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# Remarks on Some Fourth Order Variational Inequalities

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*dedicated to Hans Lewy*

## 1. – Introduction.

Some years ago the Authors studied the regularity of a variational inequality for a second order variational inequality [1], which turned out to be useful in the solution of the elastic-plastic torsion of a bar ([2], [3]).

We recall briefly the result we obtained there:

Let $\boldsymbol{K}$ be the convex of $H_0^1(\Omega)$

$$\boldsymbol{K} = \{v \in H_0^1(\Omega) \colon |\mathrm{grad}\, v| \leqslant 1 \ \text{ in } \ \Omega\}$$

where $\Omega$ is a domain of $\boldsymbol{R}^N$ satisfying suitable conditions and let $f$ be a function in $L^p(\Omega)$. $(2 \leqslant p < +\infty)$.

The solution $u$ of the variational inequality

$$u \in \boldsymbol{K} \colon \int_\Omega \mathrm{grad}\, u\, \mathrm{grad}(v-u)\, dx \geqslant \int_\Omega f(v-u)\, dx \qquad \text{for all } v \in \boldsymbol{K}$$

belongs to $H^{2,p}(\Omega)$.

The purpose of this paper is to consider an analogous problem related to the biharmonic operator in place of the laplacian. We shall consider two types of boundary conditions.

The variational inequalities we consider are the following

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(I) Let $K_1$ be the closed convex set

$$K_1 = \{v \in H_0^1(\Omega) \cap H^2(\Omega) : \alpha \leqslant \Delta v \leqslant \beta \text{ a.e. in } \Omega\}$$

where, for sake of simplicity, $\alpha$ and $\beta$ are two constants such that

$$\alpha < 0 < \beta \text{ (}^1\text{)}$$

and $\Omega$ is a bounded open set of $\boldsymbol{R}^N$. We assume that $\Omega$ is smooth; so that in particular the bilinear form

$$a(u, v) = \int_\Omega \Delta u \, \Delta v \, dx$$

is coercive in $H_0^1(\Omega) \cap H^2(\Omega)$.

Let $f$ be an element of $H^{-1,p}$ $(2 \leqslant p < +\infty)$ and denote by $u \in K_1$ the solution of the variational inequality

$$u \in K_1 : \int_\Omega \Delta u \, \Delta(v - u) \, dx \geqslant \langle f, v - u \rangle \quad \text{for all } v \in K_1 . \tag{1.1}$$

There exists one and only one solution of (1.1).

We shall prove the following

THEOREM 1. *The solution $u$ of the variational inequality* (1.1) *belongs to $H^{3,p}(\Omega)$, $\Delta u \in H_0^1(\Omega)$. $\Delta u \in H^{1,\infty}(\Omega)$ if $f \in L^p(\Omega)$ with $p > N$.*

The second variational inequality we shall consider is the following

(II) Let $K_2$ be the closed convex set

$$K_2 = \{v \in H_0^2(\Omega) : \alpha \leqslant \Delta v \leqslant \beta, \text{ a.e. in } \Omega\}$$

where again $\alpha < 0 < \beta$ and $\Omega$ is a bounded open set of $\boldsymbol{R}^N$ with smooth boundary; in particular the bilinear form $a(u, v)$ is coercive in $H_0^2(\Omega)$.

Let $f$ be an element of $H^{-1,p}(\Omega)$ and denote by $u$ the solution of the

([1]) There is no difficulty to consider two functions $\alpha(x)$ and $\beta(x)$ instead of constants.

variational inequality:

$$u \in \boldsymbol{K}_2\colon \int_\Omega \Delta u\, \Delta(v-u)\, dx \geqslant \langle f, v-u\rangle \qquad \text{for all } v \in \boldsymbol{K}_2 . \tag{1.2}$$

There exists a unique solution to (1.2). We shall prove the following

THEOREM 2. *The solution $u$ of the variational inequality* (1.2) *belongs to $H^{3,p}_{\mathrm{loc}}(\Omega)$. $\Delta u \in H^{1,\infty}_{\mathrm{loc}}(\Omega)$ if $f \in L^p(\Omega)$ with $p > N$.*

## 2. – Proof of theorem 1.

First of all we consider a very simple variational inequality. Let $F$ be a function in $H^{1,p}$ $(2 \leqslant p \leqslant +\infty)$ and $\mathcal{K}_0$ the closed convex set

$$\mathcal{K}_0 = \{V \in L^2(\Omega)\colon \alpha \leqslant V \leqslant \beta\} .$$

The (unique) solution of the variational inequality

$$U \in \mathcal{K}_0\colon \int_\Omega U(V-U)\, dx \geqslant \int_\Omega F(V-U)\, dx \qquad \text{for all } V \in \mathcal{K}_0 \tag{2.1}$$

is given by

$$U = \begin{cases} \alpha & \text{if } F \leqslant \alpha , \\ F & \text{if } \alpha \leqslant F \leqslant \beta , \\ \beta & \text{if } F \geqslant \beta . \end{cases}$$

Thus we have the following

LEMMA 2.1. *The solution of the variational inequality* (2.1) *is the projection of $F$ on $\mathcal{K}_0$ and is given by*

$$U = \mathrm{Pr}_{\mathcal{K}_0} F = \tau \circ F \tag{2.2}$$

*where $\tau(t)$ is the truncation function*

$$\tau(t) = \begin{cases} \alpha & \text{for } t \leqslant \alpha \\ t & \text{for } \alpha \leqslant t \leqslant \beta \\ \beta & \text{for } t \geqslant \beta \end{cases} \tag{2.2$'$}$$

*and hence, for* $F \in H^{1,p}(\Omega)$ $(2 \leqslant p \leqslant +\infty)$, *we have* $U \in H^{1,p}(\Omega)$. *In particular if* $F \in H_0^{1,p}(\Omega)$ *we have* $U \in H_0^{1,p}(\Omega)$.

PROOF OF THEOREM 1. Let $F$ be the solution of the boundary value problem

$$F \in H_0^1(\Omega): \Delta F = f \quad \text{in } \Omega . \tag{2.3}$$

Since $f \in H^{-1,p}(\Omega)$ it follows that $F \in H_0^{1,p}(\Omega)$.

The variational inequality (1.1) may be written

$$\int_\Omega \Delta u \, \Delta(v-u)\, dx \geqslant \int_\Omega F \Delta(v-u)\, dx$$

and consequently if we set $U = \Delta u$, $V = \Delta v$, we get

$$U \in \mathcal{K}_0: \int_\Omega U(V-U)\, dx \geqslant \int_\Omega F(V-U)\, dx \quad \text{for all } V \in \mathcal{K}_0$$

where $\mathcal{K}_0$ is the convex set defined above. It follows from lemma 2.1 that

$$U = \tau \circ F \in H_0^{1,p}(\Omega) \qquad 2 \leqslant p \leqslant +\infty .$$

Therefore the function $u$ is the solution to the boundary value problem

$$u \in H_0^1(\Omega): \Delta u = U$$

and consequently $\Delta u \in H_0^{1,p}(\Omega)$ and $u \in H^{3,p} (2 \leqslant p < +\infty)$.

Thus theorem 1 is proved.

## 3. – Proof of theorem 2.

We begin by introducing some notation.

In addition to the convex set $\mathcal{K}_0$ that we have already defined, we consider the closed linear space $\mathcal{K}_1$

$$\mathcal{K}_1 = \left\{ V \in L^2(\Omega): \int_\Omega V\zeta \, dx = 0 \text{ for all } \zeta \in C^2(\bar{\Omega}) \text{ such that } \Delta\zeta = 0 \right\}$$

and we set

$$\mathcal{K} = \mathcal{K}_0 \cap \mathcal{K}_1 \neq \emptyset .$$

The proof of theorem 2 is based on an approximation argument by a penality method.

Let $\gamma(t)$ be the function defined by

$$\gamma(t)=\begin{cases} t-\alpha & \text{for } t\leqslant\alpha \\ 0 & \text{for } \alpha\leqslant t\leqslant\beta \\ t-\beta & \text{for } t\geqslant\beta \end{cases}$$

and set, for $\lambda>0$:

$$\gamma_\lambda(t)=\frac{1}{\lambda}\gamma(t)\,.$$

Let $F$ be the solution of problem (2.3); we recall that $F\in H_0^{1,p}(\Omega)$.

We prove the following:

LEMMA 3.1. *There exists a unique function* $U_\lambda\in\mathcal{K}_1$ *such that*

$$U_\lambda+\gamma_\lambda(U_\lambda)-F\in\mathcal{K}_1^\perp\,. \tag{3.1}$$

*Moreover, as* $\lambda\to 0$, $U_\lambda$ *converges in* $L^2(\Omega)$ *to a function* $U$ *such that* $U\in\mathcal{K}=\mathcal{K}_0\cap\mathcal{K}_1$, *and*

$$\int_\Omega U(V-U)\,dx\geqslant\int_\Omega F(V-U)\,dx\qquad\text{for all } V\in\mathcal{K}\,. \tag{3.2}$$

PROOF. Since the operator $I+\gamma_\lambda-F$ is monotone hemicontinuous and strongly coercive in $L^2(\Omega)$, there exists a unique solution to the variational inequality

$$U_\lambda\in\mathcal{K}_1\colon\ (U_\lambda+\gamma_\lambda(U_\lambda)-F,\,V-U_\lambda)\geqslant 0\qquad\text{for all } V\in\mathcal{K}_1$$

i.e.

$$U_\lambda\in\mathcal{K}_1\colon\ (U_\lambda+\gamma_\lambda(U_\lambda)-F,\,\varphi)=0\qquad\text{for all } \varphi\in\mathcal{K}_1$$

and this means that (3.1) holds.

Multiplying (3.1) by $U_\lambda$ and integrating we have

$$\int_\Omega U_\lambda^2\,dx+\int_\Omega\gamma_\lambda(U_\lambda)\,U_\lambda\,dx=\int_\Omega F\cdot U_\lambda\,dx$$

and thus, since $\gamma_\lambda(U_\lambda)\,U_\lambda\geqslant 0$, it follows that

$$\|U_\lambda\|_{L^2(\Omega)}\leqslant\|F\|_{L^2(\Omega)}\,. \tag{3.3}$$

Now let $V$ be any function in $\mathcal{K} = \mathcal{K}_0 \cap \mathcal{K}_1$; then

$$\int_\Omega [\gamma_\lambda(U_\lambda) - \gamma_\lambda(V)](U_\lambda - V)\, dx = \int_\Omega \gamma_\lambda(U_\lambda)(U_\lambda - V)\, dx \geqslant 0$$

since $\gamma_\lambda(V) = 0$. Therefore

$$(3.4) \qquad \int_\Omega (F - U_\lambda)(U_\lambda - V)\, dx \geqslant 0 .$$

By (3.3) there exists a subsequence $\{U_{\lambda_n}\}$ of $U_\lambda$ such that

$$U_{\lambda_n} \to U \quad \text{weakly in } L^2(\Omega) .$$

Since $\liminf \| U_{\lambda_n} \|_{L^2(\Omega)} \geqslant \| U \|_{L^2(\Omega)}$, we get at the limit

$$\int_\Omega (F - U)(U - V)\, dx \geqslant 0 \quad \text{for all } V \in \mathcal{K}$$

and (3.2) is proved.

In order to show that $U \in \mathcal{K}$ it is enough to prove that $U \in \mathcal{K}_0$.

Let $\Gamma(t)$ be a primitive of $\gamma$, i.e.; $\Gamma'(t) = \gamma(t)$, $\Gamma(0) = 0$. Since $\Gamma(t)$ is a convex function, we have

$$-\Gamma(U_\lambda) = \Gamma(0) - \Gamma(U_\lambda) \geqslant \gamma(U_\lambda)(-U_\lambda)$$

and thus

$$\frac{1}{\lambda} \int_\Omega \Gamma(U_\lambda)\, dx \leqslant \int_\Omega (F - U_\lambda)\, U_\lambda\, dx \leqslant C = \text{const. indep. of } \lambda ,$$

which implies

$$\int_\Omega \Gamma(U_\lambda)\, dx \leqslant C\lambda .$$

Passing to the limit as $\lambda \to 0$ we get

$$\int_\Omega \Gamma(U)\, dx = 0 .$$

From this relation it follows that $\alpha \leqslant U \leqslant \beta$ a.e. in $\Omega$, i.e.

$$U \in \mathcal{K}_0 .$$

By uniqueness of the solution of (3.2) it follows that $U_\lambda$ converges weakly to $U$ as $\lambda$ goes to 0.

In order to complete the proof of the lemma we remark, first, that by (3.4), for any $V \in \mathcal{K}$

$$\limsup \int_\Omega U_\lambda^2\, dx \leqslant \int_\Omega F U\, dx - \int_\Omega F V\, dx + \int_\Omega U \cdot V\, dx$$

and then, choosing $V = U$, we get

$$\limsup \int_\Omega U_\lambda^2\, dx \leqslant \int_\Omega U^2\, dx .$$

Hence

$$\lim_{\lambda \to 0} U_\lambda = U \qquad \text{in } L^2(\Omega) .$$

Now we prove the following

LEMMA 3.2. *Let $U$ be the solution of the variational inequality* (1.2). *There exists a function $z \in L^1(\Omega)$ such that $\Delta z = 0$ and*

$$U = \tau \circ (F + z)$$

*where $\tau$ is given by* (2.2) *and $F$ is the solution of* (2.3).

PROOF. *a*) Set

$$z_\lambda = U_\lambda + \gamma_\lambda(U_\lambda) - F ,$$

where $z_\lambda \in \mathcal{K}_1^\perp$. Therefore $\int_\Omega z_\lambda \Delta\varphi\, dx = 0$ for all $\varphi \in \mathcal{D}(\Omega)$, indeed $\Delta\varphi \in \mathcal{K}_1$.

It follows that

$$\Delta z_\lambda = 0$$

in the sense of distribution and thus $z_\lambda \in C^\infty(\Omega)$.

*b*) Since for all $V \in \mathcal{K}_0$, $\gamma_\lambda(V) = 0$ we have

$$\int_\Omega \gamma_\lambda(U_\lambda)(U_\lambda - V)\, dx \geqslant 0$$

i.e.

$$\int_\Omega (z_\lambda - U_\lambda + F)(U_\lambda - V)\, dx \geqslant 0 \qquad \text{for all } V \in \mathcal{K}_0 .$$

Consequently

$$\int_\Omega (z_\lambda - U_\lambda + F)\, V\, dx \leqslant \int_\Omega (F - U_\lambda)\, U_\lambda\, dx \leqslant \tfrac{1}{2}\|F\|^2_{L^2(\Omega)}\,.$$

From this we deduce first that

$$\|z_\lambda - U_\lambda + F\|_{L^1(\Omega)} \leqslant \text{Const} \qquad (\text{indep. of } \lambda)$$

and then that

$$\|z_\lambda\|_{L^1(\Omega)} \leqslant \text{Const} \qquad (\text{indep. of } \lambda)$$

and in particular, since $\Delta z_\lambda = 0$, we have $\|z_\lambda\|_{C^1(K)}$ is bounded by a constant depending only on $K \subset\subset \Omega$.

Thus, there exists a subsequence $z_{\lambda_n}$ converging uniformly on any compact if $\Omega$ to a function $z$ such that

$$\Delta z = 0\,.$$

Moreover, by Fatou's lemma, $z \in L^1(\Omega)$.

On the other hand

$$U_\lambda + \gamma_\lambda(U_\lambda) = F + z_\lambda$$

and hence

$$U_\lambda = \omega_\lambda(F + z_\lambda) \tag{3.5}$$

where $\omega_\lambda$ is the inverse function of $t + \gamma_\lambda(t)$. It is easy to check that

$$\lim_{\lambda\to 0} \omega_\lambda(t) = \tau(t)$$

uniformly on bounded intervals, and passing to the limit in (3.5), we have

$$U = \tau \circ (F + z) \tag{3.6}$$

and the lemma is proved.

Remark. Since $z \in C^\infty(\Omega)$ and $F \in H^{1,p}(\Omega)$ it follows that

$$U \in H^{1,p}_{\mathrm{loc}}(\Omega)\,.$$

Proof of Theorem 2. Let $u$ be the solution of the boundary value problem

$$u \in H^1_0(\Omega) : \Delta u = U \quad \text{in } \Omega \tag{3.7}$$

where $U$ is the function (3.6).

From the remark above, $u \in H^{3,p}_{\mathrm{loc}}(\Omega)$ and for $\zeta \in C^2(\bar{\Omega})$ such that $\Delta\zeta = 0$, we have

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} \zeta \, d\sigma = \int_{\Omega} \Delta u \zeta \, dx = \int_{\Omega} U \zeta \, dx = 0 \qquad (\nu = \text{exterior normal})$$

since $U \in \mathcal{K} \subset \mathcal{K}_1$. Since the trace of $\zeta$ on $\partial\Omega$ is arbitrary it follows that $\partial u/\partial \nu = 0$ i.e. $u \in H^2_0(\Omega)$.

Let $v$ be any function in $\boldsymbol{K}_2$, then $V = \Delta v \in \mathcal{K}_0$ and from (3.2) and (3.7) we get

$$\int_{\Omega} \Delta u \, \Delta(v - u) \, dx \geqslant \int_{\Omega} F \Delta(v - u) \, dx = \langle f, v - u \rangle \, .$$

Since the solution of the variational inequality (1.2) is unique, theorem 2 is proved.

Remark. Two natural questions remain open:

1) It is true that $u \in H^{3,p}(\Omega)$ (instead of $H^{3,p}_{\mathrm{loc}}(\Omega)$?

2) It is true that $u \in H^{3,\infty}_{\mathrm{loc}}(\Omega)$?

Remark. It is clear from the representation formulas (2.2) and (3.6) that the solutions of the variational inequalities (1.1) and (1.2) do not belong in general to $C^3(\Omega)$!

Remark. It would be interesting to study the regularity of solutions of variational inequalities (1.1) and (1.2) when the constraints involve quadratic functions of the second derivatives.

## REFERENCES

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