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UNIQUENESS OF SOLUTIONS OF THE INITIAL-VALUE PROBLEM FOR $u_t - \Delta\varphi(u) = 0$ (*)

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Introduction

This paper is concerned with the uniqueness of solutions of the initial value problem

$$(1)\quad \begin{cases} \text{(i)} & u_t - \Delta\varphi(u) = 0, \quad 0 < t < T, \quad x \in \mathbf{R}^N, \end{cases}$$

where

$$(2)\qquad \varphi : \mathbf{R} \to \mathbf{R} \text{ is nondecreasing, continuous and } \varphi(0) = 0.$$

Equations of this sort arise in many applications. These include heat flow in materials with a temperature dependent conductivity, flow in a porous medium, the Stefan problem, biological models, etc.

The main result is formulated below. We have set $Q = (0, T) \times \mathbf{R}^N$ and the expression "in $\mathscr{D}'(Q)$" means in the sense of distributions on Q.

THEOREM 1. — *Let* (2) *hold and* u, $\hat{u}$ *satisfy*

$$(3)\qquad u,\ \hat{u} \in L^\infty(Q),$$

$$(4)\qquad u_t - \Delta\varphi(u) = \hat{u}_t - \Delta\varphi(\hat{u}) \quad \textit{in } \mathscr{D}'(Q),$$

$$(5)\qquad u - \hat{u} \in L^1(Q),$$

and

$$(6)\qquad \underset{t \downarrow 0}{\textit{essential limit}} \int_{\mathbf{R}^N} |u(t, x) - \hat{u}(t, x)|\, dx = 0.$$

Then $u = \hat{u}$ *a. e. on* Q.

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Theorem 1 implies that bounded solutions u of (1) (i) in the sense of distributions which further satisfy $w(t, x)=u(t, x)-u_0(x)\in L^1(Q)$ and $w(t,.)\to 0$ in $L^1(\mathbf{R}^N)$ as $t\downarrow 0$ are unique. Among the earlier uniqueness results we mention the works of Sabinina [11] and Vol'pert and Hudjaev [14]. Sabinina announces a Theorem which can be proved by the method exposed in [10] while Vol'pert and Hudjaev consider a broad class of equations (including first order ones) and use the relatively deep theory of BV spaces both in the formulation of their results and the proofs. In any case, as applied to (1), these results assume significantly more regularity of φ than mere continuity as well as conditions on grad $\varphi(u)$ which we do not impose. Here "grad" denotes the gradient with respect to $(x_1, \ldots, x_N)$. On the other hand, given (3), our conditions (5) and (6) are somewhat more stringent than those of [11]. This will be rectified in the remarks ending Section 1. Other works concerning the uniqueness question for (1) and variants of it are, for the most part, concerned with one space variable. See, for example, Gilding and Peletier [6], Kalašnikov [7], Kamin [8], and Kershner [9].

In some circumstances of interest we can weaken (3) [which corresponds to $u_0\in L^\infty(\mathbf{R}^N)$ in (1)]. In particular, we have:

THEOREM 2. — *Let* $\alpha>\max((N-2)/N, 0)$ *and* $\varphi(r)=|r|^\alpha \operatorname{sign} r$. *Then for each* $u_0\in L^1(\mathbf{R}^N)$ *there is exactly one function u satisfying*

$$u\in C([0, \infty): L^1(\mathbf{R}^N))\cap L^\infty([a, \infty)\times\mathbf{R}^N) \quad \textit{for every } a>0, \tag{7}$$

$$u_t-\Delta\varphi(u)=0 \quad \textit{in } \mathscr{D}'((0, \infty)\times\mathbf{R}^N), \tag{8}$$

and

$$u(0,.)=u_0(.). \tag{9}$$

Theorem 1 is proved in Section 1 and Theorem 2 is proved in Section 2. Both sections include remarks concerning variations of these results.

Section 1. The Proof of Theorem 1.

Let u, $\hat{u}$ be as in Theorem 1. Then the functions $z=u-\hat{u}$, $h=\varphi(u)-\varphi(\hat{u})$ satisfy the conditions of the following Lemma, which therefore implies Theorem 1.

PROPOSITION 1. — *Let* $z\in L^1(Q)\cap L^\infty(Q)$ *and* $h\in L^\infty(Q)$. *Let*

$$z_t-\Delta h=0 \quad \textit{in } \mathscr{D}'(Q), \tag{1.1}$$

$$zh\geqq 0 \quad \textit{a. e. in } Q, \tag{1.2}$$

$$\operatorname{meas}\{(t, x)\in Q: |h(t, x)|>\xi\}<\infty \quad \textit{for each } \xi>0, \tag{1.3}$$

where meas A *is the Lebesgue measure of* A, *and*

$$\underset{t\downarrow 0}{\textit{essential limit}} \int_{\mathbf{R}^N} |z(t, x)|\,dx=0. \tag{1.4}$$

Then $z=0$ *a. e. on* Q.

It is obvious that $z = u - \hat{u}$ and $h = \varphi(u) - \varphi(\hat{u})$ satisfy the conditions of Lemma 1, except perhaps for (1.3). In order to verify (1.3) observe that since u, $\hat{u} \in L^\infty(Q)$ and φ is continuous, for each $\xi > 0$ there is a $\delta > 0$ such that $|\varphi(u(t, x)) - \varphi(\hat{u}(t, x))| > \xi$ implies $|u(t, x) - \hat{u}(t, x)| > \delta$. But $u - \hat{u} \in L^1(Q)$ implies meas $\{(t, x) : |u(t, x) - \hat{u}(t, x)| > \delta\} < \infty$ and so (1.3) holds.

Proof of Proposition 1. — It is well-known (e. g. [3], [12]) that for each $\varepsilon > 0$ and $g \in L^p(\mathbf{R}^N)$, $1 \leqq p \leqq \infty$, the problem

$$\varepsilon v_\varepsilon - \Delta v_\varepsilon = g \quad \text{in } \mathscr{D}'(\mathbf{R}^N), \tag{1.5}$$

has a unique solution $v_\varepsilon \in L^p(\mathbf{R}^N)$. Defining B_ε by $B_\varepsilon g = v_\varepsilon$ one also has the estimate

$$\varepsilon \| B_\varepsilon g \|_p \leqq \| g \|_p, \tag{1.6}$$

where $\| \quad \|_p$ will denote either the norm of $L^p(\mathbf{R}^N)$ or the norm of $L^p(Q)$ depending on the context. Because of (1.6), B_ε defines mappings $B_\varepsilon : L^p(Q) \to L^p(Q)$ for $1 \leqq p \leqq \infty$ and (1.6) holds equally for $g \in L^p(\mathbf{R}^N)$ and $g \in L^p(Q)$. Under the assumptions of Proposition 1 we have

$$\int_0^T \int_{\mathbf{R}^N} (z \psi_t + h \Delta\psi)\, dx\, dt = 0 \qquad \text{for} \quad \psi \in \mathscr{D}(Q), \tag{1.7}$$

where $\mathscr{D}(Q)$ is the space of C^∞ functions with compact support in Q. Fixing $\gamma \in \mathscr{D}(Q)$ we wish to set $\psi = B_\varepsilon \gamma$ in (1.7). Clearly $B_\varepsilon \gamma \in C^\infty(Q)$ (since B_ε commutes with differentiations) and $(B_\varepsilon \gamma)(t, x) = 0$ for t near 0 or T. Moreover, since $z, h \in L^\infty(Q)$, (1.7) clearly continues to hold for $\psi \in C^\infty(Q) \cap L^1(Q)$ with $\psi(t, x) = 0$ for t near 0 and T provided that ψ_t, $\Delta\psi$ and $|\operatorname{grad} \psi| \in L^1(Q)$. $B_\varepsilon \gamma$ has these properties. Moreover, $\Delta B_\varepsilon \gamma = \varepsilon B_\varepsilon \gamma - \gamma$ and $(B_\varepsilon \gamma)_t = B_\varepsilon(\gamma_t)$. Thus

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}^N} (z B_\varepsilon(\gamma_t) + h(\varepsilon B_\varepsilon \gamma - \gamma))\, dx\, dt \\ &\qquad = \int_0^T \int_{\mathbf{R}^N} ((B_\varepsilon z)(\gamma_t) + (\varepsilon B_\varepsilon h - h)\gamma)\, dx\, dt = 0 \qquad \text{for} \quad \gamma \in \mathscr{D}(Q), \end{aligned} \tag{1.8}$$

where the first equality is due to the obvious symmetry of B_ε and the absolute convergence of all integrals involved. Thus

$$(B_\varepsilon z)_t = \varepsilon B_\varepsilon h - h \quad \text{in } \mathscr{D}'(Q). \tag{1.9}$$

For notational convenience we denote $z(t, .)$ by $z(t)$ and $\int_{\mathbf{R}^N} p(x) q(x)\, dx$ by (p, q) when $pq \in L^1(\mathbf{R}^N)$. Since z, $B_\varepsilon z \in L^1(Q) \cap L^\infty(Q)$:

$$g_\varepsilon(t) = (B_\varepsilon z(t), z(t))$$

is defined for almost all $t \in [0, T]$. Assume we can demonstrate that

$$\lim_{\varepsilon \downarrow 0} g_\varepsilon(t) = \lim_{\varepsilon \downarrow 0} (B_\varepsilon z(t), z(t)) = 0 \quad \text{a. e. } t \in [0, T]. \tag{1.10}$$

It will follow that $z(t)=0$ a. e. Indeed, if $w \in L^2(\mathbf{R}^N)$ then $\varepsilon B_\varepsilon w - \Delta B_\varepsilon w = w$ and so

$$(B_\varepsilon w, w) = (B_\varepsilon w, \varepsilon B_\varepsilon w - \Delta B_\varepsilon w) = \varepsilon \| B_\varepsilon w \|_2^2 + \| \, | \operatorname{grad} B_\varepsilon w | \, \|_2^2 .$$

Thus $(B_\varepsilon w, w) \to 0$ as $\varepsilon \to 0$ implies $\varepsilon B_\varepsilon w \to 0$ in $L^2(\mathbf{R}^N)$ and $\Delta B_\varepsilon w = \operatorname{div}(\operatorname{grad} B_\varepsilon w) \to 0$ in $\mathscr{D}'(\mathbf{R}^N)$ [since $\operatorname{grad} B_\varepsilon w \to 0$ in $L^2(\mathbf{R}^N)^N$]. Therefore $\varepsilon B_\varepsilon w - \Delta B_\varepsilon w = w \to 0$ in $\mathscr{D}'(\mathbf{R}^N)$ and $w=0$ a. e. In this way Proposition 1 will follow if we can verify (1 . 10). This will involve two main steps. From (1 . 9) and the various properties of h and z we will deduce that g_ε is absolutely continuous (upon correction on a set of measure zero) and

$$g_\varepsilon'(t) = 2(\varepsilon B_\varepsilon h(t) - h(t), z(t)) \quad \text{a.e. } t \in [0, T], \tag{1.11}$$

where $h(t)$ abbreviates $h(t, .)$. We assume these facts for the moment and show how to complete the proof of (1 . 10). From $z \in L^\infty(Q)$, (1 . 4) and (1 . 6) it follows that

$$g_\varepsilon(0+) = \operatorname*{essential\,limit}_{t \downarrow 0} (B_\varepsilon z(t), z(t)) = 0. \tag{1.12}$$

This equality together with (1 . 2), (1 . 11) and the symmetry of B_ε imply

$$g_\varepsilon(t) \leqq 2 \int_0^t (\varepsilon B_\varepsilon h(s), z(s))\, ds = 2 \int_0^t (h(s), \varepsilon B_\varepsilon z(s))\, ds. \tag{1.13}$$

Now

$$|(\varepsilon B_\varepsilon h(s), z(s))| \leqq \| \varepsilon B_\varepsilon h \|_\infty \| z(s) \|_1 \leqq \| h \|_\infty \| z(s) \|_1$$

by (1 . 6). Since $s \to \| z(s) \|_1 \in L^1(0, T)$, (1 . 10) will follow from the dominated convergence Theorem and (1 . 13) if:

$$\lim_{\varepsilon \downarrow 0} (h(s), \varepsilon B_\varepsilon z(s)) = 0 \quad \text{a.e. } s \in [0, T].$$

In view of our various assumptions, this last equality follows from:

LEMMA 1. — *Let* $p \in L^\infty(\mathbf{R}^N)$ *and* $\operatorname{meas}\{x \in \mathbf{R}^N : |p(x)| > \xi\} < \infty$ *for* $\xi > 0$. *Let* $q \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. *Then* $\lim_{\varepsilon \downarrow 0} (p, \varepsilon B_\varepsilon q) = 0$.

Proof of Lemma 1. — We have

$$\left| \int \varepsilon p B_\varepsilon q \, dx \right| \leqq \left| \int_{\{x : |p(x)| > \xi\}} \varepsilon p B_\varepsilon q \, dx \right| + \xi \int |\varepsilon B_\varepsilon q| \, dx \tag{1.14}$$

$$\leqq \operatorname{meas}\{x : |p(x)| > \xi\} \| p \|_\infty \| \varepsilon B_\varepsilon q \|_\infty + \xi \| \varepsilon B_\varepsilon q \|_1$$

$$\leqq \operatorname{meas}\{x : |p(x)| > \xi\} \| p \|_\infty \| \varepsilon B_\varepsilon q \|_\infty + \xi \| q \|_1 .$$

To proceed, we verify that

$$\lim_{\varepsilon \downarrow 0} \| \varepsilon \mathrm{B}_\varepsilon q \|_\infty = 0. \tag{1.15}$$

In fact, scaling arguments show that

$$(\varepsilon \mathrm{B}_\varepsilon q)(x) = \varepsilon^{N/2} \int k(\sqrt{\varepsilon}(x-y))\, q(y)\, dy, \tag{1.16}$$

where k is the kernel associated with B_1;

$$\mathrm{B}_1 q(x) = \int k(x-y)\, q(y)\, dy. \tag{1.17}$$

The properties of this kernel we use below can be obtained from [12]. Simple estimations now yield

$$|\varepsilon \mathrm{B}_\varepsilon q(x)| \leqq \varepsilon^{N/2} \mathrm{C}(r) \|q\|_1 + \varepsilon^{N/2} \|q\|_\infty \int_{\{\sqrt{\varepsilon}|x-y| \leqq r\}} k(\sqrt{\varepsilon}(x-y))\, dy$$

for $r > 0$, where $\mathrm{C}(r) = \sup\{k(x) : |x| \geqq r\}$. Since $\mathrm{C}(r) < \infty$ for $r > 0$, this last estimate shows that

$$\limsup_{\varepsilon \downarrow 0} \| \varepsilon \mathrm{B}_\varepsilon q \|_\infty \leqq \|q\|_\infty \int_{\{|y| \leqq r\}} k(y)\, dy$$

for $r > 0$. But $k \in \mathrm{L}^1(\{x : |x| \leqq 1\})$ and the right hand side above therefore tends to zero as $r \downarrow 0$, establishing (1.15). Returning to (1.14), we find now that

$$\limsup_{\varepsilon \downarrow 0} \left| \int \varepsilon p \mathrm{B}_\varepsilon q\, dx \right| \leqq \xi \|q\|_1 \qquad \text{for} \quad \xi > 0$$

and Lemma 1 follows on letting ξ tend to 0. ■

It remains to verify the absolute continuity of g_ε and (1.11). For notational simplicity we set $\varepsilon = 1$ and write $\mathrm{B}g$ instead of $\mathrm{B}_1 g$. Let

$$z_\delta(t, x) = (\rho_\delta \star \bar{z})(t, x) = \int_{-\infty}^{\infty} \rho_\delta(t-s)\, \bar{z}(s, x)\, ds$$

where $\bar{z} = z$ on Q, $\bar{z} = 0$ outside Q and ρ_δ is a standard family of mollifiers in t with ρ_δ supported in $[-\delta, \delta]$. It is clear that z_δ, $\mathrm{B}z_\delta$ and $(\mathrm{B}z_\delta(t), z_\delta(t))$ are smooth in t and we have

$$\frac{d}{dt}(\mathrm{B}z_\delta, z_\delta) = 2\left(\frac{\partial}{\partial t}\mathrm{B}z_\delta, z_\delta\right) = 2\left(\mathrm{B}z_\delta, \frac{\partial}{\partial t} z_\delta\right) \text{ on } \mathbf{R}. \tag{1.18}$$

Next we claim that almost everywhere on $(\delta, T-\delta)\times \mathbf{R}^N$:

$$\frac{\partial}{\partial t}\mathrm{B}\, z_\delta = \rho_\delta \star (\mathrm{B}\,\bar{h}-\bar{h}), \tag{1.19}$$

where $\bar{h}=h$ on Q and $\bar{h}=0$ outside Q. Indeed if $\gamma\in\mathscr{D}((\delta, T-\delta)\times\mathbf{R}^N)$ and $\check{\rho}_\delta(s)=\rho_\delta(-s)$ we find:

$$\int_{\mathbf{R}}\int_{\mathbf{R}^N}\left(\frac{\partial}{\partial t}\mathrm{B}\, z_\delta\right)\gamma\, dx\, dt = \int_{\mathbf{R}}\int_{\mathbf{R}^N}(-\mathrm{B}\,\bar{z})\frac{\partial}{\partial t}(\check{\rho}_\delta\star\gamma)\, dx\, dt$$
$$= \int_Q (\mathrm{B}\, h-h)(\check{\rho}_\delta\star\gamma)\, dx\, dt = \int_{\mathbf{R}}\int_{\mathbf{R}^N}(\rho_\delta\star(\mathrm{B}\,\bar{h}-\bar{h}))\gamma\, dx\, dt$$

by (1.9) and the fact $\check{\rho}_\delta\star\gamma\in\mathscr{D}(Q)$. Using (1.18) and (1.19) we see that for $\zeta\in\mathscr{D}(0, T)$ and sufficiently small δ,

$$-\int_0^T(\mathrm{B}\, z_\delta(s), z_\delta(s))\zeta'(s)\, ds = 2\int_0^T(\rho_\delta\star(\mathrm{B}\,\bar{h}-\bar{h})(s), z_\delta(s))\zeta(s)\, ds.$$

Since $z_\delta\to z$ in $L^1(Q)$ and $\|z_\delta\|_\infty\leqq\|z\|_\infty$ it follows easily that

$$-\int_0^T g(s)\zeta'(s)\, ds = 2\int_0^T((\mathrm{B}\,\bar{h}-\bar{h})(s), z(s))\zeta(s)\, ds.$$

The last result shows that g is absolutely continuous and $g'(t)=2(\mathrm{B}\, h(t)-h(t)), z\, t))$ a.e.

The proof of Proposition 1, and hence Theorem 1, is complete. ■

REMARKS ON VARIATIONS:

(1.20) *The inhomogeneous equation:* the way we have formulated Theorem 1 it is directly applicable to the generalization $u_t-\Delta\varphi(u)=f(t, x)$ of (1) (i).

(1.21) *Discontinuous* φ: the continuity or even the single-valuedness of φ was used only to establish that $h=\varphi(u)-\varphi(\hat{u})$ satisfied (1.3). This can be arrived at in other ways. For example, if u, $\hat{u}\in L^p(Q)$ for some p, $1\leqq p<\infty$, and φ is continuous at 0 we have (1.3) satisfied.

(1.22) *Assumption of the initial-value:* we have assumed that the initial condition is satisfied in the strong form (1) (ii) which corresponds to (1.4). This was to simplify the presentation and is justified by the existence theory which we have in mind (*see* Section 2) which provides solutions satisfying (1) (ii). However, it is quite interesting to weaken (1.4) to the requirement

$$\int_0^T\int_{\mathbf{R}^N}(z\psi_t+h\Delta\psi)\, dx\, dt=0, \qquad \forall\psi\in C_0^\infty([0, T)\times\mathbf{R}^N), \tag{1.23}$$

where $C_0^\infty([0, T)\times\mathbf{R}^N)$ means the C^∞ functions vanishing for t near T and large $|x|$, especially in view of the existing literature. In fact, the entire proof remains intact under this change of hypotheses except for the verification of (1.12). We briefly describe how to verify (1.12) under the assumption (1.23).

First, if $\psi\in C_0^\infty([0, T)\times\mathbf{R}^N)$, $0\leqq a<b<T$ and

$$g(t)=\begin{cases} 1 & \text{if } b<t, \\ \dfrac{1}{b-a}(t-a) & \text{if } a<t<b, \\ 0 & \text{if } t<a, \end{cases}$$

we can approximate $g(t)$ by smooth functions and establish that

$$\int_0^T\int_{\mathbf{R}^N}(z(g'\psi+g\psi_t)+hg\Delta\psi))\,dx\,dt=0=\int_0^T\int_{\mathbf{R}^N}(z\psi_t+h\Delta\psi)\,dx\,dt.$$

As $a, b\to 0$, this implies that

$$\lim_{\substack{b,\,a\to 0+\\ b>a}}\frac{1}{b-a}\int_a^b\int_{\mathbf{R}^N}z(t, x)\psi(t, x)\,dx\,dt=0 \tag{1.24}$$

for $\psi\in C_0^\infty([0, T): \mathbf{R}^N)$. Taking ψ independent of t (as we may clearly do) and recalling $z\in L^\infty(Q)$, we deduce that

$$\lim_{\substack{b,\,a\to 0+\\ b>a}}\frac{1}{b-a}\int_a^b\int_{\mathbf{R}^N}z(t, x)\psi(x)\,dx\,dt=0, \qquad \forall\psi\in L^1(\mathbf{R}^N). \tag{1.25}$$

From (1.9) it follows that

$$B_\varepsilon z(t, x)-B_\varepsilon z(s, x)=\int_s^t(\varepsilon B_\varepsilon h(\tau, x)-h(\tau, x))\,d\tau$$

for almost all $(t, s, x)\in(0, T)\times(0, T)\times\mathbf{R}^N$. Multiplying this result by $\psi\in C_0^\infty(\mathbf{R}^N)$, integrating over $x\in\mathbf{R}^N$ and then averaging over s, $a<s<b$, produces

$$\int_{\mathbf{R}^N}B_\varepsilon z(t, x)\psi(x)\,dx-\frac{1}{b-a}\int_a^b\int_{\mathbf{R}^N}z(s, x)B_\varepsilon\psi(x)\,dx\,ds$$
$$=\frac{1}{b-a}\int_a^b\int_{\mathbf{R}^N}\int_s^t(\varepsilon B_\varepsilon h(\tau, x)-h(\tau, x))\,d\tau\psi(x)\,dx\,ds.$$

Using (1.25), we can pass to the limit above as $b, a\to 0+$ to obtain

$$\int_{\mathbf{R}^N}(B_\varepsilon z)(t, x)\psi(x)\,dx=\int_0^t\int_{\mathbf{R}^N}(\varepsilon B_\varepsilon h(\tau, x)-h(\tau, x))\psi(x)\,dx\,d\tau.$$

This relation holds for all $\psi \in L^1(\mathbf{R}^N)$ for almost all t. Hence

$$B_\varepsilon z(t, x) = \int_0^t (\varepsilon B_\varepsilon h(\tau, x) - h(\tau, x))\, dx$$

for almost all $(t, x) \in [0, T) \times \mathbf{R}^N$ and $\| B_\varepsilon z(t) \|_\infty \leqq t\, 2 \| h \|_\infty$.

Finally, $|(B_\varepsilon z(t), z(t))| \leqq 2t \| h \|_\infty \| z(t) \|_1$, so $\| z(t) \|_1 \in L^1(0, T)$ and the existence of $g_\varepsilon(0+)$ imply (1.12).

(1.26) *Other integrability conditions:* we note that Proposition 1 remains valid if $z \in L^1(Q) \cap L^\infty(Q)$ and (1.3) are replaced by z, $h \in L^2(Q)$ and (1.4) is replaced by (1.23). The proof of this assertion consists of mild adaptations of the above arguments (several points being easier). Recalling the relationships of Proposition 1, Theorem 1 and (1), this proves uniqueness of weak solutions of (1) which satisfy $u \in L^\infty(Q)$ and $u - u_0 \in L^2(Q)$ *if* φ is locally Lipschitzian. Indeed, if u, $\hat{u}$ are two such solutions and $h = \varphi(u) - \varphi(\hat{u})$, then $|h| \leqq C|u - \hat{u}| \in L^2(Q)$ for some constant C. This result strongly generalizes the uniqueness assertion of [11].

Section 2. The proof of Theorem 2

The abstract theory of evolution equations governed by accretive operators (*see*, e. g., [1], [5]) in conjunction with [3] provides a great deal of information concerning the solution of (1). The basic idea is that for each $g \in L^1(\mathbf{R}^N)$ the problem

$$v - \Delta\varphi(v) = g \text{ in } \mathscr{D}'(\mathbf{R}^N) \tag{2.1}$$

has a solution $v \in L^1(\mathbf{R}^N)$ which is unique within a suitable class (*see* [3]). To reduce (2.1) to the problem studied in [3], put $u = \varphi(v)$, $\beta = \varphi^{-1}$ and rewrite (2.1) as $\beta(u) - \Delta u \ni g$. The mapping A defined by A: $v \to g - v$ when $g \in L^1(\mathbf{R}^N)$ and v is the unique solution of (2.1) is m-accretive in $L^1(\mathbf{R}^N)$ and $\overline{D(A)} = L^1(\mathbf{R}^N)$. Thus (1) has a solution in the sense of the abstract theory if $u_0 \in L^1(\mathbf{R}^N)$ (*see* [1], [5]). Let $u(t, .) = S(t) u_0$ denote this solution; in particular $u \in C([0, \infty); L^1(\mathbf{R}^N))$. Under the assumptions of Theorem 1 it is easy to see that if also $u_0 \in L^\infty(\mathbf{R}^N)$ then $u \in L^\infty([0, \infty) \times \mathbf{R}^N)$ and $u_t - \Delta\varphi(u) = 0$ in $\mathscr{D}'((0, \infty) \times \mathbf{R}^N)$. Thus the existence theory complements the uniqueness theory. With some further restrictions on φ (*see* [2], [13] for precise conditions) which are satisfied in the special case of Theorem 2, we have $u \in L^\infty([a, \infty) \times \mathbf{R}^N)$ for $a > 0$ if only $u_0 \in L^1(\mathbf{R}^N)$. Thus the *existence* claim of Theorem 2 is clear. We now prove the *uniqueness*. Assume u is any solution of (7), (8), (9). Then for $h > 0$ the functions $u(t + h, .)$ and $S(t) u(h)$ are two solutions of (1) (i) with the same initial value $u(h, .)$. It follows from Theorem 1 that $S(t) u(h) = u(t + h, .)$ for $t \geqq 0$. As $h \to 0$ we see that $S(t) u_0 = u(t, .)$ and the uniqueness is proved. ■

To illustrate the use of the existence theory in extending uniqueness results in a slightly more complex way (but by no means the most complex), we indicate the proof of one more result.

THEOREM 3. — *Let* $T>0$, (2) *hold and* p: $\mathbf{R}\to\mathbf{R}$ *be Lipschitz continuous with* $p(0)=0$. *Let* $u_0 \in L^1(\mathbf{R}^N)\cap L^\infty(\mathbf{R}^N)$. *Then there is exactly one function* u *satisfying*

$$u \in C([0, T]: L^1(\mathbf{R}^N)) \cap L^\infty([0, T]: \mathbf{R}^N), \tag{2.2}$$

$$u(0, .) = u_0(.), \tag{2.3}$$

and

$$u_t - \Delta\varphi(u) + p(u) = 0 \quad \text{in } \mathscr{D}'(Q). \tag{2.4}$$

Proof of Theorem 3. — Assume u and $\hat{u}$ satisfy (2.2)–(2.4). Let $f(t, x) = p(u(t, x))$, so $f \in L^1(Q)\cap L^\infty(Q)$ (by the restrictions on p). The theory mentioned above guarantees the existence of a $v \in L^\infty(Q)\cap C([0, T]: L^1(\mathbf{R}^N))$ such that $v_t - \Delta\varphi(v) + f = 0$ in $\mathscr{D}'(Q)$ and $v(0, x) = u_0(x)$. Theorem 1 implies [*see* Remark (1.20)] that $v \equiv u$. Similarly we can construct $\hat{v}$ from $\hat{u}$ and $\hat{v} \equiv \hat{u}$. But the existence theory which provided v and $\hat{v}$ also implies that if $w(t) = v(t, .) - \hat{v}(t, .) = u(t, .) - \hat{u}(t, .)$ then

$$\|w(t)\|_1 \leqq \|w(0)\| + \int_0^t \|f(\tau, .) - \hat{f}(\tau, .)\|_1 \, d\tau \leqq 0$$

$$+ K\int_0^t \|u(\tau, .) - \hat{u}(\tau, .)\|_1 \, d\tau = K\int_0^t \|w(\tau)\|_1 \, d\tau,$$

where K is a I ipschitz constant for p. Thus $w = 0$ and uniqueness is proved. Existence follows from the considerations mentioned above. ■

Remark. — A result comparable to Theorem 3 in bounded domains has been obtained in [4] (Proposition 5.2).

Added in proof. — We recently learned of the article *On the equation of Boussinesq* (*Topics in Numerical Analysis*, Vol. 3, 1977, Acad. Press, London) by J. Descloux in which a uniqueness theorem similar to ours is proved in the case of a Dirichlet problem on a bounded interval in $\mathbf{R}$. It seems worth pointing out that this and other problems on bounded domains can be obtained from simple abstract results. We formulate a sample result of this type below and leave the straight u forward proof to the reader. (To fix the ideas consider the particular case $X = L^1(\Omega)$, $X^* = L^\infty(\Omega)$, Ω a smoothly bounded subset of $\mathbf{R}^N$ and $A = -\Delta$ on $D(A) = \{u \in W_0^{1,1}(\Omega);\ \Delta u \in L^1(\Omega)\}$, in what follows, and recall the previous proof).

Let X be a real Banach space whose dual space X^* is continuously unbedded in X, i.e.:

$$X^* \subset X. \tag{A.1}$$

Let A: $D(A) \subset X \to X$ be a linear operator with a continuous inverse, A^{-1}: $X \to D(A)$. Assume moreover that

$$A^{-1}(X^*) \subset X^* \tag{A.2}$$

and

$$\langle Ax, y\rangle = \langle y, Ax\rangle \quad \text{for} \quad x, y \in D(A)\cap X^*, \tag{A.3}$$

where $\langle x, y \rangle$ is the pairing of $x \in X$ and $y \in X^*$. Let X^*_w denote X^* with the topology $\sigma(X^*, X)$.

THEOREM. — *Let the above assumptions hold and*

$$z, h \in C([0, T]; X) \cap C([0, T]; X^*_w).$$

Let

$$\text{(A.4)} \quad \frac{d}{dt}\langle z(t), x^* \rangle + \langle h(t), A^* x^* \rangle = 0 \quad \textit{in } \mathscr{D}'(0, T) \quad \textit{for every} \quad x^* \in D(A^*).$$

Then, for $0 \leqq t \leqq T$:

$$\text{(A.5)} \qquad \langle z(t), A^{-1} z(t) \rangle = \langle z(0), A^{-1} z(0) \rangle - 2 \int_0^t \langle z(s), h(s) \rangle \, ds.$$

Remark. — If $\langle z(s), h(s) \rangle \geqq 0$ and $z(0) = 0$ we find $\langle z(t), A^{-1} z(t) \rangle \leqq 0$ which implies $z \equiv 0$ in applications. If $X = L^1(\Omega)$, etc. as in the example and $z(t) = u(t, .) - \hat{u}(t, .)$;

$$h(t) = \varphi(u(t, .)) - \varphi(\hat{u}(t, .)),$$

the continuity of z, h into X^*_w follows from continuity into $L^1(\Omega)$ and boundedness. (Note that we do not have continuity into $L^\infty(\Omega)$ with the strong topology.) Finally, the Neumann boundary condition can be treated by choosing $X = L^1(\Omega)$ modulo the constants.

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