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Positive solutions of nonlinear elliptic equations in the case of critical Sobolev exponent

We report on a recent work with L. Nirenberg [5]. Consider the following problem. Assume \( \Omega \subset \mathbb{R}^N \), \( N \geq 3 \), is a bounded (smooth) domain. Find a (smooth) function \( u \) satisfying

\[
\begin{align*}
  u &> 0 \quad \text{in } \Omega \\
  -\Delta u &= u^p + f(u) \quad \text{in } \Omega \\
  u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

(1)

where \( p = \frac{N+2}{N-2} \) and \( f(u) \) is a "lower order perturbation" with \( f(0) = 0 \); a typical example is \( f(u) = \lambda u \) \( (\lambda \in \mathbb{R}) \). The exponent \( p = \frac{N+2}{N-2} \) is critical from the point of view of the variational formulation. Indeed, solutions of (1) correspond to critical points of the functional

\[
\frac{1}{2} \int |\nabla u|^2 - \frac{1}{p+1} \int u^{p+1} - \int f(u)
\]

where \( F \) is a primitive of \( f \) and \( p+1 = \frac{2N}{N-2} \) is the Sobolev exponent for the embedding \( H^1_0(\Omega) \subset L^{p+1}(\Omega) \).

Our lecture is divided as follows. First we recall some results concerning the easy case where \( p < \frac{N+2}{N-2} \) and \( f(u) = \lambda u \). Then, we consider the case where \( p = \frac{N+2}{N-2} \) and \( f(u) = \lambda u \). Finally we turn to the case where \( f \) is nonlinear.

Our interest in problem (1) comes from the fact that it presents some similarities with the Yamabe problem in geometry; see e.g. Trudinger [11] and Th. Aubin [2].
1. THE CASE $p < \frac{N+2}{N-2}$.

Throughout Section 1 we assume that $p < \frac{N+2}{N-2}$. Clearly, there is a solution $u$ of

$$
\begin{align*}
  u &> 0 \quad \text{in } \Omega \\
  -\Delta u &= u^p \quad \text{in } \Omega \\
  u &= 0 \quad \text{on } \partial \Omega \\
\end{align*}
$$

(2)

Indeed, consider the following minimization problem

$$
\inf_{v \in H^1_0} \left\{ \frac{\|v_v\|_2^2}{\|v\|_{L^{p+1}}^2} \right\}
$$

(3)

Since the injection $H^1_0 \subset L^{p+1}$ is compact, the infimum in (3) is achieved by some $v_0$. We may always assume that $v_0 \geq 0$ (otherwise replace $v_0$ by $|v_0|$) and that $\|v_0\|_{L^{p+1}} = 1$. Thus we obtain a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$
-\Delta v_0 = \mu v_0^p
$$

(4)

and

$$
\mu = \int |v_0|^2 > 0.
$$

By stretching $v_0$ we find a function $u$ satisfying

$$
\begin{align*}
  u &\geq 0 \quad \text{on } \Omega, \quad u \not\equiv 0 \\
  -\Delta u &= u^p \quad \text{on } \Omega \\
  u &= 0 \quad \text{on } \partial \Omega \\
\end{align*}
$$

(5)

[more precisely $u = kv_0$ satisfies (5) provided $k = \frac{1}{\mu^{p-1}}$. It follows from the strong maximum principle that $u > 0$ in $\Omega$.]

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(The question of uniqueness for problem (2) is open when $\Omega$ is starshaped. When $\Omega$ is an annulus and $p$ is close to $\frac{N+2}{N-2}$ the solution of (2) need not be unique; in fact there exist both spherical and non-spherical solutions, see [5]).

Let $\lambda_1$ denote the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. Consider now the following problem: find $u$ such that

\[
\begin{align*}
    u &> 0 & \text{in } \Omega \\
    -\Delta u &= u^p + \lambda u & \text{in } \Omega \\
    u &= 0 & \text{on } \partial \Omega
\end{align*}
\]  

(6)

Then for each $\lambda \in (-\infty, \lambda_1)$ there is a solution of (6). Indeed

\[
\inf_{v \in H_0^1} \left\{ \frac{\|\nabla v\|_2^2 - \lambda \|v\|_2^2}{\|v\|_{L^{p+1}}^2} \right\}
\]

(7)

is achieved by some $v_0$ satisfying $v_0 \geq 0$ on $\Omega$ and $\|v_0\|_{L^{p+1}} = 1$. Moreover there is a Lagrange multiplier $\mu \in \mathbb{R}$ such that

\[-\Delta v_0 - \lambda v_0 = \mu v_0^p.
\]

Thus $\mu = \int |\nabla v_0|^2 - \lambda \int v_0^2 > 0$ (since $\lambda < \lambda_1$). By stretching $v_0$ as above we obtain a solution of (6).

The restriction $\lambda \in (-\infty, \lambda_1)$ is essential. Indeed suppose $u$ is a solution of (6). Let $\phi_1 > 0$ in $\Omega$ be an eigenfunction corresponding to $\lambda_1$. We have

\[
\lambda_1 \int u \phi_1 = \int u^p \phi_1 + \lambda \int u \phi_1 > \lambda \int u \phi_1
\]

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and thus $\lambda < \lambda_1$.

2. **THE CASE** $p = \frac{N+2}{N-2}$ AND $f(u) = \lambda u$

Throughout Sections 2 and 3 we assume that $p = \frac{N+2}{N-2}$. We consider now the following problem: find $u$ such that

$$
\begin{cases}
u > 0 & \text{in } \Omega \\
-\Delta u = u^p + \lambda u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases} \quad (\lambda \in \mathbb{R})
$$

(8)

The argument we have used on Section 1 does not hold anymore since the injection $H^1_0 \subset L^{p+1}$ is not compact. In fact we know, by a result of Pohozaev [9], that if $\Omega$ is starshaped and $\lambda = 0$ there is no solution of (8). Using the same argument as in Pohozaev [9] one proves:

**Theorem 0**: Assume $\Omega$ is starshaped and $\lambda \leq 0$. Then there is no solution of (8).

**Remark 1**: On the other hand if $\Omega$ is an annulus then for every $\lambda \in (-\infty, \lambda_1)$ there is a spherical solution of (8). Indeed consider

$$
\inf_{v \in H^1_r} \left\{ \frac{\|\nabla v\|_2^2 - \lambda \|v\|_2^2}{\|v\|_{L^{p+1}}^2} \right\}
$$

(9)

where $H^1_r = \{ v \in H^1_0(\Omega) \text{ and } v \text{ is spherically symmetric}\}$.

The infimum in (9) is achieved since the injection of $H^1_r$ into $L^{p+1}$ is compact. Thus, after stretching, we obtain a spherical solution of (8).
Our main results are the following.

**Theorem 1**: Assume $\Omega \subset \mathbb{R}^N$, $N \geq 4$, is any (smooth) bounded domain. Then for every $\lambda \in (0, \lambda_1)$ there exists a solution of (8). Moreover

$$
\inf_{v \in H_0^1} \left\{ \frac{||v||^2_{L^2} - \lambda ||v||^2_{L^2}}{||v||^2_{L^{p+1}}} \right\}
$$

is achieved.

**Theorem 2**: Assume $\Omega$ is a ball in $\mathbb{R}^N$. Then for every $\lambda \in \left( \frac{\lambda_1}{4}, \lambda_1 \right)$ there exists a solution of (8); moreover the infimum in (10) is achieved. When $\lambda \leq \frac{\lambda_1}{4}$ there is no solution of (8).

**Remark 2**: The difference between dimension $N = 3$ and dimension $N \geq 4$ is quite surprising. We have no simple explanation for it.

**Remark 3**: When $N \geq 3$ and $\lambda \geq \lambda_1$ there is no solution of (8) (see Section 1). When $N \geq 3$ and $\Omega$ is starshaped there is no solution of (8) for $\lambda \leq C$ (by Theorem 0).

**Remark 4**: The generalization of Theorem 2 for starshaped domains is not known.

Before we sketch the proofs we present some facts about Sobolev spaces:

a) Define the best Sobolev constant $S$ to be
\[ S = \inf_{v \in H^1_0} \left\{ \frac{\|\nabla v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} \]  

(11)

In principle \( S \) depends on \( \Omega \); but in fact \( S \) depends only on \( N \). This is an easy consequence of the invariance under scaling of the ratio \( \frac{\|\nabla v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \) (that is, the ratio is unchanged if we replace \( u(x) \) by \( u_k(x) = u(kx) \)).

b) The infimum in (11) is never achieved, on any bounded domain. Indeed, suppose that the infimum in (11) is achieved by some function \( v_0 \geq 0 \). Let \( \tilde{\Omega} \) be a ball containing \( \Omega \) and set

\[
\tilde{v}_0 = \begin{cases} 
  v_0 & \text{in } \Omega \\
  0 & \text{in } \tilde{\Omega} \setminus \Omega 
\end{cases}
\]

Thus, for \( \tilde{\Omega} \) the infimum in (11) is achieved at \( \tilde{v}_0 \) and we find

\[
\begin{cases} 
  -\Delta \tilde{v}_0 = \mu v_0^p & \text{on } \Omega \\
  \tilde{v}_0 = 0 & \text{on } \partial \Omega 
\end{cases}
\]

for some constant \( \mu > 0 \). This contradicts Pohozaev's Theorem.

c) When \( \Omega = \mathbb{R}^N \) the infimum in (11) is achieved by the function

\[
u(x) = \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}} \quad \text{or - after scaling - by any of the functions}
\]

\[
u(x) = \frac{1}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}} \quad (\varepsilon > 0).
\]
see Aubin [3], Talenti [10], Lieb [8].

The following Lemma plays a crucial role in the proof of Theorem 1.

**Lemma 1**: Assume $N \geq 4$. Then, for every $\lambda \in (0, \lambda_1)$ we have

$$
S_\lambda \equiv \inf_{v \in H_0^1} \left\{ \frac{\|\nabla v\|^2_{L^2} - \lambda \|v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} < S
$$

(12)

The proof of Lemma 1 is rather technical; for details see [51]. The main idea - borrowed from Aubin [1] - consists of estimating the ratio

$$
Q(u) = \frac{\|\nabla u\|^2_{L^2} - \lambda \|u\|^2_{L^2}}{\|u\|^2_{L^{p+1}}}
$$

for $u(x) = u_\varepsilon(x) = \frac{\phi(x)}{N-2} \left( \frac{\varepsilon^2}{\varepsilon + |x|^2} \right)^{N-2}$ where $\phi \in \mathcal{D}_+(\Omega)$ is a fixed function such that $\phi(x) \equiv 1$ near $0$ (assuming $0 \in \Omega$).

A straightforward computation gives the following expansion as $\varepsilon \to 0$:

- $Q(u_\varepsilon) = S + O(\varepsilon^{\frac{N}{2}-1}) - \lambda C \varepsilon$ when $N \geq 5$

- $Q(u_\varepsilon) = S + O(\varepsilon) - \lambda C \varepsilon |\log \varepsilon|$ when $N = 4$

where $C > 0$ is a constant. In both cases we see that $Q(u_\varepsilon) < S$ for $\varepsilon > 0$ sufficiently small. We shall also use the following measure theoretic lemma.
Lemma 2: (Brezis-Lieb [4]) Suppose \((v_j)\) is a sequence in \(L^q(\Omega)\) with 
\[1 \leq q < \infty\] such that \(\|v_j\|_{L^q}\) remains bounded and \(v_j(x) \to v(x)\) a.e. on \(\Omega\). Then

\[
\lim_{j \to \infty} \left( \int |v_j|^q - \int |v_j - v|^q \right) = \int |v|^q
\]

(13)

Proof of Theorem 1: Choose a minimizing sequence \((v_j)\) for (10) such that

\[v_j \geq 0 \text{ on } \Omega, \quad \|v_j\|_{L^{p+1}} = 1\]

(14)

\[
\int |v_j|^2 - \lambda \int v_j^2 = S_\lambda + o(1)
\]

(15)

Since \(v_j\) is bounded in \(H^1_0\) we may assume, for a subsequence, that

\[v_j \rightharpoonup v \text{ weakly in } H^1_0
\]

\[v_j \to v \text{ a.e. on } \Omega
\]

\[v_j \to v \text{ strongly in } L^2
\]

Set \(w_j = v_j - v\) so that \(w_j \rightharpoonup 0\) weakly in \(H^1_0\).

By Lemma 2 (applied with \(q = p+1\)) we have

\[
\lim_{j \to \infty} \|w_j\|_{L^{p+1}}^{p+1} = 1 - \|v\|_{L^{p+1}}^{p+1}.
\]

Thus

\[
1 = \left( \|v\|_{L^{p+1}}^{p+1} + \|w_j\|_{L^{p+1}}^{p+1} \right)^{\frac{2}{p+1}} + o(1) \leq \|v^2\|_{L^{p+1}}^{p+1} + \|w_j\|_{L^{p+1}}^2 + o(1)
\]

(16)
On the other hand (since $w_j \rightharpoonup 0$ weakly in $H^1_0$) we have

$$\int |\nabla v_j|^2 = \int |\nabla v|^2 + \int |\nabla w_j|^2 + o(1) \quad (17)$$

Combining (15) (16) and (17) we obtain

$$\int |\nabla v|^2 - \lambda \int |v|^2 + \int |\nabla w_j|^2 \leq S \lambda \|v\|_{L^{p+1}}^2 + \|w_j\|_{L^{p+1}}^2 + o(1) \quad (18)$$

By definition of $S \lambda$ we have $\int |\nabla v|^2 - \lambda \int |v|^2 \geq S \lambda \|v\|_{L^{p+1}}^2$ and therefore

$$\int |\nabla w_j|^2 \leq S \lambda \|w_j\|_{L^{p+1}}^2 + o(1)$$

$$\leq S \lambda \int |\nabla v_j|^2 + o(1).$$

Since $S \lambda < S$ (by Lemma 1), it follows that $\int |\nabla w_j|^2 \to 0$. Consequently $v_j \rightharpoonup v$ strongly in $H^1_0$ (and in $L^{p+1}$). Passing to the limit in (14) and (15) we conclude that the infimum in (11) is achieved by $v$. After stretching we obtain a solution of (8).

In the proof of Theorem 2 we use

**Lemma 3**: Assume $\Omega = \{x \in \mathbb{R}^3; |x| < 1\}$. Then for each $\lambda \in \left(\frac{\lambda_1}{4}, \lambda_1\right)$ we have

$$S \lambda \equiv \inf_{v \in H^1_0} \left\{ \frac{\|v\|_{L^2}^2 - \lambda \|v\|_{L^2}^2}{\|v\|_{L^6}^2} \right\} < S \quad (19)$$
Proof : We estimate the ratio

\[ Q(u) = \frac{\|\nabla u\|_2^2 - \lambda \|u\|_2^2}{\|u\|_6^2} \]

for \( u(x) = u_\varepsilon(x) = \frac{\cos\left(\frac{\pi}{2}|x|\right)}{(\varepsilon + |x|^2)^{1/2}} \).

A technical computation (see [5]) gives the following expansion as \( \varepsilon \to 0 \):

\[ Q(u_\varepsilon) = S + C\sqrt{\varepsilon} \left( \frac{\pi^2}{4} - \lambda \right) + O(\sqrt{\varepsilon}) \]

where \( C > 0 \) is a constant. Therefore if \( \lambda > \frac{\lambda_1}{4} \) (note that here \( \lambda_1 = \pi^2 \)) we see that \( Q(u_\varepsilon) < S \) for \( \varepsilon > 0 \) sufficiently small.

Proof of Theorem 2 : The same argument as in the proof of Theorem 1 shows that for every \( \lambda \in (\frac{\lambda_1}{4}, \lambda_1) \) the infimum in (19) is achieved. Consequently we obtain a solution of (8) for each \( \lambda \in (\frac{\lambda_1}{4}, \lambda_1) \). Next we must show that no solution of (8) exists for \( \lambda \geq \frac{\lambda_1}{4} \). By a result of Gidas-Ni-Nirenberg [7] we know that any solution \( u \) of (8) in a ball must be spherically symmetric. We write \( u(x) = u(r) \) (\( r = |x| \)) and so we have

\[ -u'' + \frac{2}{r} u' = u^5 + \lambda u \quad \text{on} \ (0,1) \quad (20) \]

\[ u'(0) = u(1) = 0. \quad (21) \]

Then we use an argument "à la Pohozaev" but with more complicated multipliers. Namely we multiply (20) through by
\[ r^2 [r \cos \pi r - b \sin \pi r] u' \]

and then by

\[ r \left[ -\frac{r}{2}(1+b\pi)\cos \pi r - \frac{\pi r^2}{2} \sin \pi r + b \sin \pi r \right] u \]

for some appropriate constant \( b \). Integrating by parts and combining the two equalities leads to \( \lambda > \frac{\pi^2}{4} \); for more details see [5].

3. **The General Case**, \( -\Delta u = u^p + f(u) \) WITH \( p = \frac{N+2}{N-2} \).

Here again we take \( p = \frac{N+2}{N-2} \). Assume \( f \) is a \( C^1 \) function on \([0, +\infty)\) such that

\[ f(0) = 0, \ f(u) \geq 0 \quad \forall u \geq 0 \] \hspace{1cm} (22)

\[ \lim_{u \to +\infty} \frac{f(u)}{u^p} = 0 \] \hspace{1cm} (23)

\[ f'(0) < \lambda_1 \] \hspace{1cm} (24)

The problem is to find a function \( u \) satisfying

\[
\begin{cases}
  u > 0 & \text{on } \Omega \\
  -\Delta u = u^p + f(u) & \text{on } \Omega \\
  u = 0 & \text{on } \partial \Omega
\end{cases}
\] \hspace{1cm} (25)
Our main results are the following.

Theorem 3: Assume $N \geq 5$, (22), (23), (24) and

$$f \neq 0$$  \hspace{1cm} (26)

Then there is a solution of (25).

Theorem 4: Assume $N = 4$, (22), (23), (24) and either

$$f'(0) > 0$$  \hspace{1cm} (27)

or

$$\lim_{u \to +\infty} \inf \frac{f(u)}{u} > 0$$  \hspace{1cm} (28)

Then there is a solution of (25).

Theorem 5: Assume $N = 3$, (22), (23), (24)

$$\lim_{u \to +\infty} \frac{f(u)}{u^3} = +\infty$$

Then there is a solution of (25).

Remark 5: Theorem 3, 4 and 5 admit appropriate extensions to the case where $f$ depends also on $x$ with $f(x,0) = 0$. They may be used in order to prove the following:
Theorem 6: Assume \( N \geq 3 \). Then there is a constant \( \lambda^* > 0 \) such that the problem

\[
\begin{align*}
-\Delta u &= \lambda (1+u)^p \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega \\
\quad u &= 0 \quad \text{in } \Omega
\end{align*}
\]

has at least two solutions for each \( \lambda \in (0, \lambda^*) \) and no solution for \( \lambda > \lambda^* \).

(A similar result for \( p < \frac{N+2}{N-2} \) had been obtained earlier by Crandall-Rabinowitz [6]).

The idea of the proof is the following. Firstly one obtains (easily) a minimal solution \( \bar{u} \) of (29) for every \( \lambda \in (0, \lambda^*) \) (see e.g. [67]). Then one looks for a second solution of (29) of the form \( u = \bar{u} + v \) with \( v > 0 \) on \( \Omega \). Thus \( v \) satisfies

\[
-\Delta v = \lambda (1+u+v)^p - \lambda (1+u)^p \equiv \lambda v^p + f(x,v)
\]

and we are reduced to a problem of the type (25).

The proofs of Theorems 3, 4 and 5 involve two ingredients. Firstly a geometrical result which is a variant of the Ambrosetti-Rabinowitz [1] mountain pass Lemma without the (PS) condition (see Lemma 4). Secondly a technical Lemma which has the same flavour as Lemma 1 or Lemma 3 (see Lemma 5).

Lemma 4: Assume \( \phi \) is a \( C^1 \) function on a Banach space \( \mathcal{E} \) such that

\[
\phi(0) = 0
\]

\[
\text{there exist constants } \rho > 0 \text{ and } r > 0 \text{ such that } \phi(u) \geq \rho \quad \text{for every } u \in \mathcal{E} \text{ with } \|u\| = r
\]
\( \Phi(v) \leq 0 \) for some \( v \in E \) with \( \|v\| > r \). \( \text{(32)} \)

Set

\[
\begin{align*}
    c &= \inf \sup_{P \in S} \phi(p) \\
    &= \inf \sup_{p \in P} \phi(p)
\end{align*}
\]  \( \text{(33)} \)

where \( S \) denotes the class of all paths joining 0 to \( v \). Then there exists a sequence \( (u_j) \) in \( E \) such that \( \phi(u_j) \rightarrow c \) and \( \phi'(u_j) \rightarrow 0 \) in \( E^* \).

The proof of Lemma 4 is essentially the same as the proof given in [1].

In order to prove Theorems 3, 4 and 5 we apply Lemma 4 in \( E = H^1_0 \) to the functional

\[
\phi(u) = \frac{1}{2} \int \|u\|^2 - \frac{1}{p+1} \int (u^+)^{p+1} - \int F(u^+)
\]  \( \text{(34)} \)

where \( F(u) = \int_0^u f(t) dt \). Property (31) is an easy consequence of assumption (24). For every \( u \in H^1_0 \), \( u \geq 0 \) in \( \Omega \), \( u \neq 0 \) we have \( \lim_{t \rightarrow \infty} \phi(tu) = -\infty \). Hence, there are many \( v \)'s satisfying (32). However it is essential to make a special choice of \( v \) in order to be able to use properly the sequence \( (u_j) \) given by Lemma 4. More precisely we have

**Lemma 5**: Under the assumptions of Theorems 3, 4 and 5 there is some \( u_0 \in H^1_0 \), \( u_0 \geq 0 \) in \( \Omega \), \( u_0 \neq 0 \) and

\[
\sup_{t \geq 0} \phi(tu_0) < \frac{1}{N} S^{N/2}.
\]  \( \text{(35)} \)
The proof is rather technical. Let $u_\epsilon(x) = \frac{\phi(x)}{(\epsilon + |x|^2)^{N/2}}$ with $\phi \in \mathcal{D}_+^+(\Omega)$ and $\phi \equiv 1$ near $x = 0$ (assuming $0 \in \Omega$). We show by an expansion method (as in Lemma 1) that $u_\epsilon$ satisfies (35) provided $\epsilon > 0$ is sufficiently small. As was already observed, the expansion technique is sensitive to the dimension $N$; the cases $N = 3$, $N = 4$ and $N \geq 5$ must be considered separately. (See the details in [5]).

Proofs of Theorem 3, 4 and 5: By Lemma 5 there is some $v \in H^1_0$ such that $\|v\| > r$, $\phi(v) \leq 0$ and

$$\sup_{t \geq 0} \phi(tv) < \frac{1}{N} S^{N/2}$$

(36)

We apply Lemma 4 with such a $v$. From (33) it follows that

$$c < \frac{1}{N} S^{N/2}$$

(37)

Let $(u_j)$ be the sequence given by Lemma 4. We have

$$\frac{1}{2} \int |v u_j|^2 - \frac{1}{p+1} \int (u_j^+)^{p+1} - \int F(u_j^+) = c + o(1)$$

(38)

$$-\Delta u_j = (u_j^+)^p + f(u_j^+) + \zeta_j$$

(39)

with $\zeta_j \to 0$ in $H^{-1}$.

Combining (38) and (39) it is easy to show that $\|u_j\|_{H^1}$ remains bounded.
Thus we may assume that

\[ u_j \rightarrow u \quad \text{weakly in } H^1_0 \]
\[ u_j \rightarrow u \quad \text{a.e.} \]

From (39) we deduce that

\[ -\Delta u = (u^+)^p + f(u^+) \quad \text{in } H^{-1} \quad (40) \]

By the maximum principle we have \( u \geq 0 \) and so

\[ -\Delta u = u^{p+1} + f(u). \]

It remains to prove that \( u \neq 0 \). Suppose by contradiction that \( u \equiv 0 \). Using (23) we obtain

\[ \int F(u^+_j) \rightarrow 0, \quad \int f(u^+_j)u_j \rightarrow 0 \quad (41) \]

We may always assume that

\[ \int |\nabla u_j|^2 \rightarrow \infty \quad (42) \]

and by (39)

\[ \int (u^+_j)^{p+1} \rightarrow \infty \quad (43) \]
From (38) we deduce that

$$\frac{1}{N} \xi = c > 0.$$  \hfill (44)

On the other hand we have (by Sobolev inequality)

$$\int |\nabla u_j|^2 \geq S \|u_j\|^2_{L^{p+1}} \geq S \|u_j^+\|^2_{L^{p+1}}$$

and therefore at the limit

$$\xi \geq \frac{2}{S \xi_{p+1}}.$$ 

Thus

$$\xi \geq S^{N/2}$$

and (by (44))

$$c \geq \frac{1}{N} S^{N/2}$$

a contradiction with (37). Therefore $u \neq 0$.

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