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NON LINEAR PERTURBATIONS OF MONOTONE OPERATORS

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Non Linear Perturbations of Monotone Operators

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The theory of monotone operators from a Banach space V into its dual space V^* was recently developed by a number of authors (for a complete bibliography see Browder [10]). The main observation, originally due to Zarantonello [20], is that non linear operators satisfying a simple algebraic condition (monotonicity) are onto provided they are "coercive". It was shown later that "bounded" monotone operators have continuity properties (pseudo-monotonicity) which are actually sufficient in order to get surjectivity results.

In §1 we present some preliminaries about monotone and pseudo-monotone operators with a brief survey of simple surjectivity theorems (for a complete exposition see Browder [10]). In §2 we extend first the usual notion of linear operators to possibly multivalued linear operators. This allows us to define in a natural way the inverse of A without assuming A is one to one, the adjoint of A without assuming A is densely defined, the closure of A without assuming A is closable etc. We establish a simple characterization of linear maximal monotone operators; next we prove several results about perturbations of linear maximal monotone operators by non linear pseudo-monotone operators. Part of these results were announced without proofs in Brezis [6]. In §3 we give some applications to the solvability in a <u>weak sense</u> of evolution equations written in the form

$$Au + Bu = f$$

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where $A = \frac{d}{dt}$ and B is a non linear "elliptic" operator.

As a particular case we get the results of Lions [13] (see also Browder [8] and Visik [19]) but we can also treat a number of problems which were not included in [13], for example, Navier Stokes equation or mixed boundary value problems for non linear parabolic equations.(1)

§1. Monotone and pseudomonotone operators

Let V be a reflexive Banach space over $\mathbb{R}$ with norm $\| \ \|$ and let V^* be its dual space with dual norm $\| \ \|_*$. The scalar product in the duality between V and V^* is denoted by (f, u). Let $B(u, r) = \{ v \in V; \|v-u\| \leq r \}$ and $S(u, r) = \{ v \in V; \|v-u\| = r \}$. We consider multivalued mappings A from V into V^* i.e. subsets (or graphs) G(A) in $V \times V^*$. As usual we set

$$Au = \{ f \in V^* ; [u, f] \in G(A) \}$$

$$D(A) = \{ u \in V ; Au \neq \emptyset \}$$

$$R(A) = \bigcup_{u \in D(A)} Au$$

$$A^{-1}f = \{ u \in V ; [u, f] \in G(A) \}$$

$$(A_1 + A_2)(u) = A_1u + A_2u$$

We say that A is

<u>bounded</u> if for every bounded set $B \subset V$, $A(B) = \bigcup_{u \in B} Au$ is bounded in V^*,

<u>monotone</u> if for every $[u_i, f_i] \in G(A)$ $i = 1, 2$ $(f_1 - f_2, u_1 - u_2) \geq 0$,

<u>maximal monotone</u> if G(A) is maximal (in the sense of inclusion) among all monotone graphs.

Clearly A^{-1} is monotone (resp. maximal monotone) iff A is monotone (resp. maximal monotone).

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(1) Part of these results were presented at the Conference on Evolution Equations held at the University of Kansas (July, 1970).

We assume that both norms $\| \ \|$ and $\| \ \|_*$ are strictly convex; this is not a restriction since we know by results of Lindenstrauss [12] and Asplund [1] that there exists an equivalent norm on V with this property. More precisely (see [7] theorem 1.1) for every $a > 1$ there exists an equivalent norm $\| \ \|_a$ which is strictly convex together with its dual norm and such that

$$a^{-1}\| \ \|_a \leqq \| \ \| \leqq a\| \ \|_a .$$

Let F be the duality map from V into V^* i.e. for every $u \in V$, Fu is uniquely determined by the relations

$$(Fu, u) = \|u\|^2 \quad \text{and} \quad \|Fu\|_* = \|u\| .$$

It is well known that F is one to one and onto V^*, F is monotone and continuous from V (with strong topology) to V^* with its weak topology. Also the convex function $u \mapsto \frac{1}{2}\|u\|^2$ (resp. $f \mapsto \frac{1}{2}\|f\|_*^2$) admits Fu (resp. $F^{-1}f$) as Gateaux differential at every point i.e.

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon}\left(\|u+\epsilon v\|^2 - \|u\|^2\right) = (Fu, v) \quad \forall u, v \in V$$

$$\left(\text{resp. } \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\left(\|f+\epsilon g\|_*^2 - \|f\|_*^2\right) = (F^{-1}f, g) \quad \forall f, g \in V^*\right) .$$

There is a very simple characterization of maximal monotone operators due to Browder [9] (Minty [16] for the case where V is a Hilbert space).

Theorem 1. Let A be monotone, then A is maximal monotone iff $R(A + F) = V^*$

We state now a useful criterion for surjectivity of maximal monotone operators.

Theorem 2. Let A be maximal monotone; then $R(A) = V^*$ iff A^{-1} is locally bounded i.e. for every $f \in V^*$, there exists $r \in \mathbb{R}$ such that $\{ u \in V ; B(f, r) \cap Au \neq \emptyset \}$ is a bounded set in V.

In particular $R(A) = V^*$ if A^{-1} is bounded; using the terminology of partial differential equations it means that A is onto provided there is an a priori estimate for the solutions of the equation $Au \ni f$, f being bounded. We deduce easily from Theorem 2 the following

Corollary 3. Let A be maximal monotone and suppose A is coercive with respect to some $v_0 \in V$ i.e.,

$$\lim_{\substack{[v,w] \in G(A) \\ \|v\| \to +\infty}} \frac{(w, v-v_0)}{\|v\|} = +\infty$$

Then $R(A) = V^*$.

Theorem 2 was proved independently by Browder and Rockafellar [17]. We establish here only the "interesting" part of theorem 2 i.e. $R(A) = V^*$ if A^{-1} is locally bounded. We use the following

Lemma 4. Let $[u_n, f_n] \in G(A)$, $u \in V$, $f \in V^*$. Assume that $u_n \rightharpoonup u$, $f_n \rightharpoonup f$[1] and $\limsup (f_n, u_n - u) \leqq 0$. Then $[u, f] \in G(A)$ and $(f_n, u_n) \to (f, u)$.

Proof of Lemma 4. we have

$$(f_n - g, u_n - v) \geqq 0$$

and thus $(f_n - g, u_n - u + u - v) \geqq 0 \ \forall [v, g] \in G(A)$.

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(1) As usual $\rightharpoonup$ denotes weak convergence.

Hence $(f-g,\ u-v) \geqq 0\ \forall [v,g] \in G(A)$ and by the maximality of A, $[u,f] \in G(A)$. Taking $v = u$ and $g = f$ we see that $\liminf (f_n, u_n) \geqq (f,u)$.

Proof of Theorem 2. We show that R(A) is closed and open.

R(A) is closed: let $f_n \to f$ with $f_n \in Au_n$; since A^{-1} is locally bounded, $\{u_n\}$ is bounded and there is a subsequence $u_{n_k} \to u$. By lemma 4, $[u,f] \in G(A)$.

R(A) is open: let $[u,f] \in G(A)$ and let $r > 0$ be such that A^{-1} is bounded on $B(f,r)$. Let $g \in B\left(f, \frac{r}{2}\right)$; we are going to prove that $g \in R(A)$. By Theorem 1 there exists, for every $\epsilon > 0$, $v_\epsilon \in D(A)$ which is a solution of

$$\epsilon F(v_\epsilon - u) + g_\epsilon = g, \quad g_\epsilon \in Av_\epsilon .$$

Applying the monotonicity of A at v_ϵ and u we have

$$(g - \epsilon F(v_\epsilon - u) - f,\ v_\epsilon - u) \geqq 0$$

Hence

$$\epsilon \|v_\epsilon - u\| \leqq \|g-f\|_* \leqq \frac{r}{2} ;$$

thus $\|g_\epsilon - g\|_* \leqq \frac{r}{2}$ and $\|g_\epsilon - f\|_* \leqq r$. Consequently v_ϵ is bounded so that $g_\epsilon \to g$ as $\epsilon \to 0$. By the closedness of R(A), $g \in R(A)$.

Examples of maximal monotone operators.

Ex. 1. Let A be a single valued monotone operator with $D(A) = V$ such that A is hemicontinuous i.e. $\lim_{t \downarrow 0} (A((1-t)u + tv), w) = (Au, w)$ $\forall u, v, w \in V$. Then A is maximal monotone (this is easily seen from the definition).

Ex. 2. Let φ be a convex lower semi-continuous function from V into $(-\infty, +\infty]$. We set $G(\partial\varphi) = \{ [u,f];\ \varphi(v) - \varphi(u) \geqq (f, v-u)\ \forall v \in V \}$.

Then $\partial\varphi$ is maximal monotone. (Obviously $\partial\varphi$ is monotone and by Theorem 2 it is sufficient to show that $R(\partial\varphi + F) = V^*$; but $f \in \partial\varphi(u) + Fu$ iff the function $\varphi(v) + \frac{1}{2}\|v\|^2 - (f, v)$ achieves its minimum at u .)

In general the sum of two maximal monotone operators is not maximal monotone. This is however true under some additional assumptions.

Ex. 3. Let A and B be maximal monotone with $(\text{Int } D(A)) \cap D(B) \neq \emptyset$. Then $A + B$ is maximal monotone.

Ex. 4. Let A and B be maximal monotone with $D(A) \subset D(B)$ and

$$\|B^\circ u\|_* \leqq k\|A^\circ u\|_* + \omega(\|u\|) \quad \forall u \in D(A)^{(1)}$$

where $k < 1$ and ω is a continuous function. Then $A + B$ is maximal monotone.

Ex. 5. Let A be maximal monotone and let φ be convex l. s. c. from V into $(-\infty, +\infty]$. For every $u \in V$ and $\lambda > 0$, let u_λ be the unique solution of

$$F(u_\lambda - u) + \lambda A u_\lambda \ni 0$$

Assume that $\varphi(u_\lambda) \leqq \varphi(u) \quad \forall u \in V$, $\forall \lambda > 0$. Then $A + \partial\varphi$ is maximal monotone.

A general criteria for $A + B$ to be maximal monotone is given in [7]. It can be used to treat Examples 3, 4, 5.

Pseudo-monotone operators.

We say that a singlevalued mapping A from $C \subset V$ into V^* is pseudo-monotone if, for every sequence $u_n \rightharpoonup u$ in C for which $\limsup (Au_n, u_n - u) \leqq 0$, we have also $\liminf (Au_n, u_n - v) \geqq (Au, u - v)$ for every $v \in C$.

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(1) $\|A^\circ u\|_* = \text{Inf}\{\|z\|_* ; z \in Au\}$

In particular if $C = V$, then $u_n \rightharpoonup u$ and $\limsup (Au_n, u_n - u) \leqq 0$ imply $Au_n \rightharpoonup Au$ in V^* and $(Au_n, u_n) \to (Au, u)$.

Example. Let $C \subset V$ be closed and convex and let A be monotone hemicontinuous from C into V^*, then A is pseudo-monotone. More generally if A is monotone hemicontinuous on C and B is pseudo-monotone on C, then $A+B$ is also pseudo-monotone.

Theorem 5. Let A be maximal monotone and let B be pseudo-monotone and bounded from $\overline{D(A)}$ into V^*. Assume that $(A+B)^{-1}$ is bounded and that there exist $v_0 \in V$, $C \in \mathbb{R}$, $r \in \mathbb{R}$ such that

$$(1) \qquad (w + Bv, v - v_0) \geqq -C\|v\| \quad \forall [v, w] \in G(A), \quad \|v\| \geqq r .$$

Then $R(A + B) = V^*$.

Proof. It is sufficient to show that $0 \in R(A + B)$. Let $\omega(r) = \operatorname{Sup}\{ \|Bv\| + 1 ; \|v\| \leqq r \}$ and let $Gu = \omega(\|u\|)Fu$.

We consider the approximate equation

$$Au_\varepsilon + Bu_\varepsilon + \varepsilon Gu_\varepsilon \ni 0 , \quad \varepsilon > 0 .$$

The operator $B + \varepsilon G$ is pseudo-monotone and coercive from $\overline{D(A)}$ into V^*. More precisely $\lim\limits_{\substack{\|u\| \to +\infty \\ u \in \overline{D(A)}}} \dfrac{(Bu + \varepsilon Gu, u - u_0)}{\|u\|} = +\infty \quad \forall u_0 \in V$. Indeed

$$(Bu + \varepsilon Gu, u - u_0) \geqq -\omega(\|u\|)\|u - u_0\| + \varepsilon\omega(\|u\|)\|u\|^2 - \varepsilon\omega(\|u\|)\|u\|\,\|u_0\| .$$

By Theorem 5[(1)] in [5] we conclude that $R(A + B + \varepsilon G) = V^*$.

(Actually the definition we gave in [5] of pseudo-montonicity involved filters instead of sequences; but using a device due to Kaplansky[(2)] (which was pointed out to us by Weiss) it is enough to make an assumption on sequences (see Proposition 7.2 in [10]).)

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(1) which is stated here as Corollary 6.

(2) If Δ is a bounded set in a reflexive Banach space, then the weak closure of Δ consists of limits of weakly convergent sequences in Δ.

We will now show that u_ε is bounded as $\varepsilon \to 0$. Assuming $\|u_\varepsilon\| \geqq r$, we have by (1)

$$\varepsilon\omega(\|u_\varepsilon\|)\|u_\varepsilon\| \leqq \varepsilon\|v_0\|\omega(\|u_\varepsilon\|) + C .$$

Consequently $\varepsilon\omega(\|u_\varepsilon\|)\|u_\varepsilon\|$ is bounded as $\varepsilon \to 0$ and so is $\varepsilon Gu_\varepsilon$. Since $(A+B)^{-1}$ is bounded, u_ε and hence Bu_ε, are bounded as $\varepsilon \to 0$. We choose a sequence $\varepsilon_n \to 0$ such that $u_{\varepsilon_n} \rightharpoonup u$, $Bu_{\varepsilon_n} \rightharpoonup f$. Using the monotonicity of A at u_ε and v we get

$$(-Bu_\varepsilon - \varepsilon Gu_\varepsilon - w,\ u_\varepsilon - v) \geqq 0 \qquad \forall [v,w] \in G(A) .$$

Hence $\limsup(Bu_{\varepsilon_n}, u_{\varepsilon_n} - u) \leqq (f+w, v-u)$ $\forall [v,w] \in G(A)$. By the maximality of A we have $\operatorname{Inf}\{(f+w, v-u); [v,w] \in G(A)\} \leqq 0$. Indeed, assume $\operatorname{Inf}\{(f+w, v-u); [v,w] \in G(A)\} = \alpha > 0$. Then in particular $(w-(-f), v-u) \geqq 0$ for all $[v,w] \in G(A)$ and so $[u,-f] \in G(A)$. Taking $v = u$ and $w = -f$ we get a contradiction. Consequently $\liminf(Bu_{\varepsilon_n}, u_{\varepsilon_n} - v) \geqq (Bu, u-v)$, and finally $(Bu+w, v-u) \geqq 0$ for all $[v,w] \in G(A)$, so $Au + Bu \ni 0$.

<u>Remark 1.</u> Theorem 5 is closely related to Theorem 15 in [11] (where B may be multivalued and bounded in a slightly weaker sense). However, we assume here that B is defined only on $\overline{D(A)}$, we do not assume any continuity of B on finite dimensional spaces; also our assumption (1) is weaker than assumption (a) in [11].

<u>Corollary 6.</u> Let A be maximal monotone and let B be pseudomonotone and bounded from $\overline{D(A)}$ into V^*. Assume B is coercive with respect to some $v_0 \in D(A)$. Then $R(A+B) = V^*$.

<u>Corollary 7.</u> Let $C \subset V$ be closed and convex and let B be pseudo-monotone and bounded from C into V^*. Assume B is coercive with respect to some $v_0 \in C$. Then for every $f \in V^*$ there exists $u \in C$ such that

$$(Bu,\ v-u) \geqq (f,\ v-u) \qquad \forall v \in C .$$

<u>Proof.</u> Apply Corollary 6 with $A = \partial\psi_c$ where $\psi_c(u) = \begin{cases} 0 & \text{if } u \in C \\ +\infty & \text{if } u \notin C \end{cases}$

<u>Corollary 8.</u> Let B be pseudo-monotone and bounded from $B(0, r)$ into V^*. Assume

$$Bv + \lambda Fv \neq 0 \qquad \forall v \in S(0, r)\ , \quad \forall \lambda \geqq 0 .$$

Then there exists $u \in B(0, r)$ such that $Bu = 0$.

<u>Proof.</u> Apply Corollary 7 with $C = B(0, r)$ and $f = 0$. There exists $u \in B(0, r)$ such that

$$(Bu,\ v-u) \geqq 0 \qquad \forall v \in B(0, r) .$$

If $u \in S(0, r)$, there exists $\lambda \geqq 0$ such that $Bu + \lambda Fu = 0$ and we get a contradiction. Hence $u \notin S(0, r)$ and $Bu = 0$.

:. Perturbations of linear graphs.

We say that A is linear if $G(A)$ is a linear subspace of $V \times V^*$, e. $f \in Au$ and $g \in Av$ imply $\lambda f + \mu g \in A(\lambda u + \mu v)$ $\forall \lambda, \mu \in \mathbb{R}$. Let $D(A) = A^{-1}0 = \{ u \in V ; 0 \in Au \}$. We say that

$\bar{A}$ is the closure of A if $G(\bar{A}) = \overline{G(A)}$,

A is closed if $\bar{A} = A$,

$f \in A^* u$ if $(u, g) = (f, v)$ $\forall [v, g] \in G(A)$,

A^* is the adjoint of A.

Obviously A^* is linear and closed since $G(A^*) = J(G(A)^\perp)$ where $J[u, f] = [-f, u]$. Clearly we have $A^{**} = \bar{A}$ and $(A^*)^{-1} = (A^{-1})^*$. From the definition of A^* we deduce directly the

Proposition 9. $A^*0 = D(A)^\perp$.

Unlike the usual singlevalued case it makes sense to consider A^{-1}, $\bar{A}$, A^* without any further assumption on A. In order to see the relationship with classical definitions it is useful to notice that A is singlevalued iff $A0 = \{0\}$ and A is closable (i.e. $\bar{A}$ is singlevalued) iff $\bar{A}0 = \{0\}$. Also A^* is singlevalued iff $A^*0 = \{0\}$ or $\overline{D(A)} = V$, and A is closable iff A^{**} is singlevalued or $D(A^*) = V$.

Proposition 10. $N(A^*) = R(A)^\perp$ and thus $N(A^*)^\perp = \overline{R(A)}$.

Proof. Applying Proposition 9 to A^{-1} we have $N(A^*) = (A^*)^{-1}0 = D(A^{-1})^\perp = R(A)^\perp$.

Linear monotone graphs.

It is clear that A^{-1} and $\bar{A}$ are linear monotone if A is linear monotone; but in general A^* is not necessarily monotone (see Theorem 14).

Proposition 11. Assume A is linear and monotone. Then $N(A) \subset R(A)^{\perp} = N(A^*)$ and $A0 \subset D(A)^{\perp} = A^*0$.

Proof. Let $u \in N(A)$; by the monotonicity of A we have

$$(\lambda g\ ,\ \lambda v-u) \geqq 0 \quad \forall [v, g] \in G(A)\ , \qquad \forall \lambda \in \mathbb{R}$$

Thus $(g,\ \lambda v-u) \geqq 0$ if $\lambda > 0$ and as $\lambda \to 0$ we get $(g, u) = 0 \quad \forall g \in R(A)$. Applying this result to $\bar{A}^1$ we have $N(\bar{A}^1) = A0 \subset R(\bar{A}^1)^{\perp} = D(A)^{\perp} = N(\bar{A}^{1*}) = A^*0$

Corollary 12. Assume A is linear and monotone with $\overline{D(A)} = V$. Then A is singlevalued.

Proposition 13. Let A be linear and monotone. Then A is maximal monotone iff A is maximal among all linear monotone graphs.

Proof. Assume $[u, f] \notin G(A)$ and

$$(g-f,\ v-u) \geqq 0 \qquad \forall [v, g] \in G(A)\ .$$

Let $G(\tilde{A})$ be the linear space spanned by $G(A) \cup [u, f]$; then $\tilde{A}$ is montone. Indeed let $[v, g] \in G(A)$ and $\lambda \in \mathbb{R}$, then $(\lambda f + g,\ \lambda u + v) = \lambda^2\left(f+\frac{g}{\lambda}\ ,\ u+\frac{v}{\lambda}\right) \geqq 0$.

There is a very simple characterization of linear maximal monotone graphs in terms of their adjoints.

Theorem 14. Let A be linear monotone. Then A is maximal monotone iff A is closed and A^* is monotone.

Proof. Assume A is maximal monotone; clearly A is closed. Let $[u, f] \in G(A^*)$ and $[v, g] \in G(A)$. We have

$$(g + f,\ v-u) = (g, v) - (g, u) + (f, v) - (f, u) = (g, v) - (f, u) \geqq -(f, u)\ .$$

If $(f,u) < 0$, it follows from the maximality of A that $[-f,u] \in G(A)$; taking $v = u$ and $g = -f$ we get a contradiction. Hence $(f,u) \geqq 0$. Assume now A is closed and A^* is monotone. Let $u \in V$ and $f \in V^*$ satisfy

$$(g-f,\ v-u) \geqq 0 \qquad \forall [v,g] \in G(A) .$$

We have to prove that $[u,f] \in G(A)$. The space $G(A)$ with the norm induced by $V \times V^*$ is reflexive. Define on $G(A)$ the function

$$\varphi[v,g] = \frac{1}{2}\|g-f\|^2 + \frac{1}{2}\|v-u\|^2 + (g-f,\ v-u)$$

It is easy to check that φ is convex, continuous and $\varphi \to +\infty$ as $[v,g] \to +\infty$. Hence φ achieves its minimum at some point $[u_0, f_0] \in G(A)$. Writing that the Gateaux differential of φ at $[u_0, f_0]$ is 0 we get

$$(F^{-1}(f_0-f), g) + (F(u_0-u),\ v) + (f_0-f,\ v) + (g, u_0-u) = 0 \quad \forall [v,g] \in G(A)$$

Thus $f-f_0 - F(u_0-u) \in A^*[F^{-1}(f_0-f) + u_0-u]$. By the monotonicity of A^* we obtain

$$(F^{-1}(f_0-f) + u_0-u,\ f-f_0-F(u_0-u)) \geqq 0 .$$

Consequently

$$\|u_0-u\|^2 + \|f_0-f\|^2 + (f_0-f,\ u_0-u) \leqq \|u_0-u\|\,\|f_0-f\|$$

Since $(f_0-f,\ u_0-u) \geqq 0$ we get $u_0 = u$ and $f_0 = f$.

<u>Corollary 15</u>. Assume A is linear maximal monotone; then A^* is maximal monotone.

<u>Corollary 16</u>. Assume A is linear maximal monotone; then $N(A) = N(A^*)$, $A0 = A^*0$, $\overline{D(A)} = \overline{D(A^*)}$ and $\overline{R(A)} = \overline{R(A^*)}$.

Proof. Applying Proposition 11 to A^* (which is monotone by Theorem 14) we get $N(A) = N(A^*)$ and $A0 = A^*0$. Thus, by Propositions 9 and 10, we have $\overline{D(A)} = \overline{D(A^*)}$ and $\overline{R(A)} = \overline{R(A^*)}$.

Corollary 17. Let A be linear monotone and singlevalued; then A is maximal monotone iff $\overline{D(A)} = V$ and A is maximal among all linear singlevalued monotone operators (i.e. maximal monotone in the sense of Lumer-Phillips [15]).

Proof. If A is maximal monotone and singlevalued then $A0 = \{0\} = A^*0 = D(A)^\perp$ and hence $\overline{D(A)} = V$. The converse follows from Corollary 12 and Proposition 13.

Remark 2. It is well known (even if V is a Hilbert space) that there exist linear operators which are not densely defined and are maximal among all linear singlevalued monotone operators. Those operators have no singlevalued maximal monotone extensions. However, they do have a unique multivalued maximal monotone extension given by $\tilde{A}u = Au + D(A)^\perp$.

Remark 3. Theorem 14 was originally proved in [4] for singlevalued operators by using Theorem 1. The simple proof we give here is based on an idea of L. Nirenberg. Previously F. Browder [5] had shown that if A is linear monotone densely defined and $A^* = \overline{A^*|_{D(A)}}$ then A is maximal monotone.

In a number of concrete examples, maximal monotone operators can be obtained in the following way. Let H be a Hilbert space such that $V \subset H \subset V^*$ with continuous and dense injections. Let L be a linear

(singlevalued) maximal monotone operator in H. We make the following assumption (see [2]). For every $u \in V$ and $f \in V^*$ satisfying

$$(u, L^* v) = (f, v) \qquad \forall v \in D(L^*) \cap V$$

there exists a sequence $u_n \in D(L) \cap V$ such that $u_n \to u$ in V and $Lu_n \to f$ in V^*

Corollary 18. Let A be the closure of $L_{|V}$ in $V \times V^*$. Then A is maximal monotone.

Proof. It follows easily from the assumption that $A = \left(L^*_{|V}\right)^*$ and hence $A^* = \overline{L^*_{|V}}$. Thus A^* is monotone and A is maximal monotone by Theorem 14.

Non linear perturbations.

Theorem 19. Let A be linear maximal monotone in $V \times V^*$. Let B be a singlevalued mapping from $D(A)$ into V^* satisfying

(2) B is A-pseudo-monotone i.e. for every sequence $u_n \in D(A)$ for which $u_n \rightharpoonup u$ in V, $\|A^o u_n\|_*$ is bounded and $\limsup (Bu_n, u_n - u) \leqq 0$, we have also $\liminf (Bu_n, u_n - v) \geqq (Bu, u-v)$ $\forall v \in D(A)$.

(3) $\|Bv\|_* \leqq k\|A^o v\|_* + \omega(\|v\|)$ $\forall v \in D(A)$ where $k<1$ and ω is non decreasing.

(4) $(A + B)^{-1}$ is bounded.

(5) There exist $v_0 \in V$, $C \in \mathbb{R}$ and $r \in \mathbb{R}$ such that $(g + Bv, v - v_0) \geqq -C\|v\|$ $\forall [v, g] \in G(A)$; $\|v\| \geqq r$

Then $R(A + B) = V^*$.

The proof of Theorem 19 is based on the following:

Lemma 20. Let $Ku = \psi(\|u\|)Fu$ where $\psi(r) = \omega(r) + r$. Then $R(A + B + \varepsilon K) = V^*$ for every $\varepsilon > 0$.

Proof of Lemma 20. It is sufficient to show that $0 \in R(A+B+\varepsilon K)$. As in the proof of Theorem 14 we consider the space $G(A)$ with the norm induced by $V \times V^*$. Let $[u,f] \in G(A)$ and consider on $G(A)$ the linear functionals $[v,g] \mapsto (f,v)$, $[v,g] \mapsto (Bu,v)$, $[v,g] \mapsto (Ku,v)$ and $[v,g] \mapsto (F^{-1}f,g) + (Fu,v)$. They define elements of $G(A)^*$ (the dual of $G(A)$) which we denote respectively by $\tilde{A}[u,f]$, $\tilde{B}[u,f]$, $\tilde{K}[u,f]$, $\tilde{F}[u,f]$. Obviously $\tilde{A}$ is linear monotone and continuous from $G(A)$ into $G(A)^*$; $\tilde{B}$ is pseudomonotone and bounded from $G(A)$ into $G(A)^*$ (use (2) and (3)). $\tilde{K}$ is monotone hemicontinuous and bounded from $G(A)$ into $G(A)^*$; hence $\tilde{B} + \varepsilon\tilde{K}$ is pseudomonotone and bounded from $G(A)$ into $G(A)^*$. $\tilde{F}$ is the duality map from $G(A)$ into $G(A)^*$. Using Corollary 8 we first show that

$$0 \in R(\tilde{A} + \tilde{B} + \varepsilon\tilde{K})$$

Thus it is sufficient to prove that for every $\lambda \geqq 0$, $\tilde{A} + \tilde{B} + \varepsilon\tilde{K} + \lambda\tilde{F} \neq 0$ if $\|u\|^2 + \|f\|_*^2 = \rho^2$ and ρ is sufficiently large. Assume we have

(6) $\quad (f,v) + (Bu,v) + \varepsilon(Ku,v) + \lambda(F^{-1}f,g) + \lambda(Fu,v) = 0 \quad \forall [v,g] \in G(A).$

i.e. $F^{-1}f \in D(A^*)$ and

(7) $\quad f + Bu + \varepsilon Ku + \lambda A^* F^{-1}f + \lambda Fu \ni 0$.

Taking $v = u$ and $g = f$ in (6) we get:

(8) $$(f,u) + (Bu,u) + \varepsilon(Ku,u) + \lambda(\|f\|_*^2 + \|u\|^2) = 0$$

Multiplying (7) by $F^{-1}f$ and using the monotonicity of A^* we obtain

(9) $$\|f\|_*^2 + (Bu, F^{-1}f) + \varepsilon(Ku, F^{-1}f) + \lambda(Fu, F^{-1}f) \leqq 0 .$$

From (9) we deduce

$$\|A^o u\|_* \leqq \|f\|_* \leqq \|Bu\|_* + \varepsilon\|Ku\|_* + \lambda\|u\| \leqq$$
$$k\|A^o u\|_* + \omega(\|u\|) + \varepsilon\psi(\|u\|)\|u\| + \lambda\|u\|$$

or $\|A^o u\|_* \leqq \frac{1}{1-k} (\omega(\|u\|) + \varepsilon \psi(\|u\|)\|u\| + \lambda\|u\|)$.

Thus

(10) $$\|f\|_* \leqq \frac{1}{1-k} (\omega(\|u\|) + \varepsilon \psi(\|u\|)\|u\| + \lambda\|u\|)$$

If $\|u\| \leqq r$, it follows from (8) that

(11) $$\lambda \|f\|_*^2 \leqq (k\|A^o u\|_* + \omega(\|u\|))\|u\| \leqq (k\|f\|_* + \omega(r))r$$

Combining (10) and (11) we obtain an estimate on $\|f\|_*$ which is independent of λ. If $\|u\| \geqq r$, we deduce from (5) and (8) that

(12) $$\varepsilon \psi(\|u\|)\|u\|^2 + \lambda(\|f\|_*^2 + \|u\|^2) \leqq C_1\|u\| + C_2\omega(\|u\|) + C_3\|f\|_*$$

Again with the help of (10), (12) leads to estimates on $\|u\|$ and $\|f\|_*$ which are independent of λ. Hence $0 \in R(\tilde{A} + \tilde{B} + \varepsilon \tilde{K})$ i.e. there exists $[u, f] \in G(A)$ such that $(f + Bu + \varepsilon Ku, v) = 0 \quad \forall v \in D(A)$ or $f + Bu + \varepsilon Ku \in D(A)^{\perp} = A0$. Consequently $Au + Bu + \varepsilon Ku \ni 0$.

<u>Proof of Theorem 19.</u> It is sufficient to prove that $0 \in R(A + B)$. Let u_ε be a solution of $0 \in Au_\varepsilon + Bu_\varepsilon + \varepsilon Ku_\varepsilon$, $\varepsilon > 0$. If $\|u_\varepsilon\| \geqq r$, we have by (5)

$$\varepsilon(Ku_\varepsilon, u_\varepsilon - v_0) \leqq C\|u_\varepsilon\|$$

so that

$$\varepsilon \psi(\|u_\varepsilon\|)\|u_\varepsilon\| \leqq \varepsilon \psi(\|u_\varepsilon\|)\|v_0\| + C$$

which shows that $\varepsilon \psi(\|u_\varepsilon\|)\|u_\varepsilon\|$ is bounded as $\varepsilon \to 0$ and so is $\varepsilon Ku_\varepsilon$. Consequently u_ε is bounded as $\varepsilon \to 0$. With the help of (2) and (3) we conclude as in the proof of Theorem 5 that $u_{\varepsilon_n} \to u$ and $0 \in Au + Bu$.

<u>Remark 4.</u> Lemma 20 could also be proved directly by a non linear "elliptic regularization". Corollary 6 applied with $G(A) = V$ shows that

the equation

$$Au_\lambda + Bu_\lambda + \varepsilon Ku_\lambda + \lambda A^* F^{-1} Au_\lambda \ni 0$$

has a solution for every $\lambda > 0$. After proving that $\|u_\lambda\|$ and $\|A^o u_\lambda\|_*$ are bounded we pass to the limit as $\lambda \to 0$.

Remark 5. It would be of interest to determine whether Theorem 14 can be extended to the case where A is nonlinear maximal monotone. Notice in particular that if B is maximal monotone and (3) holds then (2) is satisfied and the conclusion of Theorem 19 holds even for nonlinear A, since A + B is maximal monotone (see example 4).

We present now some extensions and corollaries of Theorem 19.

Corollary 21. Let L be a linear monotone graph (not necessarily maximal) in $V \times V^*$. Let B be a bounded pseudo-monotone operator from V into V^*. Assume B is coercive with respect to some $v_o \in D(L)$ i.e.

$$\lim_{\|v\| \to +\infty} \frac{(Bv,\ v-v_o)}{\|v\|} = +\infty$$

Then for every $f \in V^*$ there exists $u \in V$ such that

$$(u, g) + (Bu, v) = (f, v) \quad \forall [v, g] \in G(L)$$

Proof. Let $\tilde{L}$ be a maximal monotone linear graph which contains L (such a graph exists by Zorn's lemma and Proposition 13). Then the assumptions of Theorem 19 hold with $A = \tilde{L}^*$. Indeed if $[v, g] \in G(\tilde{L}^*)$ we have

$$\frac{(g + Bv,\ v-v_o)}{\|v\|} \geqq -\|L^o v_o\|_* + \frac{(Bv, v-v_o)}{\|v\|} \to +\infty$$

as $\|v\| \to +\infty$. Thus $R(\tilde{L}^* + B) = V^*$.

Non homogeneous boundary conditions.

Let M be a linear singlevalued operator from $D(M) \subset V$ into V^*. Let $\mathcal{K}$ be a Banach space with norm $|\ |$ and let γ be a linear mapping from $D(M)$ into $\mathcal{K}$. We make the following assumptions:

(13) $\gamma(D(M))$ is dense in $\mathcal{K}$.

(14) the graph of $M \times \gamma$ i.e. $\{[u, Mu, \gamma(u)]; u \in V\}$ is closed in $V \times V^* \times \mathcal{K}$.

(15) there exists $C \geqq 0$ such that

$$(Mv, v) + C|\gamma(v)|^2 \geqq 0 \quad \forall v \in D(M)$$

Corollary 22. Assume (13), (14), (15) hold and M restricted to $\{v \in V ; \gamma(v) = 0\}$ is maximal monotone. Let B be a nonlinear operator from $D(M)$ into V^* such that

(16) $\|Bv\|_* \leqq k\|Mv\|_* + \omega(\|v\|, |\gamma(v)|)$ where $k < 1$ and ω is continuous

(17) B is M-pseudo-monotone

(18) B is coercive with respect to some $v_o \in D(M)$ i.e.

$$\lim_{\|v\| \to +\infty} \frac{(Bv, v-v_o)}{\|v\|} = +\infty$$

Then for every $f \in V^*$ and for every $\gamma_o \in \mathcal{K}$ there exists a $u \in D(M)$ such that

$$(19) \begin{cases} Mu + Bu = f \\ \gamma(u) = \gamma_o . \end{cases}$$

Proof. Assume first $\gamma_o \in \gamma(D(M))$ and let $u_o \in D(M)$ be such that $\gamma(u_o) = \gamma_o$. If we set $\tilde{u} = u - u_o$, equation (19) becomes

$$M\tilde{u} + B(\tilde{u} + u_o) = f + Mu_o , \quad \gamma(\tilde{u}) = 0 .$$

Let $\tilde{B}v = B(v + u_o)$ and let M_o be the restriction of M to $\{v \in D(M); \gamma(v) = 0\}$. We have to solve $M_o\tilde{u} + \tilde{B}\tilde{u} = f + Mu_o$. The assumptions of Theorem 19

hold with $A = M_o$ and $B = \tilde{B}$, since we have

$$\frac{(M_o v + \tilde{B}v,\ v+u_o-v_o)}{\|v\|} \geqq \frac{-C|\gamma_o-\gamma(v_o)|^2 + \|Mu_o-Mv_o\|_* \|v+u_o-v_o\|}{\|v\|}$$

$$+ \frac{(B(v+u_o),\ v+u_o-v_o)}{\|v\|} \to +\infty \quad \text{as} \quad \|v\| \to +\infty .$$

Assume now $\gamma_o \in \mathcal{H}$ and let $\gamma_n \in \gamma(D(M))$ be such that $\gamma_n \to \gamma_o$. Let u_n be the corresponding solution of

$$(20) \qquad Mu_n + Bu_n = f , \qquad \gamma(u_n) = \gamma_n .$$

Multiplying (20) by $u_n - v_o$ we see that $\|u_n\|$ is bounded and so are Mu_n and Bu_n. Let $u_{n_k} \rightharpoonup u$ with $Mu_{n_k} \rightharpoonup Mu$ and $\gamma(u) = \gamma_o$. We have

$$(Bu_n , u_n-u) = (f, u_n-u)-(Mu_n , u_n-u) \leqq (f, u_n-u)-(Mu, u_n-u) + C|\gamma_n-\gamma_o|^2$$

Thus $\limsup (Bu_{n_k} , u_{n_k}-u) \leqq 0$. We conclude that $Bu_{n_k} \rightharpoonup Bu$ and consequently $Mu + Bu = f$.

<u>Theorem 23.</u> Let A be linear maximal monotone in $V \times V^*$ and let B be a singlevalued mapping from $D(A)$ into V^*. Let T be maximal monotone (not necessarily linear) in $V \times V^*$. Assume (2) and (3) hold, and there exist C_1, $C_2 \in \mathbb{R}$ and $v_o \in \operatorname{Int} D(T)$ such that

$$(21) \qquad (g + Bv,\ v-v_o) \geqq -C_1\|v\| - C_2 \qquad \forall [v, g] \in G(A)$$

$$(22) \qquad (A + B + T)^{-1} \text{ is bounded.}$$

Then $R(A + B + T) = V^*$.

<u>Proof.</u> It suffices to show that $0 \in R(A + B + T)$. Let T_λ be the Yosida approximation of T (see [7]) i.e. for every $u \in V$, let $u_\lambda = J_\lambda u$ be the solution of

$$F(u_\lambda - u) + \lambda\, Tu_\lambda \ni 0 .$$

We set $T_\lambda u = \frac{1}{\lambda} F(u-u_\lambda)$. T_λ is monotone hemicontinuous and bounded from V into V^*.

Let $\varepsilon > 0$ be <u>fixed</u> ; Theorem 19 applied with $B + T_\lambda + \varepsilon F$ shows that there exists a solution u_λ of

$$Au_\lambda + Bu_\lambda + T_\lambda u_\lambda + \varepsilon F u_\lambda \ni 0 \tag{23}$$

Multiplying (23) by $u_\lambda - v_o$ and using (21) we see that $\|u_\lambda\|$ is bounded as $\lambda \to 0$. We use now the same device as in the proof of Theorem 2.2 in [7] in order to show that $\|T_\lambda u_\lambda\|_*$ is bounded as $\lambda \to 0$. Since $v_o \in \operatorname{Int} D(T)$, T is locally bounded at v_o (see [17]) i.e. there exists $\rho > 0$ such that T is bounded on $B(v_o, \rho)$. Using the monotonicity of T at $J_\lambda u_\lambda$ and $v_o + \rho \dfrac{F^{-1}(T_\lambda u_\lambda)}{\|T_\lambda u_\lambda\|_*}$ we have

$$\left(T_\lambda u_\lambda - g\,,\ J_\lambda u_\lambda - v_o - \rho \frac{F^{-1}(T_\lambda u_\lambda)}{\|T_\lambda u_\lambda\|_*} \right) \geqq 0 \quad \text{or}$$

$$\left(T_\lambda u_\lambda - g\,,\ -\lambda F^{-1}(T_\lambda u_\lambda) + u_\lambda - v_o - \rho \frac{F^{-1}(T_\lambda u_\lambda)}{\|T_\lambda u_\lambda\|_*} \right) \geqq 0 ,$$

with $g \in T\left(v_o + \rho \dfrac{F^{-1}(T_\lambda u_\lambda)}{\|T_\lambda u_\lambda\|_*} \right)$ and $\|g\| \leqq C$. Thus

$$\lambda \|T_\lambda u_\lambda\|_*^2 + \rho \|T_\lambda u_\lambda\|_* \leqq C_1 \|u_\lambda\| + C_2 + \varepsilon \|v_o\| \|u_\lambda\|$$
$$+ C(\lambda \|T_\lambda u_\lambda\|_* + \|u_\lambda - v_o\| + \rho) .$$

Consequently $\|T_\lambda u_\lambda\|_*$ is bounded as $\lambda \to 0$ and so are $\|A^0 u_\lambda\|$ and $\|Bu_\lambda\|_*$. We pass to the limit as $\lambda \to 0$ using standard devices and we get a solution u_ε of the equation

$$Au_\varepsilon + Bu_\varepsilon + Tu_\varepsilon + \varepsilon F u_\varepsilon \ni 0 .$$

With the same techniques we show that

$$\epsilon \|u_\epsilon\| \ , \ \|u_\epsilon\| \ , \ \|Tu_\epsilon\|_* \ , \ \|A^0 u_\epsilon\|_* \ , \ \|Bu_\epsilon\|_*$$

are bounded as $\epsilon \to 0$, and we pass to the limit as $\epsilon \to 0$.

Finally we would like to mention the following result concerning a different kind of non linear perturbation.

<u>Theorem 24.</u> Let A be linear maximal monotone in $V \times V^*$ and let B be a singlevalued bounded operator from V into V^* . Assume

(24) <u>B is of type M</u> i.e. for every sequence $u_n \in V$ such that $u_n \rightharpoonup u$ in V and $\limsup (Bu_n , u_n - u) \leqq 0$ then $Bu_n \rightharpoonup Bu$ (1)

(25) $$\lim_{\|v\| \to +\infty} \frac{(Bv, v)}{\|v\|} = +\infty$$

Then $R(A + B) = V^*$.

We do not present here the proof which is rather tricky and quite simila to the proof of Theorem 19 in [3] .

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(1) The main difference with pseudo-monotone operators is that we do not assume $(Bu_n , u_n) \to (Bu, u)$.

§3. Applications.

Application 1.

Let $\mathcal{V}$ be a reflexive Banach space and let $\mathcal{H}$ be a Hilbert space such that

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^*$$

with continuous and dense injections. Let $V = L^p(0, T; \mathcal{V})$ and $V^* = L^{p'}(0, T; \mathcal{V}^*)$ with $1 < p < +\infty$ and $1/p + 1/p' = 1$. Let $\mathcal{L}$ be a singlevalued linear maximal monotone operator from $\mathcal{V}$ into $\mathcal{V}^*$. We set

$$D(L) = \left\{ u \in V;\ u(t) \in D(\mathcal{L}) \text{ a.e. on }]0, T[\text{ and } \mathcal{L}u(t) \in V^* \right\}$$

$$(Lu)(t) = \mathcal{L}(u(t)).$$

Proposition 25. L is a maximal monotone operator from V into V^*. The adjoint L^* of L is given by

$$D(L^*) = \left\{ u \in V;\ u(t) \in D(\mathcal{L}^*) \text{ a.e. on }]0, T[\text{ and } \mathcal{L}^* u(t) \in V^* \right\}$$

$$(L^* u)(t) = \mathcal{L}^*(u(t))$$

Proof. Let $\mathcal{F}$ be the duality map from $\mathcal{V}$ into $\mathcal{V}^*$ and let $\mathcal{F}_p u = \|u\|^{p-2} \mathcal{F} u$. We set $(F_p u)(t) = \mathcal{F}_p(u(t))$; clearly F_p maps V into V^* and satisfies

$$\int_0^T (F_p u, u)dt = \|u\|_V^p, \quad \|F_p u\|_{V^*} = \|u\|_V^{p-1} .$$

Then $R(L + F_p) = V^*$, since for every $f \in V^*$ one can solve "point wise" the equation

$$\mathcal{L}u(t) + \mathcal{F}_p u(t) \ni f(t) \quad \text{and} \quad u \in D(L) .$$

Let $D(M) = \{ u \in V;\ u(t) \in D(\mathcal{L}^*)$ a.e. on $]0,T[$ and $\mathcal{L}^* u(t) \in V^* \}$

$$(Mu)(t) = \mathcal{L}^*(u(t)).$$

Since $\mathcal{L}^*$ is maximal monotone in $\mathcal{V} \times \mathcal{V}^*$ (by Corollary 15), M is maximal monotone in $V \times V^*$. We have $M \subset L^*$; indeed, let $u \in D(M)$ and $v \in D(L)$, then

$$\int_0^T (u, \mathcal{L}v)dt = \int_0^T (\mathcal{L}^* u, v)dt .$$

Consequently $M = L^*$ since L^* is monotone (by Theorem 14).

<u>Theorem 26.</u> Let B be a bounded operator from V into V^* satisfying

(26) for every sequence u_n in V for which $u_n \rightharpoonup u$ in V, $\frac{du_n}{dt}$ is bounded in $D(L^*)^*$ (1) and $\limsup \int_0^T (Bu_n, u_n - u)dt \leq 0$, we have also $Bu_n \rightharpoonup Bu$ in V^* and $\int_0^T (Bu_n, u_n)dt \to \int_0^T (Bu, u)dt$.

$$\lim_{\|v\| \to +\infty} \frac{\int_0^T (Bv, v)dt}{\|v\|} = +\infty . \tag{27}$$

Then for every $f \in V^*$ and every $u_0 \in \mathcal{H}$ there exists $u \in V \cap C([0,T]; \mathcal{H})$ such that $u(0) = u_0$ and

$$-\int_0^T \left(u, \frac{dv}{dt}\right)dt + \int_0^T (u, \mathcal{L}^* v)dt + \int_0^T (Bu, v)dt = \int_0^T (f, v)dt + (u_0, v(0)) \tag{28}$$

$$\forall v \in D(L^*);\ \frac{dv}{dt} \in V^* \quad \text{and} \quad v(T) = 0 .$$

---

(1) $D(L^*)^*$ is the dual space of $D(L^*)$ with its graph norm.

In addition, if B is monotone, the solution u of (28) is unique.

The proof of Theorem 26 is based on the following:

Lemma 27. Let $u \in V$ and let $f \in V^*$. The following conditions are equivalent:

(29) $$-\int_0^T \left(u, \frac{dv}{dt}\right) dt + \int_0^T (u, \mathcal{L}^* v) dt = \int_0^T (f, v) dt \qquad \forall v \in D(L^*);\ \frac{dv}{dt} \in V^*,\ v(T) = 0.$$

(30) There exists a sequence $u_n \in D(L)$ with $\frac{du_n}{dt} \in V^*$ and $u_n(0) = 0$ such that $u_n \to u$ in V and $\frac{du_n}{dt} + Lu_n \to f$ in V^*.

In particular $u \in C([0,T]; \mathcal{H})$ and $u(0) = 0$.

Proof of Lemma 27. Since it is obvious that (30) $\Longrightarrow$ (29) we have only to prove that (29) $\Longrightarrow$ (30).

We set

$$\bar{u}(t) = \begin{cases} 0 & \text{for } t < 0 \\ u(t) & \text{for } 0 < t < T \\ 0 & \text{for } t > T \end{cases}$$

Let ρ_n be a sequence of C^∞ functions such that $\rho_n \geqq 0$, $\operatorname{supp} \rho_n \subset (0, 1/n)$ and $\int_{-\infty}^{+\infty} \rho_n(t) dt = 1$. Let $\check{\rho}_n(t) = \rho_n(-t)$; let $\bar{u}_n = \rho_n * \bar{u}$ and let u_n be the restriction of $\bar{u}_n$ to $[0,T]$. Similarly let $\check{\bar{v}}_n = \check{\rho}_n * \bar{v}$ and let $\check{v}_n$ be the restriction of $\check{\bar{v}}_n$ to $[0,T]$. Clearly we have $u_n(0) = 0$, $\frac{du_n}{dt} \in V^*$ and also $u_n \to u$ in V.

Let $v \in D(L^*)$; we have

$$\int_0^T (u_n, \mathcal{L}^* v) dt = \int_{-\infty}^{+\infty} (\bar{u}_n, \mathcal{L}^* \bar{v}) dt = \int_{-\infty}^{+\infty} (\bar{u}, \check{\rho}_n * \mathcal{L}^* \bar{v}) dt = \int_{-\infty}^{+\infty} (\bar{u}, \mathcal{L}^* (\check{\rho}_n * \bar{v})) dt =$$

$$= \int_0^T (u, \mathcal{L}^* \check{\bar{v}}_n) dt .$$

Thus by assumption (29)

$$\int_0^T (u_n, \mathcal{L}^* v)dt = \int_0^T (f, \check{v}_n)dt + \int_0^T \left(u, \frac{d}{dt}\check{v}_n\right)dt = \int_{-\infty}^{+\infty} (\rho_n * \bar{f}, \bar{v})dt + \int_{-\infty}^{+\infty}\left(\bar{u}, \frac{d}{dt}\check{\rho}_n * \bar{v}\right)d$$

$$= \int_0^T (f_n, v)dt - \int_0^T \left(\frac{du_n}{dt}, v\right) dt .$$

(We use here the fact that $\check{v}_n \in D(L^*)$ $\frac{d\check{v}_n}{dt} \in V^*$ and $\check{v}_n(T) = 0$.)

Consequently $u_n \in D(L)$ and $\frac{du_n}{dt} + Lu_n = f_n \to f$.

<u>Proof of Theorem 26.</u> We consider the operator $\mathring{M}u = \frac{du}{dt} + Lu$ with domain

$$D(\mathring{M}) = \{ u \in D(L) ; du/dt \in V^* \} .$$

$\mathring{M}$ is closable in $V \times V^*$; indeed assume $u_n \to 0$ in V and $\frac{du_n}{dt} + Lu_n \to f$ in V^*. In particular u_n is a Cauchy sequence in $C([0,T]; \mathcal{H})$ and thus $u_n \to 0$ in $C([0,T]; \mathcal{H})$. We have

$$\int_0^T \left(\frac{du_n}{dt} + Lu_n , v\right)dt = (u_n(T), v(T)) - (u_n(0), v(0))$$

$$- \int_0^T \left(u_n , \frac{dv}{dt}\right) dt + \int_0^T (u_n , L^* v)dt .$$

Hence $\int_0^T (f, v)dt = 0 \quad \forall v \in D(L^*)$ such that $\frac{dv}{dt} \in V^*$; and so $f = 0$.

Let M be the closure of $\mathring{M}$. Clearly $D(M) \subset C([0,T]; \mathcal{H})$. If we define $\gamma(u) = u(0)$, the assumptions of Corollary 22 are satisfied since $\gamma(D(M)) \supset D(L)$, which is dense in $\mathcal{H}$, and

$$\int_0^T (Mu, u)dt + \frac{1}{2} |u(0)|^2 \geqq 0 \quad \forall v \in D(M).$$

Also the restriction M_0 of M to $\{ u \in D(M); u(0) = 0 \}$ is maximal monotone Indeed Lemma 27 asserts that $u \in D(M_0)$ and $M_0 u = f$ if and only if

$$-\int_0^T \left(u\,,\ \frac{dv}{dt}\right) dt + \int_0^T (u,\ L^* v)dt = \int_0^T (f, v)dt \quad \forall v \in D(L^*);\ \frac{dv}{dt} \in V^* \text{ and } v(T) = 0\ .$$

In other words $M_0 = N^*$, where

$$D(N) = \{\ u \in D(L^*)\ ;\ \frac{du}{dt} \in V^*\ ,\ u(T) = 0\ \}$$

and $N = -\frac{d}{dt} + L^*$.

Consequently $M_0^* = N^{**} = \overline{N}$ is monotone and by Theorem 14, M_0 is maximal monotone. Finally notice that

(31) $\qquad u \in D(M)\ ,\ Mu = f$ and $u(0) = u_0$

if and only if

(32) $$-\int_0^T \left(u,\ \frac{dv}{dt}\right) dt + \int_0^T (u,\ L^* v)dt = \int_0^T (f, v)dt + (u_0\ ,\ v(0))$$
$$\forall v \in D(L^*);\ \frac{dv}{dt} \in V^*\ ,\ v(T) = 0\ .$$

Obviously (31) $\Longrightarrow$ (32); so we have only to prove that (32) $\Longrightarrow$ (31). Let $\overline{u} \in D(M)$ be such that $\overline{u}(0) = u_0$ ($\overline{u}$ exists since $\gamma(D(M)) = \mathcal{K}$). We have

$$-\int_0^T \left(\overline{u}\ ,\ \frac{dv}{dt}\right) dt + \int_0^T (\overline{u},\ L^* v)dt = \int_0^T (g,v)dt + (u_0\ , v(0))$$
$$\forall v \in D(L^*)\ ;\ \frac{dv}{dt} \in V^*\ ,\ v(T) = 0\ .$$

where $g = M\overline{u}$.

Hence $u-\overline{u}$ satisfies (29); by Lemma 27 $u-\overline{u} \in D(M)$ with $M(u-\overline{u}) = f-g$ and $\gamma(u-\overline{u}) = 0$.

<u>Remark 6.</u> Let $\mathcal{M}$ be a singlevalued densely defined monotone operator from V into V^* (not necessarily maximal) and let B be a bounded operator from V into V^* satisfying (27) and

(33) for every sequence u_n in V for which $u_n \rightharpoonup u$ in V, $\frac{du_n}{dt}$ is bounded

in $D(\mathcal{M})^*$ and $\limsup \int_0^T (Bu_n, u_n - u)dt \leq 0$, we have also $Bu_n \rightharpoonup Bu$ in V^* and $\int_0^T (Bu_n, u_n)dt \to \int_0^T (Bu, u)dt$.

Then for every $f \in V^*$ and every $u_0 \in \mathcal{H}$ there exists $u \in V \cap C([0,T], \mathcal{H})$ such that

$$-\int_0^T \left(u, \frac{dv}{dt}\right) + \int_0^T (u, \mathcal{M}v)dt + \int_0^T (Bu, v)dt = \int_0^T (f, v)dt + (u_0, v(0))$$

$$\forall v \in D(M); \frac{dv}{dt} \in V^*, \ v(T) = 0 .$$

Proof. Let $\tilde{\mathcal{M}}$ be a linear maximal monotone extension of $\mathcal{M}$ and apply Theorem 26 with $\mathcal{L} = (\tilde{\mathcal{M}})^*$.

Remark 7. Theorem 26 generalized Theorem 3.3.1. in Strauss [18]. Assumptions (4), (5), (6) of [18] (p. 76) imply that A is maximal monotone from X into X^*. Actually one should replace $V = L^p(0, T, \mathcal{V})$ by a reflexive Banach space of functions with values in $\mathcal{V}$ which is invariant by translations and multiplication by L^∞ scalar functions.

Application 2. A Navier Stokes type equation.

Let Ω be a bounded domain in $\mathbb{R}^N$. For $u = \{u_1, u_2 \cdots u_N\}$ we set

$$|\nabla u|^2 = \sum_{i,j=1}^{N} \left| \frac{\partial u_i}{\partial x_j} \right|^2$$

and

$$Bu = -\nu \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{N} u_i \frac{\partial u}{\partial x_i}$$

Let $\mathcal{V}$ be the closure in $[W^{1,p}(\Omega)]^N$ of $\{ u \in \mathcal{D}(\Omega); \operatorname{div} u = 0 \}$. Let $V = L^p(0, T; \mathcal{V})$, and let φ be a convex lower semi-continuous function

from V into $(-\infty, +\infty]$ such that $\varphi(0) < +\infty$ and φ is continuous at 0 .

Theorem 28. Assume $p \geqq 1 + \frac{2N}{N+2}$. For every $f \in V^*$ there exists a $u \in V$ with $\frac{du}{dt} \in V^*$ and $u(0) = 0$ satisfying

$$\int_0^T \left(\frac{du}{dt} + Bu,\ v-u \right) dt + \varphi(v) - \varphi(u) \geqq \int_0^T (f,\ v-u)\,dt \qquad \forall v \in V .$$

Proof. We apply Theorem 23 with

$$A = \frac{d}{dt}\ ,\ D(A) = \{\ u \in V\ ;\ \frac{du}{dt} \in V^*\ ,\ u(0) = 0\ \} \text{ and } T = \partial\varphi .$$

Using the assumption $p \geqq 1 + \frac{2N}{N+2}$ one sees easily that B maps D(A) into V^* and satisfies (2) and (3) (c.f. Remark 2.3 in [14], p. 325). Also we have

$$\int_0^T \left(\frac{du}{dt} + Bu\ ,\ u \right) dt \geqq \nu \|u\|_V^p \qquad \forall u \in D(A)$$

which implies both (21) and (22).

Application 3: A mixed boundary value problem.

Let Ω be a bounded domain in $\mathbb{R}^N$ and let $\Sigma = \partial\Omega \times]0, T[$. Let $\Sigma_0 \subset \Sigma$. The equation

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + |u|^{p-2} u = f & \text{in} \quad \Omega \times]0, T[\\ u(x,t) = 0 & \text{on} \quad \Sigma_0 \\ \frac{\partial u}{\partial n}(x,t) = 0 & \text{on} \quad \Sigma - \Sigma_0 \\ u(x,0) = 0 & \text{on} \quad \Omega \end{cases}$$

has a weak solution in the following sense:

$$u \in L^2(0,T;\ H^1(\Omega)) \cap L^p(0,T;\ L^p(\Omega))\ ,\ u = 0 \text{ on } \Sigma_0\ ,$$

$$-\int_0^T \int_\Omega u \frac{\partial v}{\partial t}\, dxdt + \sum_{i=1}^N \int_0^T \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dxdt + \int_0^T \int_\Omega |u|^{p-2} u\, v\, dxdt$$

$$= \int_0^T \int_\Omega f\, v\, dx$$

$$\forall v \in L^2(0,T;H^1(\Omega)) \cap L^p(0,T;L^p(\Omega)); \frac{\partial v}{\partial t} \in L^2(0,T;L^2(\Omega))$$

$$v(x,t) = 0 \quad \text{on} \quad \Sigma_0 \quad \text{and} \quad v(x,T) = 0 \quad \text{on} \quad \Omega .$$

We use here Corollary 21 with

$$V = L^2(0,T;H^1(\Omega)) \cap L^p(0,T;L^p(\Omega)); \; v(x,t) = 0 \quad \text{on} \quad \Sigma_0$$

$$Lv = -\frac{\partial v}{\partial t} \quad \text{with } D(L) = \{ v \in V; \frac{\partial v}{\partial t} \in L^2(0,T;L^2(\Omega)), v(x,T) = 0 \quad \text{on } \Omega$$

We mention briefly some other examples which satisfy the assumptions of Corollary 18 (see [2]).

Consider the non-linear hyperbolic equation

$$Au + Bu = f$$

where $Au = \sum_{i=1}^{N} L^i(x) \frac{\partial u}{\partial x_i}$ is a first order hyperbolic system in the sense of Friedrichs, Lax, Phillips. Under appropriate boundary conditions A defines a maximal monotone operator from $[L^p(\Omega)]^m$ into $[L^{p'}(\Omega)]^m$.

One can also treat non-linear perturbations of Schrodinger equation

$$\frac{\partial u}{\partial t} + i\,\Delta u + Bu = f$$

or nonlinear perturbations of degenerate parabolic equations:

$$x\frac{\partial u}{\partial t} + Bu = f \qquad -1 < x < +1, \quad 0 < t < T$$

$$u(x,t) = 0 \qquad x = \pm 1, \quad 0 < t < T$$

$$u(x,0) = 0 \qquad 0 < x < 1$$

$$u(x,T) = 0 \qquad -1 < x < 0$$

where B is a pseudo-monotone operator from $W_0^{m,p}(-1,+1)$ into $W^{-m,p'}(-1,+1)$.

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