REMARKS ON THE SCHRÖDINGER OPERATOR
WITH SINGULAR COMPLEX POTENTIALS (*)

By Haïm BREZIS and Tosio KATO

1. Introduction

Let $A = -\Delta + V(x)$ be a Schrödinger operator on an (arbitrary) open set $\Omega \subset \mathbb{R}^n$, where $V \in L^1_{\text{loc}}(\Omega)$ is a complex valued function. We consider the "maximal" realization of $A$ in $L^2(\Omega)$ under Dirichlet boundary condition, that is

$$D(A) = \{ u \in H_0^2(\Omega); \ V u \in L^1_{\text{loc}}(\Omega) \text{ and } -\Delta u + V u \in L^2(\Omega) \}.$$

When $\Omega = \mathbb{R}^n$ we also consider the operator

$$A_1 = -\Delta + V,$$

with domain

$$D(A_1) = \{ u \in L^2(\Omega); \ V u \in L^1_{\text{loc}}(\Omega) \text{ and } -\Delta u + V u \in L^2(\Omega) \}.$$

We state now our main results (see Th. 3.1 and 3.2) in a special case.

**Theorem.** Let $m \geq 3$; assume that the function $\max \{-\text{Re} \ V, 0\}$ belongs to $L^m(\Omega) + L^{m/2}(\Omega)$ and also to $L^{(m/2)+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. Then $A$ (resp. $A_1$) is closable and $\bar{A} + \lambda$ (resp. $\bar{A}_1 + \lambda$) is $m$-accretive for some real constant $\lambda$.

We emphasize the fact that $\max \{-\text{Re} \ V, 0\}$ and $\text{Im} \ V$ could be arbitrary functions in $L^1_{\text{loc}}(\Omega)$.

Our methods rely on some measure theoretic arguments and standard techniques of DeGiorgi-Moser-Stampacchia type, related to the weak form of the maximum principle.

The distributional inequality

$$\Delta |u| \geq \text{Re} [\Delta u \text{ sign } u]$$

proved in [3] plays a crucial role. We also use a result from [1] concerning a property of Sobolev spaces.

(*) Partly sponsored by the United States Army under Contract DAAG29-75-C-0024 and (for the second author) by the N.S.F. under Grant MCS76-04655.
In order to describe our method in a simple case we begin in Section 2 with real valued potentials. The main results in Section 2 are essentially known (see [3], [4], [8])—except perhaps for Theorem 2.2 when \( m \leq 4 \).

In Section 3 we turn to the case of complex potentials. Schrödinger operators with complex potentials have been studied by Nelson [6]. His results were extended in [5]. Here we allow more general singularities.

We thank Professors R. Jensen and B. Simon for useful suggestions and discussions (with the first author).

2. Real valued potentials

Let \( \Omega \) be an (arbitrary) open subset of \( \mathbb{R}^m \) and let \( H = L^2 = L^2(\Omega; \mathbb{C}) \). Let \( q \in L^1_{\text{loc}}(\Omega) \) be a real valued function. Set

\[
q^+ = \max(q, 0), \quad q^- = \max(-q, 0).
\]

Assume

\[
(1) \quad q^- \in L^\infty(\Omega) + L^p(\Omega),
\]

with

\[
\begin{cases}
  p = \frac{m}{2} & \text{when } m \geq 3, \\
  p > 1 & \text{when } m = 2, \\
  p = 1 & \text{when } m = 1.
\end{cases}
\]

Consider the operator \( A \) defined in \( H \) by

\[
A = -\Delta + q(x),
\]

with

\[
\mathcal{D}(A) = \{ u \in H^1_0(\Omega); qu \in L^1_{\text{loc}}(\Omega) \text{ and } -\Delta u + qu \in L^2(\Omega) \}.
\]

The main results are the following:

Theorem 2.1. \( A \) is self-adjoint and \( A + \lambda_1 \) is \( m \)-accretive for some real constant \( \lambda_1 \). Furthermore \( u, v \in \mathcal{D}(A) \) imply \( q|u|^2 \in L^1(\Omega), q|v|^2 \in L^1(\Omega) \) and

\[
(2) \quad (Au, v) = \int \nabla u \nabla \bar{v} + \int q uv.
\]

When \( \Omega = \mathbb{R}^m \) we also consider the operator \( A_1 \) defined in \( H \) by

\[
A_1 = -\Delta + q(x),
\]

with

\[
\mathcal{D}(A_1) = \{ u \in L^2(\Omega); qu \in L^1_{\text{loc}}(\Omega) \text{ and } -\Delta u + qu \in L^2(\Omega) \}.
\]
Only when \( m = 3 \) or \( m = 4 \) we will make the additional assumption:

\[
q^{-} \in L^{p+ \epsilon}_{\infty}(\Omega) \quad \text{with} \quad p = \frac{3}{2} \quad \text{when} \quad m = 3 \quad \text{and} \quad p = 2 \quad \text{when} \quad m = 4,
\]

for some arbitrarily small \( \epsilon > 0 \).

More precisely we assume that for each \( x_0 \in \mathbb{R}^m \) there exists a neighborhood \( U \) of \( x_0 \) and some \( \epsilon > 0 \) (depending on \( x_0 \)) such that \( q^{-} \in L^{p+ \epsilon}(U) \).

**Theorem 2.2.** — Under the assumptions (1) and (3), \( A_1 = A \).

Our first lemma is well known:

**Lemma 2.1.** — Assume (1). Then for every \( \epsilon > 0 \), there exists a constant \( \lambda_\epsilon \) such that

\[
\int q^{-} |u|^2 \leq \epsilon \| \text{grad } u \|_{L^2}^2 + \lambda_\epsilon \| u \|_{L^2}^2, \quad \forall u \in H_0^1(\Omega).
\]

In particular

\[
\int q^{-} |u|^2 \leq \| \text{grad } u \|_{L^2}^2 + \lambda_1 \| u \|_{L^2}^2, \quad \forall u \in H_0^1(\Omega).
\]

**Proof.** — Write \( q^{-} = q_1 + q_2 \) with \( q_1 \in L^\infty(\Omega) \) and \( q_2 \in L^p(\Omega) \). Then for each \( k > 0 \) we have

\[
\int q^{-} |u|^2 \leq \| q_1 \|_{L^\infty} \| u \|_{L^2}^2 + \int_{|u| \geq k} |q_2| \cdot |u|^2 + k \int_{|u| \leq k} |u|^2 \\
\leq (\| q_1 \|_{L^\infty} + k^p \| q_2 \|_{L^p(\Omega)} \geq \| u \|_{L^2}^2 + k \| q_2 \|_{L^p(\Omega)} \geq \| u \|_{L^2}^2.
\]

with

\[
\frac{1}{p} + \frac{2}{t} = 1.
\]

In case \( m \geq 3 \) we find \( t = 2^* \) where \( 2^* \) is the Sobolev exponent, that is \( 1/2^* = (1/2) - (1/m) \). By the Sobolev imbedding theorem we have

\[
\| u \|_{L^t} \leq C \| \text{grad } u \|_{L^2}, \quad \forall u \in H_0^1(\Omega).
\]

When \( m = 2 \) we find \( 2 < t < \infty \) and it is known that

\[
\| u \|_{L^t} \leq C (\| \text{grad } u \|_{L^2} + \| u \|_{L^2}), \quad \forall u \in H_0^1(\Omega).
\]

When \( m = 1 \) we find \( t = \infty \) and it is known that

\[
\| u \|_{L^\infty} \leq C (\| \text{grad } u \|_{L^2} + \| u \|_{L^2}), \quad \forall u \in H_0^1(\Omega).
\]

We reach the conclusion of Lemma 2.1 in all the cases by choosing \( k \) large enough so that

\[
C^2 \| q_2 \|_{L^p(\Omega)} \geq \| u \|_{L^2} < \epsilon.
\]
Remark 2.1. — Assumption (1) is used in all the results of this paper only through Lemma 2.1 and it may in fact be weakened to a "locally uniform $L^p$-condition":

\[(1') \quad \| q^- \|_{L^p(\Omega \cap B_r(0))} \to 0 \quad as \quad r \to 0 \quad uniformly \quad in \quad y \in \Omega,\]

where

\[B_r(y) = \{ x \in \mathbb{R}^n; |x - y| \leq r \}.\]

Indeed let $\varphi \in \mathcal{D}_+ (\mathbb{R}^n)$ with supp $\varphi \subset B_r(0)$ and $\| \varphi \|_{L^1} = 1$. Then, writing $\varphi_y(x) = \varphi(x - y)$,

\[\int q^- |u|^2 = \int dy \int q^- |u \varphi_y|^2 \leq \int \| q^- \|_{L^p(B_r(0))} \| u \varphi_y \|_{L^2}^2 \, dy.\]

Here $\| q^- \|_{L^p(B_r(0))} \leq \delta$ for any small $\delta$ by (1') if $r$ is chosen small. So

\[\int q^- |u|^2 \leq \delta \int \| u \varphi_y \|_{L^2}^2 \, dy \leq C \delta \int \| \text{grad} \ (u \varphi_y) \|_{L^2}^2 \, dy \]

\[\leq 2C \delta \int (\| \varphi_y \|_{L^2}^2 + \| \text{grad} \ u \|_{L^2}^2 + C_{r} \| u \|_{L^2}^2) \, dy = 2C \delta (\| \text{grad} \ u \|_{L^2}^2 + C_{r} \| u \|_{L^2}^2).\]

Choosing $\delta$ so that $2C \delta = \epsilon$, one gets the conclusion of Lemma 2.1. Such a locally uniform $L^p$-condition was used by Simader [7].

We recall a result of [1] which will be used in the proof of Theorem 2.1 (3).

**Lemma 2.2.** — Let $T \in H^{-1}(\Omega) \cap L^1_{loc}(\Omega)$ and let $u \in H^1_0(\Omega)$ be such that a.e. on $\Omega$:

\[\text{Re} \ T . \bar{u} \geq f\]

for some real valued function $f \in L^1(\Omega)$ (4). Then Re $T . \bar{u} \in L^1(\Omega)$ and

\[\text{Re} \langle T, u \rangle = \int \text{Re} \ T . \bar{u},\]

where $\langle T, u \rangle$ denotes the Hermitian scalar product in the duality between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$.

The proof of Theorem 2.1 is divided into 4 steps.

**Step 1.** — $A + \lambda$ is onto for $\lambda > \lambda_1$. Set $q^+_n = \min (q^+, n)$; by a Theorem of Lax-Milgram there exists a unique function $u_n \in H^1_0(\Omega)$ which satisfies

\[(4) \quad - \Delta u_n + (q^+_n - q^-) u_n + \lambda u_n = f.\]

(note that by Lemma 2.1 the sesquilinear form $\int q \overline{w}$ is continuous on $H^1_0(\Omega)$).

Multiplying (4) by $\bar{u}_n$ we find a constant $C$ independent of $n$ such that

\[(5) \quad \| u_n \|_{H^1} \leq C,\]

\[(6) \quad \int q^+_n |u_n|^2 \leq C.\]

(3) The use of this sort of Lemma in this context was suggested by M. Crandall.

(4) An example showing that such an assumption is necessary was given by J. Dieudonne (personal communication).
Remarks on the Schrödinger Operator

Choose a subsequence denoted again by \( u_n \) such that \( u_n \rightarrow u \) weakly in \( H^1_0(\Omega) \) and \( u_n \rightarrow u \) a.e. on \( \Omega \). It follows from Fatou's Lemma and (6) that \( q^+ |u|^2 \in L^1(\Omega) \). We deduce that \( qu \in L^1_{\text{loc}}(\Omega) \); indeed

\[
q^+ |u| \leq \frac{1}{2} q^+ \left( |u|^2 + 1 \right) \in L^1_{\text{loc}}(\Omega),
\]

\[
q^- |u| \leq \frac{1}{2} q^- \left( |u|^2 + 1 \right) \in L^1_{\text{loc}}(\Omega).
\]

We pass now to the limit in (4) and prove that \( -\Delta u + qu + \lambda u = f \) in \( \mathcal{D}'(\Omega) \). It suffices to show that

\[
(q^+_n - q^-) u_n \rightarrow qu \quad \text{in} \quad L^1_{\text{loc}}(\Omega).
\]

For this purpose we adapt a device due to W. Strauss [9] and extensively used in the study of strongly nonlinear equations. In view of Vitali’s convergence theorem, it suffices to verify that given \( \omega \subset \subset \Omega \), then \( \forall \varepsilon > 0, \exists \delta > 0 \) such that \( E \subset \omega \) and \( |E| < \delta \) imply

\[
\int_E |q^+_n - q^-| \cdot |u_n| < \varepsilon \quad \text{for all} \quad n.
\]

But for every \( R > 0 \) we have

\[
q^+_n |u_n| \leq \frac{1}{2} q^+_n \left( R + \frac{1}{R} |u_n|^2 \right),
\]

and thus, by (6),

\[
\int_E q^+_n |u_n| \leq \frac{1}{2} R \int_E q^+ + \frac{1}{2} R C.
\]

We fix \( R \) large enough so that \( C/R < \varepsilon \) and then \( \delta > 0 \) so small that \( \int q^+ < \varepsilon \). We proceed similarly with \( q^- |u_n| \).

**Step 2.** \( A + \lambda_1 \) is accretive. Let \( u \in \mathcal{D}(A) \) and set \( T = qu \). Since \( T \in H^{-1}(\Omega) \cap L^1_{\text{loc}}(\Omega) \) and

\[
\text{Re} \langle T, u \rangle = q |u|^2 \geq -q^- |u|^2 \in L^1(\Omega),
\]

it follows from Lemma 2.2 that \( q |u|^2 \in L^1 \) and

\[
\text{Re} \langle T, u \rangle = \int q |u|^2.
\]

But \( qu = A u + \Delta u \) and so

\[
\text{Re} \langle A u, u \rangle - \int |\nabla u|^2 = \int q |u|^2.
\]
Since \( Au \in L^2(\Omega) \) we have in fact
\[
\text{Re}(Au, u) = \int |\text{grad } u|^2 + \int q|u|^2 \geq -\lambda_1 \int |u|^2
\]
by Lemma 2.1.

**Step 3.** \(- u \in D(A) \) implies \( q|u|^2 \in L^1(\Omega) \) and (2) holds. We have just seen in Step 2 that \( u \in D(A) \) implies \( q|u|^2 \in L^1(\Omega) \). Now let \( u, v \in D(A) \) and set \( T = qu \). We have \( T \in H^{-1}(\Omega) \cap L^1_{\text{loc}}(\Omega) \) and
\[
\text{Re } T \cdot \bar{v} = \text{Re } quv \geq -\frac{1}{2} |q| \cdot |u|^2 - \frac{1}{2} |q| \cdot |v|^2 \in L^1(\Omega)
\]
and therefore
\[
\text{Re } \langle T, v \rangle = \int \text{Re } quv.
\]
Thus
\[
\text{Re } (Au, v) - \int \text{grad } u \text{ grad } \bar{v} = \text{Re } \int quv.
\]
Changing \( u \) into \( iu \) we find
\[
(Au, v) = \int \text{grad } u \text{ grad } \bar{v} + \int quv.
\]

**Step 4.** \(- A \) is self-adjoint. Indeed \( A + \lambda_1 \) is \( m \)-accretive and symmetric. Therefore \( A + \lambda_1 \) is self-adjoint and so is \( A \).

**Proof of Theorem 2.2.** Clearly \( A \subset A_1 \). Let \( u \in D(A_1) \) and set \( f = A_1 u + \lambda u \) with some \( \lambda > \lambda_1 \). Let \( u^* \in D(A) \) be the unique solution of
\[
Au^* + \lambda u^* = f.
\]
We have
\[
A_1 (u - u^*) + \lambda (u - u^*) = 0.
\]
Since \( (u - u^*) \in L^1_{\text{loc}}(\mathbb{R}^n) \) and \( \Delta (u - u^*) \in L^1_{\text{loc}}(\mathbb{R}^n) \) we may apply Lemma A in [3] to conclude that
\[
\Delta |u - u^*| \geq \text{Re} \left[ \Delta (u - u^*) \text{ sign } (\bar{u} - u^*) \right] \text{ in } D'(\mathbb{R}^n),
\]
and thus in \( D'(\mathbb{R}^n) \) we find,
\[
\Delta |u - u^*| \geq \text{Re} \left( [(q + \lambda)|u - u^*|] \geq (-q^- + \lambda)|u - u^*| \right).
\]
Using the next Lemma we conclude that \( u = u^* \) [and hence \( D(A_1) = D(A) \)].
LEMMA 2.3. — Assume (1) and (3). Let \( v \in L^2(\mathbb{R}^m) \) be a real valued function with \( q^- v \in L^1_{\text{loc}}(\mathbb{R}^m) \) satisfying
\[
-\Delta v - q^- v + \lambda v \leq 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^m),
\]
with some \( \lambda > \lambda_1 \). Then \( v \leq 0 \) a.e. on \( \mathbb{R}^m \).

The proof of Lemma 2.3 relies on the following crucial result. Since we shall need it in Section 3 for a general domain \( \Omega \subset \mathbb{R}^m \) we work now again in \( \Omega \).

THEOREM 2.3. — Assume (1). Let \( g \in L^2(\Omega) \cap L^\infty(\Omega) \) and let \( \psi \in H^1_0(\Omega) \) be the unique solution of
\[
(7) \quad -\Delta \psi - q^- \psi + \lambda \psi = g \quad \text{in} \quad \Omega \quad (\lambda > \lambda_1).
\]

Then:

(a) \( g \geq 0 \) a.e. on \( \Omega \) implies \( \psi \geq 0 \) a.e. on \( \Omega \);

(b) \( \psi \in \bigcap_{1 \leq p < \infty} L^p(\Omega) \).

Proof of Theorem 2.3. — (a) Multiplying (7) by \( -\psi^- \) we find
\[
\int |\nabla \psi^-|^2 - \int q^- |\psi^-|^2 + \lambda \int |\psi^-|^2 \leq 0
\]
and thus \( \psi^- = 0 \).

(b) We have to consider only the case \( m \geq 3 \) [when \( m \leq 2 \), \( \psi \in H^1_0(\Omega) \) implies \( \psi \in \bigcap_{1 \leq p < \infty} L^p(\Omega) \)].

We can always assume that \( g \geq 0 \) a.e. on \( \Omega \) so that \( \psi \geq 0 \) a.e. on \( \Omega \). We truncate \( q^- \) by \( q^-_k = \min(q^-, k) \) and define \( \psi_k \) to be the unique solution of
\[
\left\{
\begin{array}{l}
\psi_k \in H^1_0(\Omega), \\
-\Delta \psi_k - q^-_k \psi_k + \lambda \psi_k = g \quad \text{in} \quad \Omega.
\end{array}
\right.
\]

It is clear that \( \psi_k \to \psi \) weakly in \( H^1_0(\Omega) \) as \( k \to \infty \). We shall prove that for every \( p \in [2, \infty) \), \( \psi_k \in L^p(\Omega) \) and
\[
(8) \quad \| \psi_k \|_{L^p} \leq C_p (\| g \|_{L^1} + \| g \|_{L^\infty}),
\]
where \( C_p \) is independent of \( k \), but it depends on \( q^- \) through the use of Lemma 2.1. For simplicity we drop now the subscript \( k \) on \( \psi_k \) and write
\[
(9) \quad -\Delta \psi - q^- \psi + \lambda \psi = g.
\]

Set \( \psi_n = \min(\psi, n) \) and let \( 2 \leq p < \infty \); since \( (\psi_n)^{p-1} \in H^1_0(\Omega) \) we can multiply (9) by \( (\psi_n)^{p-1} \) and we get
\[
(p-1) \int |\nabla \psi_n|^{p-2} |\nabla \psi_n| \leq \int g (\psi_n)^{p-1} + \int q^- (\psi_n)^p + \int_{\{\psi > n\}} k n^{p-1} \psi.
\]
that is
\[
\frac{4(p-1)}{p^2} \int |\nabla \psi_n|^2 \leq \|g\|_{L^p} \|\psi_n\|_{L^{p^{-1}}} + \int q^{-\frac{1}{2}} (\psi_n)^{p-1} + \int q^{-\frac{1}{2}} (\psi_n)^{p-1} + \epsilon \|\nabla \psi_n\|_{L^2}^2 + \lambda \|\psi_n\|_{L^2} + k \int_{\{\psi_n > \eta\}} \psi^p
\]

by Lemma 2.1 (here \(\int_{\{\psi_n > \eta\}} \psi^p\) is possibly infinite). Choosing \(\epsilon > 0\) small enough [for example \(\epsilon = (2(p-1))/p^2\)] we see that
\[
\int |\nabla \psi_n|^2 \leq C_p \left( \|g\|_{L^p} + \|\psi\|_{L^2} + \epsilon \int_{\{\psi_n > \eta\}} \psi^p \right)
\]

where \(C_p\) is independent of \(k\) and \(n\). Using Sobolev's inequality we find
\[
\|\psi\|_{L^{p^{*}}} \leq C_p \left( \|g\|_{L^p} + \|\psi\|_{L^2} + k \int_{\{\psi_n > \eta\}} \psi^p \right)
\]

Assuming now that \(\psi \in L^p(\Omega)\) and passing to the limit in (10) as \(n \to \infty\) we obtain that
\(\psi \in L^{p^{*}}(\Omega)\) and
\[
\|\psi\|_{L^{p^{*}}} \leq C_p \left( \|g\|_{L^p} + \|\psi\|_{L^2} \right).
\]

Iterating this process from \(p=2\) we obtain finally for every \(p \in [2, \infty)\):
\[
\|\psi\|_{L^p} \leq C_p \left( \|g\|_{L^p} + \|\psi\|_{L^2} \right).
\]

More precisely we have proved (8). The conclusion of Theorem 2.3 follows since \(\psi_k \to \psi\) weakly in \(H^1_0(\Omega)\) as \(k \to \infty\).

**Proof of Lemma 2.3.** By assumption \(q^- v \in L^1_{\text{loc}}(\mathbb{R}^m)\) and
\[
\int v(-\Delta \varphi - q^- \varphi + \lambda \varphi) \leq 0, \quad \forall \varphi \in \mathcal{D}_+(\mathbb{R}^m).
\]

An easy density argument (smoothing by convolution) shows that
\[
\int v(-\Delta \varphi - q^- \varphi + \lambda \varphi) \leq 0, \quad \forall \varphi \in H^2(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m), \text{ supp } \varphi \text{ compact}, \quad \varphi \geq 0 \text{ a.e.}
\]

Fix \(g \in \mathcal{D}_+(\mathbb{R}^m)\) and let \(\psi_k \in H^1(\mathbb{R}^m)\) be the unique solution of
\[
-\Delta \psi_k - q^- \varphi + \lambda \psi_k = g \quad \text{in } \mathbb{R}^m.
\]

We know by Theorem 2.3 that \(\psi_k \geq 0 \text{ a.e.}\)
\[
\psi_k \in \bigcap_{2 \leq p < \infty} L^p(\mathbb{R}^m) \quad \text{with } \|\psi_k\|_{L^p} \leq C_p.
\]
and also $\|\text{grad } \psi_k\|_{L^1} \leq C$. In addition we derive from (12) that

$$
\psi_k \in H^2(\mathbb{R}^m) \cap L^\infty_{\text{loc}}(\mathbb{R}^m).
$$

Fix $\zeta \in \mathcal{D}_+^1(\mathbb{R}^m)$ satisfying $\zeta(x) = 1$ for $|x| \leq 1$ and set $\zeta_n(x) = \zeta(x/n)$. In (11) we choose $\varphi = \psi_k \zeta_n$. Note that by (12):

$$
-\Delta \varphi - q^- \varphi + \lambda \varphi = \zeta_n g - (\Delta \zeta_n) \psi_k - 2 \text{ grad } \zeta_n \text{ grad } \psi_k - \zeta_n \psi_k (q^--q^-).
$$

and therefore

$$
\int v \zeta_n g \leq \frac{C}{n^2} + \frac{C}{n} + \int v \zeta_n \psi_k (q^- - q^-).
$$

First we fix $n$ and let $k \to \infty$. We distinguish two cases:

(a) $m \geq 5$;
(b) $m < 5$.

(a) When $m \geq 5$ we have $q^- - q^- \to 0$ in $L^{m/2}_{\text{loc}}(\mathbb{R}^m)$. Let $\rho \in [2, \infty)$ be such that $(1/2)(2/m) + (1/\rho) = 1$; we have

$$
\left\| \int v \zeta_n \psi_k (q^- - q^-) \right\| \leq \|v\|_{L^2} \|\psi_k\|_{L^2} \|\zeta_n (q^- - q^-)\|_{L^\infty} \to 0.
$$

Consequently

$$
\int v \zeta_n g \leq \frac{C}{n^2} + \frac{C}{n}.
$$

(b) When $m < 5$ we use the assumption (3) [or (1)]: $q^- \in L^{m/2+\varepsilon}_{\text{loc}}(\mathbb{R}^m)$ with some $\varepsilon > 0$. It follows from (12) that $\psi_k$ remains bounded in $W^{1,q}_{\text{loc}}(\mathbb{R}^m)$ for some $q > m/2$ (when $m \geq 2$) as $k \to \infty$. We conclude that $\psi_k$ remains bounded in $L^\infty_{\text{loc}}(\mathbb{R}^m)$ as $k \to \infty$ [in case $m = 1$, $\psi_k$ is bounded in $L^\infty(\mathbb{R})$ since it is bounded in $H^1(\mathbb{R})$]. Therefore

$$
\int v \zeta_n \psi_k (q^- - q^-) \to 0 \quad \text{as } k \to \infty,
$$

since $\|\zeta_n v (q^- - q^-)\|_{L^1} \to 0$ by the dominated convergence theorem [recall that $q^- v \in L^1_{\text{loc}}(\mathbb{R}^m)$]. In both cases we find

$$
\int v \zeta_n g \leq \frac{C}{n^2} + \frac{C}{n}, \quad \forall n.
$$

As $n \to \infty$ we see that

$$
\int v g \leq 0, \quad \forall g \in \mathcal{D}_+^1(\mathbb{R}^m)
$$

and therefore $v \leq 0$ a.e. on $\mathbb{R}^m$.

Remark 2.2. — The conclusion of Lemma 2.3 fails in $\mathbb{R}^3$ and in $\mathbb{R}^4$ if we do not assume (3). Ancona (personal communication) has constructed in $\mathbb{R}^3$ and in $\mathbb{R}^4$ functions.
$q^- \in L^{m/2}(\mathbb{R}^m)$ and $u \in L^{m/m-2}(\mathbb{R}^m) \cap L^2(\mathbb{R}^m)$ such that

$$-\Delta u - q^- u + u = 0 \quad \text{in } \mathcal{D}'.$$

with $\|q^-\|_{L^{m/2}}$ as small as we please and $u \neq 0$.

3. Complex potentials

Let $\Omega$ be an (arbitrary) open subset of $\mathbb{R}^m$. Assume $q(x)$ and $q'(x)$ are real valued functions such that $q, q' \in L^1_{\text{loc}}(\Omega)$ and set

$$V(x) = q(x) + iq'(x).$$

We assume

(13) \hspace{1cm} \text{either } q' \in L^{1+}(\Omega) \text{ or } q^- \in L^{m(2)+}(\Omega) \quad \text{when } m \geq 2,$$

for some arbitrarily small $\varepsilon > 0$. Define

$$A = -\Delta + V(x),$$

with

$$D(A) = \{ u \in H^1_0(\Omega); \, V u \in L^1_{\text{loc}}(\Omega) \text{ and } -\Delta u + V u \in L^2(\Omega) \}.$$  

The main results are the following:

**Theorem 3.1.** Assume (1) and (13). Then $A$ is closable in $L^2(\Omega)$ and $\overline{A} + \lambda I$ is $m$-accretive. In addition $u \in D(A)$ implies that $u \in H^1_0(\Omega), q|u|^2 \in L^1(\Omega)$ and

(14) \hspace{1cm} \text{Re}(\overline{A} u, u) = \int \text{grad } u|^2 + \int q|u|^2.$$

**Remark 3.1.** In case we assume

(15) \hspace{1cm} |q'(x)| \leq M q^+(x) + h(x) \quad \text{for a.e. } x \in \Omega,$$

with $h \in L^{2m(m+2)}_{\text{loc}}(\Omega)$ and $m \geq 3$ then $A$ is closed in $L^2(\Omega)$. (Note that (15) corresponds essentially with the assumption made in [5]). Indeed let $u_n \in D(A)$ be such that $u_n \rightharpoonup u$ in $L^2(\Omega)$ and $A u_n \rightarrow f$ in $L^2(\Omega)$. It follows from Lemma 2.1 and (14) that $u_n \rightarrow u$ in $H^1_0(\Omega)$ and $q^- u_n \rightarrow \sqrt{q^+} u$ in $L^2(\Omega)$. From (15) we deduce easily that $V u \in L^1_0(\Omega)$ and that $-\Delta u + V u = f$ in $\mathcal{D}'(\Omega)$. Therefore $u \in D(A)$ and $A u = f$.

When $\Omega = \mathbb{R}^m$ we consider also the operator $A_1$ defined in $L^2(\mathbb{R}^m)$ by

$$A_1 = -\Delta + V(x),$$

with

$$D(A_1) = \{ u \in L^2(\mathbb{R}^m); \, V u \in L^1_{\text{loc}}(\mathbb{R}^m) \text{ and } -\Delta u + V u \in L^2(\mathbb{R}^m) \}.$$
THEOREM 3.2. — Assume (1), (3) and (13). Then $A_1 = A$.

In the proof of theorem 3.1 we shall use the following:

LEMMA 3.1. — Let $v \in H^1_0(\Omega)$ be a real valued function. Assume (1) and

$$ -\Delta v - q^- v + \lambda v \leq 0 \text{ in } D'(\Omega), $$

with $\lambda > \lambda_1$. Then $v \leq 0$ a.e. on $\Omega$.

Proof of Lemma 3.1. — We have, for every $\phi \in D_+(\Omega)$:

$$ \int \text{grad } v \text{ grad } \phi - \int q^- v \phi + \lambda \int v \phi \leq 0. $$

Now we use the fact (pointed out by G. Stampacchia) that $D_+(\Omega)$ is dense in $\{ u \in H^1_0(\Omega); u \geq 0 \text{ a.e. on } \Omega \}$ for the $H^1$ norm (*) to derive that

$$ \int \text{grad } v \text{ grad } \phi - \int q^- v \phi + \lambda \int v \phi \leq 0, \quad \forall \phi \in H^1_0(\Omega), \quad \phi \geq 0. $$

Choosing $\phi = v^+$ we obtain

$$ \int |\text{grad } v^+|^2 - \int q^- |v^+|^2 + \lambda \int |v^+|^2 \leq 0 $$

and therefore $v^+ = 0$.

The proof of Theorem 3.1 is divided into five steps.

STEP 1. — $R(A + \lambda) \supset L^2(\Omega) \cap L^{\infty}(\Omega)$ for $\lambda > \lambda_1$.

Indeed let $f \in L^2(\Omega) \cap L^{\infty}(\Omega)$ and let $u_n \in H^1_0(\Omega)$ be the unique solution of

$$ -\Delta u_n + V_n u_n + \lambda u_n = f, $$

where $V_n = q^+_n - q^- + iq'_n$ and

$$ q'_n = \begin{cases} 
  n & \text{if } q' > n, \\
  q' & \text{if } |q'| \leq n, \\
  -n & \text{if } q' \leq -n.
\end{cases} $$

The existence of $u_n$ follows from a Theorem of Lax-Milgram. Multiplying (16) by $\bar{u}_n$ we find

$$ \|u_n\|_{H^1} \leq C, $$

(17)

$$ \int q^+_n |u_n|^2 \leq C. $$

(18)

(*) Indeed let $u \in H^1_0(\Omega)$ with $u \geq 0$ a.e. on $\Omega$; let $u_n \in D(\Omega)$ be such that $u_n \to u$ in $H^1(\Omega)$. We claim that $|u_n| \to |u|$ in $H^1(\Omega)$ because $\|u_n\|_{H^1} \to \|u\|_{H^1}$ and $\|u_n\|_{H^1}$ and $\|u_n\|$ weakly in $H^1(\Omega)$. On the other hand $|u_n|$ can be smoothed by convolution and for fixed $n, \rho_n |u_n| \to |u|$ in $H^1(\Omega)$ as $\varepsilon \to 0$. 

JOURNAL DE MATHÉMATIQUES PURES ET APPLIQUÉES
On the other hand we have
\[ \Delta |u_n| \geq \text{Re} \left( \Delta u_n \, \text{sign} \, u_n \right) \quad \text{in } \mathcal{D}'(\Omega), \]
which leads to
\[ -\Delta |u_n| - q^- |u_n| + \lambda |u_n| \leq |f| \quad \text{in } \mathcal{D}'(\Omega). \]

Let \( \psi \in H^1_0(\Omega) \) be the solution of
\[ -\Delta \psi - q^- \psi + \lambda \psi = |f|. \tag{19} \]

It follows from Lemma 3.1 that
\[ |u_n| \leq \psi \quad \text{a. e. on } \Omega. \tag{20} \]

By Theorem 2.3 we know that \( \psi \in L^p(\Omega) \) for every \( p \in [2, \infty] \). We extract a subsequence, denoted again by \( u_n \) such that \( u_n \to u \) weakly in \( H^1_0(\Omega) \), \( u_n \to u \) a. e. on \( \Omega \). We see as in the proof of Theorem 2.1 (Step 1) that \( (q^- u_n) \to q u \) in \( L^1_{\text{loc}}(\Omega) \). Therefore we have only to verify that \( q^- u_n \to q' u \) in \( L^1_{\text{loc}}(\Omega) \). We distinguish two cases:

(a) \( q' \in L^{1+\epsilon}_{\text{loc}}(\Omega) \);

(b) \( q' \in L^{m/2+\epsilon}_{\text{loc}}(\Omega) \).

\textbf{Case (a).} From (20) we deduce that \( u_n \to u \) in every \( L^p \) space, \( 2 \leq p < \infty \) and so \( q^- u_n \to q' u \) in \( L^1_{\text{loc}}(\Omega) \).

\textbf{Case (b).} Since \( q^- \psi \in L^q_{\text{loc}}(\Omega) \) for some \( q > m/2 \), it follows from (19) that \( \psi \in L^\infty_{\text{loc}}(\Omega) \). We deduce from the dominated convergence theorem that \( q^- u_n \to q' u \) in \( L^1_{\text{loc}}(\Omega) \).

\textbf{STEP 2.} \( A + \lambda_1 \) is accretive. Let \( u \in D(A) \) and set \( T = Vu \). We have \( T \in H^{-1}(\Omega) \cap L^1_{\text{loc}}(\Omega) \) and
\[ \text{Re} \langle T, \bar{u} \rangle = q \int |u|^2 \geq -q^- \int |u|^2 \in L^1(\Omega). \]

It follows from Lemma 2.2 that \( q |u|^2 \in L^1(\Omega) \) and
\[ \int q |u|^2 = \text{Re} \langle T, u \rangle = \text{Re} \langle A u + \Delta u, u \rangle. \]

Therefore
\[ \text{Re} \langle A u, u \rangle = \int |\text{grad } u|^2 + \int q |u|^2 \geq -\lambda_1 \int |u|^2. \tag{21} \]

\textbf{STEP 3.} \( D(A) \) is dense in \( L^2(\Omega) \). Given \( f \in L^2(\Omega) \cap L^\infty(\Omega) \) we solve for large \( n \) the equation
\[ u_n + \frac{1}{n} A u_n = f. \tag{22} \]

We shall prove that \( u_n \to f \) in \( L^2(\Omega) \) as \( n \to \infty \) — and as a consequence \( D(A) \) is dense in \( L^2(\Omega) \). By (21) we have
\[ \int |u_n|^2 + \frac{1}{n} \int |\text{grad } u_n|^2 + \frac{1}{n} \int q |u_n|^2 = \text{Re} \langle f, u_n \rangle. \]
In particular we deduce that

\begin{equation}
\limsup_{n \to \infty} \| u_n \|_{L^2} \leq \| f \|_{L^2},
\end{equation}

\begin{equation}
\frac{1}{n} \int q^+ |u_n|^2 \leq C,
\end{equation}

\begin{equation}
\frac{1}{n} \int |\nabla u_n|^2 \leq C.
\end{equation}

Next we have (as in the proof of Step 1):

\[ |u_n - \frac{1}{n} \Delta u_n - \frac{1}{n} q^- |u_n| \leq |f| \quad \text{in } \mathcal{D}'(\Omega). \]

On the other hand let \( \psi \in H^1_0(\Omega) \) be the solution of

\[ -\Delta \psi - q^- \psi + \lambda \psi = |f| \]

for some fixed \( \lambda > \lambda_1 \). Since \( |u_n| \leq \lambda |u_n/n| \) for \( n \geq \lambda \), we deduce from Lemma 3.1 that \( |u_n/n| \leq \psi \) a.e. Choose a subsequence, denoted again by \( u_n \) such that \( u_n \rightharpoonup u \) weakly in \( L^2(\Omega) \), \( (1/n)u_n \to 0 \) a.e. [this is possible since \((1/n)u_n \to 0\) in \( L^2(\Omega) \)]. For every \( \varphi \in \mathcal{D}(\Omega) \) we have

\begin{equation}
\int u_n \varphi - \frac{1}{n} \int u_n \Delta \varphi + \frac{1}{n} \int V u_n \varphi = \int f \varphi.
\end{equation}

We claim that \( 1/n \int V u_n \varphi \to 0 \) as \( n \to \infty \). Indeed by (24) and (25) we have

\[ \frac{1}{n} \int q^+ u_n \varphi \leq \frac{C}{\sqrt{n}} \quad \text{and} \quad \frac{1}{n} \int q^- u_n \varphi \leq \frac{C}{\sqrt{n}}. \]

Thus we have only to verify that \( 1/n \int q^+ u_n \varphi \to 0 \). We distinguish two cases:

(a) if \( q^- \in L^{(m/2)+\epsilon}(\Omega) \), we have \( \psi \in L^\infty(\Omega) \) and we deduce from the dominate convergence theorem that \( 1/n \int q^+ u_n \varphi \to 0 \);

(b) if \( q^\prime \in L^{1+\epsilon}(\Omega) \) we use the fact \( |u_n/n| \leq \psi \in L^p(\Omega) \) for every \( 2 \leq p < \infty \) to deduce that \( u_n/n \to 0 \) in \( L^p(\Omega) \) and so \( 1/n \int q^+ u_n \varphi \to 0 \).

In all the cases, we derive from (26) that

\[ \int u \varphi = \int f \varphi, \quad \forall \varphi \in \mathcal{D} \]

and consequently \( u = f \). We conclude using (23) that \( u_n \rightharpoonup f \) in \( L^2(\Omega) \).

**Step 4.** A is closable and \( \overline{A + \lambda_1} \) is \( m \)-accretive. This is a standard fact, see e.g. Theorem 3.4 in [2].
STEP 5. — $u \in D(\overline{A})$ implies that $u \in H^1_0(\Omega), q \left| v \right|^2 \in L^1(\Omega)$ and (14). We already know (Step 2) that $v \in D(A)$ implies $q \left| v \right|^2 \in L^1(\Omega)$ and

$$\text{Re}(A v, v) = \int \left| \nabla v \right|^2 + \int q \left| v \right|^2. \tag{27}$$

Now let $u \in D(\overline{A})$ and let $u_n \in D(A)$ be such that $u_n \to u$, $Au_n \to \overline{A}u$. It follows from (27) applied to $v = u_n - u_m$ that $u_n \to u$ in $H^1_0(\Omega)$ and $\int q^+ \left| u_n - u \right|^2 \to 0$ [since $u_n$ is a Cauchy sequence in $H^1_0(\Omega)$ and in $L^2(\Omega)$ with weight $q^+$. In particular $q \left| u \right|^2 \in L^1(\Omega)$ and (14) holds.

Proof of Theorem 3.2. — Clearly $A \subset A_1$. Now let $u \in D(A_1)$ and let $\lambda > \lambda_1$. Set $f = A_1 u + \lambda u$, and let $u^*$ be the unique solution of

$$\overline{A} u^* + \lambda u^* = f.$$

Thus, there exists a sequence $u^*_n \to u^*$ in $L^2(\mathbb{R}^m)$ with $u^*_n \in D(A)$ and

$$A u^*_n + \lambda u^*_n = f_n \to f \quad \text{in} \quad L^2(\mathbb{R}^m).$$

In particular we have

$$A_1 (u^*_n - u) + \lambda (u^*_n - u) = f_n - f$$

and therefore

$$-\Delta \left| u^*_n - u \right| - q^- \left| u^*_n - u \right| + \lambda \left| u_n - u \right| \leq |f_n - f| \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^m).$$

We deduce from Lemma 2.3 that $|u^*_n - u| \leq \psi_n$ a.e. on $\mathbb{R}^m$ where $\psi_n \in H^1(\mathbb{R}^m)$ is the solution of

$$-\Delta \psi_n - q^- \psi_n + \lambda \psi_n = |f_n - f|.$$

Hence $\|\psi_n\|_{H^1} \to 0$ and in particular $u^*_n - u \to 0$ in $L^2(\mathbb{R}^m)$. It follows that $u^* = u$ and so $u \in D(\overline{A})$. We conclude by Theorem 3.1 that $u \in H^1_0(\Omega)$ and therefore $u \in D(A_1) \cap H^1_0(\Omega) = D(A)$.

REFERENCES


(Manuscrit reçu en juin 1978.)

Haim BREZIS,
Département de Mathématiques,
Université Paris-VI,
4, place Jussieu,
75230 Paris 05;

Tosio KATO,
Department of Mathematics,
University of California,
Berkeley, CA 94720.