An Estimate Related to the Strong Maximum Principle.

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Sunto. — Con metodi elementari vengono dimostrate alcune stime di tipo fine per la norma del massimo della soluzione di una equazione lineare ellittica del secondo ordine.

Let \( \Omega \subset \mathbb{R}^N (N \geq 1) \) be an open set with finite measure \( |\Omega| \).

Let \( A \) be a second order elliptic operator in divergence form

\[
Au = -\sum_{i,j} \frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial u}{\partial x_i} \right) + \sum_i b_i \frac{\partial u}{\partial x_i} + cu,
\]

with \( a_{ij}, b_i, c \in L^\infty(\Omega), c > 0 \) a.e. on \( \Omega \),

\[
\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geq \nu |\xi|^2 \quad \forall \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega, \nu > 0.
\]

Set

\[
M = \left( \sum_i \|b_i\|_{L^\infty} \right)^{\frac{1}{2}}.
\]

For every \( \lambda > 0 \), let \( u_\lambda \in H_0^1(\Omega) \cap L^\infty(\Omega) \) be the unique solution of

\[
(1) \quad Au_\lambda + \lambda u_\lambda = 1 \quad \text{in } \Omega.
\]

(The existence and uniqueness of \( u_\lambda \) follows from Theorem 8.3 in [4] when \( \Omega \) is bounded; for unbounded domains see [6]. Note that \( A \) is not necessarily coercive, and one can not use Lax-Milgram's theorem). It follows from the maximum principle that \( u_\lambda < 1/\lambda \); in fact the strong maximum principle leads to \( u_\lambda < 1/\lambda \). Our purpose is to provide an explicit bound from below for \( |u_\lambda - 1/\lambda| \) in terms of \( \lambda, \nu, M, |\Omega| \) and \( N \).

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More precise estimates may be obtained by more delicate methods (using symmetrisation techniques and isoperimetric inequalities): the reader is referred for example to [1], [2], [3]. On the other hand, our method is very elementary.

We thank Prof. H. Weinberger for drawing our attention to the paper [3]. We start with an easy estimate for $u_\lambda$.

**Proposition 1.** We have

$$u(x) \leq \frac{1}{\lambda} (1 - \exp [-\lambda u_\lambda(x)]) \leq \frac{1}{\lambda} \left(1 - \exp [-\lambda \|u_\lambda\|_{L^\infty}]\right),$$

where $u_\lambda \in H^1_0(\Omega) \cap L^\infty(\Omega)$ is the unique solution of

$$Au_\lambda = 1.$$  

(2)

**Proof.** Set $v(x) = 1/\lambda (1 - \exp [-\lambda u_\lambda(x)])$; so that $v \in H^1_0(\Omega) \cap L^\infty(\Omega)$ (since $u_\lambda > 0$ by Theorem 8.1 in [4]). It is easy to verify that

$$Av + \lambda v \geq 1 \quad \text{in} \quad \Omega.$$

Thus

$$A(u_\lambda - v) + \lambda(u_\lambda - v) \leq 0 \quad \text{in} \quad \Omega,$$

and consequently $u_\lambda \leq v$ in $\Omega$.

We now provide a bound for $\|u_\lambda\|_{L^\infty}$.

**Proposition 2.** Let $u_\lambda$ be the solution of (2). We have

$$u_\lambda \leq \frac{|\Omega|^{2/N}}{\nu} \varphi_N \left(\frac{M|\Omega|^{1/N}}{\nu}\right),$$

where, for each $N$, $\varphi_N$ is a continuous function on $\mathbb{R}_+$. In particular when $b = 0$ we find

$$u_\lambda \leq c_N \frac{|\Omega|^{2/N}}{\nu},$$

when $c_N$ is a constant which depends only on $N$.

**Remark.** Combining Propositions 1 and 2 when $A = -\Delta$ we find that

$$\sup_{\Omega} u_\lambda \leq \frac{1}{\lambda} \left(1 - \exp [-\lambda c_N |\Omega|^{2/N}]\right).$$

A sharper estimate has been obtained in this case by C. Bandle.
(3), Theorem 1.1) using symmetrization techniques. For example, if $N = 3$ she finds
\[
\sup_{\Omega} u_\lambda < \frac{1}{\lambda} \left( 1 - \frac{\lambda^{1/3} R}{\sinh \lambda^{1/3} R} \right)
\]
where $\# \pi R^2 = |\Omega|$; such an estimate is optimal (equality holds when $\Omega$ is a ball).

PROOF OF PROPOSITION 2. – We start with the case where $b_i \equiv 0$ and we use a technique of Hartman-Stampacchia [5]. Multiplying (2) by $(u_0 - t)^+$ with $t > 0$ leads to
\[
\nu \int |\nabla (u_0 - t)^+|^2 < \int (u_0 - t)^+.
\]
Set
\[
\alpha(t) = \text{meas}\{x \in \Omega; u_0(x) > t\}.
\]
In what follows we denote by $c$ various constants depending only on $N$. We claim that
\[
\| (u_0 - t)^+ \|_{L^1} < c \| \nabla (u_0 - t)^+ \|_{L^\alpha(\Omega)}^{1/\alpha + 1}.
\]
Indeed recall an inequality of Nirenberg [7]
\[
\| \varphi \|_{L^{N/N-1}} < c \| \nabla \varphi \|_{L^1} \quad \forall \varphi \in H_0^1(\Omega)
\]
with bounded support. From (5) we deduce that
\[
\| \varphi \|_{L^1} < \| \varphi \|_{L^{N/N-1}} |\text{Supp } \varphi|^{1/N} < c \| \nabla \varphi \|_{L^1} |\text{Supp } \varphi|^{1/N} < c \| \nabla \varphi \|_{L^1} |\text{Supp } \varphi|^{1/\alpha + 1/N}.
\]
Multiplying $\varphi$ by cut-off functions we see easily that (6) holds for every $\varphi \in H_0^1(\Omega)$. In particular if we choose $\varphi = (u_0 - t)^+$ we deduce from (3) that
\[
\nu \| (u_0 - t)^+ \|_{L^1} < c \alpha(t)^{1+1/N}.
\]
But
\[
\| (u_0 - t)^+ \|_{L^1} = \int_0^\infty \alpha(s) ds = \beta(t)
\]
and therefore we obtain
\[
\beta' \beta^{-N/N+2} + c \beta^{N/N+2} < 0.
\]
Integrating (7) on the interval \((0, \|u_0\|_{L^\infty})\) leads to
\[- \beta(0)^{2N+2} + \|u_0\|_{L^\infty} \|u_0\|_{L^\infty} < 0\]
i.e.
\[\|u_0\|_{L^\infty} \|u_0\|_{L^\infty} < \|u_0\|_{L^\infty}^{2N+2} .\]

We conclude using the fact that \(\|u_0\|_{L^1} < |\Omega|\|u_0\|_{L^\infty} .\)

**Proof of Proposition 2 in the General Case.** Step 1. – Suppose first we have an estimate of the form \(\|u_0\|_{L^\infty} < \varphi(M)\) when \(|\Omega| = \nu = 1\). The conclusion of Proposition 2 for the case \(|\Omega| \neq 1, \nu \neq 1\) follows from a simple homogeneity argument. Therefore we may assume from now on that, for example, \(M = 1\) and \(\nu = 1\).

Step 2. – We shall prove the following:

**Lemma 3.** – Let \(u \in H^1_0(\Omega) \cap L^\infty(\Omega)\) be the solution of
\[(8) \quad Au + u = 1 .\]

Then \(\|u\|_{L^\infty} < k\) for some constant \(k < 1\) which depends only on \(|\Omega|\) and \(N\).

**Proof of Lemma 3.** – Multiplying (8) by \((u - t)^+\), \(0 < t < 1\) we find
\[
\int \nabla(u - t)^+ + \int u(u - t)^+ < \int (u - t)^+ + \int |\nabla(u - t)^+|(u - t)^+ \]
and consequently
\[\frac{1}{2} \int |\nabla(u - t)^+|^2 < (1 - t) \int (u - t)^+ .\]

As in the case where \(b_s \equiv 0\) we derive that
\[\|u - t\|^2 \leq c(1 - t)\alpha(t)^{1+2N} .\]

But
\[\|u - t\|^2 \leq 1 \int \alpha(s) ds \equiv \beta(t) \]
and therefore
\[(9) \quad \frac{\beta'}{\beta^{2N+2}} + \frac{c}{(1 - t)^{2N+2}} < 0 .\]

Integrating (9) on the interval \((t, k)\) where \(k = \|u\|_{L^\infty}\) we obtain
\[\beta(t)^{2N+2} > c[(1 - t)^{2N+2} - (1 - k)^{2N+2}] .\]
On the other hand if we multiply (8) by \( u/(1 - u) \) \(^{(1)}\) we find

\[
\int \frac{\lvert \nabla u \rvert^2}{(1 - u)^2} < \int \frac{\lvert \nabla u \rvert^2}{(1 - u)^2} + \int u \leq \frac{1}{2} \int \frac{\lvert \nabla u \rvert^2}{(1 - u)^2} + \frac{3}{2} |\Omega| \quad (\text{since } u < 1),
\]

and so

\[
\int \lvert \nabla \log (1 - u) \rvert^2 < 3|\Omega|.
\]

Thus (by (6))

\[
\lVert \log (1 - u) \rVert_{L^2} < c|\Omega|^{1+1/N},
\]

that is

\[
\int_0^k \frac{1 - s}{\alpha(s)} ds < c|\Omega|^{1+1/N}.
\]

Since \( \alpha = -\beta' \) we deduce that

\[(11) \int_0^k \frac{\beta(s)}{(1 - s)^2} ds < c|\Omega|^{1+1/N}.
\]

Set \( \theta = 2/(N + 2) \); by Hölder's inequality we have

\[(12) \int_0^k \frac{\beta(s)}{(1 - s)^{1+\theta}} ds \leq \left[ \int_0^k \frac{\beta(s)}{(1 - s)^2} ds \right]^\theta \lVert \log (1 - k) \rVert^{1-\theta} \leq c|\Omega|^{(\theta+1/N)} |\log (1 - k)|^{1-\theta}.
\]

Using (10) and (12) we find

\[
\left[ \lvert \log (1 - k) \rvert - \frac{\theta}{1} (1 - (1 - k)^\theta) \right] < c|\Omega|^{(\theta+1/N)} |\log (1 - k)|^{1-\theta}
\]

and finally

\[
|\log (1 - k)| < c|\Omega|^{1+1/N} + c.
\]

Step 3. - We want to estimate \( \lVert u_0 \rVert_{L^\infty} \) where \( u_0 \) is the solution of (2). We have

\[
Au_0 + u_0 = 1 + u_0
\]

\(^{(1)}\) Such a test function was introduced by Trudinger (it is used for example in the proof of Theorem 8.1 in [4]); to make the argument rigorous we should multiply by \( u/(1 + \varepsilon - u) \) and let \( \varepsilon \to 0 \).
and so, by Lemma 3,

\[ \|u_0\|_{L^\infty} < k(1 + \|u_0\|_{L^\infty}). \]

Hence

\[ \|u_0\|_{L^\infty} < \frac{k}{1 - k} < \frac{1}{1 - k} < c \exp \left( c|\Omega|1 + 1/N \right). \]

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