Strongly nonlinear parabolic variational inequalities
(pseudomonotone operators/compactness lemma)

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ABSTRACT. An existence and uniqueness result is established for a general class of variational inequalities for parabolic partial differential equations of the form \( \partial u/\partial t + A(u) + g(u) = f \) with \( g \) nondecreasing but satisfying no growth condition. The proof is based upon a type of compactness result for solutions of variational inequalities that should find a variety of other applications.

We consider the strongly nonlinear parabolic partial differential equation of order 2m

\[
\frac{\partial u}{\partial t} + A(u) + g(x,u) = f(x,t)
\]

on the finite cylinder \( Q = \Omega \times [0,T] \), in which \( \Omega \) is a bounded open subset of \( \mathbb{R}^n \) for which the Sobolev imbedding theorem holds (\( n \geq 1 \)). In a preceding note (1), we derived an existence theorem for the solution of the initial-boundary value problem in \( L^p(0,T;L^p(\Omega)) \cap C^0(0,T,L^q(\Omega)) \) with \( u(0) = 0 \), \( p \geq 2 \). In the present note, we obtain existence and uniqueness results for a general class of variational inequalities for Eq. 1 in \( L^p(0,T;W^{m,p}(\Omega)) \), which includes as a special case variational boundary value problems in the usual sense.

We impose the hypotheses i, ii, and iii of ref. 1 on the elliptic term \( A(u) \), i.e.,

\[
A(u) = \sum_{|\beta|\leq m} (-1)^{|\beta|} D^\beta A_\beta(x,t,u,\ldots,D^mu),
\]

with the corresponding bounds, ellipticity, and coerciveness in \( W^{m,p}(\Omega) \). For the strongly nonlinear term \( g(x,u) \), we impose a somewhat stronger hypothesis than in ref. 1, namely:

\( g(x,r) \) is measurable in \( x \), continuous in \( r \);
\( g(x,0) = 0 \) for all \( x \);
\( g(x,r) \) is nondecreasing in \( r \) for fixed \( x \).

For \( u \) and \( v \) in \( L^p(0,T;W^{m,p}(\Omega)) \), we define

\[
a(u,v) = \sum_{|\beta|\leq m} \int_0^T (A_\beta(\xi(u)),D^\beta v) dt.
\]

Let

\[
G(x,r) = \int_0^r g(x,s)ds.
\]

By \( \varphi(\cdot,\cdot) \), \( C \) as a function of \( r \) is convex, nonnegative, and once differentiable. Let \( X = L^p(0,T;W^{m,p}(\Omega)) \), \( X^* \) its conjugate space. For any \( u \) in \( X \), we set

\[
\gamma(u) = \int_Q G(x,u(x,t))dx dt.
\]

Let \( C \) be a closed convex subset of \( W^{m,p}(\Omega) \). We define the nonnegative proper lower semicontinuous function \( \varphi \) from \( X \) to \( R^1 \cup \{+\infty\} \) by setting

\[
\varphi(u) = \begin{cases}
\gamma(u) & \text{if } u \in C \text{ a.e. (almost everywhere)}, \\
+\infty & \text{otherwise}.
\end{cases}
\]

Let \( L \) be the realization of \( \partial/\partial t \) in \( L^q(Q) \) with domain given by

\[
D(L) = \left\{ v \in L^q(Q), \frac{\partial v}{\partial t} \in L^q(Q), \quad v \in C(0,T,L^q(\Omega)),
\right\}
\]

\( v(0) = 0 \).

THEOREM 1. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^n \) such that the imbedding map of \( W^{m,2}(\Omega) \) into \( W^{m-1,2}(\Omega) \) is compact, \( p \geq 2 \). Suppose that the parabolic equation \( |1| \) satisfies the regularity conditions i, ii, and iii of ref. 1 for \( A(u) \) in \( W^{m,p}(\Omega) \) and that the strongly nonlinear term \( g(x,r) \) satisfies iv'. Let \( f \) be a given element of \( X^* \).

Then there exists \( u \) in \( \Omega \cap C(0,T,L^q(\Omega)) \) with \( u(0) = 0, \( g(u) \) in \( L^q(\Omega) \), \( u(x) \) in \( L^q(\Omega) \) for which the following conditions are satisfied:

(1) \( u(\cdot,t) \) lies in \( C \) for almost all \( t \) in \( [0,T] \);

(2) for every \( v \) in \( \Omega \cap D(L) \cap L^q(Q) \) such that \( v(\cdot,t) \) lies in \( C \) a.e.

\( (LX,v-u) + a(u,v-u) + \int_0^T g(u,v-u) - (f,v-u) \geq 0 \); and

(3) for every \( \varphi \in \Omega \cap D(L) \) for which \( \varphi(v) < +\infty \),

\( (LX,v-u) + a(u,v-u) + \varphi(v) - \varphi(u) \geq 0 \); (1, v – u).

THEOREM 2. If, in addition, \( A \) is monotone (i.e.,

\[
a(u,v-u) - a(v,u-v) \geq 0
\]

for all \( u \) and \( v \) in \( X \), then the solution \( u \) of Theorem 1 is uniquely determined by \( f \).

The proof of Theorem 1 rests upon the compactness result given below for solutions of variational inequalities in Theorem 3. It rests upon simple properties of functional spaces and the operator \( L \) given in the following proposition.

PROPOSITION 1. Let \( X \) be the Banach space \( L^p(0,T;W^{m,p}(\Omega)) \), \( Y = L^q(T;W^{m-1,p}(\Omega)), H = L^q(Q) = L^q(0,T;L^q(\Omega)) \).

Let \( L \) be the realization of \( \partial/\partial t \) defined above, \( \varphi \) the proper l.s.c. (lower semicontinuous) convex function defined above. Then the following properties hold:

(1) \( L \) is a linear maximal monotone operator in the Hilbert space \( H \).

(2) \( X \subset Y \subset H \), in the sense that each inclusion map is continuous. For each \( \delta > 0 \), and for each \( M > 0 \), there exists \( M_M > 0 \) such that if \( u \) and \( v \) are elements of \( X \) with \( \|u\| \leq M, \|v\| \leq M \), then

\[
\|u - v\|_Y \leq \delta + k_M \|u - v\|_H.
\]

(3) For each \( \varepsilon > 0 \), \( (1 + \varepsilon L)^{-1} \) maps \( X \) into \( X \), and for a given constant \( c \) and \( \varepsilon < 1 \), then for all \( u \) in \( X \)

\[
\|u\|_X \leq \|u\|_X + \varepsilon \|u\|_X.
\]

(4) For each \( u \) in \( X \), \( \varphi((1 + c L)^{-1}u) \leq \varphi(u) \) for all \( c > 0 \).

(5) For each \( M > 1 \), the set

\[
\{u \in X \cap D(L), \|u\|_X \leq M, \|Lu\|_H \leq M \}
\]

is strongly relatively compact in \( Y \).
THEOREM 5. Let $X \subset Y \subset H$ be a triple of spaces with $X$ and $Y$ Banach spaces, $H$ a Hilbert space. Let $L$ be a linear densely defined operator in $H$, $J$ a proper l.s.c. convex function from $X$ to $\mathbb{R} ^{+} \cup \{+\infty\}$. Suppose that properties 1 to 5 of Proposition 1 hold. Suppose that $\{u_{k}\}$ is a bounded sequence in $X$ with $J(u_{k}) \leq \varepsilon$ for each $k$ and that for each $k$ there exists a subset $S_{k}$ of $X \cap D(L)$ that contains $(I + \varepsilon l)^{-1/2}(u_{k})$ for all $l > 0$ such that for all $v$ in $S_{k}$,

$$\langle L_{v}, v - u_{k} \rangle + \psi(v) - \psi(u_{k}) \geq -c_{0}$$

for a constant $c_{0}$ independent of $k$.

Then the sequence $\{u_{k}\}$ is strongly relatively compact in $Y$.

Proof of Theorem 5: Because the sequence $\{u_{k}\}$ is bounded, it follows from property 3 that if we set $v_{k} = (I + \varepsilon l)^{-1/2}(u_{k})$, then $\|v_{k}\|_{Y} \leq M$. By assumption $v_{k} \in S_{k}$. By property 4,

$$\psi(v_{k}) \leq \psi(u_{k}).$$

Hence

$$\langle L_{v_{k}}v_{k} - u_{k} \rangle \geq -c_{0}$$

Because $L_{v_{k}}v_{k}$ is bounded for all $k$, for each fixed $x$, $\{v_{k}\}$ lies in the set $\{x \mid \|x\|_{Y} \leq M, \|L_{v}x\|_{L} \leq 2M^{-1}\}$. Hence by property 5, for each fixed $x$ and for each $k$, $\|v_{k}\|_{Y}$ lies in a relatively strongly compact subset of $Y$. On the other hand, by property 2,

$$\|u_{k} - u\|_{Y} \leq \|v_{k} - u\|_{Y} \leq \delta + k_{M}(c_{0})^{1/2},$$

which can be made uniformly small by taking $\delta$ and $c_{0}$ small. Hence $\{u_{k}\}$ is strongly relatively compact in $Y$. q.e.d.

PROPOSITION 2. Let $T$ be the mapping of $X$ into $X^{*}$ given by

$$\langle Tu, v \rangle = a(u, v).$$

Under assumptions i, ii, and iii of ref. 1, $T$ is a bounded continuous coercive mapping of $X$ into $X^{*}$ that is $Y$-pseudo-monotone in the following sense: Let $\{u_{k}\}$ be a sequence in $X$ converging weakly to $u$ in $X$ and strongly to $u$ in $Y$ and such that $\lim \langle Tu_{k}, u_{k} - u \rangle \leq 0$. Then:

(a) $Tu_{k}$ converges weakly to $Tu$ in $X^{*}$.

(b) $Tu_{k}$ converges to $Tu$ in $L_{u}$.

THEOREM 4. Suppose that $X$ and $Y$ are reflexive Banach spaces, $H$ a Hilbert space, $X \subset Y \subset H$. Let $L$ be a linear maximal monotone mapping of $H$, $\psi$ a nonnegative l.s.c. convex function from $X$ to $\mathbb{R} ^{+} \cup \{+\infty\}$ with $\psi(0) = \varepsilon$, and suppose that properties 1 to 5 hold. Let $T$ be a bounded, continuous, coercive mapping of $X$ into $X^{*}$ that is $Y$-pseudo-monotone from $X$ to $X^{*}$.

Then for each $x$ in $X^{*}$, there exists $u$ in $X$ with $\phi(u) \leq \varepsilon$ and such that for every $v$ in $D(L) \cap X$,

$$\langle Lv, v - u \rangle + \psi(v) - \psi(u) \geq (f, v - u).$$

Moreover, if $A$ is monotone, $u$ is determined by $I$.

Proof of Theorem 4: Let $C_{k}$ be the closed convex subset of $X$ given by

$$C_{k} = \{x \mid \|x\|_{H} \leq M, \|L_{x}\|_{L} \leq M\}.$$

Then $C_{k}$ is strongly relatively compact in $Y$, and for each $x > 0$, $(1 + l)^{-1/2}$ maps $C_{k}$ into itself. Moreover, $T$ is pseudo-monotone from $C_{k}$ to $X^{*}$ in the standard sense, whereas $L$ is a bounded monotone mapping from $C_{k}$ to $X^{*}$. Hence, by the classical theory of variational inequalities, for each $k$ there exists $u_{k}$ in $C_{k}$ such that for all $v$ in $C_{k}$,

$$\langle Lu_{k} + Tu_{k}, v - u_{k} \rangle + \phi(v) - \phi(u_{k}) \geq (f, v - u_{k})$$

with $\phi(u_{k}) < +\infty$. Because $L$ is monotone,

$$\langle Lv, v - u_{k} \rangle \geq (Lv, v - u_{k}).$$

Therefore, for all $v$ in $C_{k}$,

$$\langle L_{v}, v - u_{k} \rangle + \psi(v) - \psi(u_{k}) \geq (f, v - u_{k}).$$

Because $0$ lies in each $C_{k}$, it follows that

$$\langle Tu_{k}, u \rangle + \psi(u_{k}) \geq (f, u - u_{k}).$$

Because $T$ is coercive while $\psi$ is nonnegative, it follows that there exists $M_{0}$ such that for all $k$, $\|u_{k}\|_{Y} \leq M_{0}$.

For each $k$, let $S_{k} = (I + \varepsilon l)^{-1/2}(u_{k})$, $l > 0$. The sets $S_{k}$ are uniformly bounded in $X$ and each $S_{k}$ is contained in $C_{k}$. For $v$ in $S_{k}$,

$$\langle L_{v}, v - u_{k} \rangle + \psi(v) - \psi(u_{k}) \geq (f, v - u_{k}) - (Tu_{k}, v - u_{k}) \geq -c_{0},$$

in which $c_{0}$ is a constant independent of $k$. Hence, by Theorem 3, $\{u_{k}\}$ is strongly relatively compact in $Y$.

Passing to an infinite subsequence, we may assume that $u_{k}$ converges weakly to $u$ in $X$ and strongly to $u$ in $Y$ while $Tu_{k}$ converges weakly to $v$ in $X^{*}$. For any $v$ in $X \cap D(L)$, $v$ lies in $C_{j}$ for some $j$. Hence for $k \geq j$,

$$\langle L_{v}, u - v \rangle + \phi(v) - \phi(u_{k}) \geq (f, v - u_{k}).$$

We obtain immediately that, for each such $v$,

$$\lim_{k \to \infty} \langle Tu_{k}, u \rangle \leq \lim_{k \to \infty} \langle L_{v}, u - v \rangle + \phi(v) - \phi(u_{k}) + \langle Tu_{k}, v - u \rangle + (f, v - u_{k}).$$

i.e.,

$$\lim_{k \to \infty} \langle Tu_{k}, u \rangle \leq \langle L_{v}, u \rangle + (f, v - u) + \phi(v) - \phi(u).$$

We introduce into this last inequality the elements $v = (I + \varepsilon l)^{-1/2}u$. Because the family $\{v_{k}\}$ is bounded in $X$ as $\varepsilon \to 0$ while $v \to u$ in $H$, $u$ converges weakly to $u$ in $X$. Moreover,

$$\phi(u_{k}) \leq \phi(u); \langle L_{u}, u_{k} - u \rangle = -\varepsilon^{-1}v_{k} - u_{k} \leq 0.$$

Hence

$$\lim_{k \to \infty} \langle Tu_{k}, u \rangle \leq \lim_{k \to \infty} (f, v_{k} - u_{k} - u) = 0.$$

If we apply the $Y$-pseudo-monotonicity of $T$ from $X$ to $X^{*}$, we see that, for all $v$ in $X$, $\langle (Tu_{k}, u_{k} - u) \rightarrow (Tu_{u}, v) \rangle$. Hence for all $v$ in $D(L) \cap X$,

$$\langle L_{v}, v - u \rangle + \phi(v) - \phi(u) \geq (f, v - u).$$

q.e.d.

Theorem 4 yields conclusions a and c of Theorem 1. Furthermore, it is necessary to obtain the facts that $u$ lies in $C(0, T; L^{a}(Q))$ with $u(0) = 0$, as well as $g(u), u(g(u))$ lies in $L^{1}(Q)$ and the validity of the variational inequality. The first fact follows from the proof of Theorem I in II on p. 78 of ref. 2. Another argument using the compactness theorem yields the second set of conclusions as follows:

Proof of Theorem 1: For each positive integer $j$, let $g^{(j)}(x, r)$ be the truncation of $g(x, r)$ at level $j$. For fixed $j$, $g(x, u)$ can be absorbed into $A(u)$. We let $G^{(j)}(x, r) = \int_{0}^{r} g^{(j)}(x, s)ds$, and let $\gamma^{(j)}u$ be the convex linear functional on $X$ given by

$$\gamma^{(j)}u(u) = \begin{cases} 0 & \text{if } u(t) \text{ lies in } C \text{ a.e.} \\ +\infty & \text{otherwise.} \end{cases}$$

Set $\phi^{(j)}(u) = \gamma^{(j)}(u) + \int_{0}^{\infty} G^{(j)}(u(t))ds$. dt.

By Theorem 4, there exists a solution $u^{(j)}$ in $X$ of the variational inequality for $v$ in $D(L) \cap X$,
\begin{align*}
(Le,v - u^{(t)}) + \langle Tu^{(t)}, v - u^{(t)} \rangle + \int_Q g^{(t)}(u) X (p - u^{(t)}) dx dt + \gamma^{(t)}(v) - \gamma^{(t)}(u) \geq \langle f,v - u^{(t)} \rangle.
\end{align*}

Using the subgradient inequality for \( G^{(t)}(x,r) \) as a function of \( r \), we see that each such \( u^{(t)} \) is also a solution of the inequality
\begin{align*}
(Le,v - u^{(t)}) + \langle Tu^{(t)}, v - u^{(t)} \rangle + \varphi^{(t)}(v) - \varphi^{(t)}(u) \geq \langle f,v - u^{(t)} \rangle.
\end{align*}

Arguing as in the proof of Theorem 4, we see that the sequence \( u^{(t)} \) is bounded in \( X \) and strongly compact in \( Y \). Moreover, the related sequence \( \int_Q u^{(t)} g^{(t)}(u^{(t)}) dx dt \) is uniformly bounded.

We may pass to an infinite subsequence (again denoted by \( u^{(t)} \)) that converges weakly in \( X \) and strongly in \( Y \) to \( u \) and \( g(u) \), respectively. It follows immediately that \( g^{(t)}(u^{(t)}) \) converges strongly in \( L^1(Q) \) to \( g(u) \), that \( g(u) \) lies in \( L^1(Q) \), and by Fatou's lemma that \( u g(u) \) lies in \( L^1(Q) \). Moreover, by the same argument as in the proof of Theorem 4,
\begin{align*}
\lim_{t} \langle Tu^{(t)}, u^{(t)} - u \rangle \leq 0.
\end{align*}

Hence, for all \( v \) in \( X \), it follows from the \( Y \)-pseudomonotonicity of \( T \) that
\begin{align*}
(Le,v - u^{(t)}) + \langle Tu^{(t)}, v - u^{(t)} \rangle \rightarrow \langle Tu, v - u \rangle.
\end{align*}

As in Theorem 4, \( u \) satisfies the variational inequality \( c \). If we let \( v \) be any element of \( X \cap D(L) \cap L^\infty(Q) \), and take the limit of inequality 3, we obtain the variational inequality b. q.e.d.

**Proof of Theorem 2:** Let \( u_0 \) and \( u_1 \) be two solutions. Let \( u = \frac{1}{2}(u_0 + u_1) \) and for \( \epsilon > 0 \) set \( v_\epsilon = (I + \epsilon L)^{-1}u \). If we insert \( v_\epsilon \) in the inequality c for \( u_0 \) and \( u_1 \), and take the limit as \( \epsilon \rightarrow 0 \), we find that
\begin{align*}
\frac{1}{2} \langle Tu_0 - Tu_1, u_1 - u_0 \rangle + [2\varphi(u) - \varphi(u_0) - \varphi(u_1)] \geq 0,
\end{align*}
from which the conclusion of Theorem 2 follows. q.e.d.