirit.

In conclusion, let me mention two natural questions which are still widely open.

QUESTION 1. Can one replace in Pohozaev's Theorem the assumption "Ω is star-shaped" by "Ω has nontrivial topology"? In other words, are there domains Ω with trivial topology on which (3) has a solution?

QUESTION 2 (P. Rabinowitz). What happens when $p > (N+2)/(N-2)$?[1] Pohozaev's Theorem still holds. On the other hand, if Ω is an annulus, it is easy to see that (3) has a (radial) solution for all $1 < p < \infty$. Assuming Ω is a domain with nontrivial topology, is there still a solution of (3) for all p?

1. The Effect of $a(x)u$

First we observe that it is essential, in order to obtain a solution of (1), to assume that the linear operator $L = -\Delta - a$ is positive, i.e.,

$$\int|\nabla\phi|^2 - a\phi^2 \geqq \delta\int\phi^2 \quad \text{for all} \quad \phi \in H_0^1, \qquad \delta > 0. \tag{4}$$

Indeed let

$$\mu_1 = \min_{\phi\in H_0^1}\left\{\frac{\int|\nabla\phi|^2 - a\phi^2}{\|\phi\|_2^2}\right\}$$

denote the first eigenvalue of L and let $\phi_1 > 0$ be the corresponding eigenfunction, so that

$$-\Delta\phi_1 - a\phi_1 = \mu_1\phi_1.$$

Multiplying (1) through by ϕ_1 and integrating by parts we find

$$\mu_1\int u\phi_1 = \int u^p\phi_1$$

[1]When $p < (N+2)/(N-2)$ it is very easy to prove that (3) has a solution on any domain Ω.

so that μ_1 is necessarily positive. In order to solve (1) it is tempting to consider

$$J = \inf_{\phi \in H_0^1} \left\{ \frac{|\nabla \phi|^2 - a\phi^2}{\|\phi\|_{2^*}^2} \right\}, \tag{5}$$

where $2^* = 2N/(N-2)$.

Note that $J > 0$ if (4) holds. *Assuming* that J is achieved by some function ϕ, we may always take $\phi \geqq 0$—otherwise we replace ϕ by $|\phi|$. It is easy to see that ϕ satisfies the Euler equation

$$-\Delta\phi - a\phi = J\phi^p \quad \text{on} \quad \Omega,$$

and that $\phi > 0$ in Ω (by the strong maximum principle). Therefore, we obtain a solution of (1) after scaling out the positive constant J.

At this point, let me emphasize two important facts:

(a) Clearly the infimum in (5) need not be achieved. This is so in particular if (1) has no solution whatsoever—for example, if Ω is star-shaped and $a(x) \equiv 0$ (by Pohozaev's Theorem).

(b) It may happen that J is *not* achieved, but nevertheless there exist solutions of (1). Suppose, for example, that Ω is an annulus and $a(x)$ is a radial function with $a \leqq 0$. Then J is *not* achieved (see subsection 1.1). However, there exist radial solutions of (1). In fact such solutions can be obtained by minimizing the functional

$$Q(\phi) = \frac{\int |\nabla\phi|^2 - a\phi^2}{\|\phi\|_{2^*}^2}$$

among radial functions in H_0^1. However these solutions are not absolute minima of Q on all of H_0^1. They correspond to critical points of Q at "high energy" levels.

In the present section, I shall investigate the existence of solutions of (1) which arise as (absolute) minima in (5). In the next section, I shall examine situations where the infimum in (5) need not be achieved, but solutions of (1) still exist.

As pointed out by Th. Aubin [1], [2], the best Sobolev constant S plays an important role; it is defined by

$$S = \inf_{p \in H_0^1} \left\{ \frac{\int |\nabla\phi|^2}{\|\phi\|_{2^*}^2} \right\}. \tag{6}$$

Here are some facts about S:

(7) S is independent of Ω; it depends only on N.

This is an easy consequence of the fact that the ratio $\|\nabla\phi\|_2/\|\phi\|_{2^*}$ is invariant under dilations.

$$S \text{ is not achieved in } \textit{any} \text{ bounded domain } \Omega. \tag{8}$$

If Ω is a ball this follows from Pohozaev's Theorem. If Ω is any domain, let $\tilde{\Omega}$ be a ball containing Ω. Suppose J is achieved in Ω by some function ϕ. Extend ϕ by 0 outside Ω and call it $\tilde{\phi}$. Then $\tilde{\phi}$ would be a minimizer for J in $\tilde{\Omega}$—a contradiction.

When $\Omega = \mathbb{R}^N$, then S is achieved by the function

$$U(x) = \frac{1}{(1+|x|^2)^{(N-2)/2}}. \tag{9}$$

Moreover, all minimizers for S are of the form $CU(k(x - x_0))$ for some constants $C \neq 0$, $k > 0$ and $x_0 \in \mathbb{R}^N$; for all these properties, see [1], [34], [17].

Note that U satisfies the equation

$$-\Delta U = N(N-2)U^p \quad \text{on} \quad \mathbb{R}^N. \tag{10}$$

A stronger uniqueness statement (see Obata [22], Gidas-Ni-Nirenberg [14] or Gidas [13]) asserts that any positive solution of (10) with $U \in L^{2^*}(\mathbb{R}^N)$ and $\nabla U \in L^2(\mathbb{R}^N)$ must be—modulo translations—of the form

$$U_\varepsilon(x) = \frac{1}{\varepsilon^{(N-2)/2}} U\left(\frac{x}{\varepsilon}\right), \qquad \varepsilon > 0. \tag{11}$$

Incidentally, it is an interesting open problem to decide whether the same conclusion holds without positivity:

QUESTION 3. Assume $V \in L^{2^*}(\mathbb{R}^N)$ with $\nabla V \in L^2(\mathbb{R}^N)$ satisfies the equation

$$-\Delta V = |V|^{p-1}V \quad \text{on} \quad \mathbb{R}^N.$$

Does V have constant sign?

We return now to the question whether J defined by (5) is achieved. The analysis below will show that one has *always*

$$J \leq S. \tag{12}$$

A very useful tool in order to prove that J is achieved is the following:

LEMMA 1. *Assume*

$$J < S; \tag{13}$$

then J is achieved.

Proof: Let (u_j) be a minimizing sequence for J, that is,

$$\int |\nabla u_j|^2 - au_j^2 = J + o(1), \tag{14}$$

$$\int |u_j|^{2^*} = \int |u_j|^{p+1} = 1. \tag{15}$$

Since (u_j) is bounded in $H_0^1(\Omega)$, we may assume that $u_j \rightharpoonup u$ weakly in H_0^1. Next, we use a device introduced by E. Lieb in [17]; we write

$$u_j = u + v_j$$

with $v_j \rightharpoonup 0$ weakly in H_0^1, $v_j \to 0$ strongly in L^2 and a.e. The relation (14) becomes

$$\int |\nabla u|^2 + \int |\nabla v_j|^2 - \int au^2 = J + o(1). \tag{16}$$

On the other hand, we deduce from a result of [10] that

$$\int |u + v_j|^{p+1} = \int |u|^{p+1} + \int |v_j|^{p+1} + o(1), \tag{17}$$

so that (15) becomes

$$1 = \int |u|^{p+1} + \int |v_j|^{p+1} + o(1)$$

and therefore we obtain (by convexity)

$$1 \leqq \|u\|_{p+1}^2 + \|v_j\|_{p+1}^2 + o(1). \tag{18}$$

Combining (16) and (18) we find

$$\int |\nabla u|^2 - \int au^2 + \int |\nabla v_j|^2 \leqq J\|u\|_{p+1}^2 + J\|v_j\|_{p+1}^2 + o(1).$$

Since (by definition of J) we have

$$\int |\nabla u|^2 - au^2 \geqq J\|u\|_{p+1}^2,$$

we conclude that

$$\int |\nabla v_j|^2 \leqq K\|v_j\|_{p+1}^2 + o(1).$$

Sobolev's inequality says that

$$\|v_j\|_{p+1}^2 \leqq \frac{1}{S}\int|\nabla v_j|^2$$

and thus we are led to

$$\left(1-\frac{J}{S}\right)\int|\nabla v_j|^2 = o(1),$$

i.e., $v_j \to 0$ strongly in H_0^1. It follows that $u_j \to u$ strongly in H_0^1 and consequently u is a minimizer for J.

Remark 1. The strong convergence was originally pointed out by F. Browder at an earlier stage when the argument was less transparent. It shows that below the level S some form of compactness holds. We shall encounter a more sophisticated version of the same phenomenon in Section 2.

In view of Lemma 1, we now look for "concrete" assumptions which guarantee that (13) holds. For some strange reason it turns out that the cases $N = 3$ and $N \geq 4$ are quite different and they will be examined separately.

1.1. The case $N \geqq 4$. The main result is the following

THEOREM 3 (H. Brezis–L. Nirenberg [11]). *Assume Ω is any bounded domain in $\mathbb{R}^N$ with $N \geqq 4$. Then, the following properties are equivalent*:

$$a(x) > 0 \quad \textit{somewhere on} \quad \Omega, \tag{19}$$

$$J < S, \tag{20}$$

$$J \quad \textit{is achieved}. \tag{21}$$

Sketch of the proof: (19) $\Rightarrow$ (20). One constructs explicitly a function ϕ such that $Q(\phi) < S$. Without loss of generality we may assume that $0 \in \Omega$ and that $a(0) > 0$. Fix any cut-off function $\zeta \in C_0^\infty(\Omega)$ such that $\zeta(x) \equiv 1$ near $x = 0$ and set

$$\phi_\varepsilon(x) = \zeta(x)U_\varepsilon(x), \qquad \varepsilon > 0,$$

where $U_\varepsilon(x)$ is defined by (11). A careful expansion as $\varepsilon \to 0$ (see [11]) leads to

$$Q(\phi_\varepsilon) = \begin{cases} S - a(0)C\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if} \quad N \geq 5, \\ S - a(0)C\varepsilon^2|\log\varepsilon| + O(\varepsilon^2) & \text{if} \quad N = 4, \end{cases}$$

where C depends only on N. The conclusion follows by choosing $\varepsilon > 0$ small enough.

(20) $\Rightarrow$ (21). See Lemma 1.

(21) $\Rightarrow$ (19). Suppose by contradiction that $a(x) \leqq 0$ everywhere on Ω. Let ϕ be a function such that $\|\phi\|_{2^*} = 1$ and $\int |\nabla\phi|^2 - a\phi^2 = J$. Therefore $\int |\nabla\phi|^2 \leqq J$. On the other hand, the expansion above—without any assumption on $a(x)$—shows that we always have $J \leqq S$. Thus ϕ would be a minimizer for the Sobolev inequality—a contradiction.

Consider now the special case where $a(x) \equiv \lambda$ is a constant, that is,

$$
(22) \qquad \begin{aligned}
-\Delta u &= u^p + \lambda u && \text{on} \quad \Omega, \\
u &> 0 && \text{on} \quad \Omega, \\
u &= 0 && \text{on} \quad \partial\Omega.
\end{aligned}
$$

Then the following assertions hold.

COROLLARY 4. *Assume Ω is any bounded domain in $\mathbb{R}^N$ with $N \geqq 4$ and assume*

$$
(23) \qquad 0 < \lambda < \lambda_1 .
$$

Then, there exists a solution of (22). *There is no solution of* (22) *for $\lambda \geqq \lambda_1$. Also, there is no solution of* (22) *for $\lambda \leqq 0$ if Ω is star-shaped.*

Proof: The first assertion follows from Theorem 3. The last assertion is a consequence of Pohozaev's identity, namely, any solution of (22) satisfies

$$
(24) \qquad \lambda \int_\Omega u^2 = \tfrac{1}{2} \int_{\partial\Omega} (x \cdot n) \left(\frac{\partial u}{\partial n} \right)^2 ,
$$

where n denotes the outward normal to Ω.

Let us return briefly to the general case $a(x)$ and let me emphasize that if $a(x) \leqq 0$ everywhere on Ω, problem (1) may still have solutions. But, in view of Theorem 3, *these solutions cannot be obtained by minimization* (and it is therefore tempting to use Morse theory as in Section 2). In fact, it is easy to construct such an example. Fix any function $f \in C_0^\infty(\Omega)$ with $f \geqq 0$, $f \not\equiv 0$, and let v be the solution of the problem

$$
\begin{aligned}
-\Delta v &= f \quad \text{on} \quad \Omega, \\
v &= 0 \quad \text{on} \quad \partial\Omega,
\end{aligned}
$$

so that $v > 0$ on Ω. Set

$$
a = \frac{f}{v} - \mu^{p-1} v^{p-1}.
$$

Note that $a(x)$ is a smooth function and $a(x) \leqq 0$ on Ω provided μ is a large constant. It is clear that $u = \mu v$ satisfies (1). This leads us to the following:

QUESTION 4. Suppose (for simplicity) that Ω is a ball and that $a(x) \leqq 0$ on Ω. Find conditions on $a(x)$ (hopefully a necessary and sufficient condition!) which guarantee that (1) has a solution.

Note that Pohozaev's identity applied to a solution u of (1) says that

$$\int_\Omega \left(a + \tfrac{1}{2} \sum x_i \frac{\partial a}{\partial x_i} \right) u^2 = \tfrac{1}{2} \int_{\partial\Omega} (x \cdot n) \left(\frac{\partial u}{\partial n} \right)^2. \tag{25}$$

An obvious necessary condition for the existence of a solution is that $(a + \frac{1}{2}\Sigma x_i\, \partial a / \partial x_i)$ should be positive somewhere on Ω.

1.2. The case $N = 3$. The situation is much more complicated; there are striking differences to the case $N \geqq 4$. Many problems are still open; here is a first natural question:

QUESTION 5. We know, by Lemma 1, that

$$(J < S) \Rightarrow (J \text{ is achieved}).$$

Is the converse true (as in dimension $N \geqq 4$)?

I shall start with some results concerning the special case where $a(x) \equiv \lambda$ is a constant, so that problem (1) takes the form

$$\begin{aligned} -\Delta u &= u^5 + \lambda u \quad && \text{on} \quad \Omega, \\ u &> 0 && \text{on} \quad \Omega, \\ u &= 0 && \text{on} \quad \partial\Omega. \end{aligned} \tag{26}$$

Set

$$J_\lambda = \inf_{\phi \in H_0^1} \left\{ \frac{|\nabla\phi|^2 - \lambda\phi^2}{\|\phi\|_6^2} \right\}. \tag{27}$$

THEOREM 5 (H. Brezis–L. Nirenberg [11]). *Assume Ω is any bounded domain in $\mathbb{R}^3$. Then, there is a constant $\lambda^* = \lambda^*(\Omega) \in (0, \lambda_1)$ such that*

$$J_\lambda = S \quad \textit{for} \quad \lambda \leqq \lambda^*, \tag{28}$$

$$J_\lambda < S \quad \textit{for} \quad \lambda > \lambda^*. \tag{29}$$

In addition, J_λ is not achieved for $\lambda < \lambda^*$.

It is an open problem whether J_{λ^*} is achieved (this is a special case of Question 5). Another open problem is:

QUESTION 6. Suppose Ω is star-shaped (or convex). Can problem (26) have a solution for some $\lambda \leqq \lambda^*$?

It is only in the case where Ω is a *ball* that we have a complete answer to all these questions:

THEOREM 6 (H. Brezis–L. Nirenberg [11]). *Assume Ω is a ball in $\mathbb{R}^3$. Then $\lambda^* = \frac{1}{4}\lambda_1$. More precisely, we have*

$$J_\lambda < S \quad \textit{for} \quad \lambda > \tfrac{1}{4}\lambda_1, \tag{30}$$

and

$$\textit{there is no solution of problem } (26) \textit{ for } \lambda \leqq \tfrac{1}{4}\lambda_1. \tag{31}$$

We return now to the case of a general domain Ω and a function $a(x)$. Recently, B. McLeod [19] and R. Schoen [27] have made an interesting contribution. Independently, they have displayed the important role of the regular part of the Green's function of the operator $L = -\Delta - a$. B. McLeod was motivated by Theorem 6, while R. Schoen was working on Yamabe's problem—but his main idea holds in our context as well. In what follows we assume that (4) holds and we consider the Green's function $G(x, y)$ for L, so that, for each fixed $y \in \Omega$, $G(\cdot, y)$ is the solution of

$$\begin{aligned} -\Delta G - aG &= \delta(\cdot - y) \quad \text{on} \quad \Omega, \\ G &= 0 \qquad\qquad\quad \text{on} \quad \partial\Omega. \end{aligned} \tag{32}$$

We write

$$G(x, y) = \frac{1}{4\pi|x-y|} + g(x, y);$$

$g(x, y)$ is the *regular part* of the Green's function and it is easy to see that g is continuous on $\Omega \times \Omega$ including the diagonal (x, x).

THEOREM 7 (B. McLeod [19], R. Schoen [27]). *Assume Ω is any bounded domain in $\mathbb{R}^3$ and*

$$g(x, x) > 0 \quad \textit{somewhere on} \quad \Omega. \tag{33}$$

Then

$$J < S, \tag{34}$$

and consequently there is a solution of (1).

We do not know whether the converse holds:

QUESTION 7. Does (34) imply (33)?

It is perhaps worth it to try Question 7 in the special case where $a(x) \equiv \lambda$ is a constant: Let G_λ and g_λ be the Green's function and its regular part for $L = -\Delta - \lambda$. It follows from the maximum principle that g_λ increases with λ and hence there is a constant $\lambda^{**} = \lambda^{**}(\Omega)$ such that

$$\max_{x\in\Omega} g_\lambda(x,x) > 0 \quad \text{for} \quad \lambda > \lambda^{**},$$

while

$$\max_{x\in\Omega} g_\lambda(x,x) \leqq 0 \quad \text{for} \quad \lambda \leqq \lambda^{**}.$$

Theorem 7 says that $\lambda^* \leqq \lambda^{**}$. Is it true that $\lambda^{**} = \lambda^*$? (Yes, if Ω is a ball, see Remark 2).

Remark 2. In practice it is not easy to decide whether (33) holds. However, there are two special cases of interest:

(i) If Ω is a ball and $a(x) \equiv \lambda$, then (33) holds if and only if $\lambda > \frac{1}{4}\lambda_1$. Indeed, for the unit ball and $\lambda > 0$, we have

$$G_\lambda(x,0) = \frac{1}{4\pi|x|}\left[\cos\sqrt{\lambda}\,|x| - \frac{\sin\sqrt{\lambda}\,|x|}{\tan\sqrt{\lambda}}\right],$$

so that

$$\max_{x\in\Omega} g_\lambda(x,x) = g_\lambda(0,0) = -\sqrt{\lambda}/4\pi\tan\sqrt{\lambda}.$$

In particular we recover assertion (30) of Theorem 6 (but not assertion (31)!)

(ii) In the Yamabe problem (see the Introduction) assumption (33) follows from the positive mass theorem (see [28] and [29]).

Proof of Theorem 7: Without loss of generality we may assume that $0 \in \Omega$ and that $g(0,0) > 0$. I shall describe a slight modification of a construction due to Bahri and Coron [5]. Namely, we shall use as testing function ϕ_ε the solution of the problem

$$(35) \qquad \begin{aligned} -\Delta\phi_\varepsilon - a\phi_\varepsilon &= -\Delta U_\varepsilon \quad \text{on} \quad \Omega, \\ \phi_\varepsilon &= 0 \qquad\quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $U_\varepsilon(x)$ is defined by (11), that is,

$$U_\varepsilon(x) = \frac{\sqrt{\varepsilon}}{\left(\varepsilon^2 + |x|^2\right)^{1/2}}.$$

I claim that, as $\varepsilon > 0$,

$$Q(\phi_\varepsilon) = S - Cg(0,0)\varepsilon + o(\varepsilon), \tag{36}$$

where C is some positive (universal) constant. We shall first check that

$$\int |\nabla \phi_\varepsilon|^2 - a\phi_\varepsilon^2 = 3\kappa\left(1 + 4\pi g(0,0)\frac{\kappa'}{\kappa}\varepsilon\right) + o(\varepsilon) \tag{37}$$

and then

$$\|\phi_\varepsilon\|_6^2 = \kappa^{1/3}\left(1 + 8\pi g(0,0)\frac{\kappa'}{\kappa}\varepsilon\right) + o(\varepsilon), \tag{38}$$

where

$$\kappa = \int_{\mathbf{R}^3} \frac{dx}{(1+|x|^2)^3} \quad \text{and} \quad \kappa' = \int_{\mathbf{R}^3} \frac{dx}{(1+|x|^2)^{5/2}}.$$

These two relations yield (36) with $C = 4\pi\kappa'/\kappa$. (Note that $S = \int |\nabla U_1|^2/\|U_1\|_6^2 = -(\int \Delta U_1 \cdot U_1)/\|U_1\|_6^2 = 3\|U_1\|_6^4 = 3\kappa^{2/3}$ since $-\Delta U_1 = 3U_1^5$.) Set

$$h_\varepsilon = (\phi_\varepsilon - U_\varepsilon)/\sqrt{\varepsilon},$$

so that, by (35), we have

$$(39) \qquad \begin{aligned} -\Delta h_\varepsilon - a h_\varepsilon &= \frac{a(x)}{(\varepsilon^2 + |x|^2)^{1/2}} \quad \text{on} \quad \Omega, \\ h_\varepsilon &= -\frac{1}{(\varepsilon^2 + |x|^2)^{1/2}} \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Since $a(x)/(\varepsilon^2 + |x|^2)^{1/2}$ remains bounded in $L^2(\Omega)$, we deduce from standard elliptic estimates that $h_\varepsilon \to h_0$ uniformly on $\overline{\Omega}$, where h_0 is the solution of

$$\begin{aligned} -\Delta h_0 - a h_0 &= \frac{a(x)}{|x|} \quad \text{on} \quad \Omega, \\ h_0 &= -\frac{1}{|x|} \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Thus, $h_0(x) = 4\pi g(x,0)$. On the other hand, it is clear that

$$\int_\Omega U_\varepsilon^6 = \kappa + O(\varepsilon^3) \tag{40}$$

and

$$\frac{1}{\sqrt{\varepsilon}} U_\varepsilon^5 \rightharpoonup \kappa' \delta \text{ weakly in the sense of measures.} \tag{41}$$

It is now easy to check (37) and (38). Indeed we have

$$\int |\nabla \phi_\varepsilon|^2 - a\phi_\varepsilon^2 = \int (-\Delta\phi_\varepsilon - a\phi_\varepsilon)\phi_\varepsilon = \int -(\Delta U_\varepsilon)\phi_\varepsilon$$

$$= 3\int U_\varepsilon^5 \big(U_\varepsilon + \sqrt{\varepsilon}\, h_\varepsilon\big) = 3\kappa + 3\kappa' 4\pi g(0,0)\varepsilon + o(\varepsilon),$$

and

$$\int |\phi_\varepsilon|^6 = \int \big(U_\varepsilon + \sqrt{\varepsilon}\, h_\varepsilon\big)^6 = \int U_\varepsilon^6 + 6U_\varepsilon^5 \sqrt{\varepsilon}\, h_\varepsilon + o(\varepsilon)$$

$$= \kappa + 6\kappa' 4\pi g(0,0)\varepsilon + o(\varepsilon).$$

2. The Impact of Topology

In this section, I would like to explain some of the new ideas introduced by Bahri and Coron for the proof of Theorem 2. The main assumption concerns the topology of Ω and it is expressed in terms of homology groups.

DEFINITION. We shall say that a smooth bounded domain Ω in $\mathbb{R}^N$ has *nontrivial topology* if there is an integer $k \geqq 1$ such that either $H_{2k-1}(\Omega; \mathbb{Q}) \neq 0$ or $H_k(\Omega; \mathbb{Z}/2\mathbb{Z}) \neq 0$.

The main result is the following theorem already stated in the introduction.

THEOREM 2 (Bahri–Coron [5]). *Assume Ω has nontrivial topology. Then, there exists a solution of* (3).

Remark 3. It is clear that any domain with nontrivial topology is not contractible (to a point). When $N = 3$, the converse is also true, that is, every domain which is not contractible has nontrivial topology (see e.g. [5]). Therefore, if $N = 3$, Theorem 2 may be stated with the assumption that Ω is not contractible. When $N \geqq 4$, the converse fails (i.e., there exist domains which are not contractible and such that $H_{2k-1}(\Omega; \mathbb{Q}) = 0$ and $H_k(\Omega; \mathbb{Z}/2\mathbb{Z}) = 0$ for all $k \geqq 1$). If $N \geqq 4$, it is an open problem whether the conclusion of Theorem 2 holds under the sole assumption that Ω is not contractible. Note that the assumption "Ω has nontrivial topology" covers a large variety of domains. For example, a domain Ω

which is (topologically) equivalent to a solid torus satisfies $H_1(\Omega;\mathbb{Q}) \neq 0$ (for all N), a domain $\Omega \subset \mathbb{R}^N$ with holes satisfies $H_{N-1}(\Omega; \mathbb{Z}/2\mathbb{Z}) \neq 0$ (for all N).

The method used in the proof of Theorem 2 is *quite flexible* and it is likely that it can be applied to problems (1) and (2) as well. In fact, it is very plausible that the following question has a positive answer:

QUESTION 8. Assume Ω has nontrivial topology and (4) holds. Is there always a solution of (1)? Likewise, if M is *any* manifold without boundary and (4) holds, is there always a solution of (2)?

In another direction, it would be interesting to know whether the topology of Ω affects the number of solutions:

QUESTION 9 (Bott). Assume Ω has several holes. Are there several solutions of (3)? How many?

The proof of Theorem 2 relies on a kind of Morse analysis. I shall first recall some elementary principles of Morse theory and then indicate what modifications have to be made for the proof of Theorem 2.

2.1. Some elementary principles of Morse theory (see e.g. [6], [21], [23], [30]). Let H be a Hilbert space (finite- or infinite-dimensional). Let F: $H \to \mathbb{R}$ be a function of class C^2. Given $a \in \mathbb{R}$ we set

$$F_a = \{u \in H;\ F(u) \leqq a\}.$$

A *critical point* is an element $u \in H$ such that $F'(u) = 0$. A *critical value* c is a real number such that $c = F(u)$ for some critical point u. A crucial assumption in Morse theory is the Palais-Smale condition (PS) (sometimes called condition (C)) which says that:

(PS) every sequence (u_n) in H such that $|F(u_n)|$ is bounded and $\|F'(u_n)\| \to 0$ is relatively compact in H.

(Of course the (PS) condition is irrelevant if the theory takes place on a compact manifold instead of the linear space H.)

For our purposes it is convenient to use a weaker form introduced in [9]. Given $c \in \mathbb{R}$ we say that $(\mathrm{PS})_c$ holds if

$(\mathrm{PS})_c$ every sequence (u_n) in H such that $F(u_n) \to c$ and $\|F'(u_n)\| \to 0$ is relatively compact in H.

In some sense the (PS) condition prevents critical points from "leaking at infinity". If the $(\mathrm{PS})_c$ condition fails at some level c, this means, roughly speaking, that c is a critical value which corresponds to "critical points at infinity"—a concept introduced by A. Bahri in [3].

THEOREM A. *Let $a < b$ and assume*

$$F \text{ has no critical value in the interval } [a, b], \tag{42}$$

$$F \text{ satisfies } (\mathrm{PS})_c \text{ for every } c \in [a, b]. \tag{43}$$

Then $F_a \simeq F_b$ (homotopy equivalence). (In fact F_a is a deformation retract of F_b.)

The next basic principle of Morse theory tells us how to "compute" the change in topology between F_a and F_b across a critical value c:

THEOREM B. *Let $a < b$ and assume (43). Suppose $c \in (a, b)$ is a critical value corresponding to a unique nondegenerate critical point u. Suppose there is no other critical value in the interval $[a, b]$. Then*

$$(F_b, F_a) \simeq (B^k, S^{k-1}) \quad (\textit{homotopy equivalence of pairs}),$$

where k is the Morse index of u, i.e., the number of negative eigenvalues of the Hessian $D^2F(u)$.

Here, $B^k = \{x \in \mathbb{R}^k;\ |x| \leq 1\}$ and $S^{k-1} = \partial B^k$. The conclusion says that F_b has the homotopy type of F_a with a k-cell "handle" attached. Of course, if F has no critical value in the interval $[a, b]$ there may still be a change in topology between F_a and F_b if the $(\mathrm{PS})_c$ condition fails at some level $c \in [a, b]$. In that case, the change in topology (F_b, F_a) is analyzed by studying the Hessian D^2F near the "critical points at infinity". Needless to say that such a study may be complicated, especially if the critical points at infinity are not isolated. Bahri and Coron have carried out the analysis on a specific example.

2.2. Sketch of the proof of Theorem 2. Set $H = H_0^1(\Omega)$ and

$$F(u) = \tfrac{1}{2}\int|\nabla u|^2 - \frac{1}{p+1}\int|u^+|^{p+1},$$

where $u^+ = \max(u, 0)$. So far, Ω can be any smooth bounded domain in $\mathbb{R}^N$—the assumption that Ω has nontrivial topology will be used only much later. Note that $u = 0$ is a critical point of F and that the nonzero critical points of F correspond precisely to the solutions of (3). Theorem 2 is established by contradiction and therefore we shall assume, in what follows, that

$$u = 0 \text{ is the } \textit{only} \text{ critical point of } F. \tag{44}$$

It is not difficult to see that the condition $(\mathrm{PS})_c$ fails at the levels $c = k\Sigma$, where $k = 1, 2, 3, \cdots$ and $\Sigma = (1/N)S^{N/2}$. Indeed, fix any point $\bar{x} \in \Omega$ (which serves as "point of concentration") and fix any sequence $\varepsilon_n \to 0$, $\varepsilon_n > 0$, (which serves as "speed of concentration").

Consider the sequence of functions

$$u_n(x) = \zeta(x) U_{\varepsilon_n}(x - \bar{x})$$

with $\zeta \in C_0^\infty(\Omega)$ (ζ is independent of n), $\zeta \geq 0$, $\zeta \equiv a$ near $\bar{x}$, the positive constant a ($= a_N$) being adjusted so that

$$-\Delta(aU_\varepsilon) = (aU_\varepsilon)^p \quad \text{on} \quad \mathbb{R}^N,$$

(in view of (10) one has to impose $a^{p-1} = N(N-2)$). A quick calculation shows that $F(u_n) \to \Sigma$ and $F'(u_n) \to 0$ (in the H^{-1} norm); for this calculation it is useful to observe that

$$\frac{\int |\nabla U_\varepsilon|^2}{\|U_\varepsilon\|_{p+1}^2} = S \quad \text{and} \quad \int |\nabla(aU_\varepsilon)|^2 = -\int \Delta(aU_\varepsilon)(aU_\varepsilon) = \int (aU_\varepsilon)^{p+1},$$

so that

$$\int |\nabla(aU_\varepsilon)|^2 = \int (aU_\varepsilon)^{p+1} = S^{N/2}$$

and

$$\tfrac{1}{2}\int \nabla(aU_\varepsilon)^2 - \frac{1}{p+1}\int (aU_\varepsilon)^{p+1} = \frac{1}{N} S^{N/2}$$

(in these equalities, $\int$ means the integral on all of $\mathbb{R}^N$). Moreover, the sequence (u_n) is *not* relatively compact in $H_0^1(\Omega)$ since $u_n \to 0$ in $L^2(\Omega)$ but *not* in $H_0^1(\Omega)$. Therefore, the condition $(\mathrm{PS})_c$ fails at the level $c = \Sigma$.

Next, we may *superimpose* k such gadgets. Namely, fix k *distinct* points of concentration $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k$ in Ω and k speeds of concentration $\varepsilon_{1,n}, \varepsilon_{2,n}, \cdots, \varepsilon_{k,n}$ (any positive sequences tending to 0). Then, the sequence of functions

$$u_n(x) = \sum_{i=1}^{k} \zeta_i(x) U_{\varepsilon_{i,n}}(x - \bar{x}_i) \tag{45}$$

satisfies $F(u_n) \to k\Sigma$ and $F'(u_n) \to 0$ while (u_n) is *not* relatively compact in $H_0^1(\Omega)$. Therefore, the condition $(\mathrm{PS})_c$ fails at the levels $k\Sigma$.

It is a striking fact that the condition $(\mathrm{PS})_c$ fails *only* at the levels $k\Sigma$ and that formula (45) provides a good representation of the "critical points at infinity" for the level $k\Sigma$.

LEMMA 2. *The condition* $(\mathrm{PS})_c$ *fails precisely when* $c = k\Sigma$, $k = 1, 2, 3, \cdots$.

Remark 4. The conclusion of Lemma 2 relies on the assumption that $u = 0$ is the only critical point of F. In general, the conclusion would be that $(\mathrm{PS})_c$ fails precisely at the levels $c = \sigma + k\Sigma$, where σ is any critical value of F and $k = 1, 2, 3, \cdots$.

Sketch of the proof of Lemma 2: It is clear that the $(\mathrm{PS})_c$ condition holds when $c < \Sigma$. Indeed let (u_n) be a sequence such that $F(u_n) \to c$ and $F'(u_n) \to 0$. Then we have

$$\tfrac{1}{2}\int|\nabla u_n|^2 - \frac{1}{p+1}\int(u_n^+)^{p+1} = c + o(1) \tag{46}$$

and

$$-\Delta u_n = (u_n^+)^p + \phi_n \quad \text{with} \quad \phi_n \to 0 \quad \text{in} \quad H^{-1}. \tag{47}$$

A standard argument shows that (u_n) is bounded in H_0^1 and thus we obtain (from (47))

$$\int|\nabla u_n|^2 = \int(u_n^+)^{p+1} + o(1). \tag{48}$$

Combining (46) and (48) we are led to

$$\int|\nabla u_n|^2 = Nc + o(1) \quad \text{and} \quad \int(u_n^+)^{p+1} = Nc + o(1).$$

Sobolev's inequality implies that

$$\int|\nabla u_n|^2 \geqq S\|u_n^+\|_{p+1}^2$$

and therefore we find

$$(Nc) \geqq S(Nc)^{2/(p+1)}$$

which says that either $c = 0$ or $c \geqq (1/N)S^{N/2} = \Sigma$. By assumption, $c < \Sigma$ so that $c = 0$ and we conclude that $u_n \to 0$ in H_0^1.

Next, we check that the condition $(\mathrm{PS})_c$ holds when $k\Sigma < c < (k+1)\Sigma$ and $k \geqq 1$ is an integer. In fact, there exists *no* sequence (u_n) such that $F(u_n) \to c$, $F'(u_n) \to 0$ and $k\Sigma < c < (k+1)\Sigma$. Suppose, by contradiction, that (u_n) is such a sequence. Again, we have (46) and (47), so that (u_n) is bounded in H_0^1 and moreover $u_n \rightharpoonup 0$ weakly in H_0^1. (Note that if $u_{n_k} \rightharpoonup u$ *weakly* in H_0^1, then u

satisfies $-\Delta u = (u^+)^p$, so that $u = 0$ by (44).) Moreover, u_n does *not* tend to 0 *strongly* in H_0^1; otherwise we would have $c = 0$.

At this stage the proof becomes rather technical and I shall be very sketchy; see [5], [8] and [33] for more details. As in [18], it is convenient to use the concentration function in order to catch a first singularity. More precisely, there exists a sequence (x_n) in Ω and a sequence $\varepsilon_n \to 0$ such that

$$\int_{B(x_n, \varepsilon_n)} |\nabla u_n|^2 = \nu \quad \text{with} \quad \nu > 0 \text{ small enough.}$$

The singular behavior of (u_n) is analyzed by a *blow-up* technique. Namely, set

$$\tilde{u}_n(x) = \varepsilon_n^{(N-2)/2} u_n(\varepsilon_n x + x_n), \qquad x \in \mathbb{R}^N,$$

$(u_n$ is extended by 0 outside $\Omega)$, so that $\tilde{u}_n$ satisfies

$$\int_{B(0,1)} |\nabla \tilde{u}_n|^2 = \nu, \qquad \int_{\mathbb{R}^N} |\nabla \tilde{u}_n|^2 \leqq C,$$

and

$$-\Delta \tilde{u}_n = (\tilde{u}_n)^p + \tilde{\phi}_n \quad \text{on} \quad \Omega_n = \frac{\Omega - x_n}{\varepsilon_n},$$

with $\tilde{\phi}_n \to 0$.

Passing to the limit on the sequence $\tilde{u}_n$ (or rather a subsequence) one can show that $\tilde{u}_n \to \omega$ strongly in $L_{\text{loc}}^{p+1}(\mathbb{R}^N)$ and $\nabla \tilde{u}_n \to \nabla \omega$ strongly in $L_{\text{loc}}^2(\mathbb{R}^N)$. It follows that

$$-\Delta \omega = (\omega^+)^p \quad \textit{on} \quad \mathbb{R}^N,$$

and $\omega \not\equiv 0$; therefore ω is an old friend!

We remove this first singularity by letting

$$v_n(x) = u_n(x) - \frac{1}{\varepsilon_n^{(N-2)/2}} \omega\left(\frac{x - x_n}{\varepsilon_n}\right), \qquad x \in \Omega.$$

One proves that (v_n) satisfies

$$F(v_n) \to c - \Sigma \quad \text{and} \quad F'(v_n) \to 0$$

(recall that $\frac{1}{2}\int_{\mathbb{R}^N} |\nabla \omega|^2 - (1/(p+1)) \int_{\mathbb{R}^N} \omega^{p+1} = \Sigma$).

In other words, the sequence (v_n) is like the sequence (u_n), except that c is replaced by $c - \Sigma$. *Iterating* k times this construction we obtain a sequence (w_n)

such that

$$F(w_n) \to c - k\Sigma < \Sigma \quad \text{and} \quad F'(w_n) \to 0.$$

I emphasize that *each singularity contributes the same amount of energy, namely* Σ. Thus, we are led to the case considered at the very beginning and we conclude that $c - k\Sigma = 0$ which is a contradiction since $c > k\Sigma$.

Remark 5. The main idea behind the proof of Lemma 2 originates in the work of Sacks–Uhlenbeck [26]. They consider a problem about harmonic maps which has the same kind of "scaling invariance" as ours. They point out—for the first time—that a sequence (u_n) which is bounded in H^1 and such that $F'(u_n) \to 0$ can have a "singular behavior" only at a *finite* number of points. Subsequently, this type of argument has been used by many authors for various problems; see Meeks–Yau [20], Sin–Yau [32], Sedaleck [31], Taubes [36], [38], Donaldson [12], P. L. Lions [18], Brezis–Coron [8], Struwe [33], Bahri–Coron [4], etc.

The next objective is to get a *good grasp of the critical points at infinity* for each level $k\Sigma$. Consider for example a sequence (u_n) in H_0^1 such that

$$F(u_n) \to \Sigma \quad \text{and} \quad F'(u_n) \to 0. \tag{49}$$

Then one can show that there exists a sequence (x_n) in Ω and a sequence of positive numbers $\varepsilon_n \to 0$ such that

$$\|u_n(x) - aU_{\varepsilon_n}(x - x_n)\|_{H^1} \to 0 \tag{50}$$

and

$$\frac{1}{\varepsilon_n}\operatorname{dist}(x_n, \partial\Omega) \to \infty. \tag{51}$$

Conversely, if (u_n) is any sequence satisfying (50) and (51), then (49) holds.

Of course, we may also assume (modulo a subsequence) that $x_n \to \bar{x}$, where $\bar{x}$ is the point of concentration of (u_n). If $\bar{x} \in \Omega$, $u_n(x)$ is roughly equivalent to $aU_{\varepsilon_n}(x - \bar{x})$. When $\bar{x} \in \partial\Omega$, the situation is more delicate; however property (51) shows that the "boundary effect" is negligible compared to the "concentration effect". This means that, for all "practical purposes", any sequence (u_n) satisfying (49) behaves like $aU_{\varepsilon_n}(x - \bar{x})$ with $\bar{x} \in \Omega$.

The situation at the level $k\Sigma$ is similar:

LEMMA 3. *Assume (u_n) is a sequence in H_0^1 satisfying*

$$F(u_n) \to k\Sigma \quad \textit{and} \quad F'(u_n) \to 0. \tag{52}$$

Then, there exist k sequences $(x_{i,n})$ in Ω and k sequences of positive numbers $(\varepsilon_{i,n})$ with $\varepsilon_{i,n} \to 0$ as $n \to \infty$ (for all $i = 1, 2, \cdots, k$) such that

$$\left\| u_n(x) - \sum_{i=1}^{k} aU_{\varepsilon_{i,n}}(x - x_{i,n}) \right\|_{H^1} \to 0, \tag{53}$$

$$\frac{1}{\varepsilon_{i,n}} \operatorname{dist}(x_{i,n}, \partial\Omega) \underset{n\to\infty}{\to} \infty, \qquad i = 1, 2, \cdots, k, \tag{54}$$

$$\textit{for all } i \neq j, \quad \max\left\{ \frac{\varepsilon_{i,n}}{\varepsilon_{j,n}}, \frac{\varepsilon_{j,n}}{\varepsilon_{i,n}}, \frac{|x_{i,n} - x_{j,n}|}{\varepsilon_{i,n} + \varepsilon_{j,n}} \right\} \underset{n\to\infty}{\to} \infty. \tag{55}$$

Conversely, if (u_n) is any sequence satisfying (53), (54) *and* (55), *then* (52) *holds.*

The proof relies on a blow-up analysis (as in the proof of Lemma 2); for the details, see [5], [8], [33]. Of course, we may also assume (modulo a subsequence) that $x_{i,n} \to \bar{x}_i$ as $n \to \infty$, where $\bar{x}_i$ is the point of concentration. If all the points $\bar{x}_i$ are distinct and belong to Ω, then, roughly speaking, we have $u_n(x) \simeq \sum_{i=1}^{k} aU_{\varepsilon_{i,n}}(x - \bar{x}_i)$. However, if some of the points $\bar{x}_i$ coincide, then the situation is more delicate. Property (55) says, for example, that if, for some $i \neq j$, $\bar{x}_i = \bar{x}_j$ and $\varepsilon_{i,n} = \varepsilon_{j,n}$, then the *speed at which the singularities coalesce is much slower than the speed of concentration*—so that one sees two *distinct* waves as in the first case. Again, for all "practical purposes", a sequence (u_n) satisfying (52) behaves like $\sum_{i=1}^{k} aU_{\varepsilon_{i,n}}(x - \bar{x}_i)$ with *distinct* points $\bar{x}_i$ in Ω.

Lemma 2 provides a complete representation of the critical points at infinity. In some sense, they can be parametrized, at the level $k\Sigma$, by $\Omega^k = \Omega \times \Omega \times \cdots \times \Omega$ (k times). The next goal is to describe the change in topology across the level $k\Sigma$.

At the first level, Σ, the answer is simply

$$(F_{\Sigma+\varepsilon}, F_{\Sigma-\varepsilon}) \simeq (\Omega \times B^1, \Omega \times S^0), \tag{56}$$

where $S^0 = \partial B^1 = \{-1, +1\}$.

At the level $k\Sigma$ the answer is more delicate. We have to introduce some notation: For each $x = (x_1, x_2, \cdots, x_k) \in \Omega^k$, let $m(x)$ denote the $k \times k$ symmetric matrix $(m_{ij}(x))$ whose coefficients are given by

$$m_{ij}(x) = \begin{cases} G(x_i, x_j) & \text{if} \quad i \neq j, \\ g(x_i, x_i) & \text{if} \quad i = j. \end{cases}$$

($G(x, y)$ is the Green's function of $-\Delta$ with zero boundary condition and $g(x, y)$ is the regular part of $G(x, y)$ as in Section 1.)

Let $\rho(x)$ denote the largest eigenvalue of $m_{ij}(x)$ ($\rho(x) = +\infty$ if $x_i = x_j$ for some $i \neq j$) and

$$I_k = \{x \in \Omega^k;\ \rho(x) \geqq 0\}$$

(note that $I_1 = \varnothing$).

Finally, σ_k denotes the group of permutations of order k and $\Delta^{k-1} = \{x \in \mathbb{R}^k;\ \sum_{i=1}^k x_i = 1,\ x_i \geqq 0 \text{ for all } i\}$.

LEMMA 4. *For every $k \geqq 1$ one has*

$$(F_{k\Sigma+\varepsilon}, F_{k\Sigma-\varepsilon}) \simeq (X \times Y, (X \times B) \cup (A \times Y)),$$

where

$$X = \frac{\Omega^k \times \Delta^{k-1}}{\sigma_k}, \qquad A = \frac{(\Omega^k \times \partial\Delta^{k-1}) \cup (I_k \times \Delta^{k-1})}{\sigma_k},$$

$Y = \Delta^1$ *and* $B = \partial\Delta^1$ *(so that $A \subset X$ and $B \subset Y$).*

Lemma 4 is proved—like Lemma B—by a deformation method, i.e., pushing down with a gradient flow. It requires a delicate Morse analysis of D^2F near the critical points at infinity and therefore it involves a study of F on a sequence like $\sum_{l=1}^k aU_{\varepsilon_{i,n}}(x - \bar{x}_i)$. This leads to expansions (as $\varepsilon_{i,n} \to 0$) similar to the expansions of Section 1—except that *interaction terms* are also present; see [5].

The last two steps deal with topological properties of the pair $(F_{k\Sigma+\varepsilon}, F_{k\Sigma-\varepsilon})$.

LEMMA 5. *There is some integer k_0 $(= k_0(\Omega))$ such that the pair $(F_{k\Sigma+\varepsilon}, F_{k\Sigma-\varepsilon})$ is trivial for every $k \geqq k_0$.*

In other words, $F_{k\Sigma+\varepsilon} \simeq F_{k\Sigma-\varepsilon}$ for $k \geqq k_0$, so that there is no change in topology across the level $k\Sigma$ for $k \geqq k_0$. The key observation is that, roughly speaking, $I_k = \Omega^k$ for k large enough. Indeed, if k is very large and x is any element in Ω^k, then at least two of the points x_i and x_j (with $i \neq j$) must be very close, and thus $G(x_i, x_j) \simeq +\infty$—so that $\rho(x) \geqq 0$; see [5].

It is only the last lemma which uses the topological assumption on Ω.

LEMMA 6. *Assume Ω has nontrivial topology. Then, for each $k \geqq 1$, the pair $(F_{k\Sigma+\varepsilon}, F_{k\Sigma-\varepsilon})$ is nontrivial.*

The argument is by induction on k (the conclusion is obvious for $k = 1$) and relies heavily on tools from algebraic topology; see [5].

We have reached a contradiction between Lemma 5 and Lemma 6. This means that assumption (44) is absurd and so F has nonzero critical points.

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