Periodic solutions of a nonlinear wave equation

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SYNOPSIS

We provide a sufficient and "almost" necessary condition for the existence of a periodic solution of the equation

$$u_{tt} - u_{xx} + F(x, t, u) = 0$$

where $F$ is nondecreasing in $u$ and has a small linear growth as $|u| \to \infty$.

1. INTRODUCTION

Consider the nonlinear wave equation

$$u_{tt} - u_{xx} \pm F(x, t, u) = 0 \quad 0 < x < \pi, \quad t \in \mathbb{R}$$

(1)

under the boundary condition

$$u(0, t) = u(\pi, t) = 0 \quad t \in \mathbb{R}.$$  

(2)

We seek solutions which are $2\pi$-periodic in time. Set $\Omega = (0, \pi) \times (0, 2\pi)$. We make the following assumptions on $F$:

1. $F(x, t, u)$ is measurable in $(x, t)$ for fixed $u$ and continuous nondecreasing in $u$ almost every $(x, t)$; $F$ is $2\pi$-periodic in $t$.

2. $|F(x, t, u)| \leq \gamma |u| + C$ for almost every $(x, t)$, for every $u \in \mathbb{R}$ where $C$ and $\gamma$ are constants such that $\gamma < 3$ or $\gamma < 1$ according to whether we have $+$ or $-$ in (1).

(3)

(4)

The generalized nullspace of the operator $A = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ acting on functions satisfying (2) and $2\pi$-periodic in $t$ consists of the functions $\psi$ of the form

$$N = \{\psi(x, t) = p(t + x) - p(t - x); \quad p \in L^2_{\text{loc}}(\mathbb{R}), \quad p \text{ is } 2\pi\text{-periodic}\}.$$  

Finally assume there exists $\varphi \in L^\infty(\Omega)$ and constants $M, \delta > 0$ such that

$$\varphi \in N^\perp, \quad \text{i.e.,} \quad \int_\Omega \varphi \psi \, dx \, dt = 0 \quad \text{for every} \quad \psi \in N$$  

(5)

$$\varphi(x, t) + F(x, t, M) \geq \delta, \quad \varphi(x, t) + F(x, t, -M) \leq -\delta \quad \text{almost everywhere on} \ \Omega.$$  

(6)

Our main result is the following:

THEOREM 1. Assume (3), (4), (5), (6). Then there exists a (weak) solution $u \in L^\infty$ of (1), (2).
Remarks. 1. Theorem 1 is a generalization of one of the results in [2]. In [2] the assumptions are (3), (4) and \(|F(x, t, u)| \to \infty\) as \(|u| \to \infty\) for every \((x, t)\); thus (5), (6) are satisfied with \(\varphi = 0\). The proof of Theorem 1 relies heavily on some of the techniques introduced in [1, 2].

2. Note that (5)–(6) is "almost" necessary for the solvability of (1), (2). Indeed if (1), (2) has a solution \(u \in L^\infty\), set \(\varphi(x, t) = -F(x, t, u), M > \|u\|_{L^\infty}\).

Then we have (5) (since \(\varphi = Au\)) and almost everywhere on \(\Omega\).

\[
\varphi(x, t) + F(x, t, M) \leq 0, \quad \varphi(x, t) + F(x, t, -M) \leq 0. \tag{6'}
\]

In fact when \(F(x, t, u)\) is continuous on \(\bar{\Omega} \times \mathbb{R}\) and strictly increasing in \(u\) for each \((x, t) \in \bar{\Omega}\), then (5) (6) is a necessary and sufficient condition for solvability.

3. Assume now \(F(x, t, u)\) has the special form \(F(x, t, u) = g(u) - f(x, t)\) where \(f\) is continuous and nondecreasing in \(u\), and \(f \in L^\infty\). Then (1) (2) has a solution provided

\[
|g(u)| \leq \gamma |u| + C \quad \text{with} \quad \gamma < 3 \quad \text{or} \quad \gamma < 1, \tag{7}
\]

\(f\) can be split as a sum

\[
f(x, t) = f'(x, t) + f''(x, t)
\]

with

\[
f' \in L^\infty \quad \int \psi dx \, dt = 0 \quad \text{for every} \quad \psi \in N, \tag{8}
\]

\[
g(-\infty) < \inf_\Omega f'' \leq \sup_\Omega f'' < g(+\infty). \tag{9}
\]

(Choose \(\varphi = f'\) and \(M\) large enough so that \(g(M) > \sup_\Omega f'', g(-M) < \inf_\Omega f''\)).

Note that (8) (9) is again an "almost" necessary and sufficient condition for solvability.

4. Assumption (5)–(6) is equivalent to

\[
F(x, t, M) - F(x, t, -M) \geq 2\delta \quad \text{almost everywhere on} \ \Omega \tag{10}
\]

\[
\int \left[ F(x, t, M)\psi^+ - F(x, t, -M)\psi^- \right] dx \, dt \geq \|\psi\|_{L^1} \quad \text{for every} \quad \psi \in N. \tag{11}
\]

It is clear that (5), (6) implies (10), (11). Conversely (11) can be written as

\[
-\int \left[ F(x, t, M) + F(x, t, -M) \right] \psi dx \, dt \leq \int \left[ F(x, t, M) - F(x, t, -M) - 2\delta \right] |\psi| dx \, dt
\]

for every \(\psi \in N\).

We deduce from the Hahn-Banach theorem that there exists \(h \in L^\infty(\Omega)\) such that

\[
\int h\psi dx \, dt \leq \int \left[ F(x, t, M) - F(x, t, -M) - 2\delta \right] |\psi| dx \, dt \quad \text{for every} \quad \psi \in L^1(\Omega) \tag{12}
\]
and
\[-\int_n \left[ F(x, t, M) + F(x, t, -M) \right] \psi \, dx \, dt = \int_n h \psi \, dx \, dt \quad \text{for every} \quad \psi \in N. \tag{13}\]

By (12) we have
\[F(x, t, M) - F(x, t, -M) - 2\delta \geq \pm h\]
and we obtain (5), (6) by choosing
\[\varphi(x, t) = -\frac{1}{2} \left[ h(x, t) + F(x, t, M) + F(x, t, -M) \right].\]

Assumption (11) is of the same nature as the assumption introduced by Landesman and Lazer [4] in order to solve some nonlinear elliptic at resonance. Our equation has similar features, except for the fact that the nullspace is infinite dimensional – this leads to some major new difficulties. Nonlinear wave equations at resonance have also been considered in [6] and [7].

2. Proof of Theorem 1

We shall suppose that we have the + sign in (1) (the argument is essentially the same for the – sign).

We first recall some well known facts concerning the operator \( A \) under (2) (see [3, 5, 8]).

For \( u \in L^2(\Omega) \) we set
\[u = u_1 + u_2 \quad \text{with} \quad u_1 \in N^+, \quad u_2 \in N.\]

We have \( R(\Lambda) = N^\perp \) and
\[
\|u_1\|_{H^1} \leq C \|Au\|_{L^2} \quad \text{for every} \quad u \in D(\Lambda) \tag{14}
\]
\[
\|u_1\|_{L^\infty} \leq C \|Au\|_{L^2} \quad \text{for every} \quad u \in D(\Lambda). \tag{15}
\]

Given a function \( u(x, t) \) which is \( 2\pi \)-periodic in \( t \) we set
\[(Qu)(t) = \frac{1}{2\pi} \int_0^\pi \left[ u(x, t-x) - u(x, t+x) \right] \, dx. \tag{16}\]

Clearly
\[
\|Qu\|_{L^1(0, 2\pi)} \leq \frac{1}{\pi} \|u\|_{L^\infty(\Omega)}. \tag{17}
\]

Also
\[Qu = 0 \quad \text{for every} \quad u \in N^+. \tag{18}\]

Indeed let \( p \in L^2_{\text{loc}}(\mathbb{R}) \) be \( 2\pi \)-periodic in \( t \): we have
\[
\int_0^{2\pi} (Qu)(t) p(t) \, dt = \frac{1}{2\pi} \int_\Omega u(x, t) \left[ p(t+x) - p(t-x) \right] \, dx \, dt = 0.
\]
Any $\psi \in N$ can be written in a unique way as

$$\psi(x, t) = p(t + x) - p(t - x) \quad \text{with} \quad \int_0^{2\pi} p(t) \, dt = 0$$

and in fact $p = Q\psi$.

Finally, recall that

$$(Au, u) \leq -\frac{1}{2} \|Au\|_{L^2}^2 \quad \text{for every} \quad u \in D(A). \quad (19)$$

We shall prove Theorem 1 by considering for $\varepsilon > 0$ small, the equation

$$\varepsilon u_\varepsilon + Au_\varepsilon + F(x, t, u_\varepsilon) = 0 \quad (20)$$

and then letting $\varepsilon \to 0$.

According to [1 Th. 1.8], there exists a solution $u_\varepsilon \in L^2$ of (20) provided $\varepsilon + \gamma < 3$.

We shall divide the proof into four steps:

**Step 1.**

$$\|Au_\varepsilon\|_{L^2} \leq C,$$

**Step 2.**

$$\|u_\varepsilon\|_{L^1} \leq C,$$

**Step 3.**

$$\|u_\varepsilon\|_{L^\infty} \leq C,$$

**Step 4.** Passage to the limit as $\varepsilon \to 0$,

where $C$ denotes various constants independent of $\varepsilon$.

**Step 1.** $\|Au_\varepsilon\|_{L^2} \leq C$.

For simplicity we drop $\varepsilon$. It follows from (6) that for some function

$$w \in L^\infty(\Omega)(\|w\|_{L^\infty} \leq M)$$

we have

$$\varphi(x, t) + F(x, t, w(x, t)) = 0.$$

Since $\varphi \in N^L$, we have $\varphi = Av$ for some $v \in L^\infty$. Therefore we obtain

$$\varepsilon u + A(u - v) + F(x, t, u) - F(x, t, w) = 0.$$

Taking the $L^2$ scalar product with $u - v$ and using (19) we find

$$\varepsilon \int_\Omega (u - v) \, dx \, dt + \int_\Omega (F(x, t, u) - F(x, t, w))(u - v) \, dx \, dt \leq \frac{1}{2} \|Au - Av\|_{L^2}^2.$$

Hence

$$\varepsilon \int_\Omega u^2 \, dx \, dt + \int_\Omega |F(x, t, u) - F(x, t, w)| \, |u - w| \, dx \, dt$$

$$\leq \varepsilon \int_\Omega uv \, dx \, dt + \frac{1}{2} \|Au - Av\|_{L^2}^2 + \|F(x, t, u) - F(x, t, w)\|_{L^2} \|v - w\|_{L^2}.$$
and consequently
\[ \frac{\varepsilon}{2} \int_\Omega u^2 \, dx \, dt + \int_\Omega |F(x, t, u) - F(x, t, w)| \, |u| \, dx \, dt \]
\[ \leq \frac{1}{2} \| u \|_{L^2}^2 + C \| F(x, t, u) \|_{L^2} + C. \]

Using now (4) and (20) we see that
\[ \frac{\varepsilon}{2} \int_\Omega u^2 \, dx \, dt + \frac{1}{\gamma} \int_\Omega |F(x, t, u)|^2 \, dx \, dt \]
\[ \leq \frac{1}{2} \| F(x, t, u) + \varepsilon u + A v \|_{L^2}^2 + C \| F(x, t, u) \|_{L^2} + C. \]

It follows easily that
\[ \| F(x, t, u) \|_{L^2} \leq C \]
\[ \varepsilon \| u \|_{L^2}^2 \leq C \]
\[ \| Au \| \leq C. \]

**Step 2.** \( \| u_e \|_{L^1} \leq C \)

From (6) we deduce that for any \( h \in L^\infty(\Omega) \) with \( \| h \|_{L^\infty} \leq \delta \), there exists some function \( w_h(x, t) \in L^\infty \) with \( \| w_h \|_{L^\infty} \leq M \) such that
\[ \varphi(x, t) + F(x, t, w_h(x, t)) = h(x, t). \]

hence
\[ \varepsilon u + A(u - v) + F(x, t, u) - F(x, t, w_h) = -h. \]

Taking the \( L^2 \) scalar product with \( u - w_h \) we obtain
\[ \varepsilon (u, u - w_h) + (A(u - v), u - w_h) \leq -(h, u - w_h). \]

By Step 1,
\[ (A(u - v), u - w_h) = (A(u - v), u - v + v - w_h) \]
\[ \geq -\frac{1}{2} \| A(u - v) \|_{L^2}^2 - \| A(u - v) \|_{L^2} \| v - w_h \| \geq -C. \]

Therefore
\[ (h, u) \leq C \quad \text{for every} \quad h \in L^\infty \quad \text{with} \quad \| h \|_{L^\infty} \leq \delta. \]

Hence
\[ \delta \| u \|_{L^1} \leq C. \]

**Step 3.** \( \| u_e \|_{L^1} \leq C. \)

Write
\[ u(x, t) = u_1(x, t) + u_2(x, t) \quad \text{with} \quad u_2(x, t) = p(t + x) - p(t - x). \]
By Step 1 and (15) we know that \( \|u_1\|_{L^1} \leq C \). By Step 2 and (17) we know that \( \|p\|_{L^1} \leq C \). Set \( G(x, t, u) = F(x, t, u) + \phi(x, t) \) for \( u \in \mathbb{R} \). Since \( Q\phi = 0 \), we deduce from (20) that
\[
\varepsilon p + QG(x, t, u) = 0,
\]
i.e.
\[
\varepsilon p(t) + \frac{1}{2\pi} \int_0^{2\pi} [G(x, t-x, u_1(x, t-x) + p(t) - p(t-2x))
-G(x, t+x, u_1(x, t+x) + p(t+2x) - p(t))] \, dx = 0.
\]
Thus for \( t \in (0, 2\pi) \),
\[
\varepsilon p(t) + \frac{1}{2\pi} \int_0^{2\pi} [G(x, t-x, -C+p(t) - p(t-2x)) - G(x, t+x, C+p(t+2x) - p(t))] \, dx \leq 0.
\]
Finally we set for \( u \in \mathbb{R} \)
\[
G(u) = \inf_{x,t} G(x, t, u),
\]
\[
\tilde{G}(u) = \sup_{x,t} G(x, t, u).
\]
We find
\[
\varepsilon p(t) + \frac{1}{2\pi} \int_0^{2\pi} \left[ G(-C+p(t) - p(t-2x)) - \tilde{G}(C+p(t+2x) - p(t)) \right] \, dx \leq 0,
\]
i.e.
\[
\varepsilon p(t) + \frac{1}{2\pi} \int_0^{2\pi} \left[ G(-C+p(t) - p(s)) - \tilde{G}(C+p(s) - p(t)) \right] \frac{ds}{2} \leq 0.
\]
Finally set for \( u \in \mathbb{R} \)
\[
H(u) = \frac{1}{2} [G(u) - \tilde{G}(-u)].
\]
So that \( H(u) \geq 0 \) for \( u \geq M \) (by (6)). We have
\[
\varepsilon p(t) + \frac{1}{2\pi} \int_0^{2\pi} H(-C+p(t) - p(s)) \, ds \leq 0.
\]
Let
\[
\mu = \sup_{t \in (0, 2\pi)} p(t) \quad (\mu \geq 0 \text{ since } \int_0^{2\pi} p(t) \, dt = 0).
\]
We obtain
\[
\int_0^{2\pi} H(-C+\mu - p(s)) \, ds \leq 0.
\] (21)
Let
\[ \Sigma = \{ s \in (0, 2\pi); \ p(s) \geq \frac{1}{2} \mu \} \]
\[ ^{c}\Sigma = \{ s \in (0, 2\pi); \ p(s) < \frac{1}{2} \mu \}. \]

Since \( \int_{0}^{2\pi} |p(s)| \, ds \leq C \) we have \( \text{meas } \Sigma \leq \frac{2C}{\mu} \) and \( \text{meas } ^{c}\Sigma \geq 2\pi - \frac{2C}{\mu} \). On the other hand we have
\[
\int_{\Sigma} H(-C + \mu - p(s)) \, ds \geq H(-C) \, \text{meas } \Sigma \geq -|H(-C)| \frac{2C}{\mu}, \tag{22}
\]
\[
\int_{^{c}\Sigma} H(-C + \mu - p(s)) \, ds \geq H\left(-C + \frac{\mu}{2}\right) \, \text{meas } ^{c}\Sigma \geq H\left(-C + \frac{\mu}{2}\right) \left(2\pi - \frac{2C}{\mu}\right). \tag{23}
\]

Combining (21), (22) (23) we find
\[
H\left(-C + \frac{\mu}{2}\right) \left(2\pi - \frac{2C}{\mu}\right) \leq |H(-C)| \frac{2C}{\mu}.
\]

Clearly this leads to a bound for \( \mu \).

**Step 4. Passage to the limit as \( \varepsilon \to 0 \).**

We repeat an argument from [1, section 1.3] to show that \( u \) is a solution of (1) (2).

For any \( \xi \in L^{2}(\Omega) \), we have by monotonicity of \( F \),
\[
(F(x, t, u_{\varepsilon}) - F(x, t, \xi), u_{\varepsilon} - \xi) \geq 0.
\]

Thus
\[
(-\varepsilon u_{\varepsilon} - Au_{\varepsilon} - F(x, t, \xi), u_{\varepsilon} - \xi) \geq 0.
\]

We have
\[
u_{n} \to u \quad \text{in } \text{w}^{*} \to L^{\infty}
\]
\[Au_{n} \to Au \quad \text{weakly in } L^{2}
\]
\[(Au_{n}, u_{n}) \to (Au, u);
\]

indeed \( (Au_{n}, u_{n}) = (Au_{n}, u_{1}) \) and \( u_{1} \to u_{1} \) in \( L^{2} \) by (14) imply \( (Au_{n}, u_{n}) \to (Au, u) \).

Hence we deduce that
\[
(-Au - F(x, t, \xi), u - \xi) \geq 0 \quad \text{for every } \xi \in L^{2}.
\]

We conclude, using Minty's trick, that \( Au + F(x, t, u) = 0 \).
REFERENCES


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