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# REMARKS ON THE SCHRÖDINGER OPERATOR WITH SINGULAR COMPLEX POTENTIALS (1)

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## 1. Introduction

Let $A = -\Delta + V(x)$ be a Schrödinger operator on an (abitrary) open set $\Omega \subset \mathbf{R}^m$, where $V \in L^1_{loc}(\Omega)$ is a complex valued function. We consider the "maximal" realization of A in $L^2(\Omega)$ under Dirichlet boundary condition, that is

$$D(A) = \{ u \in H^1_0(\Omega);\ V u \in L^1_{loc}(\Omega) \text{ and } -\Delta u + V u \in L^2(\Omega) \}.$$

When $\Omega = \mathbf{R}^m$ we also consider the operator

$$A_1 = -\Delta + V,$$

with domain

$$D(A_1) = \{ u \in L^2(\Omega);\ V u \in L^1_{loc}(\Omega) \text{ and } -\Delta u + V u \in L^2(\Omega) \}.$$

We state now our main results (*see* Th. 3.1 and 3.2) in a special case.

THEOREM. — *Let* $m \geqq 3$; *assume that the function* $\max\{-\mathrm{Re}\, V, 0\}$ *belongs to* $L^\infty(\Omega) + L^{m/2}(\Omega)$ *and also to* $L^{(m/2)+\varepsilon}_{loc}(\Omega)$ *for some* $\varepsilon > 0$. *Then* A (*resp.* $A_1$) *is closable and* $\bar{A} + \lambda$ (*resp.* $\bar{A}_1 + \lambda$) *is m-accretive for some real constant* $\lambda$.

We emphasize the fact that $\max\{\mathrm{Re}\, V, 0\}$ and $\mathrm{Im}\, V$ could be arbitrary functions in $L^1_{loc}(\Omega)$.

Our methods rely on some measure theoretic arguments and standard techniques of DeGiorgi-Moser-Stampacchia type, related to the weak form of the maximum principle.

The distributional inequality

$$\Delta |u| \geqq \mathrm{Re}[\Delta u \operatorname{sign} \bar{u}]$$

proved in [3] plays a crucial role. We also use a result from [1] concerning a property of Sobolev spaces.

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In order to describe our method in a simple case we begin in Section 2 with real valued potentials. The main results in Section 2 are essentially known (*see* [3], [4], [8]) – except perhaps for Theorem 2.2 when $m \leqq 4$.

In Section 3 we turn to the case of complex potentials. Schrödinger operators with complex potentials have been studied by Nelson [6]. His results were extended in [5]. Here we allow more general singularities.

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## 2. Real valued potentials

Let $\Omega$ be an (arbitrary) open subset of $\mathbf{R}^m$ and let $H = L^2 = L^2(\Omega; \mathbf{C})$. Let $q \in L^1_{loc}(\Omega)$ be a real valued function. Set

$$q^+ = \max(q, 0), \qquad q^- = \max(-q, 0).$$

Assume

$$q^- \in L^\infty(\Omega) + L^p(\Omega), \tag{1}$$

with

$$\begin{cases} p = \dfrac{m}{2} & \text{when} \quad m \geqq 3, \\ p > 1 & \text{when} \quad m = 2, \\ p = 1 & \text{when} \quad m = 1. \end{cases}$$

Consider the operator A defined in H by

$$A = -\Delta + q(x),$$

with

$$D(A) = \{ u \in H^1_0(\Omega);\ qu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + qu \in L^2(\Omega) \}.$$

The main results are the following:

THEOREM 2.1. – *A is self-adjoint and* $A + \lambda_1$ *is m-accretive for some real constant* $\lambda_1$. *Furthermore* $u, v \in D(A)$ *imply* $q|u|^2 \in L^1(\Omega)$, $q|v|^2 \in L^1(\Omega)$ *and*

$$(Au, v) = \int \operatorname{grad} u \operatorname{grad} \bar{v} + \int qu\bar{v}. \tag{2}$$

When $\Omega = \mathbf{R}^m$ we also consider the operator $A_1$ defined in H by

$$A_1 = -\Delta + q(x),$$

with

$$D(A_1) = \{ u \in L^2(\Omega);\ qu \in L^1_{loc}(\Omega) \text{ and } -\Delta u + qu \in L^2(\Omega) \}.$$

*Only when* $m=3$ *or* $m=4$ we will make the additional assumption:

$$q^- \in L^{p+\varepsilon}_{loc}(\Omega) \text{ with } p=\frac{3}{2} \text{ when } m=3 \text{ and } p=2 \text{ when } m=4, \tag{3}$$

for some arbitrarily small $\varepsilon>0$.

More precisely we assume that for each $x_0 \in \mathbf{R}^m$ there exists a neighborhood U of $x_0$ and some $\varepsilon>0$ (depending on $x_0$) such that $q^- \in L^{p+\varepsilon}(U)$.

THEOREM 2.2. — *Under the assumptions* (1) *and* (3), $A_1=A$.

Our first lemma is well known:

LEMMA 2.1. — *Assume* (1). *Then for every* $\varepsilon>0$, *there exists a constant* $\lambda_\varepsilon$ *such that*

$$\int q^- |u|^2 \leqq \varepsilon \| \operatorname{grad} u \|^2_{L^2} + \lambda_\varepsilon \| u \|^2_{L^2}, \qquad \forall u \in H^1_0(\Omega).$$

*In particular*

$$\int q^- |u|^2 \leqq \| \operatorname{grad} u \|^2_{L^2} + \lambda_1 \| u \|^2_{L^2}, \qquad \forall u \in H^1_0(\Omega).$$

*Proof.* — Write $q^- = q_1 + q_2$ with $q_1 \in L^\infty(\Omega)$ and $q_2 \in L^p(\Omega)$. Then for each $k>0$ we have

$$\int q^- |u|^2 \leqq \| q_1 \|_{L^\infty} \| u \|^2_{L^2} + \int_{[|q_2|>k]} |q_2| \cdot |u|^2 + k \int_{[|q_2| \leqq k]} |u|^2$$
$$\leqq (\| q_1 \|_{L^\infty} + k) \| u \|^2_{L^2} + \| q_2 \|_{L^p([|q_2|>k])} \| u \|^2_{L^t},$$

with

$$\frac{1}{p} + \frac{2}{t} = 1.$$

In case $m \geqq 3$ we find $t=2^*$ where $2^*$ is the Sobolev exponent, that is $1/2^* = (1/2) - (1/m)$. By the Sobolev imbedding theorem we have

$$\| u \|_{L^t} \leqq C \| \operatorname{grad} u \|_{L^2}, \qquad \forall u \in H^1_0(\Omega).$$

When $m=2$ we find $2<t<\infty$ and it is known that

$$\| u \|_{L^t} \leqq C(\| \operatorname{grad} u \|_{L^2} + \| u \|_{L^2}), \qquad \forall u \in H^1_0(\Omega).$$

When $m=1$ we find $t=\infty$ and it is known that

$$\| u \|_{L^\infty} \leqq C(\| \operatorname{grad} u \|_{L^2} + \| u \|_{L^2}), \qquad \forall u \in H^1_0(\Omega).$$

We reach the conclusion of Lemma 2.1 in all the cases by choosing $k$ large enough so that

$$C^2 \| q_2 \|_{L^p([|q_2|>k])} < \varepsilon.$$

*Remark* 2.1. — Assumption (1) is used in all the results of this paper only through Lemma 2.1 and it may in fact be weakened to a "locally uniform $L^p$-condition":

$$(1') \qquad \| q^- \|_{L^p(\Omega \cap B_r(y))} \to 0 \quad \text{as } r \to 0 \text{ uniformly in } y \in \Omega,$$

where

$$B_r(y) = \{ x \in \mathbf{R}^m ; |x-y| \leqq r \}.$$

Indeed let $\varphi \in \mathscr{D}_+(\mathbf{R}^m)$ with supp $\varphi \subset B_r(0)$ and $\| \varphi \|_{L^2} = 1$. Then, writing $\varphi_y(x) = \varphi(x-y)$,

$$\int q^- |u|^2 = \int dy \int q^- |u \varphi_y|^2 \leqq \int \| q^- \|_{L^p(B_r(y))} \| u \varphi_y \|^2_{L^{p'}} dy.$$

Here $\| q^- \|_{L^p(B_r(y))} \leqq \delta$ for any small $\delta$ by (1′) if $r$ is chosen small. So

$$\int q^- |u|^2 \leqq \delta \int \| u \varphi_y \|^2_{L^{p'}} dy \leqq C \delta \int \| \operatorname{grad} (u \varphi_y) \|^2_{L^2} dy$$

$$\leqq 2 C \delta \int (\| \varphi_y \operatorname{grad} u \|^2_{L^2} + \| u \operatorname{grad} \varphi_y \|^2_{L^2}) dy = 2 C \delta (\| \operatorname{grad} u \|^2_{L^2} + C_r \| u \|^2_{L^2}).$$

Choosing $\delta$ so that $2 C \delta = \varepsilon$, one gets the conclusion of Lemma 2.1. Such a locally uniform $L^p$-condition was used by Simader [7].

We recall a result of [1] which will be used in the proof of Theorem 2.1 (²).

LEMMA 2.2. — *Let* $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ *and let* $u \in H^1_0(\Omega)$ *be such that a.e. on* $\Omega$ :

$$\operatorname{Re} T . \bar{u} \geqq f$$

*for some real valued function* $f \in L^1(\Omega)$ (³). *Then* $\operatorname{Re} T . \bar{u} \in L^1(\Omega)$ *and*

$$\operatorname{Re} \langle T, u \rangle = \int \operatorname{Re} T . \bar{u},$$

*where* $\langle T, u \rangle$ *denotes the Hermitian scalar product in the duality between* $H^{-1}(\Omega)$ *and* $H^1_0(\Omega)$.

The proof of Theorem 2.1 is divided into 4 steps.

STEP 1. — $A + \lambda$ is onto for $\lambda > \lambda_1$. Set $q_n^+ = \min(q^+, n)$; by a Theorem of Lax-Milgram there exists a unique function $u_n \in H^1_0(\Omega)$ which satisfies

$$(4) \qquad -\Delta u_n + (q_n^+ - q^-) u_n + \lambda u_n = f.$$

$\left[\text{note that by Lemma 2.1 the sesquilinear form } \int \bar{q} u \bar{v} \text{ is continuous on } H^1_0(\Omega)\right]$.

Multiplying (4) by $\bar{u}_n$ we find a constant C independent of $n$ such that

$$(5) \qquad \| u_n \|_{H^1} \leqq C,$$

$$(6) \qquad \int q_n^+ |u_n|^2 \leqq C.$$

(²) The use of this sort of Lemma in this context was suggested by M. Crandall.

(³) An example showing that such an assumption is necessary was given by J. Dieudonné (personal communication).

Choose a subsequence denoted again by $u_n$ such that $u_n \to u$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ a. e. on $\Omega$. It follows from Fatou's Lemma and (6) that $q^+|u|^2 \in L^1(\Omega)$. We deduce that $qu \in L^1_{loc}(\Omega)$; indeed

$$q^+|u| \leqq \frac{1}{2} q^+ (|u|^2+1) \in L^1_{loc}(\Omega),$$

$$q^-|u| \leqq \frac{1}{2} q^- (|u|^2+1) \in L^1_{loc}(\Omega).$$

We pass now to the limit in (4) and prove that $-\Delta u + qu + \lambda u = f$ in $\mathscr{D}'(\Omega)$. It suffices to show that

$$(q_n^+ - q^-) u_n \to qu \quad \text{in } L^1_{loc}(\Omega).$$

For this purpose we adapt a device due to W. Strauss [9] and extensively used in the study of strongly nonlinear equations. In view of Vitali's convergence theorem, it suffices to verify that given $\omega \subset\subset \Omega$, then $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $E \subset \omega$ and $|E| < \delta$ imply $\int_E |q_n^+ - q^-| \cdot |u_n| < \varepsilon$ for all $n$. But for every $R > 0$ we have

$$q_n^+ |u_n| \leqq \frac{1}{2} q_n^+ \left( R + \frac{1}{R} |u_n|^2 \right),$$

and thus, by (6),

$$\int_E q_n^+ |u_n| \leqq \frac{1}{2} R \int_E q^+ + \frac{1}{2R} C.$$

We fix R large enough so that $C/R < \varepsilon$ and then $\delta > 0$ so small that $R \int_E q^+ < \varepsilon$. We proceed similarly with $q^- |u_n|$.

STEP 2. – $A + \lambda_1$ is accretive. Let $u \in D(A)$ and set $T = qu$. Since $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ and

$$\operatorname{Re} T \bar{u} = q|u|^2 \geqq -q^- |u|^2 \in L^1(\Omega),$$

it follows from Lemma 2.2 that $q|u|^2 \in L^1$ and

$$\operatorname{Re} \langle T, u \rangle = \int q |u|^2.$$

But $qu = Au + \Delta u$ and so

$$\operatorname{Re} \langle Au, u \rangle - \int |\operatorname{grad} u|^2 = \int q |u|^2.$$

Since $A u \in L^2(\Omega)$ we have in fact

$$\operatorname{Re}(A u, u) = \int |\operatorname{grad} u|^2 + \int q |u|^2 \geqq -\lambda_1 \int |u|^2$$

by Lemma 2.1.

STEP 3. — $u \in D(A)$ implies $q|u|^2 \in L^1(\Omega)$ and (2) holds. We have just seen in Step 2 that $u \in D(A)$ implies $q|u|^2 \in L^1(\Omega)$. Now let $u, v \in D(A)$ and set $T = qu$. We have $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ and

$$\operatorname{Re} T . \bar{v} = \operatorname{Re} qu\bar{v} \geqq -\frac{1}{2}|q| . |u|^2 - \frac{1}{2}|q| . |v|^2 \in L^1(\Omega)$$

and therefore

$$\operatorname{Re}\langle T, v \rangle = \int \operatorname{Re} qu\bar{v}.$$

Thus

$$\operatorname{Re}(A u, v) - \operatorname{Re} \int \operatorname{grad} u \operatorname{grad} \bar{v} = \operatorname{Re} \int qu\bar{v}.$$

Changing $u$ into $iu$ we find

$$(A u, v) = \int \operatorname{grad} u \operatorname{grad} \bar{v} + \int qu\bar{v}.$$

STEP 4. — A is self-adjoint. Indeed $A + \lambda_1$ is $m$-accretive and symmetric. Therefore $A + \lambda_1$ is self-adjoint and so is A.

*Proof of Theorem* 2.2. — Clearly $A \subset A_1$. Let $u \in D(A_1)$ and set $f = A_1 u + \lambda u$ with some $\lambda > \lambda_1$. Let $u^* \in D(A)$ be the unique solution of

$$A u^* + \lambda u^* = f.$$

We have

$$A_1(u - u^*) + \lambda(u - u^*) = 0.$$

Since $(u - u^*) \in L^1_{loc}(\mathbf{R}^m)$ and $\Delta(u - u^*) \in L^1_{loc}(\mathbf{R}^m)$ we may apply Lemma A in [3] to conclude that

$$\Delta|u - u^*| \geqq \operatorname{Re}[\Delta(u - u^*) \operatorname{sign}(\bar{u} - \bar{u}^*)] \quad \text{in } \mathscr{D}'(\mathbf{R}^m),$$

and thus in $\mathscr{D}'(\mathbf{R}^m)$ we find,

$$\Delta|u - u^*| \geqq \operatorname{Re}[(q + \lambda)|u - u^*|] \geqq (-q^- + \lambda)|u - u^*|.$$

Using the next Lemma we conclude that $u = u^*$ [and hence $D(A_1) = D(A)$].

LEMMA 2.3. — *Assume* (1) *and* (3). *Let* $v \in L^2(\mathbf{R}^m)$ *be a real valued function with* $q^- v \in L^1_{loc}(\mathbf{R}^m)$ *satisfying*

$$-\Delta v - q^- v + \lambda v \leqq 0 \quad \text{in } \mathscr{D}'(\mathbf{R}^m),$$

*with some* $\lambda > \lambda_1$. *Then* $v \leqq 0$ *a.e. on* $\mathbf{R}^m$.

The proof of Lemma 2.3 relies on the following crucial result. Since we shall need it in Section 3 for a general domain $\Omega \subset \mathbf{R}^m$ we work now again in $\Omega$.

THEOREM 2.3. — *Assume* (1). *Let* $g \in L^2(\Omega) \cap L^\infty(\Omega)$ *and let* $\psi \in H^1_0(\Omega)$ *be the unique solution of*

$$-\Delta\psi - q^- \psi + \lambda\psi = g \quad \text{in } \Omega \qquad (\lambda > \lambda_1). \tag{7}$$

*Then:*

(*a*) $g \geqq 0$ *a.e. on* $\Omega$ *implies* $\psi \geqq 0$ *a.e. on* $\Omega$;

(*b*) $\psi \in \bigcap_{2 \leqq p < \infty} L^p(\Omega)$.

*Proof of Theorem* 2.3. — (*a*) Multiplying (7) by $-\psi^-$ we find

$$\int |\operatorname{grad} \psi^-|^2 - \int q^- |\psi^-|^2 + \lambda \int |\psi^-|^2 \leqq 0$$

and thus $\psi^- = 0$.

(*b*) We have to consider only the case $m \geqq 3$ [when $m \leqq 2$, $\psi \in H^1_0(\Omega)$ implies $\psi \in \bigcap_{2 \leqq p < \infty} L^p(\Omega)$].

We can always assume that $g \geqq 0$ a.e. on $\Omega$ so that $\psi \geqq 0$ a.e. on $\Omega$. We truncate $q^-$ by $q_k^- = \min(q^-, k)$ and define $\psi_k$ to be the unique solution of

$$\begin{cases} \psi_k \in H^1_0(\Omega), \\ -\Delta\psi_k - q_k^- \psi_k + \lambda\psi_k = g \quad \text{in } \Omega. \end{cases}$$

It is clear that $\psi_k \to \psi$ weakly in $H^1_0(\Omega)$ as $k \to \infty$. We shall prove that for every $p \in [2, \infty)$, $\psi_k \in L^p(\Omega)$ and

$$\|\psi_k\|_{L^p} \leqq C_p(\|g\|_{L^2} + \|g\|_{L^\infty}), \tag{8}$$

where $C_p$ is independent of $k$, but it depends on $q^-$ through the use of Lemma 2.1. For simplicity we drop now the subscript $k$ on $\psi_k$ and write

$$-\Delta\psi - q_k^- \psi + \lambda\psi = g. \tag{9}$$

Set $\psi_n = \min(\psi, n)$ and let $2 \leqq p < \infty$; since $(\psi_n)^{p-1} \in H^1_0(\Omega)$ we can multiply (9) by $(\psi_n)^{p-1}$ and we get

$$(p-1)\int (\psi_n)^{p-2} |\operatorname{grad} \psi_n|^2 \leqq \int g(\psi_n)^{p-1} + \int q_k^- (\psi_n)^p + \int_{[\psi > n]} kn^{p-1}\psi,$$

that is

$$\frac{4(p-1)}{p^2}\int|\operatorname{grad}\psi_n^{p/2}|^2\leqq\|g\|_{L^p}\|\psi_n\|_{L^p}^{p-1}+\int q^-(\psi_n)^p+kn^{p-1}\int_{[\psi>n]}\psi$$

$$\leqq\|g\|_{L^p}\|\psi_n\|_{L^p}^{p-1}+\varepsilon\|\operatorname{grad}\psi_n^{p/2}\|_{L^2}^2+\tilde{\lambda}_\varepsilon\|\psi_n\|_{L^p}^p+k\int_{[\psi>n]}\psi^p$$

by Lemma 2.1 $\left(\text{here } \int_{[\psi>n]}\psi^p \text{ is possibly infinite}\right)$. Choosing $\varepsilon>0$ small enough [for example $\varepsilon=(2(p-1))/p^2$] we see that

$$\int|\operatorname{grad}\psi_n^{p/2}|^2\leqq C_p\left[\|g\|_{L^p}^p+\|\psi\|_{L^p}^p+k\int_{[\psi>n]}\psi^p\right],$$

where $C_p$ is independent of $k$ and $n$. Using Sobolev's inequality we find

$$\|\psi\|_{L^{p2^*/2}}^p\leqq C_p\left[\|g\|_{L^p}^p+\|\psi\|_{L^p}^p+k\int_{[\psi>n]}\psi^p\right]. \tag{10}$$

*Assuming* now that $\psi\in L^p(\Omega)$ and passing to the limit in (10) as $n\to\infty$ we obtain that $\psi\in L^{p2^*/2}(\Omega)$ and

$$\|\psi\|_{L^{p2^*/2}}^p\leqq C_p[\|g\|_{L^p}+\|\psi\|_{L^p}].$$

Iterating this process from $p=2$ we obtain finally for every $p\in[2,\infty)$ :

$$\|\psi\|_{L^p}\leqq C_p[\|g\|_{L^2}+\|g\|_{L^\infty}].$$

More precisely we have proved (8). The conclusion of Theorem 2.3 follows since $\psi_k\to\psi$ weakly in $H_0^1(\Omega)$ as $k\to\infty$.

*Proof of Lemma* 2.3. — By assumption $q^-v\in L^1_{loc}(\mathbf{R}^m)$ and

$$\int v(-\Delta\varphi-q^-\varphi+\lambda\varphi)\leqq 0,\qquad \forall\varphi\in\mathscr{D}_+(\mathbf{R}^m).$$

An easy density argument (smoothing by convolution) shows that

$$\int v(-\Delta\varphi-q^-\varphi+\lambda\varphi)\leqq 0,\quad \forall\varphi\in H^2(\mathbf{R}^m)\cap L^\infty(\mathbf{R}^m),\quad \operatorname{supp}\varphi \text{ compact},\quad \varphi\geqq 0 \text{ a.e.} \tag{11}$$

Fix $g\in\mathscr{D}_+(\mathbf{R}^m)$ and let $\psi_k\in H^1(\mathbf{R}^m)$ be the unique solution of

$$-\Delta\psi_k-q_k^-\psi_k+\lambda\psi_k=g\quad\text{in } \mathbf{R}^m. \tag{12}$$

We know by Theorem 2.3 that $\psi_k\geqq 0$ a.e.

$$\psi_k\in\bigcap_{2\leqq p<\infty}L^p(\mathbf{R}^m)\qquad\text{with}\quad\|\psi_k\|_{L^p}\leqq C_p,$$

and also $\|\operatorname{grad} \psi_k\|_{L^2} \leqq C$. In addition we derive from (12) that

$$\psi_k \in H^2(\mathbf{R}^m) \cap L^\infty_{loc}(\mathbf{R}^m).$$

Fix $\zeta \in \mathscr{D}_+(\mathbf{R}^m)$ satisfying $\zeta(x)=1$ for $|x| \leqq 1$ and set $\zeta_n(x)=\zeta(x/n)$. In (11) we choose $\varphi=\psi_k \zeta_n$. Note that by (12):

$$-\Delta\varphi - q^- \varphi + \lambda\varphi = \zeta_n g - (\Delta\zeta_n)\psi_k - 2 \operatorname{grad} \zeta_n \operatorname{grad} \psi_k - \zeta_n \psi_k (q^- - q_k^-),$$

and therefore

$$\int v \zeta_n g \leqq \frac{C}{n^2} + \frac{C}{n} + \int v \zeta_n \psi_k (q^- - q_k^-).$$

First we *fix* $n$ and let $k \to \infty$. We distinguish two cases:

(*a*) $m \geqq 5$;

(*b*) $m < 5$.

(*a*) When $m \geqq 5$ we have $q^- - q_k^- \to 0$ in $L^{m/2}_{loc}(\mathbf{R}^m)$. Let $p \in [2, \infty)$ be such that $(1/2)+(2/m)+(1/p)=1$; we have

$$\left| \int v \zeta_n \psi_k (q^- - q_k) \right| \leqq \|v\|_{L^2} \|\psi_k\|_{L^p} \|\zeta_n (q^- - q_k)\|_{L^{m/2}} \to 0.$$

Consequently

$$\int v \zeta_n g \leqq \frac{C}{n^2} + \frac{C}{n}.$$

(*b*) When $m<5$ we use the assumption (3) [or (1)]: $q^- \in L^{m/2+\varepsilon}_{loc}(\mathbf{R}^m)$ with some $\varepsilon>0$. It follows from (12) that $\psi_k$ remains bounded in $W^{2,q}_{loc}(\mathbf{R}^m)$ for some $q>m/2$ (when $m \geqq 2$) as $k \to \infty$. We conclude that $\psi_k$ remains bounded in $L^\infty_{loc}(\mathbf{R}^m)$ as $k \to \infty$ [in case $m=1$, $\psi_k$ is bounded in $L^\infty(\mathbf{R})$ since it is bounded in $H^1(\mathbf{R})$]. Therefore

$$\int v \zeta_n \psi_k (q^- - q_k^-) \to 0 \quad \text{as } k \to \infty,$$

since $\|\zeta_n v (q^- - q_k^-)\|_{L^1} \to 0$ by the dominated convergence theorem [recall that $q^- v \in L^1_{loc}(\mathbf{R}^m)$]. In both cases we find

$$\int v \zeta_n g \leqq \frac{C}{n^2} + \frac{C}{n}, \qquad \forall n.$$

As $n \to \infty$ we see that

$$\int v g \leqq 0, \qquad \forall g \in \mathscr{D}_+(\mathbf{R}^m)$$

and therefore $v \leqq 0$ a.e. on $\mathbf{R}^m$.

*Remark* 2.2. — The conclusion of Lemma 2.3 fails in $\mathbf{R}^3$ and in $\mathbf{R}^4$ if we do not assume (3). Ancona (personal communication) has constructed in $\mathbf{R}^3$ and in $\mathbf{R}^4$ functions

$q^- \in L^{m/2}(\mathbf{R}^m)$ and $u \in L^{m/m-2}(\mathbf{R}^m) \cap L^2(\mathbf{R}^m)$ such that

$$-\Delta u - q^- u + u = 0 \quad \text{in } \mathscr{D}',$$

with $\|q^-\|_{L^{m/2}}$ as small as we please and $u \neq 0$.

## 3. Complex potentials

Let $\Omega$ be an (arbitrary) open subset of $\mathbf{R}^m$. Assume $q(x)$ and $q'(x)$ are real valued functions such that $q, q' \in L^1_{\text{loc}}(\Omega)$ and set

$$V(x) = q(x) + iq'(x).$$

We assume

$$\text{(13)} \qquad \textit{either } q' \in L^{1+\varepsilon}_{\text{loc}}(\Omega) \textit{ or } q^- \in L^{(m/2)+\varepsilon}_{\text{loc}}(\Omega) \qquad \text{when} \quad m \geqq 2,$$

for some arbitrarily small $\varepsilon > 0$. Define

$$A = -\Delta + V(x),$$

with

$$D(A) = \{u \in H^1_0(\Omega);\ Vu \in L^1_{\text{loc}}(\Omega) \text{ and } -\Delta u + Vu \in L^2(\Omega)\}.$$

The main results are the following:

THEOREM 3.1. — *Assume* (1) *and* (13). *Then* A *is closable in* $L^2(\Omega)$ *and* $\bar{A} + \lambda_1$ *is m-accretive. In addition* $u \in D(\bar{A})$ *implies that* $u \in H^1_0(\Omega)$, $q|u|^2 \in L^1(\Omega)$ *and*

$$\text{(14)} \qquad \operatorname{Re}(\bar{A}u, u) = \int |\operatorname{grad} u|^2 + \int q|u|^2.$$

*Remark* 3.1. — In case we assume

$$\text{(15)} \qquad |q'(x)| \leqq M q^+(x) + h(x) \qquad \text{for a.e.} \quad x \in \Omega,$$

with $h \in L^{2m/(m+2)}_{\text{loc}}(\Omega)$ and $m \geqq 3$ then A is closed in $L^2(\Omega)$. (Note that (15) corresponds essentially with the assumption made in [5]). Indeed let $u_n \in D(A)$ be such that $u_n \to u$ in $L^2(\Omega)$ and $Au_n \to f$ in $L^2(\Omega)$. It follows from Lemma 2.1 and (14) that $u_n \to u$ in $H^1_0(\Omega)$ and $q^+ u_n \to \sqrt{q^+ u}$ in $L^2(\Omega)$. From (15) we deduce easily that $Vu \in L^1_{\text{loc}}(\Omega)$ and that $-\Delta u + Vu = f$ in $\mathscr{D}'(\Omega)$. Therefore $u \in D(A)$ and $Au = f$.

When $\Omega = \mathbf{R}^m$ we consider also the operator $A_1$ defined in $L^2(\mathbf{R}^m)$ by

$$A_1 = -\Delta + V(x),$$

with

$$D(A_1) = \{u \in L^2(\mathbf{R}^m);\ Vu \in L^1_{\text{loc}}(\mathbf{R}^m) \text{ and } -\Delta u + Vu \in L^2(\mathbf{R}^m)\}.$$

THEOREM 3.2. — *Assume* (1), (3) *and* (13). *Then* $A_1 = A$.

In the proof of theorem 3.1 we shall use the following:

LEMMA 3.1. — *Let* $v \in H_0^1(\Omega)$ *be a real valued function. Assume* (1) *and*

$$-\Delta v - q^- v + \lambda v \leqq 0 \quad \text{in } \mathscr{D}'(\Omega),$$

*with* $\lambda > \lambda_1$. *Then* $v \leqq 0$ *a.e. on* $\Omega$.

*Proof of Lemma* 3.1. — We have, for every $\varphi \in \mathscr{D}_+(\Omega)$:

$$\int \operatorname{grad} v \operatorname{grad} \varphi - \int q^- v \varphi + \lambda \int v \varphi \leqq 0.$$

Now we use the fact (pointed out by G. Stampacchia) that $\mathscr{D}_+(\Omega)$ is dense in $\{u \in H_0^1(\Omega);\ u \geqq 0 \text{ a.e. on } \Omega\}$ for the $H^1$ norm (⁴) to derive that

$$\int \operatorname{grad} v \operatorname{grad} \varphi - \int q^- v \varphi + \lambda \int v \varphi \leqq 0, \qquad \forall \varphi \in H_0^1(\Omega), \quad \varphi \geqq 0.$$

Choosing $\varphi = v^+$ we obtain

$$\int |\operatorname{grad} v^+|^2 - \int q^- |v^+|^2 + \lambda \int |v^+|^2 \leqq 0$$

and therefore $v^+ = 0$.

The proof of Theorem 3.1 is divided into five steps.

STEP 1. — $R(A+\lambda) \supset L^2(\Omega) \cap L^\infty(\Omega)$ for $\lambda > \lambda_1$.

Indeed let $f \in L^2(\Omega) \cap L^\infty(\Omega)$ and let $u_n \in H_0^1(\Omega)$ be the unique solution of

$$-\Delta u_n + V_n u_n + \lambda u_n = f, \tag{16}$$

where $V_n = q_n^+ - q^- + i q_n'$ and

$$q_n' = \begin{cases} n & \text{if } q' > n, \\ q' & \text{if } |q'| \leqq n, \\ -n & \text{if } q' \leqq -n. \end{cases}$$

The existence of $u_n$ follows from a Theorem of Lax-Milgram. Multiplying (16) by $\bar{u}_n$ we find

$$\|u_n\|_{H^1} \leqq C, \tag{17}$$

$$\int q_n^+ |u_n|^2 \leqq C. \tag{18}$$

(⁴) Indeed let $u \in H_0^1(\Omega)$ with $u \geqq 0$ a. e. on $\Omega$; let $u_n \in \mathscr{D}(\Omega)$ be such that $u_n \to u$ in $H^1(\Omega)$. We claim that $|u_n| \to |u| = u$ in $H^1(\Omega)$ because $\| |u_n| \|_{H^1} = \|u_n\|_{H^1}$ and $|u_n| \to |u|$ weakly in $H^1(\Omega)$. On the other hand $|u_n|$ can be smoothed by convolution and for fixed $n$, $\rho_\varepsilon * |u_n| \to |u_n|$ in $H^1(\Omega)$ as $\varepsilon \to 0$.

On the other hand we have

$$\Delta|u_n| \geqq \operatorname{Re}[\Delta u_n \operatorname{sign} \bar{u}_n] \quad \text{in } \mathscr{D}'(\Omega),$$

which leads to

$$-\Delta|u_n| - q^-|u_n| + \lambda|u_n| \leqq |f| \quad \text{in } \mathscr{D}'(\Omega).$$

Let $\psi \in H_0^1(\Omega)$ be the solution of

$$-\Delta\psi - q^-\psi + \lambda\psi = |f|. \tag{19}$$

It follows from Lemma 3.1 that

$$|u_n| \leqq \psi \quad \text{a. e. on } \Omega. \tag{20}$$

By Theorem 2.3 we know that $\psi \in L^p(\Omega)$ for every $p \in [2, \infty]$. We extract a subsequence, denoted again by $u_n$ such that $u_n \to u$ weakly in $H_0^1(\Omega)$, $u_n \to u$ a. e. on $\Omega$. We see as in the proof of Theorem 2.1 (Step 1) that $(q_n^+ - q^-)u_n \to qu$ in $L^1_{\mathrm{loc}}(\Omega)$. Therefore we have only to verify that $q_n' u_n \to q' u$ in $L^1_{\mathrm{loc}}(\Omega)$. We distinguish two cases:

(*a*) $q' \in L^{1+\varepsilon}_{\mathrm{loc}}(\Omega)$;

(*b*) $q^- \in L^{(m/2)+\varepsilon}_{\mathrm{loc}}(\Omega)$.

*Case* (*a*). — From (20) we deduce that $u_n \to u$ in every $L^p$ space, $2 \leqq p < \infty$ and so $q_n' u_n \to q' u$ in $L^1_{\mathrm{loc}}(\Omega)$.

*Case* (*b*). — Since $q^-\psi \in L^q_{\mathrm{loc}}(\Omega)$ for some $q > m/2$, it follows from (19) that $\psi \in L^\infty_{\mathrm{loc}}(\Omega)$. We deduce from the dominated convergence theorem that $q_n' u_n \to q' u$ in $L^1_{\mathrm{loc}}(\Omega)$.

STEP 2. — $A + \lambda_1$ is accretive. Let $u \in D(A)$ and set $T = Vu$. We have $T \in H^{-1}(\Omega) \cap L^1_{\mathrm{loc}}(\Omega)$ and

$$\operatorname{Re} T . \bar{u} = q|u|^2 \geqq -q^-|u|^2 \in L^1(\Omega).$$

It follows from Lemma 2.2 that $q|u|^2 \in L^1(\Omega)$ and

$$\int q|u|^2 = \operatorname{Re}\langle T, u\rangle = \operatorname{Re}\langle Au + \Delta u, u\rangle.$$

Therefore

$$\operatorname{Re}(Au, u) = \int |\operatorname{grad} u|^2 + \int q|u|^2 \geqq -\lambda_1 \int |u|^2. \tag{21}$$

STEP 3. — $D(A)$ is dense in $L^2(\Omega)$. Given $f \in L^2(\Omega) \cap L^\infty(\Omega)$ we solve for large $n$ the equation

$$u_n + \frac{1}{n} A u_n = f. \tag{22}$$

We shall prove that $u_n \to f$ in $L^2\Omega)$ as $n \to \infty -$ and as a consequence $D(A)$ is dense in $L^2(\Omega)$. By (21) we have

$$\int |u_n|^2 + \frac{1}{n}\int |\operatorname{grad} u_n|^2 + \frac{1}{n}\int q|u_n|^2 = \operatorname{Re}(f, u_n).$$

In particular we deduce that

$$\limsup_{n\to\infty} \|u_n\|_{L^2} \leqq \|f\|_{L^2}, \tag{23}$$

$$\frac{1}{n}\int q^+ |u_n|^2 \leqq C, \tag{24}$$

$$\frac{1}{n}\int |\operatorname{grad} u_n|^2 \leqq C. \tag{25}$$

Next we have (as in the proof of Step 1) :

$$|u_n| - \frac{1}{n}\Delta|u_n| - \frac{1}{n}q^- |u_n| \leqq |f| \quad \text{in } \mathscr{D}'(\Omega).$$

On the other hand let $\psi \in H_0^1(\Omega)$ be the solution of

$$-\Delta\psi - q^- \psi + \lambda\psi = |f|$$

for some fixed $\lambda > \lambda_1$. Since $|u_n| \geqq \lambda |u_n/n|$ for $n \geqq \lambda$, we deduce from Lemma 3.1 that $|u_n/n| \leqq \psi$ a.e. Choose a subsequence, denoted again by $u_n$ such that $u_n \to u$ weakly in $L^2(\Omega)$, $(1/n)u_n \to 0$ a.e. [this is possible since $(1/n)u_n \to 0$ in $L^2(\Omega)$]. For every $\varphi \in \mathscr{D}(\Omega)$ we have

$$\int u_n \overline{\varphi} - \frac{1}{n}\int u_n \Delta\overline{\varphi} + \frac{1}{n}\int V u_n \overline{\varphi} = \int f \overline{\varphi}. \tag{26}$$

We claim that $1/n \int V u_n \overline{\varphi} \to 0$ as $n \to \infty$. Indeed by (24) and (25) we have

$$\frac{1}{n}\left|\int q^+ u_n \overline{\varphi}\right| \leqq \frac{C}{\sqrt{n}} \quad \text{and} \quad \frac{1}{n}\left|\int q^- u_n \overline{\varphi}\right| \leqq \frac{C}{\sqrt{n}}.$$

Thus we have only to verify that $1/n \int q' u_n \overline{\varphi} \to 0$. We distinguish two cases:

(*a*) if $q^- \in L_{\text{loc}}^{(m/2)+\varepsilon}(\Omega)$, we have $\psi \in L_{\text{loc}}^{\infty}(\Omega)$ and we deduce from the dominate convergence theorem that $1/n \int q' u_n \overline{\varphi} \to 0$;

(*b*) if $q' \in L_{\text{loc}}^{1+\varepsilon}(\Omega)$ we use the fact $|u_n/n| \leqq \psi \in L^p(\Omega)$ for every $2 \leqq p < \infty$ to deduce that $u_n/n \to 0$ in $L^p(\Omega)$ and so $1/n \int q' u_n \overline{\varphi} \to 0$.

In all the cases, we derive from (26) that

$$\int u \overline{\varphi} = \int f \overline{\varphi}, \qquad \forall \varphi \in \mathscr{D}$$

and consequently $u = f$. We conclude using (23) that $u_n \to f$ in $L^2(\Omega)$.

STEP 4. — A is closable and $\overline{A} + \lambda_1$ is $m$-accretive. This is a standard fact, *see* e.g. Theorem 3.4 in [2].

STEP 5. — $u \in D(\overline{A})$ implies that $u \in H_0^1(\Omega)$, $q|u|^2 \in L^1(\Omega)$ and (14).

We already know (Step 2) that $v \in D(A)$ implies $q|v|^2 \in L^1(\Omega)$ and

$$\operatorname{Re}(A v, v) = \int |\operatorname{grad} v|^2 + \int q|v|^2. \tag{27}$$

Now let $u \in D(\overline{A})$ and let $u_n \in D(A)$ be such that $u_n \to u$, $A u_n \to \overline{A} u$. It follows from (27) applied to $v = u_n - u_m$ that $u_n \to u$ in $H_0^1(\Omega)$ and $\int q^+ |u_n - u|^2 \to 0$ [since $u_n$ is a Cauchy sequence in $H_0^1(\Omega)$ and in $L^2(\Omega)$ with weight $q^+$]. In particular $q|u|^2 \in L^1(\Omega)$ and (14) holds.

*Proof of Theorem* 3.2. — Clearly $A \subset A_1$. Now let $u \in D(A_1)$ and let $\lambda > \lambda_1$. Set $f = A_1 u + \lambda u$, and let $u^*$ be the unique solution of

$$\overline{A} u^* + \lambda u^* = f.$$

Thus, there exists a sequence $u_n^* \to u^*$ in $L^2(\mathbf{R}^m)$ with $u_n^* \in D(A)$ and

$$A u_n^* + \lambda u_n^* = f_n \to f \quad \text{in } L^2(\mathbf{R}^m).$$

In particular we have

$$A_1(u_n^* - u) + \lambda(u_n^* - u) = f_n - f$$

and therefore

$$-\Delta|u_n^* - u| - q^-|u_n^* - u| + \lambda|u_n - u| \leqq |f_n - f| \quad \text{in } \mathscr{D}'(\mathbf{R}^m).$$

We deduce from Lemma 2.3 that $|u_n^* - u| \leqq \psi_n$ a.e. on $\mathbf{R}^m$ where $\psi_n \in H^1(\mathbf{R}^m)$ is the solution of

$$-\Delta\psi_n - q^- \psi_n + \lambda\psi_n = |f_n - f|.$$

Hence $\|\psi_n\|_{H^1} \to 0$ and in particular $u_n^* - u \to 0$ in $L^2(\mathbf{R}^m)$. It follows that $u^* = u$ and so $u \in D(\overline{A})$. We conclude by Theorem 3.1 that $u \in H_0^1(\Omega)$ and therefore $u \in D(A_1) \cap H_0^1(\Omega) = D(A)$.

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