Proceedings of the International Congress of Mathematicians
Vancouver, 1974

Monotone Operators, Nonlinear Semigroups and Applications

Haim Brezis

Our purpose is to discuss some recent progress in the area of monotone operators and nonlinear semigroups as well as some applications to nonlinear partial differential equations. The first papers on this subject (by G. Minty, F. Browder, J. Leray and J. L. Lions, M. Visik; see [3], [8], [13] for a complete bibliography) were mainly concerned with the question of existence of a solution for nonlinear equations of Hammerstein type or elliptic type or parabolic type. These existence results are by now well known and I would like to describe further properties.

Essentially we are going to consider operators in a real Hilbert space H (for the latest news in Banach spaces, the reader is referred to the article of M. Crandall). Let A be a multivalued mapping from H into H, i.e., for every $u \in H$, Au is a subset of H; let $D(A) = \{u \in H;\ Au \neq \emptyset\}$ and $R(A) = \bigcup_{u \in H} Au$. A is monotone if

$$(f_1 - f_2, u_1 - u_2) \geqq 0, \quad f_1 \in Au_1, \quad f_2 \in Au_2,$$

and *maximal monotone* if A has no proper monotone extension. A characterization due to G. Minty asserts that a monotone operator A is maximal monotone iff $(I + \lambda A)$ is surjective for every $\lambda > 0$, in which case $J_\lambda = (I + \lambda A)^{-1}$ (the resolvent of A) is an everywhere defined contraction on H.

A fundamental class of maximal monotone operators consists of *subdifferentials of convex functions*. Let $\phi : H \to (-\infty, +\infty]$ be a convex l.s.c. function such that $\phi \not\equiv +\infty$; let $D(\phi) = \{u \in H;\ \phi(u) < +\infty\}$ and for $u \in D(\phi)$ let $\partial\phi(u) = \{f \in H;\ \phi(v) - \phi(u) \geqq (f, v - u),\ \forall\, v \in H\}$. Then $\partial\phi$ is maximal monotone.

We recall that when A is maximal monotone then, for every $u \in D(A)$, Au is closed and convex (A^0u denotes the projection of 0 on Au) and also $\overline{D(A)}$ is a convex set.

1. A strange property of $R(A + B)$. Let A and B be maximal monotone operators; we have always $R(A + B) \subset R(A) + R(B)$ and in general $R(A) + R(B)$ is a much larger set. However it turns out that in many cases we have "almost" equality. Here is a typical result in that direction.

THEOREM 1. *Let A be maximal monotone and let $B = \partial\phi$. Assume that*

$$\phi((I + \lambda A)^{-1}u) \leqq \phi(u), \qquad \lambda > 0,\ u \in H. \tag{1}$$

Then $\operatorname{Int}[R(A + B)] = \operatorname{Int}[R(A) + R(B)]$ *and* $\overline{R(A + B)} = \overline{R(A) + R(B)}$.

SKETCH OF THE PROOF. Given $f \in \operatorname{Int}[R(A) + R(B)]$ we would like to solve $Au + Bu \ni f$. It follows from (1) that $A + B$ is maximal monotone (see [2]). Therefore there exists u_ε such that $\varepsilon u_\varepsilon + Au_\varepsilon + Bu_\varepsilon \ni f$.

Using (1) it is easy to verify that $|Au_\varepsilon| \leqq C$, $|Bu_\varepsilon| \leqq C$ with C independent of ε. We have now to show that $|u_\varepsilon|$ remains bounded as $\varepsilon \to 0$. Instead of obtaining a bound on $|u_\varepsilon|$ (as usually done) we rely on the uniform boundedness principle. Let $r > 0$ be such that $B(f, r) \subset R(A) + R(B)$. Given $h \in H$ with $|h| < r$ we have $f + h \in Av + Bw$. By the monotonicity of A and B we get

$$(Au_\varepsilon - Av, u_\varepsilon - v) \geqq 0, \qquad (Bu_\varepsilon - Bw, u_\varepsilon - w) \geqq 0.$$

Therefore $(h, u_\varepsilon) \leqq (Av - Au_\varepsilon, v) + (Bw - Bu_\varepsilon, w) \leqq C'$ and so $|u_\varepsilon|$ remains bounded.

In case we assume just $f \in R(A) + R(B)$, then we can only prove that $\sqrt{\varepsilon}\,|u_\varepsilon|$ remains bounded and at the limit $f \in \overline{R(A + B)}$.

REMARK. The conclusion of Theorem 1 still holds true if (1) is replaced by one of the following assumptions:

(2) Both A and B are subdifferentials of convex functions and $A + B$ is maximal monotone.

(3) $D(B) = H$.

The last result can be used to solve equations of *Hammerstein type*. Let K be a maximal monotone operator with $D(K) = H$ and let $B = \partial\phi$ with $D(B) = H$; then $R(I + KB) = H$. Indeed for $f \in H$, the equation $u + KBu \ni f$ can be written as $Au + Bu \ni 0$ where $Au = -K^{-1}(u - f)$ and $R(A) = H$. So $\operatorname{Int} R(A + B) = \operatorname{Int}[R(A) + R(B)] = H$. A similar result has been proved in [5] for mappings in Banach spaces.

We now illustrate Theorem 1 by two examples.

EXAMPLE 1. Consider a periodic evolution equation of the form

$$du/dt + \mathscr{B}u = f(t) \quad \text{on } (0, T) \text{ with } u(0) = u(T) \tag{4}$$

where $u(t)$ takes its values in a Hilbert space $\mathscr{H}$ and $\mathscr{B}$ is the subdifferential of a convex function in $\mathscr{H}$. It is clear that if (4) has a solution, then necessarily

$$\frac{1}{T}\int_0^T f(t)\,dt \in \overline{\operatorname{conv}}\, R(\mathscr{B}) = \overline{R(\mathscr{B})}.$$

Conversely we have

THEOREM 2 (HARAUX). *Suppose*

$$\frac{1}{T}\int_0^T f(t)\,dt \in \operatorname{Int} R(\mathscr{B});$$

then (4) *has a solution.*

It suffices to apply Theorem 1 with $H = L^2(0, T; \mathscr{H})$, $Au = du/dt$, $D(A) = \{u \in H; u' \in H \text{ and } u(0) = u(T)\}$ and B is the canonical extension of $\mathscr{B}$ to H. It is clear that $R(A) + R(B) \supset \{f \in H;\ T^{-1}\int_0^T f(t)\,dt \in R(\mathscr{B})\}$.

EXAMPLE 2. Consider the nonlinear boundary value problem

$$\Delta u = 0 \quad \text{on } \Omega, \qquad \partial u/\partial n + \beta(u) \ni f \quad \text{on } \partial\Omega, \tag{5}$$

where Ω is a smooth bounded domain and β is a maximal monotone graph in $R \times R$. It is clear that if (5) has a solution then $|\partial\Omega|^{-1}\int_{\partial\Omega} f(\sigma)\,d\sigma \in \overline{R(\beta)}$. Conversely we have

THEOREM 3 (SCHATZMAN, HESS). *Suppose*

$$\frac{1}{|\partial\Omega|}\int_{\partial\Omega} f(\sigma)\,d\sigma \in \operatorname{Int} R(\beta);$$

then (5) *has a solution.*

REMARKS. (1) In case $D(\beta) = [0, +\infty)$, $\beta(r) = 0$ for $r > 0$, $\beta(0) = (-\infty, 0]$, the boundary condition in (5) can be written $u \geqq 0$, $\partial u/\partial n - f \geqq 0$, $u(\partial u/\partial n - f) = 0$ on $\partial\Omega$ and a solution exists provided $\int_{\partial\Omega} f(\sigma)\,d\sigma < 0$ (a similar result can be found in [**15**]).

(2) Theorems 2 and 3 are comparable to some results of Landesman and Lazer and L. Nirenberg (see [**16**]). However the techniques are totally different.

2. Evolution equations and nonlinear semigroups. Let A be a maximal monotone operator and consider the evolution equation

$$du/dt + Au \ni 0 \quad \text{on } [0, +\infty),\ u(0) = u_0. \tag{6}$$

We recall first a well-known result

THEOREM 4 (KATO [**11**], KOMURA [**12**]). *Given $u_0 \in D(A)$, there exists a unique solution of* (6) *such that $u(t) \in D(A)$ for all $t \geqq 0$ and u is Lipschitz continuous on* $[0, +\infty)$.

REMARK. The same method can be used to solve $du/dt + Au \ni f(t)$, $u(0) = u_0$, provided f is smooth enough. In addition $u(t)$ is differentiable from the right at every $t \geqq 0$ and $d^+u/dt + A^0u = 0$ for all $t \geqq 0$ (see Kato [**11**], Crandall and Pazy [**9**], Dorroh [**10**]). If $u_0, \hat{u}_0 \in D(A)$, the corresponding solutions $u(t)$ and $\hat{u}(t)$ satisfy $|u(t) - \hat{u}(t)| \leqq |u_0 - \hat{u}_0|$ for all $t \geqq 0$. Thus the mapping $u_0 \to u(t)$ can be extended by continuity to $\overline{D(A)}$. We denote the extension by $S(t)$; $S(t)$ is called the *semigroup generated by* $-A$.

A number of results about linear semigroups are still valid for nonlinear semigroups. For instance we have nonlinear analogues of the theorems of Hille-Yosida-Phillips and Trotter-Kato-Neveu.

THEOREM 5 (KOMURA [12], CRANDALL AND PAZY [9]). *Let C be a closed convex set in H and let $S(t)$ be a semigroup of contractions on C (i.e., $S(0) = I$, $S(t_1 + t_2) = S(t_1) \circ S(t_2)$, $|S(t)u - S(t)v| \leqq |u - v|$ and $|S(t)u - u| \to 0$ as $t \to 0$). Then there exists a unique maximal monotone operator A such that $\overline{D(A)} = C$ and $S(t)$ coincides with the semigroup generated by $-A$.*

THEOREM 6. *Let A_n, A be maximal monotone operators and let $S_n(t)$, $S(t)$ be the corresponding semigroups. The following properties are equivalent:*

(7) $\forall\, x \in D(A)$, $\exists\, x_n \in D(A_n)$ *such that* $x_n \to x$ *and* $S_n(t)\, x_n \to S(t)x$ *uniformly on bounded t intervals.*

(8) $\forall\, x \in D(A)$, $\forall\, \lambda > 0$, $(I + \lambda A_n)^{-1} x \to (I + \lambda A)^{-1}$ x.

(9) $\forall\, x \in D(A)$, $\exists\, x_n \in D(A_n)$ *such that* $x_n \to x$ *and* $A_n^0 x_n \to A^0 x$.

Related results were obtained by Miyadera and Oharu, Brezis and Pazy, Benilan, Goldstein, Kurtz, etc.

3. Smoothing action and asymptotic behavior of semigroups generated by subdifferentials. In general when $u_0 \in \overline{D(A)}$, (6) has no "real" solution and $S(t)u_0$ represents a generalized solution of (6). But in case $A = \partial\phi$, $S(t)u_0$ turns out to be a "classical" solution of (6) even for $u_0 \in \overline{D(A)}$. More precisely

THEOREM 7. *Let $A = \partial\phi$ and let $u_0 \in \overline{D(A)}$. Then $u(t) = S(t)u_0 \in D(A)$ for all $t > 0$, $u(t)$ is Lipschitz continuous on every interval $[\delta\ ,\ +\infty)$ $(\delta > 0)$ and $u(t)$ satisfies (6). In addition one has*

$$|A^0 S(t)u_0| = \left|\frac{d^+u}{dt}(t)\right| \leqq \frac{1}{t}\,|u_0 - u(t)| \quad \textit{for all } t > 0. \tag{10}$$

For the proof see [2] (estimate (10) is new).

Since $S(t)$ maps $\overline{D(A)}$ into $D(A)$ for positive t we can say that $S(t)$ has a smoothing action on the initial data. To illustrate the smoothing action, consider the following nonlinear heat equation

$$\begin{aligned} &\partial u/\partial t - \Delta u + \beta(u) \ni 0 \text{ on } \Omega x(0, +\infty), \\ &u = 0 \text{ on } \partial\Omega x(0, +\infty), \qquad u(x, 0) = u_0(x) \text{ on } \Omega, \end{aligned} \tag{11}$$

where β is a monotone function (or graph). Given $u_0 \in L^2(\Omega)$, the solution $u(\cdot, t)$ lies in the Sobolev space $H^2(\Omega)$ for every $t > 0$.

REMARKS. (1) In fact (11) has a stronger smoothing action. Starting with $u_0 \in L^1(\Omega)$ one can show that $u(\cdot, t) \in W^{2,p}(\Omega)$ for every $p < +\infty$ and $t > 0$. The proof is quite technical and is not a consequence of an abstract result about smoothing action in Banach spaces.

(2) When β is not smooth there is also an "unsmoothing" action of $S(t)$ (a typically nonlinear phenomenon, well known in variational inequalities): Even if $u_0 \in C^\infty$, it may happen that $u(\cdot, t) \notin C^2$.

Suppose now ϕ achieves its minimum and let $K = \{v \in H;\ \phi(v) = \operatorname{Min} \phi\}$. The trajectories $S(t)u_0$ are orthogonal to the level curves of ϕ (as in the steepest descent method) and it is natural to conjecture that $S(t)u_0$ converges to some limit in K

as $t \to +\infty$. So far it is not known whether the strong limit exists. There are only partial answers:

(a) For every $u_0 \in \overline{D(A)}$, $S(t)u_0$ converges weakly as $t \to +\infty$ to some limit in K (R. Bruck).

(b) If $(I + A)^{-1}$ is compact (i.e., maps bounded sets into compact sets), then $S(t)u_0$ converges strongly as $t \to +\infty$.

(c) If ϕ is even, then $S(t)u_0$ converges strongly as $t \to +\infty$ (R. Bruck).

4. Interpolation classes. Let A be maximal monotone; for $0 < \alpha \leqq 1$ and $1 \leqq p \leqq +\infty$ define

$$\mathscr{B}_{\alpha,p} = \left\{u_0 \in \overline{D(A)};\ \left|\frac{(I + tA)^{-1}u_0 - u_0}{t^\alpha}\right| \in L^p_* = L^p\left(0, 1; \frac{dt}{t}\right)\right\}.$$

When A is a linear operator the $\mathscr{B}_{\alpha,p}$'s coincide with the interpolation spaces between $D(A)$ and H of Lions and Peetre [14]. These intermediate classes (not spaces!) can be characterized in various ways (D. Brezis [1]):

(a) *The trace method.*

$$\mathscr{B}_{\alpha,p} = \{v(0); v \in C([0,1]; H) \text{ with } t^{1-\alpha}\,|dv/dt| \in L^p_*,\ t^{1-\alpha}|A^0 v| \in L^p_*\}.$$

(b) *The method K.* Let $K(t, u_0) = \operatorname{Inf}_{v \in D(A)} \{|v - u_0| + t|A^0 v|\}$; then

$$\mathscr{B}_{\alpha,p} = \{u_0 \in \overline{D(A)};\ t^{-\alpha}K(t, u_0) \in L^p_*\}.$$

(c) *The semigroup method.* Let $S(t)$ be the semigroup generated by $-A$; then

$$\mathscr{B}_{\alpha,p} = \left\{u_0 \in \overline{D(A)};\ \left|\frac{S(t)u_0 - u_0}{t^\alpha}\right| \in L^p_*\right\}.$$

To prove the last result one can use the following simple inequalities:

$$\begin{aligned} |S(t)u_0 - u_0| &\leqq 3|J_t u_0 - u_0|, \\ |J_t u_0 - u_0| &\leqq \frac{2}{t}\int_0^t |S(\tau)u_0 - u_0|\,d\tau. \end{aligned} \tag{12}$$

In case $A = \partial\phi$ one has also

$$|J_t u_0 - u_0| \leqq (1 + 2^{-1/2})|S(t)u_0 - u_0|. \tag{13}$$

Rephrasing this fact we can say that if we consider the singular perturbation problem $u_\varepsilon + \varepsilon A u_\varepsilon = u_0$ and the evolution problem $du/dt + Au = 0$, $u(0) = u_0$, then the rate of convergence of $|u_\varepsilon - u_0|$ as $\varepsilon \to 0$ is the same as the rate of $|u(t) - u_0|$ as $t \to 0$. When $Au = -\Delta u + |u|^k \operatorname{sign} u$ one can describe $\mathscr{B}_{\alpha,p}$ in terms of Besov and Lorentz spaces.

5. Compact supports. We conclude with a surprising property which is specific to nonlinear problems and has no analogue in the linear theory. Consider the evolution equation

$$du/dt + Au \ni f \quad \text{on } (0, +\infty), \qquad u(0) = u_0. \tag{14}$$

Question. Under what conditions does u have a compact support, i.e., $u(t) \equiv 0$ for $t \geqq T$? A necessary condition is that $f(t) \in A0$ for $t \geqq T$, but in general this is not a sufficient condition. If A is linear it just means that f has a compact support, and of course this will not imply that u has a compact support. When A is multivalued one can give a sufficient condition which is very close to the necessary condition.

THEOREM 8. *Suppose there is some $\rho(t) \geqq 0$ such that $B(f(t), \rho(t)) \subset A0$ for $t \geqq t_0$ and $\int_{t_0}^{\infty} \rho(t)\, dt = \infty$. Then the solution of* (14) *has a compact support.*

We illustrate this fact by an example. Suppose $g(t)$ is the given trajectory of a gangster chased by a policeman $p(t)$. The strategy of the policeman is simple: He runs with speed V (as fast as he can!) towards $g(t)$. Thus we have

$$\frac{dp}{dt} = V \frac{g(t) - p(t)}{|g(t) - p(t)|} \quad \text{as long as } p(t) \neq g(t),$$

i.e., $dp/dt \in A(g(t) - p(t))$ where $Au = Vu/|u|$ for $u \neq 0$ and $A0 = \bar{B}(0, V)$ (so that A is maximal monotone). Let $u(t) = g(t) - p(t)$, and we get $du/dt + Au \ni dg/dt$. Here Theorem 8 tells us that if $|dg(t)/dt| \leqq V$ and $\int_0^{\infty} V - |dg(t)/dt|\, dt = +\infty$, then p reaches g in a finite time.

A similar property holds true in nonlinear parabolic equations even though we cannot apply Theorem 8. Consider for example

$$\frac{\partial t}{\partial u} - \Delta u + \beta(u) \ni f \quad \text{on } R^n \times (0, +\infty), \qquad u(x, 0) = u_0(x) \text{ on } R^n, \tag{15}$$

where β is a monotone function with a jump at 0, $\beta(0) = [\gamma^-, \gamma^+]$.

In [6] we show that if $\gamma^- + \varepsilon \leqq f \leqq \gamma^+ - \varepsilon$ for some $\varepsilon > 0$, and if u_0 has a compact support, then (15) has a solution with compact support (both in x and t). If u_0 does not have a compact support, but $u_0 \to 0$ at infinity then, for every $t > 0$, $u(\cdot, t)$ has a compact support in x. In other words the support of u "shrinks" instantaneously (in sharp contrast with what happens for the linear heat equation!). Similar results for elliptic problems are considered in [4]. The original motivation for looking at the compact support property came from the study of the flow past a given profile. In [7] we show that in the hodograph plane the problem can be stated as a free boundary value problem which turns out to be solvable by the techniques of variational inequalities. The size of the support corresponds to the maximum velocity of the flow and plays an important role.

References

1. D. Brezis, *Classes d'interpolation associées à un opérateur monotone*, C. R. Acad. Sci. Paris Sér. A-B **276** (1973), 1553–1556.

2. H. Brezis, *Monotonicity methods in Hilbert spaces...*, Contributions to Nonlinear Functional Analysis (E. Zarantonello ed.), Academic Press, New York, 1971.

3. ———, *Opérateurs maximaux monotones et semigroupes de contractions dans les espace de Hilbert*, Math. Studies, no. 5, North-Holland, Amsterdam, 1973.

4. ———, *Solutions with compact support of variational inequalities*, Uspehi Mat. Nauk **29** (1974), 103–108.

5. H. Brézis and F. Browder, *Équations intégrales non linéaires du type Hammerstein*, C. R. Acad. Sci. Paris **279** (1974), 1–2.

6. H. Brézis and A. Friedman, *Estimates on the support of solutions of parabolic variational inequalities* (to appear).

7. H. Brézis and G. Stampacchia, *Une nouvelle méthode pour l'étude d'écoulements stationnaires*, C. R. Acad. Sci. Paris Sér. A-B **276** (1973), A129-A132. MR **47** #4521.

8. F. E. Browder, *Existence theorems for nonlinear partial differential equations*, Proc. Sympos. Pure Math., vol. 16, Amer. Math. Soc., Providence, R. I., 1970, pp. 1–60. MR **42** #4855.

9. M. G. Crandall and A. Pazy, *Semi groups of nonlinear contractions and dissipative sets*, J. Functional Analysis **3** (1969), 376–418; *On accretive sets in Banach spaces*, J. Functional Analysis **5** (1970), 204–217. MR **39** #4705; **41** #4201.

10. J. R. Dorroh, *A nonlinear Hille-Yosida-Phillips theorem*, J. Functional Analysis **3** (1969), 345–353. MR **39** #2019.

11. T. Kato, *Accretive operators and nonlinear evolution equations in Banach spaces*, Proc. Sympos. Pure Math., vol. 18, part I, Amer. Math. Soc., Providence, R. I., 1970, pp. 138–161. MR **42** #6663.

12. Y. Komura, *Nonlinear semi groups in Hilbert space*, J. Math. Soc. Japan **19** (1967), 493–507; ibid. **21** (1969), 375–402. MR **35** #7176.

13. J. -L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod; Gauthier-Villars, Paris, 1969. MR **41** #4326.

14. J. -L. Lions and J. Peetre, *Sur une classe d'espaces d'interpolation*, Inst. Hautes Études Sci. Publ. Math. No. 19 (1964), 5–68. MR **29** #2627.

15. J. -L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493–519. MR **35** #7178.

16. L. Nirenberg, *Generalized degree and nonlinear problems*, Contributions to Nonlinear Functional Analysis (Zarantonello ed.), Academic Press, New York, 1971.

UNIVERSITÉ PARIS VI
PARIS 5e, FRANCE