Remarks on the Euler Equation*

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INTRODUCTION

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \) and outward normal \( n \). The motion of an incompressible perfect fluid is described by the Euler equation

\[
\frac{\partial u_i}{\partial t} + \sum_{j=1}^{N} u_j \frac{\partial u_i}{\partial x_j} = f_i + \frac{\partial \bar{\omega}}{\partial x_i}, \quad 1 \leq i \leq N,
\]
on \( \Omega \times (0, T) \),

(1)

\[
\text{div } u = 0 \quad \text{on } \Omega \times (0, T),
\]

(2)

\[
u \cdot n = 0 \quad \text{on } \partial \Omega \times (0, T),
\]

(3)

\[
u \mid_{t=0} = u_0 \quad \text{on } \Omega,
\]

(4)

where \( f(x, t) \) and \( u_0(x) \) are given, while the velocity \( u(x, t) \) and the pressure \( \bar{\omega}(x, t) \) are to be determined.

The Euler equation has been considered by several authors including L. Lichtenstein (1925–30), J. Leray (1932–37), M. Wolibner (1938). T. Kato proved the existence of a global solution for \( N = 2 \) \([3]\) and of a local solution for \( \Omega = \mathbb{R}^3 \) \([4]\). Recently, D. Ebin and J. Marsden \([2]\) have proved the existence of a local solution in the general case. Their proof relies heavily on techniques of Riemannian geometry on infinite dimensional manifolds. Our purpose is to present

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a more “classical” proof of their result by reducing (1)–(4) to an ordinary differential equation on a closed set of a Banach space; actually, we get a slightly more general result valid for $L^p$ data instead of $L^2$ data.

The main theorem is the following

**Theorem 1.** Let $1 < p < +\infty$, and let $s > (N/p) + 1$ be an integer. Suppose $u_0 \in W^{s,p}(\Omega; \mathbb{R}^n)$ with $\text{div} \, u_0 = 0$ on $\Omega$ and $u_0 \cdot n = 0$ on $\partial \Omega$. Suppose $f \in C([0, T]; C^{s+1+\alpha}(\Omega; \mathbb{R}^n))$ with $0 < \alpha < 1$. Then there exists $0 < T_0 < T$ and a unique function

$$u \in C([0, T_0]; W^{s,p}(\Omega; \mathbb{R}^n))$$

satisfying (1)–(4).

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1. Notations and Preliminaries

Let $W^{s,p}$ be the Sobolev space of real-valued functions in $L^p$ such that all their derivatives up to order $s$ are in $L^p$. In the following we assume that $s > (N/p) + 1$ so that by the Sobolev embedding theorem $W^{s,p}(\Omega) \subset C^{1+s}(\bar{\Omega})$ with $\alpha = s - 1 - N/p$. The norm in $W^{s,p}$ is denoted by $\| \cdot \|_{s,p}$. Let

$$\mathcal{D}_{s,p} = \{ \eta \in W^{s,p}(\Omega; \mathbb{R}^n); \eta \text{ is bijective from } \overline{\Omega} \text{ onto } \bar{\Omega} \text{ and } \eta^{-1} \in W^{s,p}(\Omega; \mathbb{R}^n) \}.$$ 

Note that $\eta \in \mathcal{D}_{s,p}$ if and only if $\eta \in W^{s,p}(\Omega; \mathbb{R}^n)$ and $\eta$ is a $C^1$ diffeomorphism with $\eta(\partial \Omega) \subset \partial \Omega$.

Let

$$\mathcal{D}_{s,p}^\mu = \{ \eta \in \mathcal{D}_{s,p}; \text{ Jac } \eta \mid = 1 \text{ on } \Omega \},$$

where $\text{Jac } \eta$ denotes the Jacobian matrix of $\eta$ and $| \text{ Jac } \eta |$ its determinant. Note that $\eta \in \mathcal{D}_{s,p}^\mu$ if and only if $\eta \in W^{s,p}(\Omega; \mathbb{R}^n)$, $| \text{ Jac } \eta | = 1$ on $\Omega$ and $\eta(\partial \Omega) \subset \partial \Omega$.

Let

$$T_{e,0}^{s,p} = \{ u \in W^{s,p}(\Omega; \mathbb{R}^n); u \cdot n = 0 \text{ on } \partial \Omega \}$$

and

$$T_{e,\mu}^{s,p} = \{ u \in T_{e}^{s,p}; \text{ div } u = 0 \text{ in } \Omega \}.$$ 

1 In fact, it is sufficient to assume $f \in C([0, T]; W^{s+1,p}(\Omega; \mathbb{R}^n))$.
Recall that if $V(x, t) \in C^1(\overline{\Omega} \times [0, T])$ is such that $V$ is tangent to the boundary, i.e., $V(x, t) \cdot n(x) = 0$ on $\partial \Omega \times [0, T]$ and if $\eta(x, t)$ is the flow generated by $V$, i.e. the solution of

$$(d\eta/dt)(x, t) = V(\eta(x, t), t),$$

then

$$(d/dt) | \text{Jac} \eta(x, t)|_{t=\tau} = (\text{div } V)(\eta(x, \tau), \tau) | \text{Jac} \eta(x, \tau)|.$$  \hspace{1cm} (5)

So that in particular if $\text{div } V = 0$ on $\Omega \times [0, T]$, then

$$| \text{Jac} \eta(x, t)| = | \text{Jac} \eta(x, 0)| \quad \text{on} \quad \Omega \times [0, T].$$

The following lemmas are well-known (see, e.g., [5]).

**Lemma 1 (Neumann problem).** Given an $f \in W^{k,p}(\Omega)$ ($k \geq 0$ an integer) and a $g \in W^{k+1-1/p,p}(\partial \Omega)$ such that

$$\int_{\Omega} f \, dx = \int_{\partial \Omega} g \, d\sigma,$$

there exists a $u \in W^{k+2,p}(\Omega)$ satisfying

$$\Delta u = f \quad \text{on} \quad \Omega,$$

$$\frac{\partial u}{\partial n} = g \quad \text{on} \quad \partial \Omega.$$

In addition,

$$\| \text{grad } u \|_{k+1,p} \leq C(\| f \|_{k,p} + \| g \|_{k+1-1/p,p}).$$

**Lemma 2.** Given an $f \in W^{k,p}(\Omega; \mathbb{R}^N)$, there exists a unique $g \in T^e_{\mu} \mathbb{R}^{k,p}$ and a $\bar{\omega} \in W^{k+1,p}(\Omega)$ such that

$$f = g + \text{grad } \bar{\omega}.$$

We set $g = P(f)$. $P$ is called the projection on divergence free vector fields; it is a bounded operator in $W^{k,p}(\Omega; \mathbb{R}^N)$. $P$ is related to the solution of the Neumann problem in the following way: let $\bar{\omega} \in W^{k+1,p}(\Omega)$ be a solution of

$$\begin{cases}
\Delta \bar{\omega} = \text{div } f \quad \text{on} \quad \Omega, \\
\frac{\partial \bar{\omega}}{\partial n} = f \cdot n \quad \text{on} \quad \partial \Omega.
\end{cases}$$

Then

$$g = Pf = f - \text{grad } \bar{\omega}.$$
2. REDUCTION OF THE EULER EQUATION TO AN ORDINARY DIFFERENTIAL EQUATION

Following an idea of V. Arnold [1], we shall work as in [2] with Lagrange variables. So, we use the configuration $\gamma$ of the fluid (i.e. the flow generated by $u$) as unknown. As we shall see, this leads us to the study of a second-order "ordinary" differential equation.

Assuming (1)–(4) has a solution $u$, let $\gamma$ be the flow of $u$:

$$
\frac{d\gamma}{dt}(x, t) = u(\gamma(x, t), t), \quad \gamma(x, 0) = x. 
$$

Let us rewrite the equation (1)–(4) in terms of $\gamma$. Equation (4) becomes

$$
(\frac{d\gamma}{dt})(x, 0) = u_0(x). 
$$

Equation (3) corresponds to the fact that, for each $t$, $\eta(\cdot, t)$ is a diffeomorphism from $\Omega$ onto itself and Eq. (2) is equivalent to

$$
|\text{Jac} \eta(x, t)| = 1 \quad \text{on} \quad \Omega \times [0, T]. 
$$

In order to write down (1) in terms of $\gamma$, we eliminate the pressure $\bar{\omega}$ by applying $P$ to (1). Using (2) we get

$$
\left( \frac{\partial u}{\partial t} \right) + P \left( \sum_j u_j \frac{\partial u_j}{\partial x_j} \right) = Pf. 
$$

On the other hand, by differentiating (6) with respect to $t$, we obtain

$$
\left( \frac{\partial^2 \gamma}{\partial t^2} \right)(x, t) = \sum_i \left( \frac{\partial u_i}{\partial x_i} \right)(\gamma(x, t), t) \left( \frac{\partial \gamma_i}{\partial t} \right)(x, t) + \left( \frac{\partial u_i}{\partial t} \right)(\gamma(x, t), t)
$$

$$
= \sum_i u_i(\gamma(x, t), t) \left( \frac{\partial u_i}{\partial x_i} \right)(\gamma(x, t), t) + \left( \frac{\partial u_i}{\partial t} \right)(\gamma(x, t), t). 
$$

Therefore,

$$
\left( \frac{\partial^2 \gamma}{\partial t^2} \right)(x, t) = \left[ (I - P) \sum_i u_i \left( \frac{\partial u_i}{\partial x_i} \right) \right] (\gamma(x, t), t) + (Pf)(\gamma(x, t), t). 
$$

If we keep in mind that

$$
u = \left( \frac{\partial \gamma}{\partial t} \right)(\gamma^{-1}, t),$$

we can consider (7) as an equation involving only $\gamma$.

A crucial observation is that (7) should not be regarded as a partial differential equation in $\gamma$ but rather as an ordinary differential equation in $\gamma$ (this fact is outlined in [2, p. 147]).
We first write (7) as a system

\[
\begin{cases}
\frac{d\eta}{dt} = v \\
\frac{dv}{dt} = \left[(I - P) \sum_{i} (v \circ \eta^{-1}_i) \frac{\partial}{\partial x_i} (v \circ \eta^{-1})\right] (\eta, t) + (Pf)(\eta, t)
\end{cases}
\]

or

\[
(d/dt)(\eta, v) = A(t; \eta, v),
\]

where

\[
A(t; \eta, v) = (v, B(v \circ \eta^{-1}) \circ \eta + (Pf)(\eta, t))
\]

and

\[
Bv = (I - P) \left( \sum_i v_i \frac{\partial v}{\partial x_i} \right).
\]

We shall work in the space \( X = W^{s,p}(\Omega; \mathbb{R}^N) \times W^{s,p}(\Omega; \mathbb{R}^N) \).

Clearly, \( A \) is not everywhere defined on \( X \) and not even on an open subset because of the additional requirement \( \eta \in \mathcal{D}^{s,p}_\mu \). Thus we cannot apply standard existence theorems for ordinary differential equations, but shall use the following theorem which is a particular case of a result of R. Martin [6].

**Theorem 2.** Let \( F \) be a closed subset of a Banach space \( X \), and let \( A(t; z): [0, T) \times F \rightarrow X \) be locally Lipschitz in \( z \) and continuous in \( t \). Suppose that for each \((t, z) \in [0, T] \times F\) the following holds

\[
\lim_{h \to 0} \frac{1}{h} \text{dist}(z + hA(t, z), F) = 0.2
\]

Then for every \( z_0 \in F \) the equation

\[
\frac{dz}{dt} = A(t, z), \quad z(0) = z_0,
\]

admits a local solution \( z \in C^1([0, T_0]; F) \).

We shall apply Theorem 2 with \( F = \{ (\eta, v) \in X; \eta \in \mathcal{D}^{s,p}_\mu \land v \circ \eta^{-1} \in T_\varepsilon \mathcal{D}^{s,p}_\mu \} \) which is clearly closed in \( X \).

The main steps in proving Theorem 1 are the following:

(a) Prove that \( A(t; \eta, v) \) is locally Lipschitz in \((\eta, v)\) from \( F \) into \( X \) (see Section 3).

\[2\] Where \( \text{dist}(\cdot, F) \) denotes the distance to \( F \).
One has to be rather careful because the mapping \( \eta \mapsto \eta^{-1} \) is not locally Lipschitz from \( D^s_p \) into itself (it is only continuous); similarly, the mapping \([\psi, \eta] \mapsto \psi \circ \eta\) is not locally Lipschitz from \( D^s_p \times D^s_p \) into \( D^s_p \).

(b) Prove that \( A(t; \eta, v) \) is tangent to \( F \) in the sense of (11) (see Section 4).

Remark. In case \( f = 0 \), Eq. (7) represents the equation of geodesics on the manifold \( D^s_p \) for an appropriate weak Riemannian metric. Since the metric is weak (i.e. the topology induced by this metric is weaker than the topology of \( D^s_p \)), the existence of a Riemannian connection and of geodesics does not follow at once, but is proved in [2].

3. \( A \) is Locally Lipschitz

First of all, we observe the following.

**Lemma 3.** Let \( f \) be as in Theorem 1. The mapping \((t, \eta) \mapsto (Pf)(\eta, t)\) is continuous in \( t \) and locally Lipschitz in \( \eta \).

**Proof.** As \( t \to t_0 \), \( f(\cdot, t) \to f(\cdot, t_0) \) in \( C^0(\Omega; \mathbb{R}^N) \), and therefore \( Pf(\cdot, t) \to Pf(\cdot, t) \) in \( W^{s,p}(\Omega; \mathbb{R}^N) \). We conclude by Lemma A.4 that \( Pf(\eta, t) \to Pf(\eta, t_0) \) in \( W^{s,p}(\Omega; \mathbb{R}^N) \).

For a fixed \( t \), \( f(\cdot, t) \in C^{s+1,0}(\Omega) \) and so \( Pf(\cdot, t) \in C^{s+1,0}(\Omega) \). Thus, by Lemma A.3, \( \eta \mapsto (Pf)(\eta, t) \) is locally Lipschitz from \( D^s_p \) into \( W^{s,p}(\Omega; \mathbb{R}^N) \).

**Remark.** It is actually sufficient to assume that \( f \in W^{s+1,p}(\Omega, \mathbb{R}^N) \) and use the remark following Lemma A.5 instead of Lemma A.3.

We shall now prove

**Theorem 3.** The mapping \((\eta, v) \mapsto B(v \circ \eta^{-1}) \circ \eta \) (\( B \) is defined in (10)) is locally Lipschitz from \( F \) into \( W^{s,p}(\Omega; \mathbb{R}^N) \).

The proof of Theorem 3 relies on an appropriate factorization of \( B \). Note that if \( u \in T_0 D^s_p \), we have by Lemma 2, \( Bu = \text{grad} \bar{\omega} \) where \( \bar{\omega} \) is a solution of

\[
\Delta \bar{\omega} = \text{div} \left( \sum_i u_i \frac{\partial u}{\partial x_i} \right) \quad \text{on} \quad \Omega,
\]

\[
\frac{\partial \bar{\omega}}{\partial n} = \left( \sum_i u_i \frac{\partial u}{\partial x_i} \right) \cdot n \quad \text{on} \quad \partial \Omega.
\]
But
\[ \text{div} \left( \sum_{i} u_i \frac{\partial u}{\partial x_i} \right) = \sum_{i,j} \frac{\partial}{\partial x_j} \left( u_i \frac{\partial u_j}{\partial x_i} \right) = \sum_{i,j} \frac{\partial u_i}{\partial x_i} \frac{\partial u_j}{\partial x_j} \]
(since \( \text{div} u = 0 \)) and
\[ \left( \sum_{i} u_i \frac{\partial u}{\partial x_i} \right) \cdot n = \sum_{i,j} u_i \frac{\partial u_j}{\partial x_i} n_j = \beta(\cdot; u, u) \]
where \( \beta(x; u, u) \) denotes the second fundamental form of \( \partial \Omega \). More precisely, let \( \delta(x) \) be a smooth function on \( \mathbb{R}^N \) such that
\[ \Omega = \{ x \in \mathbb{R}^N; \delta(x) > 0 \}, \]
\[ \partial \Omega = \{ x \in \mathbb{R}^N; \delta(x) = 0 \}, \]
and \( \text{grad} \delta = -n \) on \( \partial \Omega \). For \( u \in T_{\partial \Omega}^{\mu,p} \), we have \( u, \text{grad} \delta = 0 \) on \( \partial \Omega \) and by differentiation we obtain
\[ u \cdot \text{grad}[u, \text{grad} \delta] = 0 \quad \text{on} \quad \partial \Omega, \]
i.e.,
\[ \sum_{i,j} u_i \frac{\partial}{\partial x_i} \left( u_j \frac{\partial \delta}{\partial x_j} \right) = 0 \quad \text{on} \quad \partial \Omega. \]
Therefore on \( \partial \Omega \) we have
\[ \sum_{i,j} u_i \frac{\partial u_j}{\partial x_i} n_j = \sum_{i,j} \frac{\partial^2 \delta}{\partial x_i \partial x_j} u_i u_j = \beta(\cdot; u, u). \quad (12) \]
Note that \( \beta \) is a quadratic form in \( u \) depending smoothly on \( x \in \partial \Omega \).
We consider first the mapping \( Q \) defined by
\[ Q(\eta, v) = \left( \eta, \sum_{i,j} \left( \frac{\partial u_j}{\partial x_j} \frac{\partial \eta}{\partial x_i} \right) \circ \eta, \beta(\eta; v, v) \right), \]
where \( u = v \circ \eta^{-1} \), which maps \( F \) into \( Z \), where
\[ Z = \left\{ (\eta, f, g) \in \mathbb{D}_\mu^{\delta, p} \times W^{s-1, p}(\Omega) \times W^{s-1, p,p}(\partial \Omega); \int_{\Omega} f \, dx = \int_{\partial \Omega} g \circ \eta^{-1} \, ds \right\}. \]
Next, let \( S(\eta, f, g) \) be defined from \( Z \) into \( W^{s, p}(\Omega; \mathbb{R}^N) \) by
\[ S(\eta, f, g) = (\text{grad} \pi) \circ \eta, \]
where $\pi$ is a solution of
\[
\Delta \pi = f \circ \eta^{-1} \quad \text{on } \Omega,
\]
\[
\frac{\partial \pi}{\partial n} = g \circ \eta^{-1} \quad \text{on } \partial \Omega.
\]
Therefore we obtain
\[
B(v \circ \eta^{-1}) \circ \eta = (S \circ Q)(\eta, v),
\]
and it is sufficient to prove the following propositions:

**Proposition 1.** The mapping $(\eta, v) \mapsto Q(\eta, v)$ is locally Lipschitz from $F$ into $Z$.

**Proposition 2.** The mapping $(\eta, f, g) \mapsto S(\eta, f, g)$ is locally Lipschitz from $Z$ into $W^{s,p}(\Omega; \mathbb{R}^n)$.

The following lemma will be very useful.

**Lemma 4.** Let $f \in W^{s,p}(\Omega)$ and $\eta \in \mathcal{D}^{s,p}_\mu$. Then
\[
\|\text{grad}(f \circ \eta^{-1}) \circ \eta - \text{grad} f\|_{s-1,p} \leq C_{\eta} \|\eta - e\|_{s,p} \|f\|_{s,p},
\]
where $e$ denotes the identity of $\Omega$ and $C_{\eta}$ a constant depending only on $\|\eta\|_{s,p}$.

**Proof of Lemma 4.** We have
\[
\text{grad}(f \circ \eta^{-1}) = \text{Jac} \eta^{-1} \cdot (\text{grad} f)(\eta^{-1})
\]
and
\[
(\text{grad}(f \circ \eta^{-1})) \circ \eta = \text{Jac} \eta^{-1}(\eta) \text{grad} f = (\text{Jac} \eta)^{-1} \cdot \text{grad} f.
\]
We deduce from Lemma A.1 that
\[
\|\text{grad}(f \circ \eta^{-1}) \circ \eta - \text{grad} f\|_{s-1,p} \leq C \|\text{Jac} \eta\|^{-1} - I \|s-1,p\| \|\text{grad} f\|_{s-1,p}
\]
\[
\leq C \|\text{Jac} \eta\|^{-1} \circ (I - \text{Jac} \eta)\|s-1,p\| \|f\|_{s,p}.
\]

**Remark.** Lemma 4 holds true for any first-order differential operator and in a particular grad can be replaced by div or by curl.

**Proof of Proposition 1.** From Lemma 4, it follows easily that $(\eta, f) \mapsto (\text{grad}(f \circ \eta^{-1})) \circ \eta$ is locally Lipschitz from $\mathcal{D}^{s,p}_\mu \times W^{s,p}(\Omega)$.
into $W^{s-1,p}(\Omega; \mathbb{R}^N)$. Indeed, by Lemma A.4 (applied with $\alpha = s - 1$ and $q = p^*$), we have

\[ \| (\text{grad}(f \circ \eta^{-1}_1)) \circ \eta_1 - (\text{grad}(f \circ \eta^{-1}_2)) \circ \eta_2 \|_{s-1,p} \]
\[ \leq C \| (\text{grad}(f \circ \eta^{-1}_1)) \circ \eta_1 \circ \eta_2^{-1} - \text{grad}(f \circ \eta_2^{-1}) \|_{s-1,p}(\| \eta_2 \|_{s,p} - 1) \]
\[ \leq C(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} \| f \|_{s,p} \]

where $C(\eta_1, \eta_2)$ is locally bounded. Hence, by Lemma A.1, the mapping

\[ (\eta, v) \mapsto \sum_{i,j} \frac{\partial (v_i \circ \eta^{-1})}{\partial x_j}(\eta) \frac{\partial (v_j \circ \eta^{-1})}{\partial x_i}(\eta) \]

is locally Lipschitz.

It remains to check that $(\eta, v) \mapsto \beta(\eta; v, v)$ is locally Lipschitz from $F$ into $W^{s-1/p,p}(\partial \Omega)$. This is clear (by Lemma A.5) since $\beta(x; v, v)$ is smooth in $x$ and quadratic in $v$.

In the proof of Proposition 2, we shall use the following:

**Lemma 5.** There is a positive constant $\alpha$ such that

\[ \alpha \| w \|_{s,p} \leq \| \text{div} w \|_{s-1,p} + \| \text{curl} w \|_{s-1,p} + \| w \cdot n \|_{s-1/p,p} + \| w \|_{s-1,p} \]

for all $w \in W^{s,p}(\Omega; \mathbb{R}^N)$, where $\text{curl} u$ denotes the matrix with coefficients $\varphi_{ij} = (\partial w_i/\partial x_j) - (\partial w_j/\partial x_i)$.

**Proof of Lemma 5.** We have

\[ (\partial^2 w_i/\partial x_i \partial x_j) - \partial^2 w_j/\partial x_i \partial x_i = \partial \varphi_{ij}/\partial x_i, \]

and thus for all $1 \leq j \leq N$,

\[ \frac{\partial}{\partial x_j}(\text{div} w) - \Delta w_j = \sum_i \frac{\partial \varphi_{ij}}{\partial x_i}. \quad (13) \]

Let $v = (v_j) \in C^\infty(\overline{\Omega}; \mathbb{R}^N)$ be such that $v = n$ on $\partial \Omega$ and let $U = \sum_j v_j w_j$. So that

\[ \Delta U = \sum_j v_j \frac{\partial}{\partial x_j}(\text{div} w) - \sum_{i,j} v_j \frac{\partial \varphi_{ij}}{\partial x_i} + 2 \sum_{i,j} \frac{\partial v_j}{\partial x_i} \frac{\partial w_j}{\partial x_i} + \sum_j (\Delta v_j) w_j. \]
Therefore, by a regularity theorem for the Dirichlet problem ((see [5]), we have
\[
\| U \|_{s,p} \leq C(\| \Delta U \|_{s-2,p} + \| U \|_{\partial \Omega} \|_{s-1/p,p})
\]
\[
\leq C'(\| \text{div } w \|_{s-1,p} + \| \text{curl } w \|_{s-1,p} + \| w \|_{s-1,p} + \| w \cdot n \|_{s-1/p,p}).
\]

Finally, for all \(1 < i < N\),
\[
V_i = \sum_j v_j \frac{\partial w_i}{\partial x_j} = \sum_j \frac{\partial}{\partial x_i}(w_j v_j) - \sum_j \frac{\partial v_j}{\partial x_i} w_j + \sum_j v_j \varphi_{ij}
\]
\[
= \frac{\partial U}{\partial x_i} - \sum_j \frac{\partial v_j}{\partial x_i} w_j + \sum_j v_j \varphi_{ij}.
\]
Hence, \(\partial w_i / \partial \eta = V_i|_{\partial \Omega} \in W^{s-1-1/p, p}(\partial \Omega)\) and we have the estimate
\[
\left\| \frac{\partial w_i}{\partial n} \right\|_{s-1-1/p, p} \leq C(\| U \|_{s,p} + \| w \|_{s-1,p} + \| \text{curl } w \|_{s-1,p}).
\]

On the other hand, by (13), \(\Delta w_i \in W^{s-2, p}(\Omega)\). Moreover,
\[
\| \text{grad } w_i \|_{s-1,p} \leq C \left( \| \Delta w_i \|_{s-2,p} + \left\| \frac{\partial w_i}{\partial n} \right\|_{s-1-1/p, p} \right)
\]
so that by (13) and the previous estimate we get
\[
\| w \|_{s,p} \leq C(\| \text{div } w \|_{s-1,p} + \| \text{curl } w \|_{s-1,p} + \| w \|_{s-1,p} + \| w \cdot n \|_{s-1/p,p}).
\]

**Remark.** For any norm \(\| \cdot \|\) on \(W^{s-1,p}\) which is weaker than \(\| \cdot \|_{s-1,p}\), there is a constant \(\alpha > 0\) such that
\[
\alpha \| w \|_{s,p} \leq \| \text{div } w \|_{s-1,p} + \| \text{curl } w \|_{s-1,p} + \| w \cdot n \|_{s-1/p,p} + \| w \|,
\]
since the injection \(W^{s,p} \subset W^{s-1,p}\) is compact.

**Proof of Proposition 2.** We have to estimate
\[
X = \| \text{grad } \pi_1 \circ \eta_1 - \text{grad } \pi_2 \circ \eta_2 \|_{s,p}
\]
where
\[
\Delta \pi_i = f_i \circ \eta_i^{-1} \text{ on } \Omega, \quad (\partial \pi_i / \partial n) = g_i \circ \eta_i^{-1} \text{ on } \partial \Omega, \quad i = 1, 2.
\]
By Lemma A.4 we know that
\[
X \leq C(\eta_2) \| (\text{grad } \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \text{grad } \pi_2 \|_{s,p}.
\]
We shall use the Remark following Lemma 5 to estimate
\[ \| (\text{grad } \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \text{grad } \pi_2 \|_{s,p}. \]

Let
\[ \begin{align*}
X_1 &= \| \text{div}[(\text{grad } \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \text{grad } \pi_2]\|_{s-1,p}, \\
X_2 &= \| \text{curl}[(\text{grad } \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \text{grad } \pi_2]\|_{s-1,p}, \\
X_3 &= \|[(\text{grad } \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \text{grad } \pi_2] \cdot n\|_{s-1/p,p}, \\
X_4 &= \|[(\text{grad } \eta_1) \circ \eta_1 \circ \eta_2^{-1} - \text{grad } \pi_2]\|. 
\end{align*} \]

where we choose
\[ \| u \| = \sup \{ \int_{\Omega} u \cdot \xi \, dx; \xi \in C^\infty(\overline{\Omega}; \mathbb{R}^N), \xi = 0 \text{ on } \partial \Omega \text{ and } \| \xi \|_{C^s} \leq 1 \}. \]

We have
\[ \text{div } \text{grad } \pi_2 = \Delta \pi_2 = f_2 \circ \eta_2^{-1} \]
and
\[ \text{div}[(\text{grad } \pi_1) \circ \eta_1 \circ \eta_2^{-1}] = [\text{div}(\text{grad } \pi_1)] \circ \eta_1 \circ \eta_2^{-1} + R \]
where, by the Remark following Lemma 4 (used with \( f = (\text{grad } \pi_1) \circ \eta \) and \( \eta = \eta_1 \circ \eta_2^{-1} \)), we have
\[ \begin{align*}
R \|_{s-1,p} &\leq C(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} \| \text{grad } \pi_1 \|_{s,p} \\
&\leq C'(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \circ \eta_1^{-1} \|_{s-1,p} + \| g_1 \circ \eta_1^{-1} \|_{s-1/p,p}) \\
&\leq C'(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \|_{s-1,p} + \| g_1 \|_{s-1/p,p}).
\end{align*} \]

Hence
\[ \begin{align*}
X_1 &\leq C'(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \|_{s-1,p} + \| g_1 \|_{s-1/p,p}) \\
&\quad + \| f_1 \circ \eta_2^{-1} - f_2 \circ \eta_2^{-1} \|_{s-1,p}
\end{align*} \]
and thus
\[ X_1 \leq C'(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \|_{s-1,p} + \| g_1 \|_{s-1/p,p}) + \| f_1 - f_2 \|_{s-1,p}. \]

Similarly, since \( \text{curl } \text{grad} = 0 \), we get
\[ X_2 \leq C(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \|_{s-1,p} + \| g_1 \|_{s-1/p,p}). \]
Next letting $\eta = \eta_1 \circ \eta_2^{-1}$ we have

$$X_3 \leq \|\text{grad } \eta_1 \circ \eta \|_{s-1,p} \cdot (n - n \circ \eta) + \|\frac{\partial \eta_1}{\partial n} \|_{s-1,p} + \|\frac{\partial \eta_2}{\partial n} \|_{s-1,p} \cdot \|\omega \|_{s-1,p} + \|\eta_1 - \eta_2 \|_{s,p} \leq C(\eta_1, \eta_2) \|\eta_1 - \eta_2 \|_{s,p}$$

Finally we estimate $X_4$; let $\zeta \in C^s(\overline{\Omega}; \mathbb{R}^N)$ be such that $\zeta = 0$ on $\partial \Omega$. Let

$$K(\zeta) = \int_{\Omega} [(\text{grad } \eta_1) \circ \eta - \text{grad } \eta_2] \cdot \zeta \, dx$$

$$= \int_{\Omega} [(\text{grad } \eta_1) \cdot (\zeta \circ \eta^{-1}) - \text{grad } \eta_2 \cdot \zeta] \, dx.$$

Let $\omega$ and $\omega_n$ be solutions of the equations

$$\begin{cases}
\Delta \omega = \text{div } \xi & \text{on } \Omega \\
\frac{\partial \omega}{\partial n} = 0 & \text{on } \partial \Omega
\end{cases}$$

$$\begin{cases}
\Delta \omega_n = \text{div}(\zeta \circ \eta^{-1}) & \text{on } \Omega \\
\frac{\partial \omega_n}{\partial n} = 0 & \text{on } \partial \Omega.
\end{cases}$$

We can always assume that

$$\|\omega\|_{C^s} \leq C \|\zeta\|_{C^s},$$

$$\|\omega_n - \omega\|_{s,p} \leq C \|\zeta \circ \eta^{-1} - \xi\|_{s-1,p} \leq C(\eta_1, \eta_2) \|\eta_1 - \eta_2\|_{s,p} \|\xi\|_{C^s},$$

by Lemma A.3. Thus

$$\|\omega_n \circ \eta - \omega\|_{s-1,p} \leq C \|\omega_n \circ \eta - \omega \|_{s-1,p} + \|\omega \circ \eta - \omega\|_{s-1,p}$$

$$\leq C(\|\omega_n - \omega\|_{s-1,p} + \|\omega\|_{C^s} \|\eta - \xi\|_{s,p})$$

by Lemma A.3 and A.4. Hence

$$\|\omega_n \circ \eta - \omega\|_{s-1,p} \leq C(\eta_1, \eta_2) \|\eta_1 - \eta_2\|_{s,p} \|\xi\|_{C^s}.$$

But

$$K(\zeta) = \int_{\Omega} [\pi_1 \cdot \Delta \omega_n - \pi_2 \cdot \Delta \omega] \, dx$$

$$= \int_{\Omega} (\Delta \pi_1 \cdot \omega_n - \Delta \pi_2 \cdot \omega) \, dx - \int_{\partial \Omega} (g_1 \circ \eta_1^{-1} \cdot \omega_n - g_2 \circ \eta_2^{-1} \cdot \omega) \, d\sigma$$

$$= \int_{\Omega} [(f_1 \circ \eta^{-1}) \cdot \omega_n - (f_2 \circ \eta^{-1}) \cdot \omega] \, dx$$

$$- \int_{\partial \Omega} [(g_1 \circ \eta^{-1}) \cdot \omega_n - (g_2 \circ \eta^{-1}) \cdot \omega] \, d\sigma.$$
The first term can be estimated by

$$
\| f_1 - f_2 \|_{L^p(\Omega)} \| \omega \|_{L^1(\Omega)} + \| f_1 \|_{L^1(\Omega)} \| \omega \circ \eta - \omega \|_{L^p(\Omega)},
$$

while the second term can be estimated by

$$
\| g_1 - g_2 \|_{L^p(\Omega)} \| \omega \|_{L^1(\Omega)} + \| g_2 \|_{L^p(\Omega)} \| \omega \circ \eta - \omega \|_{L^1(\Omega)}
$$

$$
+ \| g_2 \|_{L^p(\eta)} \| \eta_1^{-1} - \eta_2^{-1} \|_{L^1(\partial \Omega)} \| \omega \|_{L^p(\eta)},
$$

So finally

$$
K(\xi) \leq C(\eta_1, \eta_2) \| \xi \|_{C^s} \left[ \| f_1 - f_2 \|_{L^p(\Omega)} + \| \eta_1 - \eta_2 \|_{s,p} \| f_1 \|_{L^1(\Omega)}
$$

$$
+ \| g_1 - g_2 \|_{L^p(\Omega)} + \| \eta_1 - \eta_2 \|_{s,p} \left( \| g_2 \|_{L^p(\partial \Omega)} + \| g_2 \|_{L^p(\eta)} \right) \right],
$$

and

$$
X_4 = \sup_{\xi \neq 0} \frac{K(\xi)}{\| \xi \|_{C^s}}.
$$

4. **A is "Tangent" to the Closed Set F**

Let \( u \) and \( \gamma \) be given so that \( u \in W^{s,p}(\Omega; \mathbb{R}^N) \) with \( \text{div} \, u = 0 \) on \( \Omega \) and \( u \cdot n = 0 \) on \( \partial \Omega \) and \( \gamma \in W^{s,p}(\Omega; \mathbb{R}^N) \) satisfying

$$
\text{div} \left( \gamma - \sum_i u_i \frac{\partial u}{\partial x_i} \right) = 0 \text{ on } \Omega, \quad \left( \gamma - \sum_i u_i \frac{\partial u}{\partial x_i} \right) \cdot n = 0 \text{ on } \partial \Omega.
$$

In order to prove that \( A \) is tangent to \( F \), we shall exhibit a curve \( \eta \in C^s(I; \mathcal{D}^{s,p}_\mu) \) \( (I = [0, t_0], t_0 \text{ small enough}) \) such that \( \eta_0 = e, \eta_0 = u, \eta_0 = \gamma \). This curve will be a "good approximation" in \( \mathcal{D}^{s,p}_\mu \) of \( e + tu + (t^2/2)\gamma \).

**Theorem 4.** Let \( u \in \mathcal{D}^{s,p}_\mu \) and \( \gamma \in W^{s,p}(\Omega; \mathbb{R}^N) \) with \( s > (N/p) + 1 \) such that

$$
\text{div} \left( \gamma - \sum_i u_i \frac{\partial u}{\partial x_i} \right) = 0 \text{ on } \Omega, \quad \left( \gamma - \sum_i u_i \frac{\partial u}{\partial x_i} \right) \cdot n = 0 \text{ on } \partial \Omega.
$$
Then there exists a curve $\eta_1$ satisfying $\eta \in C^2(I; \mathbb{D}_\mu^{s,p})$

$$\eta_0 = e,$$  

$$\dot{\eta}_0 = u,$$  

$$\ddot{\eta}_0 = \gamma.$$  

**Remark.** Conversely, if $\eta$ is a curve satisfying (14), then $u = \dot{\eta}_0 \in T_\mu \mathbb{D}_\mu^{s,p}$ and $\gamma = \ddot{\eta}_0 \in W^{6,p}(\Omega; \mathbb{R}^N)$ verify

$$\text{div} \left( \gamma - \sum_i u_i \frac{\partial u}{\partial x_i} \right) = 0 \quad \text{on } \Omega \quad \text{and} \quad \left( \gamma - \sum_i u_i \frac{\partial u}{\partial x_i} \right) \cdot n = 0 \quad \text{on } \partial \Omega.$$

The proofs of Theorem 4 and its Remark are based on the following lemma.

**Lemma 6.** Let $\mathcal{C}$ and $\mathcal{B}$ be Banach spaces, and let $\varphi$ be a $C^2$ mapping defined on a neighborhood of 0 in $\mathcal{C}$ with values into $\mathcal{B}$, such that $\varphi(0) = 0$ and $D_0 \varphi$ is a split surjection (i.e. $D_0 \varphi$ is onto $\mathcal{B}$ and $\ker D_0 \varphi$ has a topological complement in $\mathcal{C}$).

Given $U, V$ in $\mathcal{C}$, there exists a curve $\zeta \in C^2(I; \mathcal{C})$ such that

$$\varphi(\zeta_t) = 0 \quad \text{for } t \in I, \quad \zeta_0 = 0,$$  

$$\dot{\zeta}_0 = U,$$  

$$\ddot{\zeta}_0 = V,$$  

if and only if $U$ and $V$ satisfy

$$D_0 \varphi \cdot U = 0,$$  

$$D_0 \varphi \cdot V + D_0^2 \varphi(U, U) = 0.$$  

**Proof of Lemma 6.** It is easy to check that $U = \dot{\zeta}_0$ and $V = \ddot{\zeta}_0$ satisfy necessarily (20) and (21) by differentiating (17). The converse relies on the implicit function theorem. Let $\mathcal{C} = \ker D_0 \varphi$, and let $P$ be a continuous projection from $\mathcal{C}$ onto $\mathcal{E}$. Define $\psi: \mathcal{C} \rightarrow \mathcal{B} \times \mathcal{C}$ by $\psi(u) = (\varphi(u), Pu)$, so that $D_0 \psi = D_0 \varphi \times P$ is an isomorphism from $\mathcal{C}$ onto $\mathcal{B} \times \mathcal{C}$. Therefore, by the implicit function theorem, $\psi$ is a $C^2$ isomorphism from a neighborhood of 0 in $\mathcal{C}$ onto a neighborhood of 0 in $\mathcal{B} \times \mathcal{C}$. For $t$ small enough, consider

$$\zeta_t = \psi^{-1}(0, tU + (t^2/2) PV).$$
Therefore, \( \varphi(\xi_t) = 0 \) and \( P_t = tU + (t^2/2) PV \). Consequently, 
\[ D_0 \varphi \cdot \xi_0 = 0 \quad \text{and} \quad P_{\xi_0} = U, \]
which implies \( \xi_0 = U \). Also,
\[ D_0 \varphi \cdot \xi_0 + D_0^2 \varphi(U, U) = 0 \]
and \( P_{\xi_0} = PV \). Hence, \( D_0 \varphi(\xi_0 - V) = 0 \) and \( P(\xi_0 - V) = 0 \), which implies \( \xi_0 = V \). ■

**Proof of Theorem 4.** Let \( \mathcal{C} = W^{s,p}(\Omega; \mathbb{R}^N) \) and let
\[ \mathcal{B} = \left\{ (f, g) \in W^{s-1,p}(\Omega) \times W^{s-1,p}((\partial \Omega)); \int_\Omega f \, dx = \int_{\partial \Omega} g \, d\sigma \right\}. \]

We consider the mapping \( \varphi \) defined on \( \mathcal{C} \) by \( \varphi(u) = (\varphi_1(u), \varphi_2(u)) \) where
\[ \varphi_1(u) = |\text{Jac}(e + u)| - \frac{1}{\text{Vol} \Omega} \int_\Omega |\text{Jac}(e + u)| \, dx \quad \text{and} \quad \varphi_2(u) = -\delta \circ (e + u)|_{\partial \Omega} \]
(recall that \( \delta \) is smooth and \( \partial \Omega = \{ x; \delta(x) = 0 \} \)). Observe that \( \varphi \) takes its values in \( \mathcal{B} \) and that \( \varphi \in C^\infty \) since \( |\text{Jac}| \) is a polynomial in the first derivatives (we suppose \( s > (N/p) + 1 \); cf. Lemma A.1) and since \( \delta \) is \( C^\infty \). For \( u \) small enough, \( \varphi(u) = 0 \) implies that \( (e + u) \in \mathbb{R}^{a,p} \). Indeed, \( \eta = (e + u) \) is a \( C^1 \) diffeomorphism and \( \eta(\partial \Omega) \subset \partial \Omega \). Therefore, \( \eta \in \mathbb{R}^{a,p} \) and since \( |\text{Jac} \eta| \) is constant on \( \Omega \), we have \( \text{Vol} \Omega = \text{Vol} \eta(\Omega) = \int_\Omega |\text{Jac} \eta| \, dx = C \text{Vol} \Omega \); so that \( C = 1 \) and \( \eta \in \mathbb{R}^{a,p} \). For \( v \in \mathcal{C} \), we have the expansion
\[ |\text{Jac}(e + tv)| = 1 + t \text{div} v + \frac{t^2}{2} \left( |\text{div} v|^2 - \sum_{i,j=1}^N \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \right) + \cdots \]
since for any matrix \( M = (m_{ij}) \) we know that
\[ |I + \epsilon M| = 1 + \epsilon \text{tr} M + \frac{\epsilon^2}{2} \left( |\text{tr} M|^2 - \sum_{i,j=1}^N m_{ij}m_{ji} \right) + \cdots \]

Hence,
\[ D_0 \varphi_1 \cdot v = \text{div} v - \frac{1}{\text{Vol} \Omega} \int_\Omega \text{div} v \, dx - \frac{1}{\text{Vol} \Omega} \int_{\partial \Omega} v \cdot n \, d\sigma = \text{div} v; \]
and $D_0\varphi_2 \cdot v = v \cdot n$. Consequently, $D_0\varphi \cdot v = (\text{div } v, v \cdot n)$ is a split surjection onto $\mathcal{B}$. Also

$$D_0^2\varphi_1(v, v) = \lim_{{\epsilon \to 0}} \frac{\varphi_1(\epsilon v) + \varphi_1(-\epsilon v)}{{\epsilon}^2} = |\text{div } v|^2 - \sum_{i,j=1}^{N} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i}$$

$$- \frac{1}{\text{Vol } \Omega} \int_{\Omega} \left( |\text{div } v|^2 - \sum_{i,j=1}^{N} \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i} \right) dx$$

$$- \frac{1}{\text{Vol } \Omega} \int_{\partial \Omega} \sum_{i,j=1}^{N} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} v_i v_j d\sigma,$$

and

$$D_0^2\varphi_2(v, v) = - \sum_{i,j=1}^{N} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} v_i v_j = -\beta(\cdot ; v, v).$$

We apply now Lemma 6 with $U = u$ and $V = \gamma$. Conditions (20) and (21) are satisfied since

$$D_0\varphi \cdot u = (\text{div } u, u \cdot n) = 0,$$

and by (12),

$$D_0\varphi_1 \cdot \gamma + D_0^2\varphi_1(u, u) = \text{div } \gamma - \sum_{i,j=1}^{N} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} + \frac{1}{\text{Vol } \Omega} \int_{\Omega} \sum_{i,j=1}^{N} \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i} dx$$

$$- \frac{1}{\text{Vol } \Omega} \int_{\partial \Omega} \sum_{i,j=1}^{N} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} u_i u_j d\sigma = 0,$$

$$D_0\varphi_2 \cdot \gamma + D_0^2\varphi_2(v, u) = \gamma \cdot n - \sum_{i,j=1}^{N} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} u_i u_j = 0.$$

**Theorem 5.** $A$ is "tangent" to $F$ in the following sense:

$$\lim_{{h \to 0}} \frac{\text{dist}((\eta, v) + hA(t; \eta, v), F)}{h} = 0 \quad \text{for all } (\eta, v) \in F,$$

where $\text{dist}(\cdot, F)$ denotes the distance to the closed set $F$ in the space $X = W^{s,p}(\Omega; \mathbb{R}^N) \times \dot{W}^{s,p}(\Omega; \mathbb{R}^N)$.

**Proof of Theorem 5.** We recall that

$$A(t; \eta, u) = (u, B(u \circ \eta^{-1}) \circ \eta + P(f_t) \circ \eta)$$
(where \( f_i \) is the given field of external forces),

\[
F = \{(\eta, u) \in X; \eta \in D^{s,p}_\mu \text{ and } u \circ \eta^{-1} \in T_e D^{s,p}_\mu \}.
\]

We start by proving (22) for the case \( \eta = e \). We observe then that \( u \in T_e D^{s,p}_\mu \) and \( \gamma = B(u) + P(f) \) meets the requirements of Theorem 4, i.e.,

\[
\text{div} \left( \gamma - \sum_i u_i \frac{\partial u}{\partial x_i} \right) = 0 \quad \text{on } \Omega \quad \text{and} \quad \left( \gamma - \sum_i u_i \frac{\partial u}{\partial x_i} \right) \cdot n = 0 \quad \text{on } \partial \Omega
\]

since \( \gamma - \sum_i u_i (\partial u/\partial x_i) = P(f - \sum_i u_i (\partial u/\partial x_i)) \) by the definition of \( B \).

From Theorem 4 we know that there exists a curve \( \eta \in C^2(I; D^{s,p}_\mu) \) with initial data \((e, u, \gamma)\). Since \((\eta_h, \dot{\eta}_h) \in F\), we have

\[
(1/h) \text{dist}[(e, u) + hA(t; e, u), F] \leq (1/h) \text{dist}[(e, u) + hA(t; e, u), (\eta_h, \dot{\eta}_h)].
\]

By construction of \( \eta \), the right-hand side tends to 0 as \( h \to 0 \), which proves Theorem 5 at \( \eta = e \). For the general case, we just have to notice that

\[
A(t; \eta, u) = A(t; e, u \circ \eta^{-1}) \circ \eta,
\]

that \( \gamma(F) = F \) for \( \eta \in D^{s,p}_\mu \), and that the map \( v \mapsto v \circ \eta \) is continuous (cf. Lemma A.4). Therefore, we can apply the result at \( e \), completing the proof of Theorem 5.

APPENDIX: PRODUCT AND COMPOSITION OF FUNCTIONS IN SOBOLEV SPACES

1. PRODUCT OF TWO FUNCTIONS

Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with smooth boundary.

**Lemma A.1.** Let \( \alpha \geq 1 \) be an integer, and let \( 1 \leq p \leq +\infty \), \( 1 \leq q \leq +\infty \).

If \( u \in W^{\alpha,p}(\Omega) \) and \( v \in W^{\alpha,q}(\Omega) \), then \( u, v \in W^{\alpha,r}(\Omega) \), where \( r \) is defined by

\[
1/r = (1/p) + (1/q) - \alpha/N \quad \text{when} \quad \max\{p, q\} < N/\alpha, \tag{1}
\]

\[
r \quad \text{arbitrary} \quad \text{if} \quad \max\{p, q\} = N/\alpha, \tag{2}
\]

\[
(r = 1 \text{ if } p = q = N = \alpha = 1),
\]

\[
r = \min\{p, q\} \quad \text{when} \quad \max\{p, q\} > N/\alpha. \tag{3}
\]
In addition, \( \| u \cdot v \|_{W^{\alpha,r}} \leq C \| u \|_{W^{\alpha,p}} \| v \|_{W^{\alpha,q}} \), where \( C \) depends only on \( \alpha, p, q, r, \) and \( \Omega \).

**Proof.** By induction on \( \alpha \), the proof is easy for \( \alpha = 1 \). In order to show that \( u \cdot v \in W^{\alpha,r}(\Omega) \), we have to prove that \( u \cdot v \in L^r(\Omega) \) (which is straightforward) and that \( Du \cdot v + u \cdot Dv \in W^{\alpha-1,r}(\Omega) \). By symmetry, it is sufficient to check that \( Du \cdot v \in W^{\alpha-1,r}(\Omega) \). But \( Du \in W^{\alpha-1,p}(\Omega) \) and \( v \in W^{\alpha,q}(\Omega) \subset W^{\alpha-1,q^*}(\Omega) \), where \( q^* \) is determined by

\[
\frac{1}{q^*} = \begin{cases} \frac{1}{q} - \frac{1}{N} & \text{when } q < N, \\ \text{arbitrarily small with} & \\ \frac{1}{q^*} < \frac{1}{q} & \text{when } q = N, \\ 0 & \text{when } q > N. 
\end{cases}
\]

We have now to distinguish three cases:

**Case 1.** \( \max\{p, q\} < N/\alpha \) and thus \( \max\{p, q^*\} < N/(\alpha - 1) \). By the induction assumption, we know that \( Du \cdot v \in W^{\alpha-1,s}(\Omega) \) where \( 1/s = (1/p) + (1/q^*) - (\alpha - 1)/N = (1/p) + (1/q) - \alpha/N \).

**Case 2.** \( \max\{p, q\} = N/\alpha \). Either \( p \leq q = N/\alpha \), so that \( q^* = N/(\alpha - 1) \). Thus, \( \max\{p, q^*\} = N/(\alpha - 1) \) and by the induction assumption we know that \( Du \cdot v \in W^{\alpha-1,s}(\Omega) \) for any \( s < \min\{p, q^*\} = p = \min\{p, q\}. \) Or \( q < p = N/\alpha \), so that \( \max\{p, q^*\} < N/(\alpha - 1) \) and by the induction assumption \( Du \cdot v \in W^{\alpha-1,s}(\Omega) \) with

\[
1/s = (1/p) + (1/q^*) - (\alpha - 1)/N = (1/p) + (1/q) - \alpha/N = 1/q.
\]

Hence \( Du \cdot v \in W^{\alpha-1,s}(\Omega) \) with \( s = \min\{p, q\} \).

**Case 3.** \( \max\{p, q\} > N/\alpha \). Either \( q > N/\alpha \) so that \( \max\{p, q^*\} > N/(\alpha - 1) \) and by the induction assumption \( Du \cdot v \in W^{\alpha-1,s}(\Omega) \) with \( s = \min\{p, q^*\} \geq \min\{p, q\}. \) Or \( p > N/\alpha \) and \( q < N/\alpha \); by the induction assumption \( Du \cdot v \in W^{\alpha-1,s}(\Omega) \), for \( s \) as follows: when

\[
\max\{p, q^*\} < N/(\alpha - 1)
\]
we have $1/s = (1/p) + (1/q^*) - (\alpha - 1)/N$ and $1/s < 1/q$. Therefore, $Du \cdot v \in W^{s-1,n}(\Omega)$ with $s = \min\{p, q\}$. When
\[
\max\{p, q^*\} \geq N/(\alpha - 1),
\]
we have $Du \cdot v \in W^{s-1,n}(\Omega)$ for any $s < \min\{p, q^*\}$ and in particular we can choose $s = \min\{p, q\}$. ■

2. COMPOSITION OF TWO MAPPINGS

Let $\Omega' \subset \mathbb{R}^M$ be a bounded domain with smooth boundary.

**Lemma A.2.** Let $\alpha \geq 1$ be an integer, and let $1 \leq p \leq +\infty$ with $\alpha > N/p$. Let $F \in C^\alpha(\Omega')$, and let $G \in W^{s,p}(\Omega; \mathbb{R}^M)$ such that $G(\Omega) \subset \Omega'$. Then $F \circ G \in W^{s,p}(\Omega)$ and
\[
\|F \circ G\|_{W^{s,p}} \leq C \|F\|_{C^\alpha} \left(\|G\|_{W^{s,p}}^{\alpha-1} + 1\right),
\]
where $C$ depends only on $\alpha, p, \Omega,$ and $\Omega'$.

**Proof.** By induction on $\alpha$, the proof is easy for $\alpha = 1$. In order to show that $F \circ G \in W^{s,p}(\Omega)$, we have to check that $F \circ G \in L^p(\Omega)$ (which is obvious) and that $(DF \circ G) \cdot DG \in W^{s-1,p}(\Omega)$.

Since $\alpha - 1 > N/p^*$, we know by the induction assumption that $DF \circ G \in W^{s-1,p^*}(\Omega)$ with
\[
\|DF \circ G\|_{W^{s-1,p^*}} \leq C \|F\|_{C^\alpha} \left(\|G\|_{W^{s-1,p^*}}^{\alpha-1} + 1\right).
\]
But $DG \in W^{s-1,p}(\Omega)$ and from Lemma A.1 (Case 3) we get $(DF \circ G) \cdot DG \in W^{s-1,p}(\Omega)$ with the corresponding estimate. ■

**Remark.** A slightly sharper version of Lemma A.2 can be found in [7].

**Lemma A.3.** Let $\alpha \geq 1$ be an integer and let $1 \leq p \leq +\infty$ with $\alpha > N/p$. Let $F \in C^{\alpha+1}(\Omega')$, and let $G \in W^{s,p}(\Omega; \mathbb{R}^M)$ and
\[
H \in W^{s,p}(\Omega; \mathbb{R}^M)
\]
such that $G(\Omega) \subset \Omega'$, $H(\Omega) \subset \Omega'$. Then
\[
\|F \circ G - F \circ H\|_{W^{s,p}} \leq C \|F\|_{C^{\alpha+1}} \|G - H\|_{W^{s,p}} \left(\|G\|_{W^{s,p}} + \|H\|_{W^{s,p}} + 1\right),
\]
where $C$ depends only on $\alpha, p, \Omega$ and $\Omega'$. 
Proof. By induction on $0_1$, the proof is easy for $0_1 = 1$. In order to show that (4) holds, we have to check that
\[ \| F \circ G - F \circ H \|_{L^p} \leq C \| G - H \|_{W^{1,p}} \]
(which is obvious) and that
\[ \|(DF \circ G) \cdot DG - (DF \circ H) \cdot DH\|_{W^{a-1,p}} \]
can be bounded by the right-hand side in (4). But
\[ (DF \circ G) \cdot DG - (DF \circ H) \cdot DH = (DF \circ G - DF \circ H) \cdot DG + (DF \circ H) \cdot (DG - DH). \]
The first term in the right-hand side is bounded in $W^{a-1,p}(\Omega)$ by
\[ C \| F \|_{C^{a+1}} \| G - H \|_{W^{a-1,p}} \left( \| G \|_{W^{a-1,p}}^{a-1} + \| H \|_{W^{a-1,p}}^{a-1} + 1 \right) \| G \|_{W^{a,p}} \]
(using the induction assumption and Lemma A.1 with $q = p^*$), while the second term in the right-hand side is bounded in $W^{a-1,p}$ by
\[ C \| G - H \|_{W^{a,p}} \| F \|_{C^a} \left( \| H \|_{W^{a-1,p}}^{a-1} + 1 \right) \]
(using Lemmas A.1 and A.2). \hfill \Box

The following result differs essentially from Lemma A.2 by the fact that we assume only that $F \in W^{a,p}(\Omega)$ (instead of $C^a$), but $G$ is here a diffeomorphism.

Lemma A.4. Let $\alpha \geq 2$ be an integer, and let $1 \leq p \leq q \leq +\infty$ such that $\alpha > (N/q) + 1$. Let $F \in W^{a,p}(\Omega)$, and let $G \in \mathcal{S}^a, q(\Omega)$ (i.e. $G \in W^{a,q}(\Omega; \mathbb{R}^N)$ and $G$ is a $C^1$ diffeomorphism from $\Omega$ onto $\bar{\Omega}$). Then $F \circ G \in W^{a,p}(\Omega)$ and
\[ \| F \circ G \|_{W^{a,p}} \leq C \| F \|_{W^{a,p}} \frac{1}{\inf \| \text{Jac} \ G \|^{1/p}} \left( \| G \|^{a-1}_{W^{a,p}} + 1 \right), \]
where $C$ depends only on $\alpha, p, q$ and $\Omega$.

Proof. By induction on $\alpha$, we consider first the case where $\alpha = 2$. It is clear that $F \circ G \in L^p(\Omega)$ and
\[ \| F \circ G \|_{L^p} \leq \frac{1}{\inf \| \text{Jac} \ G \|^{1/p}} \| F \|_{L^p}. \]
Also, \( D(F \circ G) = (DF \circ G) \cdot DG \) belongs to \( W^{1,p}(\Omega) \) by Lemma A.1 since \( DG \in W^{1,q}(\Omega) \) \((q > N)\) and \( DF \circ G \in W^{1,p}(\Omega) \) with
\[
\| DF \circ G \|_{W^{1,p}} \leq \frac{1}{\inf |\text{Jac} G |^{1/p}} (\| DF \|_{L^p} + \| D^2 F \|_{L^p} \| DG \|_{L^p}).
\]

In the general case, we have to check that \( F \circ G \in L^p(\Omega) \) and that \((DF \circ G) \cdot DG \in W^{\alpha-1,p}(\Omega)\). By the induction assumption, we know that \( DF \circ G \in W^{\alpha-1,p}(\Omega) \) (since \( \alpha - 1 > (N/q) + 1 \)) and
\[
\| DF \circ G \|_{W^{\alpha-1,p}} \leq C \| F \|_{W^{\alpha-1,p}} \frac{1}{\inf |\text{Jac} G |^{1/p}} (\| G \|_{W^{\alpha-1,q}}^{\alpha-1} + 1).
\]

From Lemma A.1, we conclude that \((DF \circ G) \cdot DG \) belongs to \( W^{\alpha-1,p}(\Omega) \) with the corresponding estimate.

**Lemma A.5.** Let \( \alpha \geq 2 \) be an integer, and let \( 1 < p < q < +\infty \) be such that \( p < +\infty \) and \( \alpha > (N/q) + 1 \). Let \( F \in W^{\alpha,p}(\Omega) \); then the mapping \( G \mapsto F \circ G \) is continuous from \( \mathcal{D}^{a,q}(\Omega) \) into \( W^{\alpha,p}(\Omega) \).

**Proof.** Given \( \delta > 0 \), there exists \( \tilde{F} \in C^{a+1}(\overline{\Omega}) \) such that
\[
\| F - \tilde{F} \|_{W^{\alpha,p}} < \delta.
\]

We have
\[
F \circ G - F \circ H = (F \circ G - \tilde{F} \circ G) + (\tilde{F} \circ G - \tilde{F} \circ H) + (\tilde{F} \circ H - F \circ H).
\]
The first and third terms in the right-hand side can be bounded in \( W^{\alpha,p}(\Omega) \) (using Lemma A.4) by
\[
C \delta \inf |\text{Jac} G |^{1/p} (\| G \|_{W^{a,v}}^{a} + 1) + C \delta \inf |\text{Jac} H |^{1/p} (\| H \|_{W^{a,v}}^{a} + 1),
\]
while the second term can be bounded in \( W^{\alpha,q}(\Omega) \) (and *a fortiori* in \( W^{\alpha,p}(\Omega) \)), using Lemma A.3, by
\[
C \| \tilde{F} \|_{C^{a+1}} (\| G \|_{W^{a,v}}^{a} + \| H \|_{W^{a,v}}^{a} + 1). \]

**Remark.** More generally, one can show, under the assumptions of Lemma A.5, that if \( F \in W^{a+b,p}(\Omega) \), then the mapping \( G \mapsto F \circ G \) is of class \( C^b \) from \( \mathcal{D}^{a,q}(\Omega) \) into \( W^{a,p}(\Omega) \)[\( \mathcal{D}^{a,q}(\Omega) \) is provided with an appropriate manifold structure].
3. Integration of Vector Fields

Let $F(x, t): \Omega \times [0, T] \rightarrow \mathbb{R}^N$ be a vector field tangent to $\partial \Omega$ on $\partial \Omega$ (i.e. $F(x, t) \cdot n(x) = 0$ for $x \in \partial \Omega$ and $t \in [0, T]$).

**Lemma A.6.** Assume $F \in C([0, T]; W^{\alpha, p}(\Omega; \mathbb{R}^N))$ with

$$\alpha > (N/p) + 1 \quad \text{and} \quad 1 \leq p < +\infty.$$

Then the differential equation

$$(du/dt)(x, t) = F(u(x, t), t)$$

$$u(x, 0) = x$$

has a solution $u \in C^1([0, T]; \mathcal{D}^{\alpha, p}(\Omega))$.

**Remark.** Lemma A.6 is not used in our paper, but it answers a question raised by Ebin and Marsden [2] who proved the same result for the case where $p = 2$ and $\alpha > (N/2) + 2$.

**Proof.** When $\alpha = 2$ (so that $p > N$), we have

$$F \in C([0, T]; C^{1, \lambda}(\Omega; \mathbb{R}^N))$$

where $\lambda = 1 - N/p$. In this case, it is well-known that there exists a solution $u \in C^1([0, T]; C^{1, \lambda}(\Omega; \mathbb{R}^N))$ and in addition $(d/dt) Du = DF(u(x, t)) Du$. On the other hand, $x \mapsto u(x, t)$ is a diffeomorphism for all $t \in [0, T]$ since

$$|\text{Jac } u(x, t)|_{t=0} = \text{div } F(u(x, t), t) = 0$$

and thus $|\text{Jac } u(x, t)| \geq e^{-Ct}$. Hence, $DF(u(x, t), t) \in W^{1, p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$ for all $t \in [0, T]$; more precisely, the mapping $t \mapsto DF(u(x, t), t)$ is continuous from $[0, T]$ into $W^{1, p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$ (as in the proof of Lemma A.5). For a fixed $u \in C^1(\Omega, \overline{\Omega})$, the operator $v \mapsto DF(u, t) \cdot v$ is bounded from $W^{1, p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$ into itself (by Lemma A.1). Therefore, the linear differential equation $dv/dt = DF(u, t) \cdot v$ (considered in the Banach space $W^{1, p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$) has a solution

$$v \in C^1([0, T]; W^{1, p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)).$$

Consequently, $Du \in C^1([0, T]; W^{1, p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N))$ and

$$u \in C^1([0, T]; W^{\alpha, p}(\Omega; \mathbb{R}^N)).$$
In the general case, the proof is by induction on \( \alpha \). Since
\[
F \in C([0, T]; W^{\alpha-1,p}(\Omega; \mathbb{R}^N)),
\]
we know from the induction assumption that \( u \in C^1([0, T]; \mathcal{D}^{\alpha-1,q}(\Omega)) \), where \( q = p^* \) for \( p \leq N \) and \( q \) is any finite number for \( p > N \).

Lemma A.4 shows that \( DF(u, t) \in W^{\alpha-1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N) \) for all \( t \in [0, T] \); more precisely, it follows from Lemma A.5 that the mapping \( t \mapsto DF(u(x, t), t) \) is continuous from \([0, T]\) into \( W^{\alpha-1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N) \). Therefore, the linear differential equation
\[
\frac{dv}{dt} = DF(u, t) \cdot v
\]
has a solution \( v \in C^1([0, T]; W^{\alpha-1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)) \). Consequently, \( Du \in C^1([0, T]; W^{\alpha-1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)) \) and \( u \in C^1([0, T]; W^{\alpha,p}(\Omega; \mathbb{R}^N)) \).

\[\square\]

**References**