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An Estimate Related to the Strong Maximum Principle.

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Sunto. – *Con metodi elementari vengono dimostrate alcune stime di tipo fine per la norma del massimo della soluzione di una equazione lineare ellittica del secondo ordine.*

Let $\Omega \subset \boldsymbol{R}^N$ $(N \geqslant 1)$ be an open set with finite measure $|\Omega|$. Let A be a second order elliptic operator in divergence form

$$Au = -\sum_{i,j} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + \sum_i b_i \frac{\partial u}{\partial x_i} + cu\,,$$

with a_{ij}, b_i, $c \in L^\infty(\Omega)$, $c \geqslant 0$ a.e. on Ω,

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqslant \nu|\xi|^2 \qquad \forall \xi \in \boldsymbol{R}\,,\ \text{a.e. } x \in \Omega,\ \nu > 0\,.$$

Set

$$M = \left(\sum_i \|b_i\|^2_{L^\infty}\right)^{\frac{1}{2}}.$$

For every $\lambda > 0$, let $u_\lambda \in H^1_0(\Omega) \cap L^\infty(\Omega)$ be the unique solution of

$$Au_\lambda + \lambda u_\lambda = 1 \quad \text{in } \Omega\,. \tag{1}$$

(The existence and uniqueness of u_λ follows from Theorem 8.3 in [4] when Ω is bounded; for unbounded domains see [6]. Note that A *is not necessarily coercive*, and one can not use Lax-Milgram's theorem). It follows from the maximum principle that $u_\lambda \leqslant 1/\lambda$; in fact the strong maximum principle leads to $u_\lambda < 1/\lambda$. Our purpose is to provide an explicit bound from below for $|u_\lambda - 1/\lambda|$ in terms of λ, ν, M, $|\Omega|$ and N.

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More precise estimates may be obtained by more delicate methods (using symmetrisation techniques and isoperimetric inequalities): the reader is referred for example to [1], [2], [3]. On the other hand, our method is very elementary.

We thank Prof. H. Weinberger for drawing our attention to the paper [3]. We start with an easy estimate for u_λ.

PROPOSITION 1. – *We have*

$$u(x) \leqslant \frac{1}{\lambda}\left(1 - \exp\left[-\lambda u_0(x)\right]\right) \leqslant \frac{1}{\lambda}\left(\left(1 - \exp\left[-\lambda \|u_0\|_{L^\infty}\right]\right)\right)$$

where $u_0 \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *is the unique solution of*

$$Au_0 = 1 . \tag{2}$$

PROOF. – Set $v(x) = 1/\lambda(1 - \exp[-\lambda u_0(x)])$; so that $v \in H_0^1(\Omega) \cap \cap L^\infty(\Omega)$ (since $u_0 \geqslant 0$ by Theorem 8.1 in [4]). It is easy to verify that

$$Av + \lambda v \geqslant 1 \quad \text{in } \Omega .$$

Thus

$$A(u_\lambda - v) + \lambda(u_\lambda - v) \leqslant 0 \quad \text{in } \Omega$$

and consequently $u_\lambda \leqslant v$ in Ω.

We now provide a bound for $\|u_0\|_{L^\infty}$.

PROPOSITION 2. – *Let* u_0 *be the solution of* (2). *We have*

$$u_0 \leqslant \frac{|\Omega|^{2/N}}{\nu} \varphi_N\left(\frac{M|\Omega|^{1/N}}{\nu}\right)$$

where, for each N, φ_N *is a continuous function on* $\mathbf{R}_+$. *In particular when* $b_i \equiv 0$ *we find*

$$u_0 \leqslant c_N \frac{|\Omega|^{2/N}}{\nu} ,$$

when c_N *is a constant which depends only on* N.

REMARK. – Combining Propositions 1 and 2 when $A = -\Delta$ we find that

$$\sup_\Omega u_\lambda \leqslant \frac{1}{\lambda}\left(1 - \exp\left[-\lambda c_N |\Omega|^{2/N}\right]\right) .$$

A sharper estimate has been obtained in this case by C. Bandle

([3], Theorem 1.1) using symmetrization techniques. For example, if $N = 3$ she finds

$$\sup_{\Omega} u_\lambda \leqslant \frac{1}{\lambda}\left(1 - \frac{\lambda^{1/2} R}{\sinh \lambda^{1/2} R}\right)$$

where $\frac{4}{3}\pi R^3 = |\Omega|$; such an estimate is optimal (equality holds when Ω is a ball).

PROOF OF PROPOSITION 2. – We start with the case where $b_i \equiv 0$ and we use a technique of Hartman-Stampacchia [5]. Multiplying (2) by $(u_0 - t)^+$ with $t \geqslant 0$ leads to

$$\nu \int |\nabla (u_0 - t)^+|^2 \leqslant \int (u_0 - t)^+ . \tag{3}$$

Set

$$\alpha(t) = \text{meas}\,\{x \in \Omega;\, u_0(x) > t\} .$$

In what follows we denote by c various constants depending only on N. We claim that

$$\|(u_0 - t)^+\|_{L^1} \leqslant c \|\nabla (u_0 - t)^+\|_{L^2} \alpha(t)^{1/2+/1} . \tag{4}$$

Indeed recall an inequality of Nirenberg [7]

$$\|\varphi\|_{L^{N/N-1}} \leqslant c \|\nabla \varphi\|_{L^1} \qquad \forall \varphi \in H_0^1(\Omega) \tag{5}$$

with bounded support. From (5) we deduce that

$$\|\varphi\|_{L^1} \leqslant \|\varphi\|_{L^{N/N-1}} |\text{Supp}\,\varphi|^{1/N} \leqslant c \|\nabla \varphi\|_{L^1} |\text{Supp}\,\varphi|^{1/N} \leqslant \tag{6}$$
$$\leqslant c \|\nabla \varphi\|_{L^2} |\text{Supp}\,\varphi|^{1/2+1/N} .$$

Multiplying φ by cut-off functions we see easily that (6) holds for every $\varphi \in H_0^1(\Omega)$. In particular if we choose $\varphi = (u_0 - t)^+$ we deduce from (3) that

$$\nu \|(u_0 - t)^+\|_{L^1} \leqslant c\alpha(t)^{1+2/N} .$$

But

$$\|(u_0 - t)^+\|_{L^1} = \int_t^\infty \alpha(s)\, ds \equiv \beta(t)$$

and therefore we obtain

$$\beta' \beta^{-N/N+2} + c\nu^{N/N+2} \leqslant 0 . \tag{7}$$

Integrating (7) on the interval $(0, \|u_0\|_{L^\infty})$ leads to

$$-\beta(0)^{2/N+2} + c\nu^{N/N+2}\|u_0\|_{L^\infty} < 0$$

i.e.

$$c\nu^{N/N+2}\|u_0\|_{L^\infty} < \|u_0\|_{L^1}^{2/N+2}.$$

We conclude using the fact that $\|u_0\|_{L^1} < |\Omega|\,\|u_0\|_{L^\infty}$.

PROOF OF PROPOSITION 2 IN THE GENERAL CASE. *Step* 1. – Suppose first we have an estimate of the form $\|u_0\|_{L^\infty} < \varphi(M)$ when $|\Omega| = \nu = 1$. The conclusion of Proposition 2 for the case $|\Omega| \neq 1$, $\nu \neq 1$ follows from a simple homogeneity argument. Therefore *we may assume from now on that*, for example, $M = 1$ *and* $\nu = 1$.

Step 2. – We shall prove the following:

LEMMA 3. – *Let* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *be the solution of*

$$Au + u = 1. \tag{8}$$

Then $\|u\|_{L^\infty} < k$ *for some constant* $k < 1$ *which depends only on* $|\Omega|$ *and* N.

PROOF OF LEMMA 3. – Multiplying (8) by $(u-t)^+$, $0 < t < 1$ we find

$$\int |\nabla(u-t)^+|^2 + \int u(u-t)^+ < \int (u-t)^+ + \int |\nabla(u-t)^+|(u-t)^+$$

and consequently

$$\tfrac{1}{2}\int |\nabla(u-t)^+|^2 < (1-t)\int (u-t)^+.$$

As in the case where $b_i \equiv 0$ we derive that

$$\|(u-t)^+\|_{L^1} < c(1-t)\alpha(t)^{1+2/N}.$$

But

$$\|(u-t)^+\|_{L^1} = \int_t^1 \alpha(s)\,ds \equiv \beta(t)$$

and therefore

$$\frac{\beta'}{\beta^{N/N+2}} + \frac{c}{(1-t)^{N/N+2}} < 0. \tag{9}$$

Integrating (9) on the interval (t, k) where $k = \|u\|_{L^\infty}$ we obtain

$$\beta(t)^{2/N+2} \geqslant c[(1-t)^{2/N+2} - (1-k)^{2/N+2}]. \tag{10}$$

On the other hand if we multiply (8) by $u/(1-u)$ (1) we find

$$\int \frac{|\nabla u|^2}{(1-u)^2} \leq \int \frac{|\nabla u| u}{(1-u)} + \int u \leq \frac{1}{2} \int \frac{|\nabla u|^2}{(1-u)^2} + \frac{3}{2}|\Omega| \qquad (\text{since } u \leq 1),$$

and so

$$\int |\nabla \log(1-u)|^2 \leq 3|\Omega| .$$

Thus (by (6))

$$\|\log(1-u)\|_{L^1} \leq c|\Omega|^{1+1/N},$$

that is

$$\int_0^k \frac{1-s}{\alpha(s)}\, ds \leq c|\Omega|^{1+1/N} .$$

Since $\alpha = -\beta'$ we deduce that

$$\int_0^k \frac{\beta(s)}{(1-s)^2}\, ds \leq c|\Omega|^{1+1/N} . \tag{11}$$

Set $\theta = 2/(N+2)$; by Hölder's inequality we have

$$\int_0^k \frac{\beta^\theta(s)}{(1-s)^{1+\theta}}\, ds \leq \left[\int_0^k \frac{\beta(s)}{(1-s)^2}\, ds\right]^\theta |\log(1-k)|^{1-\theta} \leq \tag{12}$$

$$\leq c|\Omega|^{\theta(1+1/N)} |\log(1-k)|^{1-\theta} .$$

Using (10) and (12) we find

$$\left[|\log(1-k)| - \frac{\theta}{1}(1-(1-k)^\theta)\right] \leq c|\Omega|^{\theta(1+1/N)} |\log(1-k)|^{1-\theta}$$

and finally

$$|\log(1-k)| \leq c|\Omega|^{1+1/N} + c .$$

Step 3. – We want to estimate $\|u_0\|_{L^\infty}$ where u_0 is the solution of (2). We have

$$Au_0 + u_0 = 1 + u_0$$

(1) Such a test function was introduced by Trudinger (it is used for example in the proof of Theorem 8.1 in [4]); to make the argument rigorous we should multiply by $u/(1+\varepsilon-u)$ and let $\varepsilon \to 0$.

and so, by Lemma 3,

$$\|u_0\|_{L^\infty} < k(1 + \|u_0\|_{L^\infty}) .$$

Hence

$$\|u_0\|_{L^\infty} < \frac{k}{1-k} < \frac{1}{1-k} < c \exp\left[c|\Omega|1 + 1/N\right] .$$

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