Asymptotic Behavior of Some Evolution Systems

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Introduction

Let \( C \) be a closed convex subset of a Hilbert space \( H \). We denote by \( S(t) \) a semi-group of nonlinear contractions on \( C \) i.e. \( \{S(t)\}_{t>0} \) is a family of mappings from \( C \) into itself satisfying: \( S(0) = I, S(t_1)S(t_2) = S(t_1 + t_2), \)
\[ |S(t)x - S(t)y| \leq |x - y| \quad \forall t > 0, \forall x, y \in C \quad \text{and} \]
\[ \lim_{t \to 0} |x - S(t)x| = 0 \quad \forall x \in C. \]

We say that \( p \in C \) is an equilibrium point of \( S(t) \) provided \( S(t)p = p \quad \forall t > 0 \) and we set
\[ F = \{p \in C; S(t)p = p \quad \forall t > 0\}. \]

It is well known that \( F \) is convex; we assume that \( F \neq \emptyset \).

Since, in general, \( S(t)x \) does not converge to a limit as \( t \to \infty \), it is of interest to consider the behavior as \( t \to \infty \) of the ergodic mean \( \sigma_t = \frac{1}{t} \int_0^t S(t)x \, dt \).

In §I we prove that \( \sigma_t \) converges weakly as \( t \to \infty \) to a limit \( \sigma \) which can be identified as the unique element in \( F \cap \overline{\text{conv}} \omega(x) \) (where \( \omega(x) \) denotes the weak \( \omega \)-limit set of \( S(t)x \)). In addition, \( \sigma \) coincides with the asymptotic center of \( S(t)x \) in the sense of Edelstein. In general \( \sigma_t \) does not converge strongly to \( \sigma \); yet in some special cases of importance, strong convergence holds.
In §II we consider gradient flows; i.e. $S(t)x$ denotes the solution of
\[
\begin{cases}
\frac{du}{dt} + \partial \varphi(u) \ni 0 & \text{a.e. on } (0, \infty) \\
u(0) = x
\end{cases}
\]
where $\varphi$ is a convex l.s.c. function on $H$. As we shall see $S(t)x$ converges weakly as $t \to \infty$ to a limit $\sigma$ such that $\varphi(\sigma) = \min \varphi$. However, strong convergence does not hold in general; this means that the method of steepest descent applied to a (smooth) convex function converges weakly—but not strongly in general.

In §III we consider the asymptotic behavior of the solution of an equation of the form
\[
\frac{du}{dt} + \partial \varphi(u) \ni f(t) \text{ a.e. on } (0, \infty), \; u(0) = x
\]
where $f(t)$ is a T-periodic forcing term. Here, the system "converges" as $t \to \infty$ to a T-periodic motion.

In §IV we conclude with a simple application.

Many of the results we discuss are due to J. B. Baillon and to R. Bruck; we also refer to the works of Dafermos-Slemrod [17], Brezis-Browder [12], Pazy [19], [20], [21], Reich [22].

I. Convergence of the ergodic mean.

We begin with a simple result

**Theorem 1 ([8])** $\sigma_t$ converges weakly as $t \to \infty$ to some $\sigma \in F$—which depends on $x$.

**Remark 1.** In general $\sigma_t$ does not converge strongly (see §II).

In order to identify the limit $\sigma$ we need some definitions:

a) $\omega(x)$ denotes the weak $\omega$-limit set of $S(t)x$ i.e. $\omega(x)$ iff there is a sequence $t_n \to \infty$ such that $S(t_n)x$ converges weakly to $y$.

b) The asymptotic center of a bounded function $u(t): [0, \omega) \to H$ is defined as follows. Given $y \in H$, set
\[
G(y) = \lim_{t \to \omega} \sup_{t} |u(t) - y|^2
\]
so that $G$ is strictly convex,
continuous, \( G(y) \to \infty \) as \( |y| \to \infty \). Therefore \( G \) achieves
its minimum on \( H \) at a unique point: \( \text{AC}(u(t)) \), called the
asymptotic center - a concept introduced by Edelstein.

**Theorem 2** ([15]). Let \( \sigma = \text{weak lim } \sigma_t \). Then

\[
\begin{align*}
(1) & \quad \sigma = F \cap \text{conv } \omega(x) \\
(2) & \quad \sigma = \text{AC}(S(t)x) .
\end{align*}
\]

**Remark 2.** Instead of \( \sigma_t \) consider now a more general aver-
ing process:

\[
\sigma_n = \int_0^\infty S(\tau)x a_n(\tau)d\tau
\]

where \( a_n \in L^1(0,\infty) \), \( a_n > 0 \) and \( \int_0^\infty a_n(\tau)d\tau = 1 \). Assume that
the functions \( a_n \) are of bounded variation and \( \int_0^\infty |da_n| \to 0 \).
Then \( \sigma_n \) converges weakly to \( \sigma \) as \( n \to \infty \), where \( \sigma \) is
the same as in Theorem 1 (\( \sigma \) is independent of \( a_n \)).

We begin with a simple Lemma

**Lemma 1.** Assume \( p_1, p_2 \in F \), \( q_1, q_2 \in \omega(x) \). Then
\( (p_1 - p_2, q_1 - q_2) = 0 \). In particular \( F \cap \text{conv } \omega(x) \) con-
tains at most one point.

**Proof of Lemma 1**

Set \( u(t) = S(t)x \). Given \( p \in F \), the function
\( |u(t) - p|^2 \) is nonincreasing and thus converges as \( t \to \infty \)
to a limit, say \( \xi(p) \).

We have
\[
|u(t) - p_1|^2 = |u(t) - p_2|^2 + 2(u(t) - p_2, p_2 - p_1) + |p_2 - p_1|^2 .
\]

Choosing \( t = t_n \) such that \( u(t_n) \to q_1 \) we find

\[
(3) \quad \xi(p_1) = \xi(p_2) + 2(q_1 - p_2, p_2 - p_1) + |p_2 - p_1|^2 .
\]

Similarly

\[
(4) \quad \xi(p_1) = \xi(p_2) + 2(q_2 - p_2, p_2 - p_1) + |p_2 - p_1|^2 .
\]

Comparing (3) and (4) leads to the conclusion.
Proofs of Theorem 1, 2 and Remark 2

Let $a_n$ and $c_n$ be as in Remark 2. Since
\[ ||a_n||_L^\infty \leq \int_0^\infty |da_n| \to 0 \]
it follows that every weak limit point of $c_n$ lies in $\text{conv} \; \omega(x)$.

We recall that $S(t)$ has a generator in the sense of Komura, Kato, Crandall-Pazy (see e.g. [11]); that is, there exists a maximal monotone operator $A$ with $\overline{D(A)} = C$ such that when $x \in D(A)$, $S(t)x$ coincides with the unique solution $u(t)$ of $\frac{du}{dt} + Au \geq 0$, a.e. on $(0, \infty)$, $u(0) = x$. Assume first $x \in D(A)$ and thus
\[ (Av + \frac{du}{dt}(t), v - u(t)) \geq 0 \quad \forall v \in D(A) \]
or
\[ (Av, v - u(t)) \geq \frac{1}{2} \frac{d}{dt} |u(t) - v|^2 . \]

Multiplying by $a_n(t)$ and integrating we find
\[ (Av, v - c_n) \geq - \frac{1}{2} a_n(0) |x - v|^2 - \frac{1}{2} \int_0^\infty |u(t) - v|^2 da_n . \]

Hence $\forall v \in D(A)$

\[ (Av, v - c_n) \geq - ||u - v||_L^2 \int_0^\infty |da_n| . \]

By a density argument, (5) holds now even when $x \in \overline{D(A)} = C$.

It follows from (5) that every weak limit point of $c_n$ lies in $F$. Indeed let $c_n \rightharpoonup \sigma$; we have

\[ (Av, v - \sigma) \geq 0 \quad \forall v \in D(A) \]
and so $0 \in A\sigma$, i.e. $\sigma \in F$.

Consequently every weak limit point of $c_n$ lies in $F \cap \text{conv} \; \omega(x)$ — which consists of a single element.

We prove now that $\sigma$ coincides with $AC(S(t)x)$.

Given $y \in H$ we have
\[ |u(t) - \sigma|^2 = |u(t) - y|^2 + 2(u(t) - y, y - \sigma) + |y - \sigma|^2 . \]

Thus
\[ \int_0^\infty |u(t) - \sigma|^2 a_n(t) dt = \int_0^\infty |u(t) - y|^2 a_n(t) dt + \\
+ 2(u_n - y, y - \sigma) + |y - \sigma|^2 . \]
Since $\sigma \in F$, $|u(t) - \sigma|^2$ converges to a limit as $t \to \infty$, which is simply $G(\sigma)$. Since on the other hand $G(y) = \limsup_{t \to \infty} |u(t) - y|^2$ we find

$$G(\sigma) \leq G(y) - |y - \sigma|^2 \quad \forall y \in H.$$ 

Consequently $\sigma = \text{AC}(S(t)x)$.

**Remark 3.** Let $T$ be a contraction in $H$ having at least one fixed point. Baillon [2] has proved the following (see also [19]):

**Theorem 3.** The Cesaro mean

$$\sigma_n = \frac{1}{n} (x + Tx + \ldots + T^{n-1}x)$$

converges weakly as $n \to \infty$ to a fixed point of $T$.

Theorem 3 can be viewed as a nonlinear version of the classical ergodic theory of von Neumann, Kakutani, Yosida. Theorem 1 can be derived from Theorem 3 by using a device due to Konishi (see [6]). In general $\sigma_n$ does not converge strongly.

Theorem 3 has been extended by Baillon to $L^p$ spaces, $1 < p < \infty$, in [6] and subsequently by Bruck [16] to more general spaces. The proofs are very tricky. In view of Konishi's device - which is valid in general Banach spaces - Theorem 1 holds true in $L^p$ spaces $1 < p < \infty$.

**Strong convergence**

In some special cases $\sigma_t$ converges strongly as $t \to \infty$. For example when the orbit $\cup S(t)x$ is relatively compact $t > 0$ this is the setting of Dafermos-Slemrod [17].

Another surprising condition which implies strong convergence of $\sigma_t$ is the oddness of $S(t)$ - the fact that oddness has an impact on strong convergence was first observed by Bruck [14] in the case of gradient flows.

**Theorem 4** ([4]). Assume $C$ is symmetric and $S(t)$ is odd i.e. $S(t)(-u) = -S(t)u \forall u \in C, \forall t > 0$. Then $\sigma_t$ converges strongly as $t \to \infty$. More generally $\sigma_n = \int S(t)x a_n(t)dt$ converges strongly as $n \to \infty$ provided $a_n^0$ satisfies the assumptions in Remark 2.

**Remark 4.** It is not known whether Theorem 4 holds in $L^p$ spaces $1 < p < \infty$. 
Remark 5. Assume \( T \) is an odd contraction on \( H \). Then

Baillon [3] has proved that

\[
\sigma_n = \frac{1}{n} (x + T x + \ldots + T^{n-1} x)
\]

converges strongly to a fixed point of \( T \). Such a result, combined with Konishi's device could be used to prove Theorem 4.

A crucial ingredient in the proof of Theorem 4 is the following:

**Lemma 2** ([15], [12], [22]). Let \( u(t) \) be a function defined on \((0, \infty)\) with values into \( H \) such that: for all \( h \geq 0 \),

\( u(t), u(t + h) \) converges as \( t \to \infty \) to a limit, say \( l(h) \), uniformly in \( h \). Then \( \sigma_n = \int_0^\infty u(\tau) a_n(\tau) d\tau \) converges strongly as \( n \to \infty \) to \( AC(u(t)) \) provided \( a_n \) satisfies the conditions of Remark 2.

For the proof of Lemma 2 we refer to [15], [12] and [22]. Note that a special case of Lemma 2 asserts that if \( (x_n) \) is a sequence such that \( (x_n, x_{n+1}) \) converges as \( n \to \infty \) to \( l(i) \), uniformly in \( i \), then \( \sigma_n = \frac{1}{n} (x_0 + x_1 + \ldots + x_{n-1}) \) converges strongly.

**Proof of Theorem 4.** Let \( t \geq s \) and \( x, y \in C \). We have

\[
|S(t+h)x - S(t)y| \leq |S(s+h)x - S(s)y|
\]

and so

\[
|S(t+h)x|^2 - 2(S(t+h)x, S(t)y) + |S(t)y|^2 \leq |S(s+h)x|^2 - 2(S(s+h)x, S(s)y) + |S(s)y|^2.
\]

Hence we find

\[
2[(S(s+h)x, S(s)y) - (S(t+h)x, S(t)y)]
\]

\[
\leq |S(s+h)x|^2 + |S(s)y|^2 - |S(t+h)x|^2 - |S(t)y|^2.
\]

Choosing \( y = \pm x \) and using the oddness of \( S(t) \) leads to

\[
2|(S(s+h)x, S(s)x) - (S(t+h)x, S(t)x)|
\]

\[
\leq 2|S(s)x|^2 - 2|S(t+h)x|^2.
\]
But $|S(t)x|$ converges as $t \to \infty$. It follows that 
$(S(t)x, S(t+h)x)$ is Cauchy as $t \to \infty$, uniformly in $h$. 
Therefore Lemma 2 can be applied.

II. Gradient flows

For the class of gradient flows corresponding to convex functions it is not necessary to consider averages of orbits: 
the orbit itself converges as $t \to \infty$. More precisely let $\varphi$ 
be a l.s.c. convex function on $H$ such that $\min \varphi$ is 
achieved. Set $F = \{p; \varphi(p) = \min \varphi\}$ and set $A = \partial \varphi$. We 
denote by $S(t)x$ the semigroup generated by $-A$, i.e. 
$S(t)x = u(t)$ is the unique solution of 
$$
\frac{du}{dt} + Au \geq 0 \text{ a.e. on } (0, \infty), u(0) = x.
$$

**Theorem 5** (Bruck [14]). For each $x \in \overline{D(A)}$ $S(t)x$ converges 
weakly as $t \to \infty$ to some $p \in F$.

Theorem 5 is an obvious consequence of Theorem 1 and the 
following

**Lemma 3.** We have for every $x \in \overline{D(A)}$

$$
\lim_{t \to \infty} \left| S(t)x - \frac{1}{t} \int_0^t S(\tau)x d\tau \right| = 0.
$$

**Proof.** We have, setting $u(t) = S(t)x$

$$
|u(t) - \frac{1}{t} \int_0^t u(\tau)d\tau| = \frac{1}{t} \int_0^t |u'(\tau)d\tau|.
$$

Therefore it suffices to prove that $\lim_{t \to \infty} t|u'(t)| = 0$. We 
know (see [10] Theorem 22) that

$$
\int_0^\infty |\frac{du}{dt}(t)|^2 t dt \leq \frac{1}{2} \text{dist} (x, F)^2.
$$

Since the function $t \mapsto |\frac{du}{dt}(t)|$ is non increasing it follows that

$$
\frac{3}{8} \int_{t/2}^t |\frac{du}{dt}(\tau)|^2 \tau d\tau = 0 \text{ as } t \to \infty.
$$

**Remark 6.** Baillon [5] has constructed an example of a convex 
l.s.c. function $\varphi$ such that $S(t)x$ does not converge 
strongly as $t \to \infty$. The existence of such an example had 
been suggested earlier by Kömura. In fact such a $\varphi$ can be
chosen to be $C^1$ with $\text{grad} \varphi$ lipschitzian (see [7]). In view of (6), $\sigma_t = \frac{1}{t} \int_0^t S(\tau)x$ does not converge strongly as $t \to \infty$, thus providing a counterexample to the strong convergence of the ergodic mean.

Strong convergence of $S(t)x$ holds in the following cases:

a) For every $M$ the set $\{u \in H, |u| \leq M$ and $\varphi(u) \leq M\}$ is compact (see [10]).

b) The function $\varphi$ is even (see [14]); here $A$ is odd and we conclude by Theorem 4 and (6) that $S(t)x$ converges strongly.

It would be of interest to find other conditions implying the strong convergence of $S(t)x$. For example, assume $\varphi$ is even and let $f \in H$ be such that $\inf_{z \in H} (\varphi(z)-(f,z))$ is achieved.

Let $u(t)$ be the solution of $\frac{du}{dt} + \partial \varphi(u) \ni f, u(0) = x$. Does $u(t)$ converge strongly as $t \to \infty$?

Remark 7. A "discrete" version of Theorem 5 has been proved in [13]: let $A = \partial \varphi$ and let $u_n$ be the sequence defined by the "implicit" scheme

$$\frac{u_{n+1} - u_n}{\tau_n} + Au_{n+1} \ni 0, \quad \tau_n > 0.$$ 

Then $u_n \to p \in F$ provided $\sum_1^\infty \tau_n = \infty$.

III. Periodic forcing

We turn now to the following question. Consider a system governed by a convex potential $\varphi$ with a periodic forcing term $f(t)$:

$$\begin{aligned}
\begin{cases}
\frac{du}{dt}(t) + \partial \varphi(u(t)) \ni f(t) \text{ a.e. on } (0,\infty) \\
u(0) = u_0.
\end{cases}
\end{aligned}$$

What happens as $t \to \infty$? The answer is given by the following Theorem 6 (Baillon-Haraux [9])

Assume $f \in L^2_{\text{loc}}(0,\infty; H)$ is $T$-periodic and assume $u(t)$ remains bounded as $t \to \infty$. Then there exists a $T$-periodic function $\bar{u}(t)$ satisfying
\[ \frac{d\bar{u}}{dt} + \beta \tilde{\psi}(\bar{u}) \geq f \text{ a.e. on } (0,\infty) \]

such that \( \lim_{t \to \infty} (u(t) - \bar{u}(t)) = 0 \).

In addition,

\[ \lim_{n \to \infty} \int_0^T \left| \frac{du}{dt}(nT + t) - \frac{d\bar{u}}{dt}(t) \right|^2 dt = 0. \]

In other words the system evolves "slowly" to a periodic motion. For the proof of Theorem 6 we refer to [9]. The proof involves an interesting Lemma which we describe in a special case:

**Lemma 4.** Let \( f \in L^2(0,T; H) \) and let \( A = \beta \tilde{\psi} \). Assume \( u_n \) and \( u \) are the solutions of

\[ \frac{du_n}{dt} + Au_n \geq f \text{ a.e. on } (0,T), \quad u_n(0) = u_0, \]

\[ \frac{du}{dt} + Au \geq f \text{ a.e. on } (0,T), \quad u(0) = u_0, \]

with \( u_0 \) being the limit of \( u_n \). Then \( u_n + u \) is a solution of

\[ \frac{d}{dt} \int_0^T \frac{du_n}{dt} - \frac{du}{dt} \parallel t \parallel dt = 0. \]

**Proof.** We know (see [10]) that \( u_n + u \) is a solution of \( \psi(u_n), \psi(u) \in L^1(0,T) \) and \( \frac{du_n}{dt}, \frac{du}{dt} \in L^2(0,T; H) \). We have

\[ (8) \psi(u(t)) - \psi(u_n(t)) \geq (f(t) - \frac{du_n(t)}{dt}, u(t) - u_n(t)) \]

and for fixed \( v \in D(\psi) \)

\[ (9) \psi(v) - \psi(u_n(t)) \geq (f(t) - \frac{du_n(t)}{dt}, v - u_n(t)). \]

From (9) we deduce

\[ (10) \int_0^\epsilon \psi(u_n) dt \leq \epsilon \psi(v) - \epsilon (f, v - u_n) dt - \frac{1}{2}v - u_n(\epsilon) \parallel \epsilon \parallel^2 + \frac{1}{2}v - u_0 \parallel^2 \]

and from (8)
\[
(11) \int_t^T \varphi(u_n) dt \leq \int_0^T \varphi(u) dt + \int_0^T (f, u_n - u) dt \\
+ \int_0^T \left( \frac{\partial u_n}{\partial t}, u - u_n \right) dt - \frac{1}{2} |u_n(T) - u(T)|^2 + \frac{1}{2} |u_n(0) - u(0)|^2.
\]

Adding (10) and (11) and using the fact that \( u_n \rightarrow u \) in \( C([0, T]; H) \) we find
\[
\limsup_{n \to \infty} \int_0^T \varphi(u_n) dt \leq \int_0^T \varphi(v) dt - \int_0^T (f, v - u) dt - \frac{1}{2} |v - u(0)|^2 \]
\[
+ \frac{1}{2} |v - u_0|^2 + \int_0^T \varphi(u) dt.
\]

As \( \epsilon \to 0 \) we obtain
\[
(12) \limsup_{n \to \infty} \int_0^T \varphi(u_n) dt \leq \int_0^T \varphi(u) dt.
\]

On the other hand we have
\[
(13) \int_0^T |\frac{\partial u_n}{\partial t}|^2 dt + T\varphi(u_n(T)) - \int_0^T \varphi(u_n) dt = \int_0^T (f, \frac{\partial u_n}{\partial t}) dt
\]
\[
(14) \int_0^T |\frac{\partial u}{\partial t}|^2 dt + T\varphi(u(T)) - \int_0^T \varphi(u) dt = \int_0^T (f, \frac{\partial u}{\partial t}) dt.
\]

Combining (12), (13), (14) and the fact that \( \sqrt{\epsilon} \frac{\partial u_n}{\partial t} \) converges weakly in \( L^2(0, T; H) \) to \( \sqrt{\epsilon} \frac{\partial u}{\partial t} \) we see that
\[
\limsup_{n \to \infty} \int_0^T |\frac{\partial u_n}{\partial t}|^2 dt \leq \int_0^T |\frac{\partial u}{\partial t}|^2 dt, \text{ and therefore}
\]
\[
\lim_{n \to \infty} \int_0^T |\frac{\partial u_n}{\partial t} - \frac{\partial u}{\partial t}|^2 dt = 0.
\]

Remark 8. Using a similar device, H. Attouch [1] has proved the following. Let \( A_n = \varphi_n \) and \( A = \varphi \) be such that \( A_n \rightarrow A \) in the sense of graphs i.e. \( (I + A_n)^{-1} \rightarrow (I + A)^{-1} \).

Assume \( u_n \) and \( u \) are the solutions of
\[
\frac{du_n}{dt} + A_n u_n = f \quad \text{a.e. on } (0, T), \quad u_n(0) = u_{0n}
\]
\[
\frac{du}{dt} + Au = f \quad \text{a.e. on } (0, T), \quad u(0) = u_0
\]

with \( u_{0n} \rightarrow u_0 \).
Then \( u_n \to u \) in \( C([0,T]; H) \) (this is just the nonlinear version of the Trotter-Kato theorem) and also
\[
\int_0^T |u_n' - u'|^2 dt \to 0.
\]

**Remark 9.** Let \( f \in L^2_{\text{loc}}(0,\infty; H) \), let \( A = \partial \psi \) and let \( u, v \) be the solutions of
\[
\begin{align*}
\frac{du}{dt} + Au &\geq f \text{ a.e. on } (0,\infty), \quad u(0) = u_0 \\
\frac{dv}{dt} + Av &\geq f \text{ a.e. on } (0,\infty), \quad v(0) = v_0.
\end{align*}
\]

Does \( u(t) - v(t) \) converge weakly to a limit as \( t \to \infty \)? Note that the answer is positive in case \( f \) is \( T \)-periodic and \( u(t) \) remains bounded. Indeed there are two \( T \)-periodic solutions \( \overline{u}(t) \) and \( \overline{v}(t) \) such that \( w\text{-}\lim_{t \to \infty} (u(t) - \overline{u}(t)) = 0 \), \( w\text{-}\lim_{t \to \infty} (v(t) - \overline{v}(t)) = 0 \). Since the difference of two \( T \)-periodic solutions is constant (see [9]) it follows that \( w\text{-}\lim_{t \to \infty} (u(t) - v(t)) \) exists.

**IV. An example**

Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^N \). Given \( \psi(x) \in L^2(\Omega) \) with \( \psi \geq 0 \) on \( \Omega \) and \( u_0 \in H^1_0(\Omega) \), find a function \( u(x,t) \) satisfying
\[
\begin{align*}
u_t &= \min\{\Delta u, \psi\} & &\text{on } \Omega \times (0,\infty) \\
u &= 0 & &\text{on } \partial \Omega \times (0,\infty) \\
u(x,0) &= u_0(x) & &\text{on } \Omega
\end{align*}
\]
(15)

(related questions occur in heat control, see [18] Chap. 2).

**Theorem 7.** There exists a unique solution of (15). In addition \( u(x,t) \) converges weakly in \( H^1_0 \), as \( t \to \infty \), to a function \( u_\infty(x) \) satisfying
\[
\min\{\Delta u_\infty, \psi\} = 0.
\]
(16)

**Remark 10.** It is clear that any function \( u_\infty \in H^1_0 \) satisfying (16) is an equilibrium and that there are many such functions. We don't know how to identify \( \lim_{t \to \infty} u(x,t) \) (in terms...
of \( u_0 \) among all the equilibria. Also we don't know whether \( u(x,t) \) converges strongly in \( H^1_0 \).

**Proof.** Let \( K = \{ v \in H^1_0(\Omega), v \leq \psi \text{ a.e.} \} \). On \( H = H^1_0(\Omega) \) we consider the scalar product

\[
a(u,v) = \int_\Omega \text{grad } u \cdot \text{grad } v \, dx.
\]

Problem (15) can be expressed in a weak form as:

\[
u_t \in K
\]

\[
\int_\Omega u_t (v - u_t) \, dx + a(u, v - u_t) \geq 0 \quad \forall v \in K, \forall t > 0
\]

or equivalently

\[
a(u, v - u_t) + \psi(v) - \psi(u_t) \geq 0 \quad \forall v \in H^1_0, \forall t > 0
\]

where \( \psi \) is a convex l.s.c. function defined on \( H \) by

\[
\varphi(v) = \begin{cases} 
\frac{1}{2} \int_\Omega |v|^2 \, dx & \text{if } v \in K \\
\infty & \text{if } v \notin K
\end{cases}
\]

But (17) can be written as \( -u \in \partial \varphi(u_t) \) or \( u_t - \partial \varphi^*(-u) \geq 0 \).

Now we are reduced to the abstract setting and we can use Theorem 5. Note that \( u_\infty \) is an equilibrium provided

\[
- \frac{1}{2} \int_\Omega v^2 \, dx \geq - a(u_\infty, v) \quad \forall v \in K
\]

or equivalently \( \int_\Omega \text{grad } u_\infty \cdot \text{grad } v \, dx \geq 0 \quad \forall v \in K \).

**Remark 11.** Instead of (15) consider now a problem with two side-constraints on \( u_t \): find \( u(x,t) \) satisfying

\[
\begin{align*}
|u_t| & \leq \psi & & \text{on } \Omega \times (0,\infty) \\
u_t - \Delta u & = 0 & & \text{on } \{|u_t| < \psi\} \\
u_t - \Delta u & \leq 0 & & \text{on } \{u_t = \psi\} \\
u_t - \Delta u & \geq 0 & & \text{on } \{u_t = - \psi\} \\
u & = 0 & & \text{on } \partial \Omega \times (0,\infty) \\
u(x,0) & = u_0(x) & & \text{on } \Omega
\end{align*}
\]

(18)
The same kind of proof as above shows that there is a unique solution \( u \) of (18) and that \( u(x, t) \) converges strongly in \( H^1_0 \) to a function \( u_\infty \) satisfying \( \Delta u_\infty = 0 \) on \( \{ \psi > 0 \} \). Here strong convergence holds since \( \psi \) is even.

REFERENCES

[5] J. B. Baillon, Un exemple concernant le comportement asymptotique de la solution du problème \( \frac{du}{dt} + \psi(u) \geq 0 \), to appear J. Funct. Anal.


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