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Mathematics

Strongly nonlinear parabolic variational inequalities

(pseudomonotone operators/compactness lemma)

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ABSTRACT An existence and uniqueness result is established for a general class of variational inequalities for parabolic partial differential equations of the form $\partial u/\partial t + A(u) + g(u) = f$ with g nondecreasing but satisfying no growth condition. The proof is based upon a type of compactness result for solutions of variational inequalities that should find a variety of other applications.

We consider the strongly nonlinear parabolic partial differential equation of order $2m$

$$\frac{\partial u}{\partial t} + A(u) + g(x,u) = f(x,t) \qquad [1]$$

on the finite cylinder $Q = \Omega \times [0,T]$, in which Ω is a bounded open subset of R^n for which the Sobolev imbedding theorem holds ($n \geq 1$). In a preceding note (1), we derived an existence theorem for the solution of the initial-boundary value problem in $L^p(0,T;W_0^{m,p}(\Omega)) \cap C^0(0,T;L^2(\Omega))$ with $u(0) = 0$, $p \geq 2$. In the present note, we obtain existence and uniqueness results for a general class of variational inequalities for Eq. 1 in $L^p(0,T;W^{m,p}(\Omega))$, which includes as a special case variational boundary value problems in the usual sense.

We impose the hypotheses *i*, *ii*, and *iii* of ref. 1 on the elliptic term $A(u)$, i.e.,

$$A(u) = \sum_{|\beta| \leq m} (-1)^{|\beta|} D^\beta A_\beta(x,t,u, \ldots D^m u),$$

with the corresponding bounds, ellipticity, and coerciveness in $W^{m,p}(\Omega)$. For the strongly nonlinear term $g(x,u)$, we impose a somewhat stronger hypothesis than in ref. 1, namely:

iv′ $g(x,r)$ is measurable in x, continuous in r;
$g(x,0) \equiv 0$ for all x;
$g(x,r)$ is nondecreasing in r for fixed x.

For u and v in $L^p(0,T;W^{m,p}(\Omega))$, we define

$$a(u,v) = \sum_{|\beta| \leq m} \int_0^T (A_\beta(\xi(u)), D^\beta v).$$

Let

$$G(x,r) = \int_0^r g(x,s)ds.$$

By *iv′*, G as a function of r is convex, nonnegative, and once differentiable. Let $X = L^p(0,T;W^{m,p}(\Omega))$, X^* its conjugate space. For any u in X, we set

$$\gamma(u) = \int_Q G(x,u(x,t))dx\,dt.$$

Let C be a closed convex subset of $W^{m,p}(\Omega)$. We define the nonnegative proper lower semicontinuous function φ from X to $R^1 \cup \{+\infty\}$ by setting

$$\varphi(u) = \begin{cases} \gamma(u) & \text{if } u(t) \text{ lies in } C \text{ a.e. (almost everywhere)}, \\ +\infty & \text{otherwise.} \end{cases}$$

Let L be the realization of $\partial/\partial t$ in $L^2(Q)$ with domain given by

$$D(L) = \left\{ v \,|\, v \in L^2(Q),\ \frac{\partial v}{\partial t} \in L^2(Q),\ v \in C(0,T;L^2(\Omega)),\ v(0) = 0 \right\}.$$

THEOREM 1. *Let Ω be a bounded open subset of R^n such that the imbedding map of $W^{m,2}(\Omega)$ into $W^{m-1,2}(\Omega)$ is compact, $p \geq 2$. Suppose that the parabolic equation [1] satisfies the regularity conditions* i, ii, *and* iii *of ref.* 1 *for* A(u) *in* $W^{m,p}(\Omega)$ *and that the strongly nonlinear term* g(x,r) *satisfies* iv′. *Let* f *be a given element of* X^*.

Then there exists u *in* $X \cap C(0,T;L^2(\Omega))$ *with* $u(0) = 0$, g(u) *in* $L^1(Q)$, ug(u) *in* $L^1(Q)$ *for which the following conditions are all satisfied:*

(a) u(t) *lies in* C *for almost all* t *in* [0,T];

(b) *for every* v *in* $X \cap D(L) \cap L^\infty(Q)$ *such that* v(t) *lies in* C *a.e.*
$\langle Lv, v-u\rangle + a(u,v-u) + \int_Q g(u)(v-u) \geq \langle f, v-u\rangle$; *and*

(c) *for every* v *in* $X \cap D(L)$ *for which* $\varphi(v) < +\infty$,
$\langle Lv, v-u\rangle + a(u,v-u) + \varphi(v) - \varphi(u) \geq \langle f, v-u\rangle$.

THEOREM 2. *If, in addition,* A *is monotone (i.e.,*

$$a(u,u-v) - a(v,u-v) \geq 0$$

for all u *and* v *in* X), *then the solution* u *of* Theorem 1 *is uniquely determined by* f.

The proof of Theorem 1 rests upon the compactness result given below for solutions of variational inequalities in *Theorem 3*. It rests upon simple properties of functional spaces and the operator L given in the following proposition.

PROPOSITION 1. *Let* X *be the Banach space* $L^p(0,T;W^{m,p}(\Omega))$, $Y = L^2(0,T;W^{m-1,2}(\Omega))$, $H = L^2(Q) = L^2(0,T;L^2(\Omega))$. *Let* L *be the realization of* $\partial/\partial t$ *defined above,* φ *the proper l.s.c. (lower semi-continuous) convex function defined above. Then the following properties hold:*

(1) L *is a linear maximal monotone operator in the Hilbert space* H.

(2) $X \subset Y \subset H$, *in the sense that each inclusion map is continuous. For each* $\delta > 0$, *and for each* $M > 0$, *there exists* $k_{M,\delta} > 0$ *such that if* u *and* v *are elements of* X *with* $\|u\|_X \leq M$, $\|v\|_X \leq M$, *then*

$$\|u - v\|_Y \leq \delta + k_{M,\delta}\|u - v\|_H.$$

(3) *For each* $\varepsilon > 0$, $(I + \varepsilon L)^{-1}$ *maps* X *into* X, *and for a given constant* c *and all* ε *with* $0 < \varepsilon < 1$, *then for all* u *in* X

$$\|(I + \varepsilon L)^{-1}u\|_X \leq c\|u\|_X.$$

(4) *For each* u *in* X, $\varphi((I + \varepsilon L)^{-1}u) \leq \varphi(u)$ *for all* $\varepsilon > 0$.

(5) *For each* $M > 0$, *the set*

$$\{u \,|\, u \in X \cap D(L),\ \|u\|_X \leq M,\ \|Lu\|_H \leq M\}$$

is strongly relatively compact in Y.

THEOREM 3. *Let* X $\subset$ Y $\subset$ H *be a triple of spaces with* X *and* Y *Banach spaces,* H *a Hilbert space. Let* L *be a linear densely defined operator in* H, φ *a proper l.s.c. convex function from* X *to* $R^1 \cup \{+\infty\}$. *Suppose that properties* 1 *to* 5 *of* Proposition 1 *hold. Suppose that* $\{u_k\}$ *is a bounded sequence in* X *with* $\varphi(u_k) < +\infty$ *for each* k, *and suppose that for each* k *there exists a subset* S_k *of* X $\cap$ D(L) *that contains* $(I + \varepsilon L)^{-1}(u_k)$ *for all* $\varepsilon > 0$ *such that for all* v *in* S_k,

$$(Lv, v - u_k) + \varphi(v) - \varphi(u_k) \geq -c_0$$

for a constant c_0 *independent of* k.

Then the sequence $\{u_k\}$ *is strongly relatively compact in* Y.

Proof of Theorem 3: Because the sequence $\{u_k\}$ is bounded, it follows from property 3 that if we set $v_{k,\varepsilon} = (I + \varepsilon L)^{-1}(u_k)$, then $\|v_{k,\varepsilon}\|_X \leq M$. By assumption $v_{k,\varepsilon} \in S_k$. By property 4, $\varphi(v_{k,\varepsilon}) \leq \varphi(u_k)$. Hence

$$(Lv_{k,\varepsilon}, v_{k,\varepsilon} - u_k) \geq -c_0.$$

Because $Lv_{k,\varepsilon} = \varepsilon^{-1}(u_k - v_{k,})$, it follows that

$$\|v_{k,\varepsilon} - u_k\|_H \leq (c_0\varepsilon)^{1/2}.$$

If M_1 is a bound for $\|u_k\|_H$ for all k, for each fixed $\varepsilon > 0$, $\{v_{k,\varepsilon}\}$ lies in the set $\{v \mid \|v\|_X \leq M,\ \|Lv\|_H \leq 2M_1\varepsilon^{-1}\}$. Hence by property 5, for each fixed $\varepsilon > 0$, $\{v_{k,\varepsilon}\}$ lies in a relatively strongly compact subset of Y. On the other hand, by property 2,

$$\|u_k - v_{k,\varepsilon}\|_Y \leq \delta + k_{M,\delta}(c_0\varepsilon)^{1/2},$$

which can be made uniformly small by making δ and then ε small. Hence $\{u_k\}$ is strongly relatively compact in Y. q.e.d.

PROPOSITION 2. *Let* T *be the mapping of* X *into* X* *given by*

$$\langle Tu, v\rangle = a(u,v).$$

Under assumptions i, ii, *and* iii *of ref.* 1, T *is a bounded continuous coercive mapping of* X *into* X* *that is* Y*-pseudomonotone in the following sense: Let* $\{u_k\}$ *be a sequence in* X *converging weakly to* u *in* X *and strongly to* u *in* Y *and such that* $\overline{\lim}\ \langle Tu_k, u_k - u\rangle \leq 0$. *Then:*

(a) Tu_k *converges weakly to* Tu *in* X*.

(b) $\langle Tu_k, u_k\rangle$ *converges to* $\langle Tu, u\rangle$.

THEOREM 4. *Suppose that* X *and* Y *are reflexive Banach spaces,* H *a Hilbert space, with* X $\subset$ Y $\subset$ H. *Let* L *be a linear maximal monotone mapping of* H, φ *a nonnegative l.s.c. convex function from* X *to* $R^1 \cup \{+\infty\}$ *with* $\varphi(0) < +\infty$, *and suppose that properties* 1 *to* 5 *hold. Let* T *be a bounded, continuous, coercive mapping of* X *into* X* *that is* Y*-pseudomonotone from* X *to* X*.

Then for each f *in* X*, *there exists* u *in* X *with* $\varphi(u) < +\infty$ *and such that for every* v *in* D(L) $\cap$ X,

$$\langle Lv, v - u\rangle + \langle Tu, v - u\rangle + \varphi(v) - \varphi(u) \geq \langle f, v - u\rangle.$$

Moreover, if A *is strictly monotone,* u *is determined by* f.

Proof of Theorem 4: Let C_k be the closed convex subset of X given by

$$C_k = \{v \mid v \in X \cap D(L),\ \|v\|_X \leq M,\ \|Lv\|_H \leq M\}.$$

Then C_k is strongly relatively compact in Y, and for each $\varepsilon > 0$, $(I + \varepsilon L)^{-1}$ maps C_k into itself. Moreover, T is pseudomonotone from C_k to X^* in the standard sense, whereas L is a bounded monotone mapping from C_k to X^*. Hence, by the classical theory of variational inequalities, for each k there exists u_k in C_k such that for all v in C_k,

$$\langle Lu_k + Tu_k, v - u_k\rangle + \varphi(v) - \varphi(u_k) \geq \langle f, v - u_k\rangle$$

with $\varphi(u_k) < +\infty$. Because L is monotone,

$$\langle Lv, v - u_k\rangle \geq \langle Lu_k, v - u_k\rangle.$$

Therefore, for all v in C_k,

$$\langle Lv, v - u_k\rangle + \langle Tu_k, v - u_k\rangle + \varphi(v) - \varphi(u_k) \geq \langle f, v - u_k\rangle.$$

Because 0 lies in each C_k, it follows that

$$\langle Tu_k, u_k\rangle + \varphi(u_k) \leq \langle f, u_k\rangle.$$

Because T is coercive while φ is nonnegative, it follows that there exists M_0 such that for all k, $\|u_k\|_X \leq M_0$.

For each k, let $S_k = \{(I + \varepsilon L)^{-1}(u_k),\ \varepsilon > 0\}$. The sets S_k are uniformly bounded in X and each S_k is contained in C_k. For v in S_k,

$$\langle Lv, v - u_k\rangle + \varphi(v) - \varphi(u_k) \geq \langle f, v - u_k\rangle - \langle Tu_k, v - u_k\rangle \geq -c_0,$$

in which c_0 is a constant independent of k. Hence, by *Theorem 3*, $\{u_k\}$ is strongly relatively compact in Y.

Passing to an infinite subsequence, we may assume that u_k converges weakly to u in X and strongly to u in Y while Tu_k converges weakly to w in X^*. For any v in $X \cap D(L)$, v lies in C_j for some j. Hence for $k \geq j$,

$$\langle Lv, v - u_k\rangle + \langle Tu_k, v - u_k\rangle + \varphi(v) - \varphi(u_k) \geq \langle f, v - u_k\rangle.$$

We obtain immediately that, for each such v,

$$\overline{\lim}\ \langle Tu_k, u_k - u\rangle \leq \overline{\lim}_k \{\langle Lv, v - u_k\rangle + \varphi(v) - \varphi(u_k) + \langle Tu_k, v - u\rangle + \langle f, u_k - v\rangle\},$$

i.e.,

$$\overline{\lim}\ \langle Tu_k, u_k - u\rangle \leq \langle Lv, v - u\rangle + \langle w - f, v - u\rangle + \varphi(v) - \varphi(u).$$

We introduce into this last inequality the elememts $v_\varepsilon = (I + \varepsilon L)^{-1}u$. Because the family $\{v_\varepsilon\}$ is bounded in X as $\varepsilon \to 0$ while $v \to u$ in H, v_ε converges weakly to u in X. Moreover,

$$\varphi(v_\varepsilon) \leq \varphi(u);\ \langle Lv_\varepsilon, v_\varepsilon - u\rangle = -\varepsilon^{-1}\|v_\varepsilon - u\|_H^2 \leq 0.$$

Hence

$$\overline{\lim}\ \langle Tu_k, u_k - u\rangle \leq \underline{\lim}\ \langle w - f, v_\varepsilon - u\rangle = 0.$$

If we apply the Y-pseudomonotonicity of T from X to X^*, we see that, for all v in X, $\langle Tu_k, u_k - v\rangle \to \langle Tu, u - v\rangle$. Hence for all v in $D(L) \cap X$,

$$\langle Lv, v - u\rangle + \langle Tu, v - u\rangle + \varphi(v) - \varphi(u) \geq \langle f, v - u\rangle.$$

q.e.d.

Theorem 4 yields conclusions *a* and *c* of *Theorem 1*. Further argument is needed to obtain the facts that u lies in $C(0,T; L^2(\Omega))$ with $u(0) = 0$, as well as $g(u), ug(u)$ lies in $L^1(Q)$ and the validity of the variational inequality *b*. The first fact follows from the proof of *Theorem II.3* on p. 72 of ref. 2. Another argument using the compactness theorem yields the second set of conclusions as follows:

Proof of Theorem 1: For each positive integer j, let $g^{(j)}(x,r)$ be the truncation of $g(x,r)$ at level j. For fixed j, $g^j(x,u)$ can be absorbed into $A(u)$. We let $G^{(j)}(x,r) = \int_0^r g^{(j)}(x,s)ds$, and let $\gamma^{(j)}$ be the convex linear functional on X given by

$$\gamma^{(j)}(u) = \begin{cases} 0 & \text{if } u(t) \text{ lies in } C \text{ a.e.,} \\ +\infty & \text{otherwise.} \end{cases}$$

Set $\varphi^{(j)}(u) = \gamma^{(j)}(u) + \int_Q G^{(j)}(u)dx\,dt$.

By *Theorem 4*, there exists a solution $u^{(j)}$ in X of the variational inequality for v in $D(L) \cap X$,

$$\langle Lv, v - u^{(j)}\rangle + \langle Tu^{(j)}, v - u_j\rangle + \int_Q g^{(j)}(u) \times (v - u^{(j)})dx\,dt + \gamma^{(j)}(v) - \gamma^{(j)}(u) \geq \langle f, v - u^j\rangle.$$

Using the subgradient inequality for $G^{(j)}(x,r)$ as a function of r, we see that each such $u^{(j)}$ is also a solution of the inequality

$$\langle Lv, v - u^{(j)}\rangle + \langle Tu^{(j)}, v - u^{(j)}\rangle + \varphi^{(j)}(v) - \varphi^{(j)}(u^{(j)}) \geq \langle f, v - u^{(j)}\rangle.$$

Arguing as in the proof of *Theorem 4*, we see that the sequence $\{u^{(j)}\}$ is bounded in X and strongly compact in Y. Moreover, the related sequence $\{\int_Q u^{(j)}g^{(j)}(u^{(j)})dx\,dt\}$ is uniformly bounded. We may pass to an infinite subsequence (again denoted by $u^{(j)}$) that converges weakly in X and strongly in Y to u and such that $u^{(j)}$ and $g^{(j)}(u_j)$ converge a.e. in Q to u and $g(u)$, respectively. It follows immediately that $g^{(j)}(u^{(j)})$ converges strongly in $L^1(Q)$ to $g(u)$, that $g(u)$ lies in $L^1(Q)$, and by Fatou's lemma that $ug(u)$ lies in $L^1(Q)$. Moreover, by the same argument as in the proof of *Theorem 4*,

$$\overline{\lim}_j \langle Tu^{(j)}, u^{(j)} - u\rangle \leq 0.$$

Hence, for all v in X, it follows from the Y-pseudomonotonicity of T that

$$\langle Tu^{(j)}, v - u^{(j)}\rangle \to \langle Tu, v - u\rangle.$$

As in *Theorem 4*, u satisfies the variational inequality c. If we let v be any element of $X \cap D(L) \cap L^\infty(Q)$, and take the limit of inequality 3, we obtain the variational inequality b. q.e.d.

Proof of Theorem 2: Let u_0 and u_1 be two solutions. Let $u = \frac{1}{2}(u_0 + u_1)$ and for $\varepsilon > 0$ set $v_\varepsilon = (I + \varepsilon L)^{-1}u$. If we insert v_ε in the inequality c for u_0 and u_1, and take the limit as $\varepsilon \to 0$, we find that

$$\frac{1}{2}(Tu_0 - Tu_1, u_1 - u_0) + \{2\varphi(u) - \varphi(u_0) - \varphi(u_1)\} \geq 0,$$

from which the conclusion of *Theorem 2* follows. q.e.d.

1. Brézis, H. & Browder, F. E. (1979) *Proc. Natl. Acad. Sci. USA* **76**, 38–40.
2. Brézis, H. (1972) *J. Math. Pures Appl.* **51**, 1–168.