[98] in Nonlinear Partial Differential Equations and their Applications, Collège de France Sem. Vol III Pitman (1982)

Haim Brezis

Positive solutions of nonlinear elliptic equations in the case of critical Sobolev exponent

We report on a recent work with L. Nirenberg [5]. Consider the following problem. Assume $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded (smooth) domain. Find a (smooth) function u satisfying

$$\begin{cases} u > 0 & \text{in } \Omega \\ -\Delta u = u^p + f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $p = \frac{N+2}{N-2}$ and f(u) is a "lower order perturbation" with $f(0) = 0$; a typical example is $f(u) = \lambda u$ $(\lambda \in \mathbb{R})$. The exponent $p = \frac{N+2}{N-2}$ is critical from the point of view of the variational formulation. Indeed, solutions of (1) correspond to critical points of the functional $\frac{1}{2}\int|\nabla u|^2 - \frac{1}{p+1}\int u^{p+1} - \int F(u)$ where F is a primitive of f and $p+1 = \frac{2N}{N-2}$ is the Sobolev exponent for the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$.

Our lecture is divided as follows. First we recall some results concerning the *easy* case where $p < \frac{N+2}{N-2}$ and $f(u) = \lambda u$. Then, we consider the case where $p = \frac{N+2}{N-2}$ and $f(u) = \lambda u$. Finally we turn to the case where f is nonlinear.

Our interest in problem (1) comes from the fact that it presents some similarities with the Yamabe problem in geometry ; see e.g. Trudinger [11] and Th. Aubin [2].

1. THE CASE $p < \frac{N+2}{N-2}$.

Throughout Section 1 we assume that $p < \frac{N+2}{N-2}$. Clearly, there is a solution u of

$$\begin{cases} u > 0 & \text{in } \Omega \\ -\Delta u = u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

Indeed, consider the following minimization problem

$$\operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|_{L^2}^2}{\|v\|_{L^{p+1}}^2} \right\} \tag{3}$$

Since the injection $H_0^1 \subset L^{p+1}$ is compact, the infimum in (3) is achieved by some v_0. We may always assume that $v_0 \geq 0$ (otherwise replace v_0 by $|v_0|$) and that $\|v_0\|_{L^{p+1}} = 1$. Thus we obtain a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$-\Delta v_0 = \mu v_0^p \tag{4}$$

and $\quad \mu = \int |\nabla v_0|^2 > 0.$

By stretching v_0 we find a function u satisfying

$$\begin{cases} u \geq 0 \quad \text{on } \Omega, \qquad u \not\equiv 0 \\ -\Delta u = u^p \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{cases} \tag{5}$$

[more precisely $u = kv_0$ satisfies (5) provided $k = \mu^{\frac{1}{p-1}}$]. It follows from the strong maximum principle that $u > 0$ in Ω.

(The question of uniqueness for problem (2) is open when Ω is starshaped. When Ω is an annulus and p is close to $\frac{N+2}{N-2}$ the solution of (2) need not be unique ; in fact there exist both spherical and non-spherical solutions, see [5]).

Let λ_1 denote the first eigenvalue of $-\Delta$ with zero Dirichlet boundary condition. Consider now the following problem : find u such that

$$\begin{cases} u > 0 & \text{in } \Omega \\ -\Delta u = u^p + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6}$$

Then for each $\lambda \in (-\infty,\lambda_1)$ there is a solution of (6). Indeed

$$\operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|^2_{L^2} - \lambda \|v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} \tag{7}$$

is achieved by some v_0 satisfying $v_0 \geq 0$ on Ω and $\|v_0\|_{L^{p+1}} = 1$. Moreover there is a Lagrange multiplier $\mu \in \mathbb{R}$ such that

$$-\Delta v_0 - \lambda v_0 = \mu v_0^p.$$

Thus $\mu = \int |\nabla v_0|^2 - \lambda \int v_0^2 > 0$ (since $\lambda < \lambda_1$). By stretching v_0 as above we obtain a solution of (6).

The restriction $\lambda \in (-\infty,\lambda_1)$ is essential. Indeed suppose u is a solution of (6). Let $\phi_1 > 0$ in Ω be an eigenfunction corresponding to λ_1. We have

$$\lambda_1 \int u\phi_1 = \int u^p \phi_1 + \lambda \int u\phi_1 > \lambda \int u\phi_1$$

and thus $\lambda < \lambda_1$.

2. THE CASE $p = \frac{N+2}{N-2}$ AND $f(u) = \lambda u$

Throughout Sections 2 and 3 we assume that $p = \frac{N+2}{N-2}$. We consider now the following problem : find u such that

$$\begin{cases} u > 0 & \text{in } \Omega \\ -\Delta u = u^p + \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad (\lambda \in \mathbb{R}) \tag{8}$$

The argument we have used on Section 1 does not hold anymore since the injection $H_0^1 \subset L^{p+1}$ is *not* compact. In fact we know, by a result of Pohozaev [9], that if Ω is *starshaped* and $\lambda = 0$ there is *no* solution of (8). Using the same argument as in Pohozaev [9] one proves :

Theorem 0 : *Assume Ω is starshaped and $\lambda \leq 0$. Then there is no solution of* (8).

Remark 1 : On the other hand if Ω is an *annulus* then for every $\lambda \in (-\infty, \lambda_1)$ there is a spherical solution of (8). Indeed consider

$$\operatorname*{Inf}_{v \in H_r^1} \left\{ \frac{\|\nabla v\|^2_{L^2} - \lambda \|v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} \tag{9}$$

where $H_r^1 = \{v \in H_0^1(\Omega)$ and v is spherically symmetric$\}$.

The infimum in (9) is achieved since the injection of H_r^1 into L^{p+1} is *compact*. Thus, after stretching, we obtain a spherical solution of (8).

Our main results are the following

Theorem 1 : *Assume* $\Omega \subset \mathbb{R}^N$, $N \geq 4$, *is any (smooth) bounded domain. Then for every* $\lambda \in (0,\lambda_1)$ *there exists a solution of* (8). *Moreover*

$$\operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|^2_{L^2} - \lambda \|v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} \tag{10}$$

is achieved.

Theorem 2 : *Assume* Ω *is a ball in* $\mathbb{R}^3$. *Then for every* $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$ *there is a solution of* (8) ; *moreover the infimum in* (10) *is achieved. When* $\lambda \leq \frac{\lambda_1}{4}$ *there is no solution of* (8).

Remark 2 : The difference between dimension $N = 3$ and dimension $N \geq 4$ is quite suprizing. We have no simple explanation for it.

Remark 3 : When $N \geq 3$ and $\lambda \geq \lambda_1$ there is *no* solution of (8) (see Section 1). When $N \geq 3$ and Ω is *starshaped* there is no solution of (8) for $\lambda \leq 0$ (by Theorem 0).

Remark 4 : The generalization of Theorem 2 for starshaped domains is not known.

Before we sketch the proofs we present some facts about Sobolev spaces :

a) Define the best Sobolev constant S to be

$$S = \operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} \tag{11}$$

In principle S depends on Ω ; but in fact S depends only on N. This is an easy consequence of the invariance under scaling of the ratio $\frac{\|\nabla v\|_{L^2}}{\|v\|_{L^{p+1}}}$ (that is, the ratio is unchanged if we replace $u(x)$ by $u_k(x) = u(kx)$).

b) The infimum in (11) is never achieved, on any bounded domain. Indeed, suppose that the infimum in (11) is achieved by some function $v_0 \geq 0$. Let $\tilde{\Omega}$ be a ball containing Ω and set

$$\tilde{v}_0 = \begin{cases} v_0 & \text{in } \Omega \\ 0 & \text{in } \tilde{\Omega} \setminus \Omega \end{cases}$$

Thus, for $\tilde{\Omega}$ the infimum in (11) is achieved at v_0 and we find

$$\begin{cases} -\Delta \tilde{v}_0 = \mu v_0^p \text{ on } \Omega \\ v_0 = 0 \text{ on } \partial\Omega \end{cases}$$

for some constant $\mu > 0$. This contradicts Pohozaev's Theorem.

c) When $\Omega = \mathbb{R}^N$ the infimum in (11) is achieved by the function $u(x) = \frac{1}{(1+|x|^2)^{\frac{N-2}{2}}}$ or - after scaling - by any of the functions

$$u(x) = \frac{1}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}} \quad (\varepsilon > 0).$$

see Aubin [3], Talenti [10], Lieb [8].

The following Lemma plays a crucial role in the proof of Theorem 1.

<u>Lemma 1</u> : *Assume* $N \geq 4$. *Then, for every* $\lambda \in (0,\lambda_1)$ *we have*

$$S_\lambda \equiv \operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|^2_{L^2} - \lambda \|v\|^2_{L^2}}{\|v\|^2_{L^{p+1}}} \right\} < S \tag{12}$$

The proof of Lemma 1 is rather technical ; for details see [5]. The main idea - borrowed from Aubin [1] - consists of estimating the ratio

$$Q(u) = \frac{\|\nabla u\|^2_{L^2} - \lambda \|u\|^2_{L^2}}{\|u\|^2_{L^{p+1}}}$$

for $u(x) = u_\varepsilon(x) = \dfrac{\phi(x)}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}}$ where $\phi \in \mathcal{D}_+(\Omega)$ is a fixed function such that $\phi(x) \equiv 1$ near 0 (assuming $0 \in \Omega$).

A straightforward computation gives the following expansion as $\varepsilon \to 0$:

$$Q(u_\varepsilon) = S + O(\varepsilon^{\frac{N}{2}-1}) - \lambda C \varepsilon \qquad \text{when } N \geq 5$$

$$Q(u_\varepsilon) = S + O(\varepsilon) - \lambda C \varepsilon |\log \varepsilon| \quad \text{when } N = 4$$

where $C > 0$ is a constant. In both cases we see that $Q(u_\varepsilon) < S$ for $\varepsilon > 0$ sufficiently small. We shall also use the following measure theoretic lemma.

Lemma 2 : (Brezis-Lieb [4]) *Suppose* (v_j) *is a sequence in* $L^q(\Omega)$ *with* $1 \le q < \infty$ *such that* $\|v_j\|_{L^q}$ *remains bounded and* $v_j(x) \to v(x)$ *a.e. on* Ω. *Then*

$$\lim_{j\to\infty} \{ \int |v_j|^q - \int |v_j - v|^q \} = \int |v|^q \tag{13}$$

Proof of Theorem 1 : Choose a minimzing sequence (v_j) for (10) such that

$$v_j \ge 0 \text{ on } \Omega, \quad \|v_j\|_{L^{p+1}} = 1 \tag{14}$$

$$\int |v_j|^2 - \lambda \int v_j^2 = S_\lambda + o\ (1) \tag{15}$$

Since v_j is bounded in H_0^1 we may assume, for a subsequence, that

$v_j \rightharpoonup v$ weakly in H_0^1

$v_j \to v$ a.e. on Ω

$v_j \to v$ strongly in L^2

Set $w_j = v_j - v$ so that $w_j \rightharpoonup 0$ weakly in H_0^1.

By Lemma 2 (applied with $q = p+1$) we have

$$\lim \|w_j\|_{L^{p+1}}^{p+1} = 1 - \|v\|_{L^{p+1}}^{p+1}.$$

Thus

$$\begin{aligned} 1 &= (\|v\|_{L^{p+1}}^{p+1} + \|w_j\|_{L^{p+1}}^{p+1})^{\frac{2}{p+1}} + o\ (1) \le \\ &\le \|v^2\|_{L^{p+1}} + \|w_j\|_{L^{p+1}}^2 + o\ (1) \end{aligned} \tag{16}$$

On the other hand (since $w_j \rightharpoonup 0$ weakly in H_0^1) we have

$$\int |\nabla v_j|^2 = \int |\nabla v|^2 + \int |\nabla w_j|^2 + o\ (1) \tag{17}$$

Combining (15) (16) and (17) we obtain

$$\int |\nabla v|^2 - \lambda \int v^2 + \int |\nabla w_j|^2 \le S_\lambda [\, \|v\|^2_{L^{p+1}} + \|w_j\|^2_{L^{p+1}}] + o\ (1) \tag{18}$$

By definition of S_λ we have $\int |\nabla v|^2 - \lambda \int v^2 \ge S_\lambda \ \|v\|^2_{L^{p+1}}$ and therefore

$$\int |\nabla w_j|^2 \le S_\lambda \ \|w_j\|^2_{L^{p+1}} + o\ (1)$$

$$\le \frac{S_\lambda}{S} \int |\nabla w_j|^2 \ + o\ (1).$$

Since $S_\lambda < S$ (by Lemma 1), it follows that $\int |\nabla w_j|^2 \to 0$. Consequently $v_j \to v$ strongly in H_0^1 (and in L^{p+1}). Passing to the limit in (14) and (15) we conclude that the infimum in (11) is achieved by v. After stretching we obtain a solution of (8).

In the proof of Theorem 2 we use

<u>Lemma 3</u> : *Assume* $\Omega = \{x \in \mathbb{R}^3 \ ; \ |x| < 1\}$. *Then for each* $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$ *we have*

$$S_\lambda \equiv \operatorname*{Inf}_{v \in H_0^1} \left\{ \frac{\|\nabla v\|^2_{L^2} - \lambda \|v\|^2_{L^2}}{\|v\|^2_{L^6}} \right\} < S \tag{19}$$

Proof : We estimate the ratio

$$Q(u) = \frac{\|\nabla u\|^2_{L^2} - \lambda \|u\|^2_{L^2}}{\|u\|^2_{L^6}}$$

for $u(x) = u_\varepsilon(x) = \dfrac{\cos(\frac{\pi}{2}|x|)}{(\varepsilon+|x|^2)^{1/2}}$.

A technical computation (see [5]) gives the following expansion as $\varepsilon \to 0$:

$$Q(u_\varepsilon) = S + C\sqrt{\varepsilon}\,(\frac{\pi^2}{4} - \lambda) + \mathcal{O}(\sqrt{\varepsilon})$$

where $C > 0$ is a constant. Therefore if $\lambda > \frac{\lambda_1}{4}$ (note that here $\lambda_1 = \pi^2$) we see that $Q(u_\varepsilon) < S$ for $\varepsilon > 0$ sufficiently small.

Proof of Theorem 2 : The same argument as in the proof of Theorem 1 shows that for every $\lambda \in (\frac{\lambda_1}{4},\lambda_1)$ the infimum in (19) is achieved. Consequently we obtain a solution of (8) for each $\lambda \in (\frac{\lambda_1}{4},\lambda_1)$. Next we must show that no solution of (8) exists for $\lambda \geq \frac{\lambda_1}{4}$. By a result of Gidas-Ni-Nirenberg [7] we know that any solution u of (8) in a ball must be spherically symmetric. We write $u(x) = u(r)$ $(r = |x|)$ and so we have

$$- u'' - \frac{2}{r} u' = u^5 + \lambda u \quad \text{on } (0,1) \tag{20}$$

$$u'(0) = u(1) = 0. \tag{21}$$

Then we use an argument "à la Pohozaev" but with more complicated multipliers. Namely we multiply (20) through by

$$r^2[r \cos \pi r - b \sin \pi r]u'$$

and then by

$$r[-\frac{r}{2}(1+b\pi)\cos \pi r - \frac{\pi r^2}{2} \sin \pi r + b \sin \pi r]\, u$$

for some appropriate constant b. Integrating by parts and combining the two equalities leads to $\lambda > \frac{\pi^2}{4}$; for more details see [5].

3. THE GENERAL CASE, $-\Delta u = u^p + f(u)$ WITH $p = \frac{N+2}{N-2}$.

Here again we take $p = \frac{N+2}{N-2}$. Assume f is a C^1 function on $[0,+\infty)$ such that

$$f(0) = 0, \quad f(u) \geq 0 \quad \forall u \geq 0 \tag{22}$$

$$\lim_{u\to\infty} \frac{f(u)}{u^p} = 0 \tag{23}$$

$$f'(0) < \lambda_1 \tag{24}$$

The problem is to find a function u satisfying

$$\begin{cases} u > 0 & \text{on } \Omega \\ -\Delta u = u^p + f(u) & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{25}$$

Our main results are the following

Theorem 3 : *Assume* $N \geq 5$, (22), (23), (24) *and*

$$f \not\equiv 0 \tag{26}$$

Then there is a solution of (25).

Theorem 4 : *Assume* $N = 4$, (22), (23), (24) *and either*

$$f'(0) > 0 \tag{27}$$

or

$$\liminf_{u \to +\infty} \frac{f(u)}{u} > 0 \tag{28}$$

Then there is a solution of (25).

Theorem 5 : *Assume* $N = 3$, (22), (23), (24)

$$\lim_{u \to +\infty} \frac{f(u)}{u^3} = +\infty$$

Then there is a solution of (25).

Remark 5 : Theorem 3, 4 and 5 admit appropriate extensions to the case where f depends also on x with $f(x,0) \equiv 0$. They may be used in order to prove the following :

Theorem 6 : *Assume* $N \geq 3$. *Then there is a constant* $\lambda^* > 0$ *such that the problem*

$$\begin{cases} u > 0 & \text{in } \Omega \\ -\Delta u = \lambda(1+u)^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad p = \frac{N+2}{N-2} \tag{29}$$

has at least two solutions for each $\lambda \in (0,\lambda^*)$ *and no solution for* $\lambda > \lambda^*$.

(A similar result for $p < \frac{N+2}{N-2}$ had been obtained earlier by Crandall-Rabinowitz [6]).

The idea of the proof is the following. Firstly one obtains (easily) a minimal solution $\underline{u}$ of (29) for every $\lambda \in (0,\lambda^*)$ (see e.g. [6]). Then one looks for a second solution of (29) of the form $u = \underline{u} + v$ with $v > 0$ on Ω. Thus v satisfies $-\Delta v = \lambda(1+\underline{u}+v)^p - \lambda(1+\underline{u})^p \equiv \lambda v^p + f(x,v)$ and we are reduced to a problem of the type (25).

The proofs of Theorems 3, 4 and 5 involve two ingredients. Firstly a geometrical result which is a variant of the Ambrosetti-Rabinowitz [1] mountain pass Lemma <u>without the (PS) condition</u> (see Lemma 4). Secondly a technical Lemma which has the same flavour as Lemma 1 or Lemma 3 (see Lemma 5).

Lemma 4 : *Assume* Φ *is a* C^1 *function on a Banach space* E *such that*

$$\Phi(0) = 0 \tag{30}$$

$$\begin{cases} \textit{there exist constants } \rho > 0 \textit{ and } r > 0 \textit{ such that } \Phi(u) \geq \rho \\ \textit{for every } u \in E \textit{ with } \|u\| = r \end{cases} \tag{31}$$

$\Phi(v) \leq 0$ *for some* $v \in E$ *with* $\|v\| > r$. (32)

Set

$$c = \inf_{P \in \mathcal{T}} \sup_{p \in P} \Phi(p) \tag{33}$$

where $\mathcal{T}$ *denotes the class of all paths joining* 0 *to* v. *Then there exists a sequence* (u_j) *in* E *such that* $\Phi(u_j) \to c$ *and* $\Phi'(u_j) \to 0$ *in* E^*.

The proof of Lemma 4 is essentially the same as the proof given in [1]. In order to prove Theorems 3,4 and 5 we apply Lemma 4 in $E = H_0^1$ to the functional

$$\Phi(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int (u^+)^{p+1} - \int F(u^+) \tag{34}$$

where $F(u) = \int_0^u f(t)dt$. Property (31) is an easy consequence of assumption (24). For every $u \in H_0^1$, $u \geq 0$ in Ω, $u \not\equiv 0$ we have $\lim_{t\to+\infty} \Phi(tu) = -\infty$. Hence, there are many v's satisfying (32). However it is essential to make a *special choice* of v in order to be able to use properly the sequence (u_j) given by Lemma 4. More precisely we have

<u>Lemma 5</u> : *Under the assumptions of Theorems 3, 4 and 5 there is some* $u_0 \in H_0^1$, $u_0 \geq 0$ *in* Ω, $u_0 \not\equiv 0$ *and*

$$\sup_{t\geq 0} \Phi(t\, u_0) < \frac{1}{N} S^{N/2}. \tag{35}$$

The proof is rather technical. Let $u_\varepsilon(x) = \dfrac{\phi(x)}{(\varepsilon+|x|^2)^{\frac{N-2}{2}}}$ with $\phi \in \mathcal{D}_+(\Omega)$ and $\phi \equiv 1$ near $x = 0$ (assuming $0 \in \Omega$). We show by an expansion method (as in Lemma 1) that u_ε satisfies (35) provided $\varepsilon > 0$ is sufficiently small. As was already observed, the expansion technique is sensitive to the dimension N ; the cases $N = 3$, $N = 4$ and $N \geq 5$ must be considered separately. (See the details in [5]).

<u>Proofs of Theorem 3, 4 and 5</u> : By Lemma 5 there is some $v \in H_0^1$ such that $\|v\| > r$, $\Phi(v) \leq 0$ and

$$\sup_{t\geq 0} \Phi(tv) < \frac{1}{N} S^{N/2} \tag{36}$$

We apply Lemma 4 with such a v. From (33) it follows that

$$c < \frac{1}{N} S^{N/2} \tag{37}$$

Let (u_j) be the sequence given by Lemma 4. We have

$$\frac{1}{2}\int|\nabla u_j|^2 - \frac{1}{p+1}\int(u_j^+)^{p+1} - \int F(u_j^+) = c + o(1) \tag{38}$$

$$-\Delta u_j = (u_j^+)^p + f(u_j^+) + \zeta_j \tag{39}$$

with $\zeta_j \to 0$ in H^{-1}.

Combining (38) and (39) it is easy to show that $\|u_j\|_{H^1}$ remains bounded.

Thus we may assume that

$$u_j \rightharpoonup u \quad \text{weakly in } H_0^1$$

$$u_j \to u \quad \text{a.e.}$$

From (39) we deduce that

$$-\Delta u = (u^+)^p + f(u^+) \qquad \text{in } H^{-1} \tag{40}$$

By the maximum principle we have $u \geq 0$ and so

$$-\Delta u = u^{p+1} + f(u).$$

It remains to prove that $u \not\equiv 0$. Suppose by contradiction that $u \equiv 0$. Using (23) we obtain

$$\int F(u_j^+) \to 0, \quad \int f(u_j^+)u_j \to 0 \tag{41}$$

We may always assume that

$$\int |\nabla u_j|^2 \to \ell \tag{42}$$

and by (39)

$$\int (u_j^+)^{p+1} \to \ell \tag{43}$$

From (38) we deduce that

$$\frac{1}{N}\,\ell = c > 0. \tag{44}$$

On the other hand we have (by Sobolev inequality)

$$\int |\nabla u_j|^2 \geq S\|u_j\|^2_{L^{p+1}} \geq S\|u_j^+\|^2_{L^{p+1}}$$

and therefore at the limit

$$\ell \geq S\ell^{\frac{2}{p+1}} .$$

Thus

$$\ell \geq S^{N/2}$$

and (by (44))

$$c \geq \frac{1}{N}\, S^{N/2}$$

a contradiction with(37). Therefore $u \not\equiv 0$.

REFERENCES

[1] A. Ambrosetti, P. Rabinowitz, Dual variational methods in critical point theory and applications. J. Funct. Anal. 14 (1973), 349-381.

[2] Th. Aubin, Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire. J. Math. Pures et Appl. 55 (1976), 269-296.

[3] Th. Aubin, Problèmes isopérimétriques et espaces de Sobolev. J. Diff. Géom. 11 (1976), 573-598.

[4] H. Brézis, E. Lieb, to appear.

[5] H. Brézis, L. Nirenberg, to appear.

[6] M. Crandall, P. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. Arch. Rat. Mech. Anal. 58 (1975), 207-218.

[7] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle. Comm. Math. Phys. 68 (1979), 209-243.

[8] E. Lieb, to appear.

[9] S.I. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$. Dokl. Akad. Nauk SSSR 165 (1965), 33-36.

[10] G. Talenti, Best constants in Sobolev inequality. Ann. Math. Pura Appl. 110 (1976), 353-372.

[11] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann. Sc. Norm. Sup. Pisa 22 (1968), 265-274.

Haim BREZIS

Laboratoire d'Analyse Numérique
Tour 55-65 - 5e étage
Université Pierre et Marie Curie
4, place Jussieu

75230 - PARIS CEDEX 05

FRANCE