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Monotonicity Methods in Hilbert Spaces and Some Applications to Nonlinear Partial Differential Equations

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We recall first some classical properties of maximal monotone operators in Hilbert spaces. In doing so we concentrate on a particular class of monotone operators, namely those which are gradients of convex functions. We emphasize their specific properties which do not hold for general monotone operators.

Next we consider evolution equations associated with gradients of convex functions: smoothing effect on the initial data, behavior at infinity, etc. . We mention some applications to non linear partial differential equations and also several open problems.

I. Maximal monotone operators in Hilbert spaces.

Let H be a real Hilbert space. Let A be a mapping from H into H which could eventually be multivalued, i.e., to every $u \in H$ we associate a subset $Au \subset H$ (which may be empty). We set $D(A) = \{u \in H;\ Au \neq \phi\}$, $R(A) = \bigcup_{u \in H} Au$, $(Au)^{-1} = \{f \in H; u \in Af\}$, $(\lambda A)(u) = \{\lambda f;\ f \in Au\}$, $(A_1 + A_2)(u) = \{f_1 + f_2;\ f_1 \in A_1 u,\ f_2 \in A_2 u\}$. One says that A is a <u>monotone operator</u> (or a monotone graph) if it satisfies

$$(1) \qquad (f_1 - f_2,\ u_1 - u_2) \geq 0 \quad \forall u_1, u_2 \in D(A),\ \forall f_1 \in Au_1,\ \forall f_2 \in Au_2 .$$

The following property is clearly equivalent to (1)

$$(2) \qquad |(u_1 + \lambda f_1) - (u_2 + \lambda f_2)| \geq |u_1 - u_2|$$

$$\forall \lambda \geq 0 \ , \ \forall u_1, u_2 \in D(A), \ \forall f_1 \in Au_1, \ \forall f_2 \in Au_2 \ .$$

Inequality (2) just says that $(I + \lambda A)^{-1}$, wherever defined, is a <u>contraction</u> in H .

Definition (1) involves only the notion of scalar product and can be extended to mappings which map a Banach space into its dual space. While property (2) makes sense for mappings which map a Banach space into itself; these mappings are usually called accretive. For simplicity we restrict ourselves to the case of a Hilbert space, but some of the results we are going to discuss could be extended in one or two ways.

Using a well known fixed point theorem for contractions, E. H. Zarantonello, in his pioneering paper [32] proved the following

<u>Theorem 1. Assume</u> A <u>is monotone, single-valued, Lipschitz continuous with</u> $D(A) = H$. <u>Then</u>

$$(3) \qquad R(I + A) = H \ .$$

Slightly later, Theorem 1 was extended by F. Browder [10] and G. Minty [24] who showed that (3) holds true if one replaces the Lipschitz by a continuity assumption. Introducing the concept of maximal monotone operators, Minty [24] was able to give a complete characterization of monotone operators satisfying (3).

One says that a monotone operator A is <u>maximal monotone</u> if it is maximal in the sense of inclusion of graphs, i.e., it admits no proper monotone extension.

<u>Theorem 2. Let</u> A <u>be monotone. Then</u> A <u>is maximal monotone if and only if</u> $R(I + A) = H$ (<u>resp.</u> $R(I + \lambda A) = H$ <u>for every</u> $\lambda > 0$).

It is easy to check that a monotone, singlevalued, continuous, everywhere defined operator, is maximal monotone, and so Theorem 2 implies Theorem 1 .

An important class of monotone operators consists of gradients of convex functions. More precisely, let φ be a convex lower semicontinuous (l. s. c.) function from H into $(-\infty, +\infty]$. We assume $\varphi \not\equiv +\infty$, and let

$$D(\varphi) = \{u \in H \; ; \; \varphi(u) < +\infty\} \; .$$

For $u \in D(\varphi)$, the set

$$\partial\varphi(u) = \{f \in H; \; \varphi(v) - \varphi(u) \geq (f, v-u) \qquad \forall\, v \in D(\varphi)\}$$

is called the <u>subdifferential</u> of φ at u . Note that $\partial\varphi(u)$ is closed and convex, and may be empty; however if φ is Gateaux differentiable at u , then $\partial\varphi(u)$ is reduced to a single point and coincides with the Gateaux differential.

<u>Theorem 3</u> (Minty [25]). <u>The operator</u> $u \mapsto \partial\varphi(u)$ <u>is maximal monotone.</u>

Since the monotonicity of $\partial\varphi$ is immediate, it is sufficient to show that for every $f \in H$, equation

$$u + \partial\varphi(u) \ni f \tag{4}$$

has a solution. One can check easily that u satisfies (4) if and only if the convex function $\psi(v) = \frac{1}{2}\,|v - f|^2 + \varphi(v)$ achieves its minimum at u . But the function ψ is convex l. s. c. and tends to $+\infty$ as $|v| \to +\infty$, thus its minimum is attained.

Maximal monotone operators have simple convexity and topological properties (see [6], [13]). Let A be maximal monotone, then

$$\overline{D(A)} \text{ is convex} \tag{5}$$

(6) for every $u \in D(A)$, Au is closed and convex; so it has a unique element of least norm which we denote by A^0u.

(7) let $f_n \in Au_n$ such that $u_n \to u$ weakly, $f_n \to f$ weakly and $\limsup (f_n, u_n) \leq (f, u)$; then $f \in Au$ and $(f_n, u_n) \to (f, u)$.

In the case where $A = \partial\varphi$, one has

$$D(A) \subset D(\varphi) \subset \overline{D(\varphi)} = \overline{D(A)}$$

and in general those 3 sets are distinct (for a simple proof of $\overline{D(\varphi)} = \overline{D(A)}$ see [4] Remark 4).

The Yosida approximation provides a convenient way of approximating maximal monotone operators by monotone operators which are Lipschitz continuous. Let A be maximal monotone; for $\lambda > 0$, $J_\lambda = (I + \lambda A)^{-1}$ is a contraction defined on all of H; it is called the resolvent of A. The Yosida approximation of A is defined by

$$A_\lambda u = \frac{u - J_\lambda u}{\lambda}$$

(note that $A_\lambda u \in AJ_\lambda u$). A_λ is monotone (everywhere defined) Lipschitz continuous (with Lipschitz constant $1/\lambda$). Also for every $u \in H$, $J_\lambda u \to \mathrm{Proj}_{\overline{D(A)}}\, u$ as $\lambda \to 0$, and for $u \in D(A)$, $|A_\lambda u| \leq |A^0 u|$ with $A_\lambda u \to A^0 u$ as $\lambda \to 0$ (see [13]).

It is of interest to notice that if A is the subdifferential of a convex function, then its Yosida approximation remains in the same class. More precisely

Theorem 4. Let $A = \partial\varphi$; the function φ_λ defined by

$$\varphi_\lambda(u) = \operatorname*{Min}_{v \in H} \left\{ \frac{1}{2\lambda} |u-v|^2 + \varphi(v) \right\} = \frac{\lambda}{2} |A_\lambda u|^2 + \varphi(J_\lambda u)$$

is convex, Frechet differentiable and $\partial\varphi_\lambda = A_\lambda$. In addition $\varphi_\lambda(u) \uparrow \varphi(u)$ as $\lambda \downarrow 0$.

The convexity and differentiability of φ_λ were proved by Moreau [26] who made extensive use of the notion of inf-convolution. For $u \in \overline{D(A)}$ we have

$$\varphi(J_\lambda u) \leq \varphi_\lambda(u) \leq \varphi(u)$$

and since $J_\lambda u \to u$, we get $\varphi_\lambda(u) \to \varphi(u)$. If $u \notin \overline{D(A)}$, $\varphi_\lambda(u) \geq \frac{1}{2\lambda} |(u - J_\lambda u)|^2 - c_1 |J_\lambda u| - c_2$ (by Hahn-Banach φ is bounded from below by an affine function), so that $\varphi_\lambda(u) \to +\infty$ as $\lambda \to 0$.

One of the main purposes of the theory of monotone operators is to obtain surjectivity results. The following theorem provides a very simple and useful sufficient condition.

Theorem 5. Let A be maximal monotone. Assume

$$(8) \qquad \lim_{\substack{|u| \to +\infty \\ u \in D(A)}} |A^0 u| = +\infty \qquad (\text{i.e. } A^{-1} \text{ is bounded}^\dagger)$$

Then $R(A) = H$.

Note that in the case where $A = \partial\varphi$, then $R(A) = H$ if and only if $\lim_{|u| \to +\infty} \{\varphi(u) - (f, u)\} = +\infty$ for every $f \in H$. Actually, if one replaces in Theorem 5 the assumption (8) by "A^{-1} is locally bounded" then we get a condition which is both necessary and sufficient for surjectivity; this result, proved independently by F. Browder [11] and Rockafellar [28] is based on the following:

Theorem 6. Let B be a monotone operator; then B is locally bounded at every point of $\operatorname{Int} D(B)$.

Using the terminology of partial differential equations,

† One says that B is bounded if for every bounded set $N \subset H$, $\bigcup_{u \in N} Bu$ is bounded.

Theorem 5 asserts that if A is maximal monotone and if one knows a priori estimates for possible solutions of the equation $Au \ni f$ with a bound of the form $|u| \leq \omega(|f|)$ (ω continuous) then $R(A) = H$. In many applications such a bound is provided by a coerciveness assumption:

(9) there exists $u_0 \in H$ such that

$$\lim_{\substack{|u| \to +\infty \\ u \in D(A)}} \frac{(A^0 u, u-u_0)}{|u|} = +\infty .$$

Surprisingly it turns out that in case $A = \partial\varphi$, property (9) is also a necessary condition.

Theorem 7. Assume $A = \partial\varphi$. The following are equivalent

(10) for every $u_0 \in D(\varphi)$, $\displaystyle\lim_{\substack{|u| \to +\infty \\ u \in D(A)}} \frac{(A^0 u, u-u_0)}{|u|} = +\infty$.

(11) there exists $u_0 \in H$ such that $\displaystyle\lim_{\substack{|u| \to +\infty \\ u \in D(A)}} \frac{(A^0 u, u-u_0)}{|u|} = +\infty$.

(12) $\displaystyle\lim_{\substack{|u| \to +\infty \\ u \in D(A)}} |A^0 u| = +\infty$.

(13) $R(A) = H$ and A^{-1} is bounded.

(14) $\displaystyle\lim_{|u| \to +\infty} \frac{\varphi(u)}{|u|} = +\infty$.

Remark. It is clear that Theorem 7 does not hold for general maximal monotone operators; for example a rotation by $\pi/2$ in $H = \mathbb{R}^2$ satisfies (12) and not (11).

Proof of Theorem 7. (10) $\Rightarrow$ (11) $\Rightarrow$ (12) $\Rightarrow$ (13) are immediate. To show that (13) $\Rightarrow$ (14), we can always reduce to the case where $\varphi \geq 0$ since φ is bounded from below by an affine function (this amounts to shift A by a constant). Let $r > 0$ be fixed. For every $z \in H$ with $|z| \leq r$ there exists (by (13)) $v \in D(A)$ such that $Av \ni z$ and $|v| \leq M$. Thus

$$\varphi(u) - \varphi(v) \geq (z, u-v) \quad \forall\, u \in D(\varphi)$$

and thus $(z,u) \leq \varphi(u) + Mr$, $\forall\, u \in D(\varphi)$, $\forall\, z \in H$; $|z| \leq r$.
Consequently, $r|u| \leq \varphi(u) + Mr$ and $\dfrac{\varphi(u)}{|u|} \geq r - \dfrac{Mr}{|u|}$.
Hence $\displaystyle\liminf_{|u| \to +\infty} \frac{\varphi(u)}{|u|} \geq r$.

Finally (14) $\Rightarrow$ (10) since we have

$$\varphi(u_0) - \varphi(u) \geq (A^0 u, u_0 - u)$$

and

$$\frac{(A^0 u, u-u_0)}{|u|} \geq \frac{\varphi(u) - \varphi(u_0)}{|u|} \to +\infty \quad \text{as} \quad |u| \to +\infty .$$

Another important topic in the theory of monotone operators concerns the sum of maximal monotone operators. If A and B are maximal monotone, $A + B$ need not be maximal monotone. So it is natural to raise the question: when is $A + B$ maximal monotone? There is no general and convenient answer to this question. However the following criteria may be used to prove that $A + B$ is maximal.

Theorem 8. (see [6]) Let $f \in H$; for every $\lambda > 0$ the equation $u_\lambda + A_\lambda u_\lambda + Bu_\lambda \ni f$ has a unique solution and $f \in R(I + A + B)$ if and only if $A_\lambda u_\lambda$ is bounded as $\lambda \to 0$. In this case $u_\lambda \to u$ as $\lambda \to 0$ and $f \in u + Au + Bu$.

Sufficient conditions have been given by Rockafellar [29] and Crandall-Pazy [13]:

if $(\mathrm{Int}(D(A)) \cap D(B) \neq \phi$, then $A + B$ is maximal monotone

if B is <u>dominated</u> by A, i.e., $D(A) \subset D(B)$ and $|B^0 u| \leq k|A^0 u| + \omega(|u|)$ for all $u \in D(A)$, where $k < 1$ and ω is continuous, then $A + B$ is maximal monotone.

One may also ask a more restricted question: when is $A + \partial\varphi$ maximal monotone ? Again, this is an open problem, however the following sufficient condition turns out to be quite useful in applications.

<u>Theorem 9.</u> <u>Let</u> A <u>be a maximal monotone and let</u> φ <u>be a convex l. s. c. function from</u> H <u>into</u> $(-\infty, +\infty]$, $\varphi \not\equiv +\infty$. <u>Assume there exists</u> C <u>such that</u>

$$\varphi((I + \lambda A)^{-1})u) \leq \varphi(u) + C\lambda, \quad \forall \lambda > 0, \ \forall u \in D(\varphi). \tag{15}$$

<u>Then</u> $A + \partial\varphi$ <u>is maximal monotone.</u> <u>In addition</u>

$$|A^0 u| \leq |(A + \partial\varphi)^0 u| + \sqrt{C} \quad \forall u \in D(A) \cap D(\partial\varphi). \tag{16}$$

<u>Proof of Theorem 9.</u> We apply Theorem 8 with $B = \partial\varphi$. Let u_λ be the solution of

$$u_\lambda + A_\lambda u_\lambda + \partial\varphi(u_\lambda) \ni f$$

i.e.,

$$\varphi(v) - \varphi(u_\lambda) \geq (f - u_\lambda - A_\lambda u_\lambda, v - u_\lambda) \qquad \forall v \in D(\varphi).$$

Taking $v = (I + \lambda A)^{-1} u_\lambda$ we get

$$C\lambda \geq (f - u_\lambda - A_\lambda u_\lambda, -\lambda A_\lambda u_\lambda)$$

so that $|A_\lambda u_\lambda|^2 \leq |f - u_\lambda|\,|A_\lambda u_\lambda| + C$ and $|A_\lambda u_\lambda| \leq |f - u_\lambda| + \sqrt{C}$. Let $v_0 \in D(A) \cap D(\varphi)$ (such v_0 exists by (15)); by the monotonicity of A_λ we have

$$\varphi(v_0) - \varphi(u_\lambda) \geq (f - u_\lambda - A_\lambda v_0, v_0 - u_\lambda) .$$

Since $\varphi(u_\lambda) \geq -C_1|u_\lambda| - C_2$ and $|A_\lambda v_0|$ is bounded, we conclude that $|u_\lambda|$ is bounded as $\lambda \to 0$. Hence $A + \partial\varphi$ is maximal monotone. For $u \in D(\partial\varphi)$ and $z \in \partial\varphi(u)$ we have

$$\varphi(J_\lambda u) - \varphi(u) \geq (z, J_\lambda u - u)$$

and thus $\lambda C \geq (z, -\lambda A_\lambda u)$. Consequently, if $u \in D(A) \cap D(\partial\varphi)$ we have $(A^0 u, z) \geq -C$, for all $z \in \partial\varphi(u)$.
Let $f = (A + \partial\varphi)^0 u$, so that $f = y + z$ with $y \in Au$ and $z \in \partial\varphi(u)$. We have $(A^0 u, f) = (A^0 u, y) + (A^0 u, z) \geq |A^0 u|^2 - C$.
Hence

$$|A^0 u| \leq |f| + \sqrt{C} .$$

Assumption (15) is convenient because it is preserved under sum of maximal monotone operators. More precisely

<u>Theorem 10.</u> <u>Let</u> A^1 <u>and</u> A^2 <u>be maximal monotone with</u> $A^1 + A^2$ <u>maximal monotone.</u> <u>Assume</u>

$$\varphi(J^1_\lambda u) \leq \varphi(u) + C_1\lambda \quad \forall\, \lambda > 0 , \quad \forall\, u \in D(\varphi)$$

$$\varphi(J^2_\lambda u) \leq \varphi(u) + C_2\lambda \quad \forall\, \lambda > 0 , \quad \forall\, u \in D(\varphi) .$$

<u>Then</u>

$$\varphi(J_\lambda u) \leq \varphi(u) + (C_1 + C_2)\lambda \quad \forall\, \lambda > 0 , \quad \forall\, u \in D(\varphi) \tag{17}$$

<u>where</u> $J^1_\lambda = (I + \lambda A^1)^{-1}$, $J^2_\lambda = (I + \lambda A^2)^{-1}$, $J_\lambda = (I + \lambda(A^1 + A^2))^{-1}$.
<u>In particular,</u> $A^1 + A^2 + \partial\varphi$ <u>is maximal monotone.</u>

Proof of Theorem 10. Let $\alpha > 0$ and let u_α be the solution of $u \in u_\alpha + \lambda A^1 u_\alpha + \lambda A^2_\alpha u_\alpha$. By Theorem 8 we know that $u_\alpha \to J_\lambda u$ as $\alpha \to 0$. We have $\alpha u \in \alpha u_\alpha + \alpha\lambda A^1 u_\alpha +$

$\lambda(u_\alpha - J^2_\alpha u_\alpha)$, or $\dfrac{\alpha u + \lambda J^2_\alpha u_\alpha}{\alpha+\lambda} \in u_\alpha + \dfrac{\alpha\lambda}{\alpha+\lambda} A^1 u_\alpha$. Thus

$$u_\alpha = J^1_{\alpha\lambda/(\alpha+\lambda)} \left(\frac{\alpha u + \lambda J^2_\alpha u_\alpha}{\alpha + \lambda} \right)$$

For a fixed u, the mapping $z \mapsto J^1_{\alpha\lambda/(\alpha+\lambda)} \left(\dfrac{\alpha u + \lambda J^2_\alpha z}{\alpha + \lambda} \right)$

maps the closed convex set

$$K = \{x \in H \; ; \varphi(x) \leq \varphi(u) + \lambda(C_1 + C_2)\}$$

into itself, and it is a strict contraction. Thus, its fixed point u_α belongs to K. Consequently, $\varphi(u_\alpha) \leq \varphi(u) + \lambda(C_1 + C_2)$; passing to the limit as $\alpha \to 0$, we get (17).

Remark. Let K be a closed convex subset of H and let $\varphi = I_K$ be the indicator function of K, i.e.,

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K \\ +\infty & \text{if } u \notin K \end{cases}$$

Assumption (15) just asserts that $(I + \lambda A)^{-1} K \subset K$, for all $\lambda > 0$ (for this case see also Brezis-Pazy [7] Theorems 2.2 and 4.2).

Theorem 11. Under the assumptions of Theorem 9 we have

$$\overline{D(A + \partial\varphi)} = \overline{D(A) \cap D(\partial\varphi)} = \overline{D(A)} \cap \overline{D(\varphi)}$$

Proof. Clearly $\overline{D(A) \cap D(\partial\varphi)} \subset \overline{D(A)} \cap \overline{D(\varphi)}$. Conversely, we first show that $\overline{D(A)} \cap \overline{D(\varphi)} \subset \overline{D(A) \cap D(\varphi)}$. Indeed let $u \in \overline{D(A)} \cap \overline{D(\varphi)}$ and let $v_\varepsilon \in D(\varphi)$ be such that $v_\varepsilon \to u$ in H. Let $u_\varepsilon = J_\varepsilon v_\varepsilon = (I + \varepsilon A)^{-1} v_\varepsilon$. By (15), $u_\varepsilon \in D(A) \cap D(\varphi)$ and

$$|u_\varepsilon - u| \leq |u_\varepsilon - (I + \varepsilon A)^{-1} u| + |(I + \varepsilon A)^{-1} u - u|$$

$$\leq |v_\varepsilon - u| \leq |(I + \varepsilon A)^{-1} u - u| .$$

Thus $u_\varepsilon \to u$ as $\varepsilon \to 0$.

We prove now that $D(A) \cap D(\varphi) \subset \overline{D(A) \cap D(\partial\varphi)}$. Let $u \in D(A) \cap D(\varphi)$ and let $u_\varepsilon \in D(A) \cap D(\partial\varphi)$ be the solution of

$$u_\varepsilon + \varepsilon (Au_\varepsilon + \partial\varphi(u_\varepsilon)) \ni u$$

(which exists by Theorem 9). We have

$$\varphi(u) - \varphi(u_\varepsilon) \geq \left(\frac{u - u_\varepsilon}{\varepsilon} - Au_\varepsilon , u - u_\varepsilon\right) \geq \frac{1}{\varepsilon} |u - u_\varepsilon|^2 - (A^0 u, u - u_\varepsilon).$$

Hence $u_\varepsilon \to u$ as $\varepsilon \to 0$.

II. Some examples of maximal monotone operators.

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary Γ.

Example 1. Let $H = L^2(\Omega)$ and let j be a convex l. s. c. function from $\mathbb{R}$ into $(-\infty, +\infty]$; assume $j \not\equiv +\infty$ and let $\beta = \partial j$. We define for $u \in H$

$$\varphi(u) = \begin{cases} \frac{1}{2} \int_\Omega |\operatorname{grad} u|^2 dx + \int_\Gamma j(u) d\Gamma & \text{if } u \in H^1(\Omega)^\dagger \text{ and } j(u) \in L^1(\Gamma) \\ +\infty & \text{otherwise .} \end{cases}$$

$^\dagger H^k$ and H_0^k denote the usual Sobolev spaces

It is easy to check that φ is convex l.s.c. on H (note that if $u_n \to u$ in $L^2(\Omega)$ and $\varphi(u_n) \leq \lambda$, then u_n is bounded in $H^1(\Omega)$, thus $u_n \to u$ weakly in $H^1(\Omega)$ and by Fatou's lemma, $\liminf \int_\Gamma j(u_n)d\Gamma \geq \int_\Gamma j(u)d\Gamma$).

<u>Theorem 12.</u> <u>We have</u> $\partial\varphi(u) = -\Delta u$ <u>with</u> $D(\partial\varphi) = \{u \in H^2(\Omega);$ $-\frac{\partial u}{\partial n} \in \beta(u)$ a.e. on $\Gamma\}$ <u>where</u> $\partial/\partial n$ <u>is the outward normal derivative.</u> <u>In addition there exist constants</u> c_1, c_2 <u>such that</u> $\|u\|_{H^2} \leq c_1 |-\Delta u + u|_{L^2} + c_2$ <u>for every</u> $u \in D(\partial\varphi)$.

<u>Remark.</u> The precise description of the domain of a subdifferential often amounts to proving a regularity theorem for some nonlinear elliptic equation, and usually it is not an easy matter. Note that $\partial\varphi$ is "apparently" a linear operator but it has to be considered as a nonlinear operator because its domain is not a linear subspace.

<u>Proof of Theorem 12.</u> Let $Au = -\Delta u$ with $D(A) = \{u \in H^2(\Omega);$ $-\frac{\partial u}{\partial n} \in \beta(u)$ a.e. on $\Gamma\}$. We are going to prove that A is maximal monotone and $A \subset \partial\varphi$. It is clear that A is monotone and also for $u \in D(A)$ and $v \in D(\varphi)$

$$\int_\Omega -\Delta u \cdot (v-u)dx = \int_\Omega \text{grad } u(\text{grad } v - \text{grad } u)dx - \int_\Gamma \frac{\partial u}{\partial n}(v-u)d\Gamma$$

$$\leq \varphi(v) - \varphi(u) .$$

Thus $A \subset \partial\varphi$, and it remains to show that $R(I + A) = H$. This is proved in [5] under slightly more general assumptions, but for completeness we sketch here the proof. Let $f \in H$ be given and let $u_\lambda \in H^2(\Omega)$, $\lambda > 0$, be the solution of

$$(18) \qquad \begin{cases} -\Delta u_\lambda + u_\lambda = f & \text{on } \Omega \\ -\dfrac{\partial u_\lambda}{\partial n} = \beta_\lambda(u_\lambda) & \text{on } \Gamma \end{cases} .$$

The existence of u_λ can be established for example by a fixed point argument for the map $u \in L^2(\Gamma) \mapsto Tu \in L^2(\Gamma)$ where $Tu = v|_\Gamma$ and v is the solution of the linear equation

$$-\Delta v + v = f \text{ on } \Omega, \quad v + \lambda \frac{\partial v}{\partial n} = (I + \lambda\beta)^{-1} u \text{ on } \Gamma,$$

(note that T is a strict contraction in $L^2(\Gamma)$).

We show that $\|u_\lambda\|_{H^2} \le c_1 |f|_{L^2} + c_2$ where c_1 and c_2 are independent of λ.

Multiplying equation (18) by $u_\lambda - v_0$ where $v_0 \in D(\beta)$ we get easily $\|u_\lambda\|_{H^1} \le c_1 |f|_{L^2} + c_2$. Interior estimates in H^2 are immediate and we have $\|u_\lambda\|_{H^2(\Omega')} \le c(|f|_{L^2(\Omega)} + \|u_\lambda\|_{H^1(\Omega)})$ for $\overline{\Omega'} \subset \Omega$.

Next we obtain estimates near the boundary by using local charts and "tangential shifts" as in the linear theory. Assume that the equation†

$$(19) \qquad -\sum_{k,\ell=1}^{N} \frac{\partial}{\partial x_\ell}\left(a_{k\ell}(x) \frac{\partial u}{\partial x_k}\right) + u = f$$

holds in $\Omega_R = \{x = (x', x_N) ; |x'| < R \text{ and } 0 < x_N < R\}$ where $x' = (x_1, x_2, \ldots, x_{N-1})$ and the equation

$$(20) \qquad -\frac{\partial u}{\partial \nu} = \beta(u)$$

holds in $\Gamma_R = \{x = (x', x_N) ; |x'| < R \text{ and } x_N = 0\}$ where $\frac{\partial u}{\partial \nu} = -\sum_{k=1}^{N} a_{kN} \frac{\partial u}{\partial x_k}$ and $a_{k\ell}(x)$ are smooth with $\sum_{k,\ell=1}^{N} a_{k\ell}\xi_k\xi_\ell \ge a|\xi|^2$ for all $\xi \in \mathbb{R}^N$, $a > 0$.

Let $\eta(x)$ be a smooth function with $\eta = 1$ on a neighborhood of $\Omega_{R/2}$ and $\eta = 0$ for $|x'| \ge R$ or $x_N \ge R$.

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†for simplicity we have dropped λ.

Multiplying equation (19) by $-\sum_{i=1}^{N-1} \frac{\partial}{\partial x_i}(\eta^2 \frac{\partial u}{\partial x_i})$ and integrating by parts we get

$$\sum_{i=1}^{N-1} \int_{\Gamma_R} \beta'(u)\eta^2 d\Gamma + \sum_{i=1}^{N-1} \sum_{k,\ell=1}^{N} \int_{\Omega_R} \frac{\partial}{\partial x_i}(a_{k\ell} \frac{\partial u}{\partial x_k}) \frac{\partial}{\partial x_\ell}(\eta^2 \frac{\partial u}{\partial x_i}) dx$$

$$= -\sum_{i=1}^{N-1} \int_{\Omega_R} f \cdot \frac{\partial}{\partial x_i}(\eta^2 \frac{\partial u}{\partial x_i})\, dx .$$

From the monotonicity of β we deduce after some rearrangements that

$$\sum_{i=1}^{N-1} \sum_{k=1}^{N} \| \eta \frac{\partial^2 u}{\partial x_i \partial x_k} \|^2_{L^2(\Omega_R)} \leq C(\|f\|_{L^2(\Omega_R)} + \|u\|_{H^1(\Omega_R)})$$

$$(\|u\|_{H^1(\Omega_R)} + \sum_{i=1}^{N-1} \sum_{k=1}^{N} \| \eta \frac{\partial^2 u}{\partial x_i \partial x_k} \|_{L^2(\Omega_R)}) .$$

Consequently

$$\sum_{i=1}^{N-1} \sum_{k=1}^{N} \| \eta \frac{\partial^2 u}{\partial x_i \partial x_k} \|_{L^2(\Omega_R)} \leq C(\|f\|_{L^2(\Omega_R)} + \|u\|_{H^1(\Omega_R)}).$$

Using the equation (19) to estimate $\partial^2 u/\partial x_N^2$ we obtain finally

$$\|u\|_{H^2(\Omega_{R/2})} \leq C(\|f\|_{L^2(\Omega_R)} + \|u\|_{H^1(\Omega_R)}) .$$

Going back to (18) we conclude that $\|u_\lambda\|_{H^2(\Omega)}$ is bounded as $\lambda \to 0$. Hence $u_{\lambda_n} \to u$ weakly in $H^2(\Omega)$, $u_{\lambda_n} \to u$ in $L^2(\Gamma)$, $(\partial u_{\lambda_n})/\partial n \to \partial u/\partial n$ in $L^2(\Gamma)$.

Also $\|u_\lambda - (I + \lambda\beta)^{-1}u_\lambda\|_{L^2(\Gamma)} \leq \lambda \|\frac{\partial u_\lambda}{\partial n}\|_{L^2(\Gamma)} \to 0$

and thus $(I + \lambda_n\beta)^{-1}u_{\lambda_n} \to u$ in $L^2(\Gamma)$.

Finally, we have $-\Delta u + u = f$ and since $-\frac{\partial u_\lambda}{\partial n} \in \beta((I + \beta)^{-1}u)$ a.e. on Γ, we have $-\frac{\partial u}{\partial n} \in \beta(u)$ a.e. on Γ.

Let now k be another convex l.s.c. function from $\mathbb{R}$ into $(-\infty, +\infty]$ and let $\gamma = \partial k$. The function ψ defined on $L^2(\Omega)$ by

$$\psi(u) = \begin{cases} \int_\Omega k(u)\,dx & \text{if } k(u) \in L^1(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

is convex l.s.c. and it is easy to check that $f \in \partial\psi$ if and only if $f(x) \in \gamma(u(x))$ a.e. on Ω. Also $\overline{D(\partial\psi)} = \overline{D(\psi)} = \{u \in L^2(\Omega);\ u(x) \in \overline{D(\gamma)}$ a.e. on $\Omega\}$.

<u>Corollary 13.</u> <u>If</u> $D(\beta) \cap D(\gamma) \neq \phi$, <u>then</u> $\partial\varphi + \partial\psi$ <u>is maximal monotone and</u>

$$\overline{D(\partial\varphi) \cap D(\partial\psi)} = \{u \in L^2(\Omega);\ u(x) \in \overline{D(\gamma)} \text{ a.e. on } \Omega\}.$$

<u>In particular, for every</u> $f \in L^2(\Omega)$ <u>there exists a unique function</u> $u \in H^2(\Omega)$ <u>satisfying</u>

$$\begin{cases} -\Delta u + \gamma(u) + u \ni f & \text{a.e. on } \Omega \\ -\frac{\partial u}{\partial n} \in \beta(u) & \text{a.e. on } \Gamma . \end{cases}$$

<u>Proof.</u> After some changes of variable and shifts we can assume that $0 \in \beta(0)$ and $0 \in \gamma(0)$. By Theorem 9 it is sufficient to show that $\varphi((I + \lambda\partial\psi)^{-1}u) \leq \varphi(u)$ for all $u \in D(\varphi)$, and all $\lambda > 0$; but this clearly holds since $(I + \lambda\gamma)^{-1}$ is a monotone contraction in $\mathbb{R}$. From Theorem 11 we deduce that $\overline{D(\partial\varphi + \partial\psi)} = \overline{D(\partial\varphi)} \cap \overline{D(\partial\psi)}$ and since $\overline{D(\partial\varphi)} = L^2(\Omega)$, (note that $\mathcal{D}(\Omega) \subset D(\partial\varphi)$), we conclude that

$$\overline{D(\partial\varphi) \cap D(\partial\psi)} = \overline{D(\partial\varphi)} = \{u \in L^2(\Omega);\ u(x) \in \overline{D(\gamma)} \text{ a.e. on } \Omega\}.$$

Let now K be the closed convex set $K = \{u \in L^2(\Omega)\ ;\ \psi_1 \leq u \leq \psi_2$ a.e. on $\Omega\}$ where $\psi_1, \psi_2 \in H^2(\Omega)$, $\psi_1 \leq \psi_2$ a.e. on Ω. We use the notation

$$\beta^+(r) = \text{Max}\{z\,;\, z \in \beta(r)\},\quad \beta^-(r) = \text{Min}\{z\,;\, z \in \beta(r)\} \text{ if } r \in D(\beta)$$

$$\beta^+(r) = \beta^-(r) = +\infty \quad \text{if } r \notin D(\beta)\ ,\ r \geq \text{Sup } D(\beta)$$

$$\beta^+(r) = \beta^-(r) = -\infty \quad \text{if } r \notin D(\beta)\ ,\ r \leq \text{Inf } D(\beta)\ .$$

<u>Corollary 14.</u> <u>Assume</u>

$$(21)\quad \frac{\partial\psi_1}{\partial n} + \beta^-(\psi_1) \leq 0 \ \underline{\text{and}}\ \frac{\partial\psi_2}{\partial n} + \beta^+(\psi_2) \geq 0 \quad \text{a.e. on } \Gamma\ .$$

<u>Then</u> $\partial\varphi + \partial I_K$ <u>is maximal monotone and</u> $\overline{D(\partial\varphi) \cap K} = K$. <u>In particular for every</u> $f \in L^2(\Omega)$, <u>there exists a unique function</u> $u \in H^2(\Omega) \cap K$ <u>satisfying</u>

$$\int_\Omega (-\Delta u + u)(v-u)\,dx \geq \int_\Omega f\cdot(v-u)dx \quad \underline{\text{for all}}\ v \in K$$

and

$$-\frac{\partial u}{\partial n} \in \beta(u) \quad \text{a.e. } \underline{\text{on}}\ \Gamma\ .$$

<u>Proof.</u> Write K as $K = K_1 \cap K_2$ where $K_1 = \{u \in L^2(\Omega);\ u \geq \psi_1$ a.e. on $\Omega\}$ and $K_2 = \{u \in L^2(\Omega);\ u \leq \psi_2$ a.e. on $\Omega\}$. We first show that $(I + \lambda(\partial\varphi + \Delta\psi_1))^{-1}K_1 \subset K_1$. Indeed let $u \in K_1$ and let $u_\lambda = (I + \lambda(\partial\varphi + \Delta\psi_1))^{-1}u$, i.e., $u_\lambda \in H^2(\Omega)$ satisfies

$$(22)\quad \begin{cases} u_\lambda - \lambda\Delta u_\lambda + \lambda\Delta\psi_1 = u & \text{a.e. on } \Omega\ . \\ -\dfrac{\partial u_\lambda}{\partial n} \in \beta(u_\lambda) & \text{a.e. on } \Gamma\ . \end{cases}$$

Multiplying (22) by $-(u_\lambda - \psi_1)^-$ and integrating by parts we get

$$\int_\Omega |(u_\lambda-\psi_1)^-|^2 dx + \int_\Omega |\mathrm{grad}(u_\lambda-\psi_1)^-|^2 dx + \int_\Gamma \left(\frac{\partial u_\lambda}{\partial n} - \frac{\partial \psi_1}{\partial n}\right)(u_\lambda-\psi_1)^- d\Gamma$$
$$= -\int_\Omega (u - \psi_1)(u_\lambda - \psi_1)^- dx \leq 0 .$$

Also $\left(\frac{\partial u_\lambda}{\partial n} - \frac{\partial \psi_1}{\partial n}\right)(u_\lambda - \psi_1)^- \geq 0$ a.e. on Γ (note that if $u_\lambda < \psi_1$ at some point, let ξ be such that $u_\lambda < \xi < \psi_1$, and then by (21) we have $-\frac{\partial u_\lambda}{\partial n} \leq \beta^+(u_\lambda) \leq \beta^-(\xi) \leq \beta^-(\psi_1) \leq \frac{\partial \psi_1}{\partial n}$). Consequently $(u_\lambda - \psi_1)^- = 0$ and $u_\lambda \in K_1$; similarly $(I + \lambda(\partial\varphi + \Delta\psi_2))^{-1}K_2 \subset K_2$.

We deduce from Theorem 9 that $\partial\varphi + \Delta\psi_1 + \partial I_{K_1}$ is maximal monotone and so is $\partial\varphi + \partial I_{K_1}$. Also, by Theorem 11, we have $\overline{D(\partial\varphi) \cap D(\partial I_{K_1})} = \overline{D(\partial\varphi) \cap K_1} = \overline{D(\partial\varphi)} \cap K_1 = K_1$. Applying Theorem 10 with $A^1 = \partial\varphi + \Delta\psi_2$ and $A^2 = \partial I_{K_1}$ and $\varphi = I_{K_2}$ (note that $(I + \lambda A^2)^{-1}K_2 \subset K_2$ is obviously satisfied since $\mathrm{Proj}_{K_1}(K_2) \subset K_2$) we obtain that $\partial\varphi + \Delta\psi_2 + \partial I_{K_1} + \partial I_{K_2}$ is maximal monotone and so is $\partial\varphi + \partial I_{K_1} + \partial I_{K_2} = \partial\varphi + \partial I_K$. And finally we have $\overline{D(A^1 + A^2)} \cap K_2 = \overline{D(A^1 + A^2) \cap K_2} = \overline{D(\partial\varphi)} \cap K_1 \cap K_2 = K$.

<u>Remark.</u> Corollary 13 and 14 are related to regularity results (proved in [5] and [8]) for unilateral problems.

<u>Example 2.</u> Let $H = L^2(\Omega)$; we define for $u \in H$

$$\varphi(u) = \begin{cases} \frac{1}{2}\int_\Omega |\mathrm{grad}\, u|^2 dx + \int_\Omega |\mathrm{grad}\, u| dx & \text{if } u \in H_0^1(\Omega) \\ +\infty & \text{otherwise} \end{cases} .$$

It is easy to check that φ is convex and l.s.c. on H.

<u>Theorem 15.</u> <u>We have</u> $f \in \partial\varphi(u)$ <u>if and only if</u> $u \in H^2(\Omega) \cap H_0^1(\Omega)$ <u>satisfies the inequality</u>

$$(23) \quad \int_\Omega -\Delta u(v-u)dx + \int_\Omega |\text{grad } v|dx - \int_\Omega |\text{grad } u|dx \geq \int_\Omega f(v-u)dx \quad \forall\, v \in H_0^1(\Omega).$$

<u>In particular, for every</u> $f \in H$, <u>there is a unique solution</u> $u \in H^2(\Omega) \cap H_0^1(\Omega)$ of (23).

The proof of Theorem 15 is based on the following

<u>Lemma 1.</u> Let $u \in H_0^1(\Omega)$ and let u_ε be the solution of

$$\begin{cases} u_\varepsilon - \varepsilon\,\Delta u_\varepsilon = u & \text{on } \Omega \\ u_\varepsilon = 0 & \text{on } \Gamma. \end{cases}$$

There is a constant C, depending only on Ω, such that

$$(24) \qquad \int_\Omega |\text{grad } u_\varepsilon|dx \leq \int_\Omega |\text{grad } u|dx + \varepsilon\, C \|u_\varepsilon\|_{H^2(\Omega)}$$

In particular if Ω is convex, one can take $C = 0$.

<u>Proof of Lemma 1.</u> Let ζ be a smooth function such that $\zeta > 0$ on Ω, $\zeta = 0$ on Γ and $\frac{\partial\zeta}{\partial n} \neq 0$ on Γ.

We first prove that, if v is a smooth function on $\bar{\Omega}$ which vanishes on Γ, then

$$(25) \qquad \Delta v - \frac{\partial^2 v}{\partial n^2} = \frac{\partial v}{\partial n}\left(\frac{\partial\zeta}{\partial n}\right)^{-1}\left(\Delta\zeta - \frac{\partial^2\zeta}{\partial n^2}\right) \quad \text{on } \Gamma.$$

We can assume that $0 \in \Gamma$ and we choose a coordinate system $(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_N)$ such that $\vec{e}_N = \vec{n}$. Locally, the equation of Γ is $x_N = f(x')$. In a neighborhood of 0 in

$\mathbb{R}^{N-1}$ we have $\zeta(x', f(x')) \equiv 0$ and $v(x', f(x')) \equiv 0$. Thus, for $1 \leq i \leq N-1$

$$\frac{\partial \zeta}{\partial x_i}(x', f(x')) + \frac{\partial \zeta}{\partial x_N}(x', f(x'))\frac{\partial f}{\partial x_i}(x') \equiv 0$$

$$\frac{\partial^2 \zeta}{\partial x_i^2}(x', f(x')) + 2\frac{\partial^2 \zeta}{\partial x_N \partial x_i}(x', f(x'))\frac{\partial f}{\partial x_i}(x') +$$

$$+ \frac{\partial \zeta}{\partial x_N}(x', f(x'))\frac{\partial^2 f}{\partial x_i^2}(x') \equiv 0 \quad .$$

Since $\frac{\partial f}{\partial x_i}(0) = 0$, we have $\frac{\partial^2 \zeta}{\partial x_i^2}(0,0) + \frac{\partial \zeta}{\partial n}(0,0)\frac{\partial^2 f}{\partial x_i^2}(0) = 0$,

and hence

$$\Delta\zeta(0,0) - \frac{\partial^2 \zeta}{\partial n^2}(0,0) + \frac{\partial \zeta}{\partial n}(0,0)\sum_{i=1}^{N-1}\frac{\partial^2 f}{\partial x_i^2}(0) = 0 \quad .$$

Similarly

$$\Delta v(0,0) - \frac{\partial^2 v}{\partial n^2}(0,0) + \frac{\partial v}{\partial n}(0,0)\sum_{i=1}^{N-1}\frac{\partial^2 f}{\partial x_i^2}(0) = 0 \quad .$$

Consequently

$$\frac{\partial v}{\partial n}\left(\Delta\zeta - \frac{\partial^2 \zeta}{\partial n^2}\right) = \frac{\partial \zeta}{\partial n}\left(\Delta v - \frac{\partial^2 v}{\partial n^2}\right) \quad \text{on } \Gamma \quad .$$

Note that if Ω is convex, ζ can be chosen to be concave so that $\Delta\zeta - \frac{\partial^2 \zeta}{\partial n^2} \leq 0$ and $\frac{\partial \zeta}{\partial n} < 0$. Define for $\xi \in \mathbb{R}^N$ and $\lambda > 0$, the convex function

$$j_\lambda(\xi) = \begin{cases} \frac{1}{2\lambda}|\xi|^2 & \text{if } |\xi| \leq \lambda \\ |\xi| - \frac{\lambda}{2} & \text{if } |\xi| \geq \lambda \end{cases} \quad .$$

By continuity it is sufficient to establish (24) for smooth u. We have

$$j_\lambda(\text{grad } u) - j_\lambda(\text{grad } u_\varepsilon) \geq \sum_{k=1}^{N} \frac{\partial j_\lambda}{\partial \xi_k}(\text{grad } u_\varepsilon) \cdot \left(\frac{\partial u}{\partial x_k} - \frac{\partial u_\varepsilon}{\partial x_k}\right)$$

$$= \sum_{k=1}^{N} \frac{\partial j_\lambda}{\partial \xi_k}(\text{grad } u_\varepsilon) \cdot -\varepsilon \Delta \frac{\partial u_\varepsilon}{\partial x_k} .$$

Hence

$$\int_\Omega j_\lambda(\text{grad } u_\varepsilon)dx \leq \int_\Omega j_\lambda(\text{grad } u)\, dx$$

$$- \varepsilon \sum_{k,\ell,m} \int_\Omega \frac{\partial^2 j_\lambda}{\partial \xi_k \partial \xi_\ell}(\text{grad } u_\varepsilon) \frac{\partial^2 u_\varepsilon}{\partial x_k \partial x_m} \cdot \frac{\partial^2 u_\varepsilon}{\partial x_\ell \partial x_m}\, dx$$

$$+ \varepsilon \int_\Gamma \frac{\partial j_\lambda}{\partial \xi_k}(\text{grad } u_\varepsilon) \frac{\partial^2 u_\varepsilon}{\partial x_k \partial n}\, d\Gamma$$

$$\leq \varepsilon \int_{[x \in \Gamma;\ |\frac{\partial u_\varepsilon}{\partial n}| < \lambda]} \frac{1}{\lambda} \frac{\partial u_\varepsilon}{\partial n} \frac{\partial^2 u_\varepsilon}{\partial n^2}\, d\Gamma$$

$$+ \varepsilon \int_{[x \in \Gamma;\ |\frac{\partial u_\varepsilon}{\partial n}| \geq \lambda]} \text{sign}\left(\frac{\partial u_\varepsilon}{\partial n}\right) \frac{\partial^2 u_\varepsilon}{\partial n^2}\, d\Gamma .$$

As $\lambda \to 0$ we get

$$\int_\Omega |\text{grad } u_\varepsilon|\, dx \leq \int_\Omega |\text{grad } u|\, dx +$$

$$+ \varepsilon \int_\Gamma \text{sign}\left(\frac{\partial u_\varepsilon}{\partial n}\right) \frac{\partial^2 u_\varepsilon}{\partial n^2}\, d\Gamma .$$

(Note that, by (25), $\frac{\partial^2 u_\varepsilon}{\partial n^2} = 0$ on the set where $\frac{\partial u_\varepsilon}{\partial n} = 0$).

Applying (25) we see that $\left|\frac{\partial^2 u_\varepsilon}{\partial n^2}\right| \le C \left|\frac{\partial u_\varepsilon}{\partial n}\right|$ on Γ (where C depends only on Ω). In the case where Ω is convex we have $\frac{\partial u_\varepsilon}{\partial n} \cdot \frac{\partial^2 u_\varepsilon}{\partial n^2} \le 0$ on Γ (by (25)) so that

$$\int_\Omega |\text{grad } u_\varepsilon| dx \le \int_\Omega |\text{grad } u| \, dx .$$

<u>Proof of Theorem 15.</u> It is immediate that if $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies (23) then $f \in \partial\varphi(u)$.

Conversely assume $u \in H_0^1(\Omega)$ and $f \in \partial\varphi(u)$. By definition, we have

$$(26) \qquad \frac{1}{2}\int_\Omega |\text{grad } v|^2 dx + \int_\Omega |\text{grad } v| dx - \frac{1}{2}\int_\Omega |\text{grad } u|^2 dx \, .$$

$$-\int_\Omega |\text{grad } u| dx \ge \int_\Omega f \cdot (v-u) dx \qquad \forall\, v \in H_0^1(\Omega) .$$

Taking $v = (1-t)u + tw$ in (26) with $t \in (0,1)$ and $w \in H^2(\Omega) \cap H_0^1(\Omega)$ we get, after letting $t \to 0$,

$$(27) \qquad \int_\Omega (\text{grad } u, \text{grad } w - \text{grad } u) dx + \int_\Omega |\text{grad } w| dx$$

$$-\int_\Omega |\text{grad } u| dx \ge \int_\Omega f \cdot (w-u) dx$$

$$\forall\, w \in H^2(\Omega) \cap H_0^1(\Omega) .$$

Thus

$$\int_\Omega (\text{grad } w, \text{grad } w - \text{grad } u) dx + \int_\Omega |\text{grad } w| dx$$

$$-\int_\Omega |\text{grad } u| dx \ge \int_\Omega f \cdot (w-u) dx .$$

Taking $w = u_\varepsilon$, as in Lemma 1, we obtain

$$\int_\Omega -\Delta u_\varepsilon \cdot \varepsilon \Delta u_\varepsilon \, dx + \int_\Omega |\operatorname{grad} u_\varepsilon| dx - \int_\Omega |\operatorname{grad} u| dx$$
$$\geq \int_\Omega f \cdot \varepsilon \Delta u_\varepsilon \, dx ,$$

and

$$\int_\Omega |\Delta u_\varepsilon|^2 dx \leq C \|u_\varepsilon\|_{H^2(\Omega)} + \int_\Omega |f| \, |\Delta u_\varepsilon| dx .$$

Consequently $|\Delta u_\varepsilon|_{L^2(\Omega)}$ is bounded as $\varepsilon \to 0$ and $u \in H^2(\Omega)$.

Let now k be a convex l.s.c. function from $\mathbb{R}$ into $(-\infty, +\infty]$ and let $\partial k = \gamma$. The function ψ is defined on $L^2(\Omega)$ by

$$\psi(u) = \begin{cases} \int_\Omega k(u) \, dx & \text{if } k(u) \in L^1(\Omega) \\ +\infty & \text{otherwise .} \end{cases}$$

<u>Corollary 16.</u> <u>If</u> $0 \in D(\gamma)$, <u>then</u> $\partial\psi + \partial\psi$ <u>is maximal monotone and</u> $\overline{D(\partial\psi + \partial\psi)} = \{u \in L^2(\Omega);\ u(x) \in \overline{D(\gamma)} \text{ a.e on } \Omega\}$. <u>In particular, for every</u> $f \in L^2(\Omega)$, <u>there exists a unique function</u> $u \in H^2(\Omega) \cap H_0^1(\Omega)$ <u>satisfying</u>

$$\int_\Omega (-\Delta u + \gamma(u)) \cdot (v-u) dx + \int_\Omega |\operatorname{grad} v| dx - \int_\Omega |\operatorname{grad} u| dx$$
$$\geq \int_\Omega f \cdot (v-u) dx \qquad \forall\, v \in H_0^1(\Omega) ;$$

<u>more precisely there exists</u> $g \in L^2(\Omega)$ <u>such that</u> $g(x) \in \gamma(u(x))$ a.e. on Ω <u>and</u>

$$\int_\Omega (-\Delta u + g)(v-u) dx + \int_\Omega |\operatorname{grad} v| dx - \int_\Omega |\operatorname{grad} u| dx$$
$$\geq \int_\Omega f(v-u) dx \qquad \forall\, v \in H_0^1(\Omega) .$$

Proof of Corollary 16. We reduce to the case where $0 \in \gamma(0)$. By Theorem 9 it is sufficient to show that

$$\varphi((I + \lambda\partial\psi)^{-1}u) \leq \varphi(u) \qquad \forall\, u \in D(\varphi)\,, \qquad \forall\, \lambda > 0$$

which clearly holds since $(I+\lambda\gamma)^{-1}$ is a contraction on $\mathbb{R}$.

Remark. Problems of similar nature appear in [27].

Example 3. Let $\Lambda = -\Delta$ be the canonical isomorphism from $H_0^1(\Omega)$ onto $H^{-1}(\Omega)$. Let $H = H^{-1}(\Omega)$ with its usual scalar product $(\bar{\Lambda}^1 f, g)$ for $f, g \in H^{-1}(\Omega)$. Let j be a convex l.s.c. function from $\mathbb{R}$ into $(-\infty, +\infty]$ with $j \not\equiv +\infty$, and let $\beta = \partial j$. We assume that $\lim\limits_{|r| \to +\infty} \frac{j(r)}{|r|} = +\infty$ (or equivalently $R(\beta) = \mathbb{R}$). For $u \in H^{-1}(\Omega)$ we define

$$\varphi(u) = \begin{cases} \int_\Omega j(u)dx & \text{if } u \in L^1(\Omega) \text{ and } j(u) \in L^1(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

Theorem 17. The function φ is convex l.s.c. on $H^{-1}(\Omega)$. Also $f \in H^{-1}(\Omega)$, $f \in \partial\varphi(u)$ if and only if $(\bar{\Lambda}^1 f)(x) \in \beta(u(x))$ a.e. on Ω.

Proof of Theorem 17. Let u_n be a sequence such that $u_n \in H^{-1}(\Omega) \cap L^1(\Omega)$, $u_n \to u$ in $H^{-1}(\Omega)$ and $\int_\Omega j(u_n(x))dx \leq \lambda$. We first prove that $u_n \to u$ weakly in $L^1(\Omega)$ by using Dunford-Pettis theorem (see e.g. [15] p. 294). The integrals $\int |u_n|$ are uniformly absolutely continuous i.e. $\forall\, \varepsilon > 0$ $\exists\, \delta > 0$ such that meas $E < \delta$ implies $\int_E |u_n| dx < \varepsilon$. Indeed, let $A > \frac{2\lambda}{\varepsilon}$ and let R be such that $\frac{j(r)}{|r|} \geq A$ for $|r| > R$. For $\delta < \frac{\varepsilon}{2R}$ we have

$$\int_E |u_n|dx \le \int_{[x \in E;\, |u_n(x)| \ge R]} |u_n|dx + \int_{[x \in E;\, |u_n(x)| < R]} |u_n|dx$$

$$\le \int_\Omega \frac{j(u_n)}{A}\, dx + R\,\delta \le \frac{\lambda}{A} + R\,\delta < \varepsilon \quad .$$

By Dunford-Pettis theorem there is a subsequence such that $u_{n_k} \to \tilde{u}$ weakly in $L^1(\Omega)$. Since we already know that $u_n \to u$ in $H^{-1}(\Omega)$, we conclude that $u_n \mapsto u$ weakly in $L^1(\Omega)$. Finally the function $u \to \int_\Omega j(u)dx$ is convex l. s. c. on $L^1(\Omega)$ (by Fatou's lemma) and thus it is weakly l. s. c. on $L^1(\Omega)$.

We define the operator A on $H^{-1}(\Omega)$ to be

$Au = \{ \Lambda w;\ w \in H_0^1(\Omega)$ and $w(x) \in \beta(u(x))$ a. e. on $\Omega \}$

with $u \in D(A)$ if and only if there is some $w \in H_0^1(\Omega)$ such that $w(x) \in \beta(u(x))$ a.e. on Ω. We prove now that $A \subset \partial\varphi$ and next that A is a maximal monotone. We need the following

<u>Lemma 2.</u> Let $F \in H^{-1}(\Omega) \cap L^1(\Omega)$ and let $w \in H_0^1(\Omega)$. Let $g \in L^1(\Omega)$ and let h be measurable with

$$F \cdot w \ge h \ge g \qquad \text{a.e. on } \Omega \ . \tag{28}$$

Then $h \in L^1(\Omega)$ and $(F, w) \ge \int_\Omega h\, dx$ (where $(,)$ denotes the scalar product in the duality between H_0^1 and H^{-1}).

<u>Proof of Lemma 2.</u> Let

$$w_n = \begin{cases} n & \text{if} \quad w \ge n \\ w & \text{if} \quad |w| \le n \\ -n & \text{if} \quad w \le -n \end{cases} \quad .$$

Let $h_n = h\,\dfrac{w_n}{w}$ and let $g_n = g\,\dfrac{w_n}{w}$. Multiplying (28) by

$\frac{w_n}{w}$ we get

$$F \cdot w_n \geq h_n \geq g_n \qquad \text{a.e. on } \Omega$$

and hence

$$0 \leq h_n - g_n \leq F \cdot w_n - g_n \qquad \text{a.e. on } \Omega .$$

The sequence $h_n - g_n \to h - g$ a.e. on Ω as $n \to +\infty$ and also

$$\int_\Omega (h_n - g_n)dx \leq \int_\Omega F \cdot w_n dx - \int_\Omega g_n dx = (F, w_n) - \int_\Omega g_n \, dx .$$

Since $w_n \to w$ in $H_0^1(\Omega)$ and $g_n \to g$ in $L^1(\Omega)$, we conclude by Fatou's lemma that $h-g \in L^1(\Omega)$ and thus $h \in L^1(\Omega)$ with

$$\int_\Omega (h-g)\, dx \leq (F, w) - \int_\Omega g(x)dx .$$

<u>Proof of Theorem 17. continued.</u>

Let $f \in Au$, i.e. $u \in H^{-1}(\Omega) \cap L^1(\Omega)$, $f = \Lambda w$ with $w \in H_0^1(\Omega)$, $w(x) \in \beta(u(x))$ a.e. on Ω. Let $v \in H^{-1}(\Omega) \cap L^1(\Omega)$ be such that $j(v) \in L^1(\Omega)$. We have $j(v) - j(u) \geq w \cdot (v - u)$ a.e. on Ω. Applying Lemma 2 with $F = u - v$, $h = j(u) - j(v)$ and $g = -C_1|u| - C_2 - j(v)$ $(j(r) \geq -C_1|r| - C_2)$, we conclude that $j(u) \in L^1(\Omega)$ and

$$\int_\Omega j(v)dx - \int_\Omega j(u)dx \geq (w, v-u) = (\bar{\Lambda}^1 f, \; v-u) .$$

Hence $f \in \partial\varphi(u)$.

We prove now that A is maximal monotone. For a given $f \in H^{-1}(\Omega)$ we have to find $u \in H^{-1}(\Omega) \cap L^1(\Omega)$ and $w \in H_0^1(\Omega)$ such that $u + \Lambda w = f$ and $w(x) \in \beta(u(x))$ a.e. on Ω. Let $\gamma = \bar{\beta}^1$ so that $D(\gamma) = \mathbb{R}$. Without loss of generality we can assume that $0 \in \gamma(0)$. Let $w_\lambda \in H_0^1(\Omega)$ be the solution of the equation

$$\gamma_\lambda(w_\lambda) + \Lambda w_\lambda = f \tag{29}$$

(which exists by standard results).

Multiplying (29) by w_λ and integrating over Ω we see that w_λ is bounded in $H_0^1(\Omega)$ as $\lambda \to 0$. Thus we can find a sequence $\lambda_n \to 0$ such that $w_{\lambda_n} \to w$ weakly in $H_0^1(\Omega)$, $w_{\lambda_n} \to w$ a.e. on Ω, $(I+\lambda_n\gamma)^{-1} w_{\lambda_n} \to w$ a.e. on Ω. The proof of Theorem 17 is completed by using the following general result.

<u>Theorem 18.</u> <u>Let</u> γ <u>be a maximal monotone graph in</u> $\mathbb{R}\times\mathbb{R}$ <u>such that</u> $D(\gamma) = \mathbb{R}$ <u>and</u> $0 \in \gamma(0)$. <u>Let</u> f_n <u>and</u> v_n <u>be measurable functions on</u> Ω <u>such that</u> $v_n \to v$ a.e. on Ω, $f_n(x) \in \gamma(v_n(x))$ a.e. on Ω <u>and</u> $f_n \cdot v_n \in L^1(\Omega)$ <u>with</u> $\int_\Omega f_n \cdot v_n \, dx \le C$. <u>Then, there is a subsequence</u> $n_k \to +\infty$ <u>such that</u> $f_{n_k} \to f$ <u>weakly in</u> $L^1(\Omega)$ <u>and</u> $f(x) \in \gamma(v(x))$ a.e. on Ω.

<u>Proof of Theorem 18.</u> Let $\beta = \gamma^{-1}$ and let j be such that $j(0) = 0$, $\partial j = \beta$. Since $R(\beta) = \mathbb{R}$ we get $\lim_{|r|\to+\infty} \frac{j(r)}{|r|} = +\infty$.

We have a.e. on Ω

$$j(0) - j(f_n(x)) \ge v \cdot -f_n(x) \qquad \text{for every } v \in \beta(f_n(x)).$$

Consequently

$$j(f_n(x)) \le v_n(x) \; f_n(x) \qquad \text{a.e. on } \Omega,$$

and so

$$\int_\Omega j(f_n(x))dx \le \int_\Omega f_n \, v_n \, dx \le C.$$

Using again Dunford-Pettis theorem (as in the beginning of the proof of Theorem 17) we get a sequence $n_k \to +\infty$ such that $f_{n_k} \to f$ weakly in $L^1(\Omega)$. We conclude the proof of Theorem 19 with the help of the following

<u>Lemma 3.</u> Let M be maximal monotone in a Hilbert space $\mathcal{H}$. Let f_n and v_n be measurable functions from Ω (a finite measure space) into $\mathcal{H}$. Assume $v_n \to v$ a.e. on Ω and $f_n \to f$ weakly in $L^1(\Omega;\mathcal{H})$. If $f_n(x) \in Mv_n(x)$ a.e. on Ω, then $f(x) \in Mv(x)$ a.e. on Ω.

Proof of Lemma 3. It is sufficient to show that, for every N, $f(x) \in Mv(x)$ a.e. on $\Omega_N = \{x \in \Omega;\ |v(x)| \leq N\}$. By Egorov's lemma, for every $\delta > 0$ there is a set $E \subset \Omega_N$ such that meas $E < \delta$ and $v_n \to v$ uniformly in $\Omega_N \setminus E$. Thus we are reduced to the case where v is bounded on Ω and $v_n \to v$ uniformly on Ω. Without loss of generality we may assume now that $D(M)$ is bounded (if $D(M)$ is not bounded, consider $\tilde{M} = M + \partial I_B$, where I_B is the indicator function of a ball centered at 0 of large radius). Let $\tilde{v} \in L^\infty(\Omega;\mathcal{H})$ and let $\tilde{f} \in L^1(\Omega;\mathcal{H})$ be such that $\tilde{f}(x) \in M\tilde{v}(x)$ a.e. on Ω. By the monotonicity of M, we have $(\tilde{f} - f_n, \tilde{v} - v_n) \geq 0$ a.e. on Ω, and thus $\int_\Omega (\tilde{f} - f_n, \tilde{v} - v_n)dx \geq 0$. Consequently $\int_\Omega (\tilde{f} - f, \tilde{v} - v)dx \geq 0$. Let now $\tilde{v} = (I + M)^{-1}(v + f)$ ($\tilde{v} \in L^\infty(\Omega;\mathcal{H})$ since $D(M)$ is bounded). We have $\tilde{v} + M\tilde{v} \ni v + f$ a.e. on Ω. Choosing $\tilde{f} = v + f - \tilde{v}$ we get $\int_\Omega |v - \tilde{v}|^2 dx \leq 0$ so that $\tilde{v} = v$ and $f \in Mv$ a.e. on Ω.

Example 4. Let $\mathcal{H}$ be a Hilbert space and let $H = L^2(0,T;\mathcal{H})$ with its usual Hilbert structure. Given $u_0 \in \mathcal{H}$ we define on H the maximal monotone operator A by $Au = \frac{du}{dt}$ with

$$D(A) = \{u \in H;\ \frac{du}{dt} \in H \text{ in the sense of distributions and } u(0) = u_0\}.$$

Let φ be a convex l.s.c. function from $\mathcal{H}$ into $(-\infty, +\infty]$ such that $\varphi \not\equiv +\infty$. We introduce on H the function Φ by

$$\Phi(u) = \begin{cases} \int_0^T \varphi(u(t))dt & \text{if } \varphi(u) \in L^1(0,T) \\ +\infty & \text{otherwise} \end{cases}.$$

It is easy to check (using Fatou's lemma) that Φ is l.s.c. and that $f \in \partial\Phi(u)$ if and only if $f(t) \in \partial\varphi(u(t))$ a.e. on $(0,T)$.

Theorem 19. If $u_0 \in D(\varphi)$ then $A + \partial\Phi$ is maximal monotone.

Proof of Theorem 19. First we reduce to the case where $\varphi \geq 0$ (this amounts to shift $\partial\Phi$ by a constant). We are going to prove that

$$(30) \qquad \Phi((I + \lambda A)^{-1}u) \leq \Phi(u) + \varphi(u_0)\,\lambda \qquad \forall u \in H\,, \quad \forall \lambda > 0\,.$$

Indeed, let $u_\lambda = (I + \lambda A)^{-1}u$ i.e.

$$\lambda \frac{du_\lambda}{dt} + u_\lambda = u \quad \text{and} \quad u_\lambda(0) = u_0\,.$$

Let φ_α be the approximation of φ introduced in Theorem 4. We have

$$\varphi_\alpha(u(t)) - \varphi_\alpha(u_\lambda(t)) \geq (\partial\varphi_\alpha(u_\lambda(t)),\ u(t) - u_\lambda(t)) =$$

$$= (\partial\varphi_\alpha(u_\lambda(t)),\ \lambda \frac{du_\lambda}{dt}(t)) = \lambda \frac{d}{dt}\ \varphi_\alpha(u_\lambda(t))\,.$$

Hence

$$\int_0^T \varphi_\alpha(u_\lambda)dt \leq \int_0^T \varphi_\alpha(u)dt - \varphi_\alpha(u_\lambda(T)) + \varphi_\alpha(u_0) \leq \Phi(u) + \varphi(u_0).$$

Passing to the limit as $\alpha \to 0$, we get (30), which implies by Theorem 9 that $A + \partial\Phi$ is maximal monotone.

Corollary 20. For every $f \in L^2(0,T;\mathcal{H})$ and $u_0 \in D(\varphi)$ there exists a unique $u \in C([0,T];\mathcal{H})$ such that $u(t) \in D(\partial\varphi)$ a.e. on $(0,T)$, $\frac{du}{dt} \in L^2(0,T;\mathcal{H})$,

$$(31) \qquad \begin{cases} \dfrac{du}{dt} + \partial\varphi(u) \ni f & \text{a.e. on } (0,T) \\ u(0) = u_0\,. \end{cases}$$

In addition the following estimates holds

$$(32) \qquad \left(\int_0^T \left|\frac{du}{dt}\right|^2 dt\right)^{1/2} \leq \left(\int_0^T |f|^2 dt\right)^{1/2} + |\varphi(u_0)|^{1/2} + C_1|u_0|^{1/2} + C_2$$

(where C_1 and C_2 depend only on φ ; in particular $C_1 = C_2 = 0$ if $\varphi \geq 0$).

Proof of Corollary 20. By Theorems 5 and 19 it is sufficient to show that $(A + \partial\Phi)^{-1}$ is bounded. Let $f \in (A + \partial\Phi)(u)$ and and let $v_0 \in D(\varphi)$ be fixed. We have

$$\varphi(v_0) - \varphi(u(t)) \geq (f(t) - \frac{du}{dt}(t), v_0 - u(t)) \quad \text{a.e. on } (0,T).$$

Let $\tilde{\varphi}(u) = \varphi(u) + (a,u) + b$, be such that $\tilde{\varphi} \geq 0$ on H. Thus $\frac{1}{2}\frac{d}{dt}|u - v_0|^2 \leq \varphi(v_0) + |f||u - v_0| + |a||u| + |b|$ and we get a bound on $|u|_H$ provided $|f|_H$ is bounded. Estimate (32) follows easily from (30) and Theorem 9 applied with $\tilde{\varphi}$ instead of φ.

This example leads us to evolution equations associated with maximal monotone operators.

III. Evolution equations associated with maximal monotone operators.

Let A be a maximal monotone in a Hilbert space H. One of the main results is the following

Theorem 21. Let f be absolutely continuous from $[0,T]$ into H and let $u_0 \in D(A)$. There exists a unique function $u \in C([0,T];H)$ satisfying

(33) $\quad u(t) \in D(A) \qquad \forall t \in [0,T]$

(34) $\quad u(t)$ is Lipschitz continuous on $[0,T]$ (and thus $u(t)$ is differentiable a.e. on $(0,T)$).

(35) $\quad \frac{du}{dt} + Au \ni f \qquad$ a. e. on $(0,T)$

(36) $\quad u(0) = u_0$.

The proofs of Theorem 21 and of the following remarks can be found in Kato [19]. But many authors, including

F. Browder [12], Crandall-Pazy [13], Dorroh [14], Komura [20], have considered related problems.

Remark. The solution of (35)-(36) has some additional properties:

(37) u is differentiable from the right at every $t \in [0,T)$ and $\frac{d^+u}{dt}(t) + (Au(t) - f(t))^0 = 0 \qquad \forall\, t \in [0,T)$

(38) $$\left|\frac{d^+u}{dt}(t)\right| \leq \left|\frac{d^+u}{dt}(0)\right| + \int_0^t \left|\frac{df}{dt}(s)\right| ds =$$

$$|(Au_0 - f(0))^0| + \int_0^t \left|\frac{df}{dt}(s)\right| ds \qquad \forall\, t \in [0,T] .$$

(39) Given $f, \hat{f}$ and $u_0, \hat{u}_0$, the corresponding solutions $u, \hat{u}$ satisfy

$$|u(t) - \hat{u}(t)| \leq |u_0 - \hat{u}_0| + \int_0^t |f(s) - \hat{f}(s)| ds.$$

From (39) it is clear that the mapping $\{u_0, f\} \mapsto u$ can be extended by continuity from $\overline{D(A)} \times L^1(0,T;H)$ into $C([0,T]; \overline{D(A)})$. In the case where $f = 0$, the mapping $u_0 \mapsto u(t)$ is called the semigroup of nonlinear contractions generated by $-A$ on $\overline{D(A)}$. It may well happen, even in the linear case, that for $u_0 \in \overline{D(A)}$ and $f \in L^1(0,T;H)$, the corresponding $u(t)$ is nowhere differentiable and $u(t) \notin D(A)$ $\forall\, t \in [0,T]$. Thus, we have to consider $u(t)$ as a generalized solution of (35)-(36). We are going to show that for some special classes of maximal monotone operators A, in particular $A = \partial\varphi$ or $\operatorname{Int} D(A) \neq \phi$ (which correspond in the linear case to self adjoint or bounded operators), the generalized solution is "almost" a strong solution.

We consider first the case where $A = \partial\varphi$, φ being a convex l.s.c. function from H into $(-\infty, +\infty]$ with $\operatorname{Min}_H \varphi = 0$. Let $K = \{v \in H;\ \varphi(v) = 0\}$.

Theorem 22. For every $u_0 \in \overline{D(\varphi)}$ and $f \in L^2(0,T;H)$ there is a unique solution $u \in C([0,T];H)$ of (35)-(36) satisfying

(40) $\quad u(t) \in D(A) \quad$ a.e. on $(0,T)$

(41) $\quad \sqrt{t}\ \dfrac{du}{dt} \in L^2(0,T;H)$.

In addition we have the estimates

(42) $$\left(\int_0^T \left|\frac{du}{dt}(t)\right|^2 t\,dt\right)^{1/2} \le \left(\int_0^T |f(t)|^2 t\,dt\right)^{1/2} + \frac{1}{\sqrt{2}}\int_0^T |f(t)|\,dt + \frac{1}{\sqrt{2}}\,\mathrm{dist}(u_0,K)$$

(43) $$\left(\int_\delta^T \left|\frac{du}{dt}(t)\right|^2 dt\right)^{1/2} \le \left(\int_0^T |f(t)|^2 dt\right)^{1/2} + \frac{1}{\sqrt{2\delta}}\int_0^\delta |f(t)|\,dt + \frac{1}{\sqrt{2\delta}}\,\mathrm{dist}\,(u_0,K)$$

$$\forall\, \delta \in (0,T)\ .$$

Remark. Note that in Theorem 22 we assume only $u_0 \in \overline{D(\varphi)}$ (instead of $u_0 \in D(\varphi)$ in Corollary 20). Theorem 22 is closely related to Proposition 5 in [4] but we give here a new proof which exploits a suggestion of P. Lax.

Proof of Theorem 22. As in [4], the crucial point is to establish the estimate (42) (or (43)) for the Yosida approximation $\dfrac{du_\lambda}{dt} + A_\lambda u_\lambda = f$, $u_\lambda(0) = u_0$ (where $A_\lambda = \partial\varphi_\lambda$, see Theorem 4). Next one can pass to the limit as $\lambda \to 0$ using standard devices. In order to simplify the notations we drop λ.

Estimate I. (energy estimate) Let $v_0 \in K$; we have

$$\varphi(v_0) - \varphi(u(t)) \ge (\partial\varphi(u(t)), v_0 - u(t)) = \left(f(t) - \frac{du}{dt}(t), v_0 - u(t)\right)\ .$$

Hence

$$\int_0^T \varphi(u(t))dt \le \int_0^T |f(t)|\,|u(t)-v_0|dt + \frac{1}{2}\,|u_0 - v_0|^2 \quad .$$

But $|u(t) - v_0| \le |u_0 - v_0| + \int_0^t |f(s)|ds$ (apply (39) with $\hat{u} \equiv v_0$ and $\hat{f} \equiv 0$). Consequently

$$(44) \qquad \int_0^T \varphi(u(t))dt \le \frac{1}{2}\left(|u_0 - v_0| + \int_0^T |f(t)|dt\right)^2 \quad .$$

<u>Estimate II.</u> Multiplying (35) by $t\,\frac{du}{dt}$ we get

$$t\left|\frac{du}{dt}(t)\right|^2 + t\frac{d}{dt}\,\varphi(u(t)) = \left(f(t),\ t\,\frac{du}{dt}(t)\right)$$

Consequently

$$(45) \qquad \int_0^T \left|\frac{du}{dt}(t)\right|^2 t\,dt + T\,\varphi(u(T))$$

$$\le \int_0^T |f(t)|\,\left|\frac{du}{dt}(t)\right| t\,dt + \int_0^T \varphi(u(t))dt \ .$$

Thus

$$(46) \qquad \left(\int_0^T \left|\frac{du}{dt}(t)\right|^2 t\,dt\right)^{1/2}$$

$$\le \left(\int_0^T |f(t)|^2 t\,dt\right)^{1/2} + \left(\int_0^T \varphi(u(t))dt\right)^{1/2} \quad .$$

Combining (44) and (46) we obtain (42). From (45) we deduce also that

$$(47) \qquad T\,\varphi(u(T)) \le \frac{1}{4}\int_0^T |f(t)|^2\,t\,dt + \int_0^T \varphi\,(u(t))dt \ .$$

Multiplying (35) by $\frac{du}{dt}$ and integrating on (δ, T) we get

$$\int_\delta^T \left|\frac{du}{dt}(t)\right|^2 dt \le \varphi(u(\delta)) + \int_\delta^T |f(t)|\,\left|\frac{du}{dt}(t)\right| dt \quad .$$

Hence using (47) with $T = \delta$ we have

$$\int_\delta^T \left(\left|\frac{du}{dt}(t)\right| - \frac{1}{2}|f(t)|\right)^2 dt \le \frac{1}{4}\int_\delta^T |f(t)|^2 dt + \frac{1}{4\delta}\int_0^\delta |f(t)|^2 t\, dt$$

$$+ \frac{1}{2\delta}\left(|u_0 - v_0| + \int_0^\delta |f(t)|dt\right)^2 .$$

The estimate (43) follows easily.

In the case where f is smooth, $u(t)$ has some further properties.

<u>Theorem 23.</u> <u>Let</u> $u_0 \in \overline{D(\varphi)}$ <u>and let</u> f <u>be absolutely continuous from</u> $[0,T]$ <u>into</u> H. <u>Then the corresponding solution of</u> (35)-(36) <u>satisfies</u>

(48) $\quad u(t) \in D(A) \qquad \forall\, t \in (0,T]$

(49) $\quad t\left|\frac{du}{dt}(t)\right| \in L^\infty(0,T)$

(50) $\quad \frac{d^+u}{dt}(t)$ <u>exists for all</u> $t \in (0,T)$ <u>and</u>

$$\frac{d^+u}{dt}(t) + (Au(t) - f(t))^0 = 0 \qquad \forall\, t \in (0,T) .$$

<u>In addition we have the estimate</u>

(51) $$\left|\frac{d^+u}{dt}(t)\right| \le \int_0^t \left|\frac{df}{dt}(s)\right| \frac{s^2}{t^2}dx + \frac{\sqrt{2}}{t}\left(\int_0^t |f(s)|^2 s\, ds\right)^{1/2}$$

$$+ \frac{1}{t}\int_0^t |f(s)|ds + \frac{1}{t}\ \mathrm{dist}\,(u_0, K) \qquad \forall\, t \in (0,T) .$$

<u>Proof of Theorem 23.</u> Since $u(t) \in D(A)$ a.e. on $(0,T)$ and f is absolutely continuous, we conclude by Theorem 21 that (48) and (50) hold. In order to establish (51) we use (42) and the fact that the function $t \mapsto \left|\frac{d^+u}{dt}(t)\right| - \int_0^t \left|\frac{df}{dt}(s)\right|ds$ is nonincreasing in t (see (38)). For $0 < t < T$ we have

$$\left|\frac{d^+u}{dt}(T)\right| \leq \left|\frac{d^+u}{dt}(t)\right| + \int_t^T \left|\frac{df}{dt}(s)\right| ds \quad .$$

Consequently

$$\int_0^T \left[\left|\frac{d^+u}{dt}(T)\right| - \int_t^T \left|\frac{df}{dt}(s)\right| ds\right] t\,dt \leq \int_0^T \left|\frac{du}{dt}(t)\right| t\,dt$$

$$\leq \frac{T}{\sqrt{2}} \left(\int_0^T \left|\frac{du}{dt}(t)\right|^2 t\,dt\right)^{1/2} .$$

Thus

$$\frac{T^2}{2}\left|\frac{d^+u}{dt}(T)\right| \leq \int_0^T t\left(\int_t^T \left|\frac{df}{dt}(s)\right| ds\right)dt + \frac{T}{\sqrt{2}}\left(\int_0^T \left|\frac{du}{dt}(t)\right|^2 t\,dt\right)^{1/2}$$

$$= \int_0^T \left|\frac{df}{dt}(t)\right| \frac{t^2}{2}\,dt + \frac{T}{\sqrt{2}}\left(\int_0^T \left|\frac{du}{dt}(t)\right|^2 t\,dt\right)^{1/2} \quad .$$

We conclude by applying (42).

The estimate (51) provides also informations about the <u>behavior of</u> $u(t)$ as $t \to +\infty$. Let φ be a convex l. s. c. function from H into $(-\infty, +\infty]$, $\varphi \not\equiv +\infty$. Let f be a function from $[0, +\infty)$ into H such that $\left|\frac{df}{dt}\right| \in L^1(0, +\infty)$, so that $\lim f(t) = f_\infty$ exists. We assume that $f_\infty \in R(\partial\varphi)$.

<u>Theorem 24.</u> <u>Let</u> $u_0 \in \overline{D(\varphi)}$ <u>and let</u> u <u>be the solution of</u> $\frac{du}{dt} + \partial\varphi(u) \ni f$ a. e. on $(0, +\infty)$, $u(0) = u_0$. <u>Then</u> $\lim_{t \to +\infty} \left|\frac{du}{dt}(t)\right| = 0$. <u>If in addition</u> $\left|\frac{df}{dt}(t)\right| = 0(t^{-\alpha})$ <u>with</u> $\alpha > 2$ <u>as</u> $t \to +\infty$, <u>then</u> $\left|\frac{du}{dt}(t)\right| = 0(t^{-1})$ <u>as</u> $t \to +\infty$.

<u>Proof of Theorem 24.</u> Let $\tilde{\varphi}(u) = \varphi(u) - (f_\infty, u) - \operatorname{Inf}_H\{\varphi(u) - (f_\infty, u)\}$, since $f_\infty \in R(\partial\varphi)$, $\operatorname{Min} \tilde{\varphi} = 0$. We have $\frac{du}{dt} + \partial\tilde{\varphi}(u) \ni f - f_\infty$ so that by (51) we obtain

$$(52)\quad \left|\frac{d^+u}{dt}(t)\right| \le \int_0^t \left|\frac{df}{dt}(s)\right| \frac{s^2}{t^2}\,ds + \frac{\sqrt{2}}{t}\left(\int_0^t |f(s)-f_\infty|^2 s\,ds\right)^{1/2}$$

$$+ \frac{1}{t}\int_0^t |f(s)-f_\infty|\,ds + \frac{1}{t}\,\mathrm{dist}\,(u_0, K)\,,$$

where now $K = \{ v \in H;\ \partial\varphi(v) \ni f_\infty \}$. Let ε be fixed and let t_0 be such that $\int_{t_0}^t \left|\frac{df}{dt}(s)\right| ds < \varepsilon$ for $t \ge t_0$. We have, for t large enough,

$$\int_0^t \left|\frac{df}{dt}(s)\right| \frac{s}{t}\,ds \le \frac{1}{t}\int_0^{t_0} \left|\frac{df}{dt}(s)\right| s\,ds + \int_{t_0}^t \left|\frac{df}{dt}(s)\right| ds < 2\varepsilon\,.$$

Since $\int_0^t \left|\frac{df}{dt}(s)\right| \frac{s^2}{t^2}\,ds \le \int_0^t \left|\frac{df}{dt}(s)\right| \frac{s}{t}\,ds$, we get

$$\lim_{t\to+\infty} \int_0^t \left|\frac{df}{dt}(s)\right| \frac{s^2}{t^2}\,ds = \lim_{t\to+\infty} \int_0^t \left|\frac{df}{dt}(s)\right| \frac{s}{t}\,ds = 0\,.$$

On the other hand

$$\frac{1}{t}\left(\int_0^t |f(s) - f_\infty|^2\, s\,ds\right)^{1/2} \le M\left(\frac{1}{t}\int_0^t |f(s) - f_\infty|\,ds\right)^{1/2}$$

where $M = \sup_{s \ge 0} |f(s) - f_\infty|$.

Thus it remains only to prove that $\lim_{t\to+\infty} \frac{1}{t}\int_0^t |f(s) - f_\infty|\,ds = 0$.

But $\frac{1}{t}\int_0^t |f(s) - f\ |\,ds \le \frac{1}{t}\int_0^t ds \int_s^{+\infty} \left|\frac{df}{dt}(\tau)\right| d\tau =$

$$\int_t^{+\infty} \left|\frac{df}{dt}(\tau)\right| d\tau + \int_0^t \left|\frac{df}{dt}(s)\right| \frac{s}{t}\,ds\,.$$

In the case where $\left|\frac{df}{dt}(t)\right| \le C\,t^{-\alpha}$, $\alpha > 2$, it is easy to check that $|f(t) - f_\infty| \le \int_t^\infty \left|\frac{df}{dt}(s)\right| ds \le C\,t^{1-\alpha}$ and that,

by (52)

$$\left|\frac{d^+u}{dt}(t)\right| \leq C\, t^{1-\alpha} + \frac{1}{t}\ \mathrm{dist}\,(u_0, K) \qquad \text{as} \qquad t \to +\infty\,.$$

Remark. In the case where $(Ax - Ay,\ x-y) \geq \gamma|x - y|^2$ for some $\gamma > 0$, it is standard that $\left|\frac{du}{dt}(t)\right|$ decays exponentially as $t \to 0$ provided $\left|\frac{df}{dt}\right| \to 0$ fast enough.

Problem. Suppose, for simplicity, that $f \equiv 0$. It would be of great interest to determine whether $\lim_{t\to+\infty} u(t)$ exists (one can prove it under some additional assumptions, for example if $\{u \in H\ ;\ |u| \leq C_1$ and $\varphi(u) \leq C_2\}$ is compact for every C_1, C_2 ; see Remark 6 in [4]). Assuming $\lim_{t\to+\infty} u(t) = u_\infty$ exists, one would like to have further information about the mapping $u_0 \mapsto u_\infty$. For example, is it true that $u_\infty = \lim_{n\to+\infty} (I + \lambda\partial\varphi)^{-n} u_0$?

Evolution equations associated with maximal monotone operators A such that $\mathrm{Int}\, D(A) \neq \phi$ have also remarkable properties. We restrict ourselves to the case where f is absolutely continuous, but further results for the case where $f \in L^1(0,T;H)$ can be found in [2].

Theorem 25. Let A be maximal monotone with $\mathrm{Int}\, D(A) \neq \phi$. Let $u_0 \in \overline{D(A)}$ and let f be absolutely continuous on $[0,T]$. There exists a unique function $u \in C([0,T];H)$ satisfying $\left|\frac{du}{dt}\right| \in L^1(0,T)$, (48), (49), (50) and $u(0) = u_0$.

Proof of Theorem 25. Let u_λ be the solution of the equation $\frac{du_\lambda}{dt} + A_\lambda u_\lambda = f$, $u_\lambda(0) = u_0$. Let $v_0 \in \mathrm{Int}\, D(A)$; by Theorem 6, there is a constant C such that $|v - v_0| < \rho$ implies $v \in D(A)$ and $|A^0 v| \leq C$. We have $(A_\lambda u_\lambda - A_\lambda v,\ u_\lambda - v) \geq 0$ which implies, for $v = v_0 + \rho w$, $|w| \leq 1$, $\rho|A_\lambda u_\lambda| \leq (A_\lambda u_\lambda,\ u_\lambda - v_0) + C|u_\lambda - v_0| + C\rho$ or

$$\rho\left|\frac{du_\lambda}{dt}\right| \leq \rho(|f| + C) + \left(f - \frac{du_\lambda}{dt},\ u_\lambda - v_0\right) + C|u_\lambda - v_0|\,.$$

Hence

$$\rho \int_0^T \left|\frac{du_\lambda}{dt}\right| dt \le \rho \int_0^T (|f| + C)dt + \int_0^T (|f| + C)|u_\lambda - v_0| dt + \frac{1}{2}|u_0 - v_0|^2 .$$

But $|u_\lambda(t) - v_0| \le |u_0 - v_0| + \int_0^t (|f(s)| + C)\, ds$ (apply (39) with $\hat{u} \equiv v_0$). Consequently

$$(53) \quad \rho \int_0^T \left|\frac{du_\lambda}{dt}\right| dt \le \rho \int_0^T (|f| + C)dt + |u_0 - v_0| \int_0^T (|f| + C)dt + \frac{1}{2}\left(\int_0^T (|f| + C)dt\right)^2 + \frac{1}{2}|u_0 - v_0|^2 =$$

$$= \rho \int_0^T (|f| + C)dt + \frac{1}{2}\left(|u_0 - v_0| + \int_0^T (|f| + C)dt\right)^2 .$$

On the other hand, we have for $t \in [0,T]$

$$\left|\frac{du_\lambda}{dt}(T)\right| \le \left|\frac{du_\lambda}{dt}(t)\right| + \int_t^T \left|\frac{df}{dt}(s)\right| ds .$$

Finally

$$(54) \quad T\left|\frac{du_\lambda}{dt}(T)\right| \le \int_0^T \left|\frac{du_\lambda}{dt}\right| dt + \int_0^T \left(\int_t^T \left|\frac{df}{dt}(s)\right| ds\right) dt$$

$$\le \int_0^T (|f| + C)dt + \frac{1}{2\rho}\left(|u_0 - v_0| + \int_0^T (|f| + C)dt\right)^2 + \int_0^T \left|\frac{df}{dt}(t)\right| t\, dt .$$

The estimates (53) and (54) are independent of λ; (54) actually shows that $t\frac{du_\lambda}{dt}(t)$ is bounded in $L^\infty(0,T;H)$. We conclude the proof by passing to the limit as $\lambda \to 0$.

Behavior as $t \to +\infty$. Assume now that $\left|\frac{df}{dt}\right| \in L^1(0,+\infty)$ so that $\lim_{t\to+\infty} f(t) = f_\infty$ exists and assume $|f - f_\infty| \in L^1(0,+\infty)$.

Theorem 26. Suppose $\mathrm{Int}\,(\bar{A}^1 f_\infty) \neq \phi$ and let u be the solution of (35), (36). Then $\left|\frac{du}{dt}\right| \in L^1(0,+\infty)$, $\lim_{t\to+\infty} u(t) = u_\infty$ exists, $f_\infty \in Au_\infty$ and $\lim_{t\to+\infty} \left|\frac{du}{dt}(t)\right| = 0$. If in addition $t\left|\frac{df}{dt}(t)\right| \in L^1(0,+\infty)$, then $\left|\frac{du}{dt}\right| = 0(t^{-1})$ as $t \to +\infty$.

Proof of Theorem 26. Let $\tilde{A}u = Au - f_\infty$; we have $\frac{du}{dt} + \tilde{A}u \ni f - f_\infty$ a.e. on $(0,+\infty)$. If we take in the proof of Theorem 25, $v_0 \in \mathrm{Int}(\bar{A}^1 f_\infty)$, we have $0 \in \tilde{A}v$ for $|v - v_0| < \rho$ and thus $C = 0$ is permissible. Hence, by (53) and (54) we have

$$\rho\int_0^T \left|\frac{du}{dt}\right| dt \leq \rho\int_0^T |f - f_\infty| dt + \frac{1}{2}\left(|u_0 - v_0| + \int_0^T |f - f_\infty| dt\right)^2$$

and

$$\left|\frac{d^+u}{dt}(t)\right| \leq \frac{1}{t}\int_0^t |f(s) - f_\infty| ds + \frac{1}{2\rho t}\left(|u_0 - v_0| + \int_0^t |f - f_\infty| ds\right)^2 + \int_0^t \left|\frac{df}{dt}(s)\right| \frac{s}{t}\, ds .$$

One can also consider sums of operators in the previous classes. Let φ be a convex l.s.c. function from H into $(-\infty,+\infty]$, $\varphi \not\equiv +\infty$ and let A be maximal monotone with $(\mathrm{Int}\, D(A)) \cap D(\partial\varphi) \neq \phi$.

Theorem 27. Let f be absolutely continuous from $[0,T]$ in into H and let $u_0 \in \overline{D(A)} \cap \overline{D(\varphi)}$. There exists a unique function $u \in C([0,T];H)$ satisfying

$$u(t) \in D(A) \cap D(\partial\varphi) \qquad \forall t \in (0,T]$$

$$t\frac{du}{dt} \in L^\infty(0,T)$$

$$\frac{du}{dt} + Au + \partial\varphi(u) \ni f \text{ a.e. on } (0,T),\ u(0) = u_0 .$$

Remark. The case $f = 0$ has been treated previously by F. Browder using a different and ingenious argument.

Proof of Theorem 27. We know already that $A + \partial\varphi$ is maximal monotone. Also for any maximal monotone B such that $(\text{Int } D(A)) \cap D(B) \neq \phi$ we have

$$\overline{D(A + B)} = \overline{D(A) \cap D(B)} = \overline{D(A)} \cap \overline{D(B)} .$$

Indeed, let $u \in \overline{D(A)} \cap \overline{D(B)}$ and let $v \in \text{Int } D(A)$ such that $|v - u| < \varepsilon$. Let $u_\lambda = (I + \lambda B)^{-1}u$ and $v_\lambda = (I + \lambda B)^{-1}v$. We have $|v_\lambda - u_\lambda| \leq |v - u| < \varepsilon$ and $|v_\lambda - u| \leq \varepsilon + |u_\lambda - u|$. Choose λ small enough so that $|u_\lambda - u| < \varepsilon$ and $v_\lambda \in$ $\in \text{Int } D(A)$. Let $v_0 \in \text{Int } D(A) \cap D(\partial\varphi)$; we can always assume that $\text{Min } \varphi = \varphi(v_0) = 0$. We consider the approximate equation

$$\frac{du_\lambda}{dt} + A_\lambda u_\lambda + \partial\varphi_\lambda(u_\lambda) = f , \qquad u_\lambda(0) = u_0$$

and then we pass easily to the limit as $\lambda \to 0$, after having established the crucial estimates. In order to simplify the notations we drop λ.

Estimate I. As in the proof of Theorem 25 we have

$$\rho\, |Au| \leq (Au,\ u - v_0) + C|u - v_0| + C\rho$$

Hence

$$\rho\, |Au| \leq (f - \frac{du}{dt} - \partial\varphi(u),\ u - v_0) + C|u - v_0| + C\rho .$$

Therefore

$$\rho\, |Au| + \varphi(u) \leq (|f| + C)\ |u - v_0| + C\rho - \frac{1}{2}\frac{d}{dt}\ |u - v_0|^2$$

Consequently

$$\rho\int_0^T |Au|\,dt + \int_0^T \varphi(u)dt \leq \int_0^T (|f| + C)\,|u - v_0|\,dt + \frac{1}{2}\,|u_0 - v_0|^2 + C\rho T.$$

But $|u(t) - v_0| \leq |u_0 - v_0| + \int_0^t (|f| + C)\, ds$ and thus

$$\rho \int_0^T |Au|\, dt + \int_0^T \varphi(u) dt \leq \frac{1}{2} (|u_0 - v_0| + \int_0^T (|f| + C) dt)^2 + C\, \rho T.$$

<u>Estimate II.</u> Multiplying the equation $\frac{du}{dt} + Au + \partial\varphi(u) = f$ by $t \frac{du}{dt}$ we get

$$\int_0^T |\frac{du}{dt}(t)|^2 t\, dt \leq \int_0^T (|Au| + |f|) |\frac{du}{dt}|\, t\, dt + \int_0^T \varphi(u) dt\ .$$

Using now the estimate

$$|\frac{du}{dt}(T)| \leq |\frac{du}{dt}(t)| + \int_0^T |\frac{df}{dt}|\, ds \quad \text{for} \quad t \in [0, T]$$

we get

$$|\frac{du}{dt}(T)|^2 \leq 2\, |\frac{du}{dt}(t)|^2 + 2\, (\int_0^T |\frac{df}{dt}|\, ds)^2$$

and

$$\frac{T^2}{2} |\frac{du}{dt}(T)|^2 \leq 2 \int_0^T |\frac{du}{dt}(t)|^2 t\, dt + T^2 (\int_0^T |\frac{df}{dt}|\, ds)^2\ .$$

Therefore

$$T^2 |\frac{du}{dt}(T)|^2 \leq 4 \int_0^T (|Au| + |f|) |\frac{du}{dt}|\, t\, dt + 4 \int_0^T \varphi(u) dt +$$

$$+ 2\, T^2 (\int_0^T |\frac{df}{dt}|\, ds)^2\ .$$

In other words we have on $(0, T)$

$$F(t)^2 \leq \int_0^t F(s)\, G(s)\, ds + H(t) \tag{55}$$

where $F(t) = t |\frac{du}{dt}|$, $G(t) = 4(|Au| + |f|)$ and $H(t) =$ $= 4 \int_0^t \varphi(t) ds + 2\, t^2 (\int_0^t |\frac{df}{dt}|\, ds)^2$. If $G \geq 0$, (55) implies

that

$$F(t) \leq \frac{1}{2}\int_0^t G(s)\,ds + (\sup_{[0,t]} |H|)^{1/2} .$$

Combined with Estimate I , we get a bound independent of λ for $t\left|\frac{du}{dt}(t)\right|$ in $L^\infty(0,T)$.

Problem. We have seen that the semigroups $S(t)$ generated by several classes of maximal monotone operators A ($A = \partial\varphi$, $\operatorname{Int} D(A) \neq \phi$ or general A if $\dim H < +\infty$ by reduction to the previous case) have a smoothing effect on the initial data.

More precisely:

$$(56)\begin{cases} S(t) \text{ maps } \overline{D(A)} \text{ into } D(A) \text{ and for every } x \in \overline{D(A)}, \\ \text{there is a constant } C \text{ such that } t\left|\frac{d^+}{dt} S(t)x\right| \leq C \text{ for} \\ t \in (0,1] . \end{cases}$$

Is it possible to find a simple characterization of maximal monotone operators A such that the semigroup $S(t)$ generated by $-A$ satisfies (56)? Note that in the linear case, we get exactly the generators of analytic semigroups of contractions.

IV Some applications to nonlinear partial differential equations.

Example 5. Let γ be a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ such that $0 \in D(\gamma)$.

Corollary 28. Let $u_0(x) \in L^2(\Omega)$ be such that $u_0(x) \in \overline{D(\gamma)}$ a.e. on Ω and let f be absolutely continuous from $[0,T]$ into $L^2(\Omega)$. There exists a unique solution u of the equation

$$\frac{\partial u}{\partial t} - \Delta u + \gamma(u) \ni f \qquad \text{a.e. on } \Omega\times(0,T)$$

$$u(x,t) = 0 \qquad \text{a.e. on } \Gamma\times(0,T)$$

$$u(x,0) = u_0(x) \qquad \text{a.e. on } \Omega$$

<u>satisfying</u> $u \in C([0,T]; L^2(\Omega))$, $tu \in L^\infty(0,T;H^2(\Omega) \cap H_0^1(\Omega))$, $t\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega))$, $u(x,t) \in H^2(\Omega) \quad \forall t \in (0,T]$.
<u>In addition we have for every</u> $t \in (0,T)$

$$(57)\quad \begin{cases} \dfrac{\partial^+ u}{\partial t} - \Delta u + \gamma(u) = f \text{ a.e. on } \{x \in \Omega; \gamma \text{ is singlevalued at } u(x,t)\} \\[2ex] \dfrac{\partial^+ u}{\partial t} + \mathrm{Proj}_{\gamma(u)} f = f \text{ a.e. on } \{x \in \Omega; \gamma \text{ is multivalued at } u(x,t)\} \end{cases}$$

(<u>where</u> $\mathrm{Proj}_I r$ = <u>projection of</u> $r \in \mathbb{R}$ <u>on the closed interval</u> $I \subset \mathbb{R}$).

<u>Proof of Corollary 28</u>. The first part is immediate by combining Theorem 23 and Corollary 13 (applied with $\beta(0) = \mathbb{R}$, $\beta(r) = \phi$ for $r \neq 0$). Thus we have only to prove (57). In view of (50) we consider $(-\Delta u + \gamma(u) - f)^0$ for a fixed t. Let $D = \{r \in \mathbb{R};\ \gamma(r)$ contains more than 1 point $\}$; obviously D is denumerable. On the set $\{x \in \Omega;\ u(x,t) \notin D\}$ we have clearly $(-\Delta u + \gamma(u) - f)^0 = -\Delta u + \gamma(u) - f$. We use now the fact that if $u(x) \in H^2(\Omega)$, then

$$(58)\qquad \Delta u = 0 \text{ a.e. on } \{x \in \Omega;\ u(x) \in D\}$$

Indeed, it is standard that for $u \in H^1(\Omega)$ and $c \in \mathbb{R}$, $\operatorname{grad} u = 0$ a.e. on $\{x \in \Omega;\ u(x) = c\}$ (this is based on the following properties: u is differentiable a.e. in the classical sense and, in each direction, a set of positive measure has one as linear density a.e.) Applying again this result to $\operatorname{grad} u$ we see that $\Delta u = 0$ a.e. on $\{x \in \Omega;\ u(x) = c\}$. Considering a denumerable union of such sets we get (58). Therefore a.e. on $\{x \in \Omega;\ u(x,t) \in D\}$ we have

$$(-\Delta u + \gamma(u) - f)^0 = (\gamma(u) - f)^0 = \mathrm{Proj}_{\gamma(u)} f - f .$$

Remark. One can prove that $A = -\Delta + \gamma$ with domain $D(A) = \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$; there is a $g \in L^p(\Omega)$ such that $g(x) \in \gamma(u(x))$ a.e. on $\Omega\}$ is m-accretive in the space $L^p(\Omega)$ (in the sense of Kato [19]; see [9]). But it is not known whether the semigroup generated by $-A$ has a smoothing effect in $L^p(\Omega)$ i.e. is it true that $u(x,t) \in W^{2,p}(\Omega)$ for $t > 0$, assuming $u_0(x) \in L^p(\Omega)$? (The answer is likely to be positive!). More generally which m-accretive operators in Banach spaces play the same role (with regard to smoothing effects) as subdifferentials of convex functions in Hilbert spaces?

Example 6. Let β be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$.

Corollary 29. Let $u_0(x) \in L^2(\Omega)$ and let f be absolutely continuous from $[0,T]$ into $L^2(\Omega)$. There exists a unique solution u of the equation

$$(59)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = f & \text{a.e. on } \Omega \times (0,T) \\ -\dfrac{\partial u}{\partial n} \in \beta(u) & \text{a.e. on } \Gamma \times (0,T) \\ u(x,0) = u_0(x) & \text{a.e. on } \Omega \end{cases}$$

satisfying $u \in C([0,T]; L^2(\Omega))$, $t\,u \in L^\infty(0,T; H^2(\Omega))$, $t\dfrac{\partial u}{\partial t} \in L^\infty(0,T; L^2(\Omega))$, $u(x,t) \in H^2(\Omega)$ $\forall\, t \in (0,T]$.

Proof of Corollary 29. It is a direct application of Theorem 23, taking in $H = L^2(\Omega)$ the function φ of Example 1 (Theorem 12).

Behavior as $t \to +\infty$. Assume $\dfrac{df}{dt} \in L^1(0,+\infty; L^2(\Omega))$ so that $\lim_{t\to+\infty} f(t) = f_\infty$ exists in $L^2(\Omega)$, and suppose

$$(60)\qquad f_\infty \in R(\partial\varphi)$$

(a necessary and sufficient condition for (60) to hold is that

$\int_\Omega f_\infty(x)dx \in (\text{meas } \Gamma)\, R(\beta)$; see [30]).

<u>Theorem 30.</u> <u>If</u> $f - f_\infty \in L^1(0,+\infty; L^2(\Omega))$, <u>then</u> $\lim_{t\to+\infty} u(x,t) = u_\infty(x)$ <u>exists in</u> $L^2(\Omega)$, and $u_\infty \in H^2(\Omega)$ <u>satisfies</u>

$$(61) \qquad -\Delta u_\infty = f_\infty \text{ a.e. on } \Omega, \quad -\frac{\partial u_\infty}{\partial n} \in \beta(u_\infty) \text{ a.e. on } \Gamma .$$

<u>Proof of Theorem 30.</u> It is based on estimates which appear in the proof of Theorem 22, and also on a <u>compactness</u> argument. Let v_∞ be such that $f_\infty \in \partial\varphi(v_\infty)$. We have

$$|u(t) - v_\infty|_{L^2(\Omega)} \le |u_0 - v_\infty|_{L^2(\Omega)} + \int_0^t |f(s) - f_\infty|_{L^2(\Omega)}\, ds .$$

On the other hand, let

$$\tilde{\varphi}(u) = \varphi(u) - \varphi(v_\infty) - \int_\Omega f_\infty (u - v_\infty)\, dx .$$

Thus

$$\tilde{\varphi}(u) = \frac{1}{2}\int_\Omega |\text{grad } u|^2 dx + \int_\Gamma j(u)d\Gamma - \frac{1}{2}\int_\Omega |\text{grad } v_\infty|^2 dx$$

$$- \int_\Gamma j(v_\infty)d\Gamma - \int_\Omega f_\infty (u - v_\infty)\, dx$$

$$= \frac{1}{2}\int_\Omega |\text{grad } u - \text{grad } v_\infty|^2 dx + \int_\Omega \text{grad } u \cdot \text{grad } v_\infty dx +$$

$$+ \int_\Omega \Delta v_\infty (u - v_\infty)dx + \int_\Gamma j(u) - j(v_\infty)\, d\Gamma$$

$$= \frac{1}{2}\int_\Omega |\text{grad } u - \text{grad } v_\infty|^2 dx + \int_\Omega |\text{grad } v_\infty|^2 dx +$$

$$\int_\Gamma \frac{\partial v_\infty}{\partial n}(u - v_\infty) + j(u) - j(v_\infty)d\Gamma \ge \frac{1}{2}\int_\Omega |\text{grad } u - \text{grad } v_\infty|^2 dx.$$

Clearly we have $\frac{du}{dt} + \partial\tilde{\varphi}(u) \ni f - f_\infty$. Therefore we deduce from (47) that

$$t\,\tilde{\varphi}(u(t)) \le \frac{1}{4}\int_0^t |f(s) - f_\infty|^2_{L^2} ds + \frac{1}{2}\left(|u_0 - v_\infty|_{L^2} + \int_0^t |f(s) - f_\infty| ds\right)^2.$$

Consequently $\|u(x,t)\|_{H^1(\Omega)}$ is bounded as $t \to +\infty$, and there is a sequence $t_n \to +\infty$ such that $u(x,t_n) \to u_\infty(x)$ in $L^2(\Omega)$. Since $f - \frac{du}{dt} \in \partial\varphi(u)$ and $f - \frac{du}{dt} \to f_\infty$ as $t \to +\infty$ in $L^2(\Omega)$ we have $f_\infty \in \partial\varphi(u_\infty)$. In addition, for $t_1 \ge t_2$, we have

$$|u(t_1) - u_\infty|_{L^2} \le |u(t_2) - u_\infty|_{L^2} + \int_{t_2}^{t_1} |f(s) - f_\infty|_{L^2} ds$$

which shows that $\lim_{t \to +\infty} u(x,t) = u_\infty(x)$ in $L^2(\Omega)$.

<u>Remark.</u> In general, the solution of (61) need not be unique. It would be of interest to "recognize" u_∞ among all solutions of (61).

<u>Comments.</u> Equations of the form (59) are of physical interest and were studied by many authors. The case where β is a continuous (or even Lipschitz continuous) function was considered by A. Friedman in [17] (results concerning the behavior as $t \to +\infty$ are given only under the restrictive assumption $\beta(r_1) - \beta(r_2) \cdot (r_1 - r_2) \ge \alpha |r_1 - r_2|^2 \quad \alpha > 0$).

The case where
$$\beta(r) = \begin{cases} 0 & \text{for } r > 0 \\ (-\infty, 0] & \text{for } r = 0 \\ \phi & \text{for } r < 0 \end{cases}$$

corresponds to variational inequalities introduced in [23].

The case where $\beta(r) = \begin{cases} +1 & \text{for } r > 0 \\ [-1,+1] & \text{for } r = 0 \\ -1 & \text{for } r < 0 \end{cases}$

appears in [16]. Further regularity results in L^p spaces for the solution of (59) have been obtained in [5] (but again, the question of smoothing effect in L^p has not been settled).

Example 7. Let β be a monotone and continuous (for simplicity!) function from $\mathbb{R}$ onto $\mathbb{R}$ i.e. $R(\beta) = \mathbb{R}$.

Corollary 31. Let $u_0(x) \in H^{-1}(\Omega)$ and let f be absolutely continuous from $[0,T]$ into $H^{-1}(\Omega)$. There exists a unique solution u of the equation

$$(62) \quad \begin{cases} \dfrac{\partial u}{\partial t} - \Delta\beta(u) = f & \text{on } \Omega \times (0,T) \\ \beta(u(x,t)) = 0 & \text{on } \Gamma \times (0,T) \\ u(x,0) = u_0(x) & \text{on } \Omega \end{cases}$$

satisfying $u \in C([0,T]; H^{-1}(\Omega))$, $t\,\beta(u) \in L^\infty(0,T: H_0^1(\Omega))$, $t\dfrac{\partial u}{\partial t} \in L^\infty(0,T; H^{-1}(\Omega))$, $t\,u \in L^\infty(0,T;L^1(\Omega))$, $u(x,t) \in L^1(\Omega)$ and $\beta(u(x,t)) \in H_0^1(\Omega)$ $\forall\, t \in (0,T]$.

Proof of Corollary 31. It is a direct application of Theorem 23 taking in $H = H^{-1}(\Omega)$ the function φ of Example 3 (Theorem 17). Note that $L^\infty(\Omega) \subset D(\varphi)$ and thus $D(\varphi)$ is dense in $H^{-1}(\Omega)$.

Behavior as $t \to +\infty$. Assume $\dfrac{df}{dt} \in L^1(0,+\infty; H^{-1}(\Omega))$ so that $\lim\limits_{t\to+\infty} f(t) = f_\infty$ exists in $H^{-1}(\Omega)$.

Corollary 32. We have $\lim\limits_{t\to+\infty} \|\beta(u(x,t)) - g_\infty(x)\|_{H_0^1} = 0$ where

$g_\infty \in H_0^1(\Omega)$ is the solution of $-\Delta g_\infty = f_\infty$. In addition if $\| \frac{df}{dt}(t) \|_{H^{-1}(\Omega)} = 0(t^{-\alpha})$ for some $\alpha > 2$ then $\| \beta(u(x,t)) - g_\infty(x) \|_{H_0^1} = 0\,(t^{-1})$ as $t \to +\infty$.

Corollary 32 is a direct application of Theorem 24.

Comments. Equations of the form (62) have been extensively studied in the literature, but the properties we present here (smoothing effect and behavior at infinity) seem to be new. Sharp regularity results were obtained for the case where $\beta(r) = |r|^{m-1}r$ and $\Omega = \mathbb{R}$ by Aronson [1]; cf also [3] and [21] for additional references.

After some transformations (see [3]), the Stefan free boundary value problem can be written in the form (62) with

$$\beta(r) = \begin{cases} a_1 r & \text{for } r \leq 0 \\ 0 & \text{for } 0 < r < k \\ a_2(r-k) & \text{for } r \geq k \end{cases}$$

$(a_1, a_2 > 0)$; cf also [18] for another approach.

Example 8. Let β be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $D(\beta) = \mathbb{R}$ and $0 \in \beta(0)$. Let $j: \mathbb{R} \to \mathbb{R}$ be such that $\partial j = \beta$.

Theorem 33. Let $u_0 \in H_0^1(\Omega)$ with $j(u_0) \in L^1(\Omega)$ and let $v_0 \in L^2(\Omega)$. Let $f \in L^1(0,T;L^2(\Omega))$. There exist two functions u and g satisfying

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \Delta u + g = f & \text{in the sense of distribution on } \Omega \times (0,T) \\ u(x,t) = 0 & \text{on } \Gamma \times (0,T) \\ u(x,0) = u_0(x), \ \dfrac{\partial u}{\partial t}(x,0) = v_0(x) & \text{on } \Omega \\ g(x,t) \in \beta(u(x,t)) & \text{a.e. on } \Omega \times (0,T) \end{cases}$$

$$u \in L^\infty(0,T;H_0^1(\Omega)),\ \frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)),\ g \in L^1(\Omega \times (0,T)) .$$

<u>Proof of Theorem 33.</u> Let $u_{0\lambda} \in H^2(\Omega) \cap H_0^1(\Omega)$ be such that $u_{0\lambda} \to u_0$ in $H_0^1(\Omega)$ and $j(u_{0\lambda}) \to j(u_0)$ in $L(\Omega)$ as $\lambda \to 0$. Let $v_{0\lambda} \in H_0^1(\Omega)$ be such that $v_{0\lambda} \to v_0$ in $L^2(\Omega)$ as $\lambda \to 0$. Let f_λ with $\frac{\partial f_\lambda}{\partial t} \in L^1(0,T;L^2(\Omega))$ be such that $f_\lambda \to f$ in $L^1(0,T;L^2(\Omega))$ as $\lambda \to 0$. Let u_λ be the solution of the equation

$$(63)\quad \begin{cases} \dfrac{\partial^2 u_\lambda}{\partial t^2} - \Delta u_\lambda + \beta_\lambda(u_\lambda) = f_\lambda & \text{on } \Omega \times (0,T) \\ u_\lambda(x,t) = 0 & \text{on } \Gamma \times (0,T) \\ u_\lambda(x,0) = u_{0\lambda}(x),\ \dfrac{\partial u_\lambda}{\partial t}(x,0) = v_{0\lambda}(x) & \text{on } \Omega . \end{cases}$$

Note that (63) can be written in the form of a system in $H = H_0^1(\Omega) \times L^2(\Omega)$ as

$$\frac{dU_\lambda}{dt} + A\,U_\lambda + B\,U_\lambda = F_\lambda\ ,\quad U_\lambda(0) = U_{0\lambda}$$

where

$$U = \begin{pmatrix} u \\ \frac{\partial u}{\partial t} \end{pmatrix},\ A = \begin{pmatrix} 0\ , & -I \\ -\Delta\ , & 0 \end{pmatrix}\ B = \begin{pmatrix} 0 \\ \beta_\lambda \end{pmatrix} F = \begin{pmatrix} 0 \\ f_\lambda \end{pmatrix},\quad U_{0\lambda} = \begin{pmatrix} u_{0\lambda} \\ v_{0\lambda} \end{pmatrix}$$

Since B is Lipschitz on H and $U_{0\lambda} \in D(A)$, $\frac{dF}{dt} \in L^1(0,T;H)$ there is a strong solution u_λ of (63)

Multiplying (63) by $\frac{\partial u_\lambda}{\partial t}$ and integrating over Ω we get

$$\frac{1}{2}\frac{d}{dt}\left|\frac{\partial u_\lambda}{\partial t}\right|^2_{L^2} + \frac{d}{dt}\left|\operatorname{grad} u_\lambda\right|^2_{L^2} + \frac{d}{dt}\int_\Omega j_\lambda(u_\lambda)\,dx \le |f_\lambda|_{L^2}\left|\frac{\partial u_\lambda}{\partial t}\right|_{L^2}$$

which implies that $\left|\frac{\partial u_\lambda}{\partial t}\right|_{L^2}$ and $\left|\text{grad } u_\lambda\right|_{L^2}$ are bounded in $L^\infty(0,T)$ as $\lambda \to 0$.

Next we multiply (63) by $v_\lambda = (I + \lambda\beta)^{-1} u_\lambda$ and we integrate over $\Omega \times (0,T)$. We obtain

$$\int_0^T\int_\Omega \beta_\lambda(u_\lambda) v_\lambda \, dxdt \le \int_0^T\int_\Omega |f_\lambda| \, |u_\lambda| dxdt + \int_0^T\int_\Omega \left|\frac{\partial u_\lambda}{\partial t}\right|^2 dxdt$$

$$+ \int_\Omega \left|\frac{\partial u_\lambda}{\partial t}(x,T)\right| |u_\lambda(x,T)| dx + \int_\Omega |v_{0\lambda}(x)| \, |u_{0\lambda}(x)| dx$$

which is bounded as $\lambda \to 0$.

Let $\lambda_n \to 0$ be such that

$$u_{\lambda_n} \to u \qquad \text{weakly in } L^\infty(0,T;\, H_0^1(\Omega))$$

$$\frac{\partial u_{\lambda_n}}{\partial t} \to \frac{\partial u}{\partial t} \qquad \text{weakly in } L^\infty(0,T;\, L^2(\Omega))$$

$$u_{\lambda_n} \to u \qquad \text{a. e. on } \Omega \times (0,T)$$

$$\beta_{\lambda_n}(u_{\lambda_n}) \to g \qquad \text{weakly in } L^1(\Omega \times (0,T)) \, ;$$

such a λ_n exists by Theorem 18 and also $g(x,t) \in \beta(u(x,t))$ a.e. on $\Omega \times (0,T)$.

<u>Example 9.</u> Let β be a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $D(\beta) = \mathbb{R}$ and $0 \in \beta(0)$. Let $j: \mathbb{R} \to \mathbb{R}$ be such that $\partial j = \beta$.

<u>Theorem 34.</u> <u>Let</u> $u_0 \in H^1(\Omega)$ <u>with</u> $j(u_0) \in L^1(\Gamma)$ <u>and let</u> $v_0 \in L^2(\Omega)$. Let $f \in L^1(0,T;L^2(\Omega))$. <u>There exist two functions</u> u <u>and</u> g <u>satisfying</u>

$$-\int_0^T\int_\Omega \frac{\partial u}{\partial t}\cdot\frac{\partial \zeta}{\partial t}dx\,dt+\int_0^T\int_\Omega \operatorname{grad} u\cdot\operatorname{grad}\zeta\,dx\,dt+\int_0^T\int_\Gamma g\,\zeta\,d\Gamma\,dt$$

$$=\int_0^T\int_\Omega f\cdot\zeta\,dx\,dt+\int_\Omega v_0(x)\,\zeta\,(x,0)dx$$

$$\forall\zeta;\ \frac{\partial\zeta}{\partial t}\in L^2(0,T;H^1(\Omega)),\ \zeta\in C(\bar\Omega\times[0,T]),\ \zeta(x,T)=0$$

$g(x,t)\in\beta(u(x,t))$ a.e. on $\Gamma\times(0,T)$

$u(x,0)=u_0(x)$ a.e. on Ω

$u\in L^\infty(0,T;H^1(\Omega))$, $\frac{\partial u}{\partial t}\in L^\infty(0,T;L^2(\Omega))$, $g\in L^1(\Gamma\times(0,T))$.

<u>Proof of Theorem 34.</u> We consider the approximate equation

$$(64)\quad\begin{cases}\dfrac{\partial^2 u_\lambda}{\partial t^2}-\Delta u_\lambda=f & \text{on } \Omega\times(0,T)\\[2ex] -\dfrac{\partial u_\lambda}{\partial n}=\beta_\lambda(u_\lambda)+\lambda\dfrac{\partial u_\lambda}{\partial t} & \text{on } \Gamma\times(0,T)\\[2ex] u_\lambda(x,0)=u_0(x),\ \dfrac{\partial u_\lambda}{\partial t}(x,0)=v_0(x) & \text{on } \Omega\end{cases}$$

which we write in the form of a system in $H^1(\Omega)\times L^2(\Omega)$ as

$$\frac{dU_\lambda}{dt}+A\,U_\lambda=F\ ,\quad U_\lambda(0)=U_0$$

where $A\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}-v\\-\Delta u\end{pmatrix}$ and

$D(A)=\{\begin{pmatrix}u\\v\end{pmatrix};\ u\in H^2(\Omega),\ v\in H^1(\Omega)$ and $-\frac{\partial u}{\partial n}=\beta_\lambda(u)+\lambda v$ on $\Gamma\}$.

There is a constant $\gamma(\lambda)$ such that $A+\gamma I$ is maximal monotone.

Indeed

$$\langle AU_1 - AU_2, U_1 - U_2 \rangle = -\int_\Omega \operatorname{grad}(v_1 - v_2) \cdot \operatorname{grad}(u_1 - u_2)\, dx$$

$$- \int_\Omega (v_1 - v_2)(u_1 - u_2)\, dx - \int_\Omega \Delta(u_1 - u_2)(v_1 - v_2)\, dx$$

$$= \int_\Gamma (\beta_\lambda(u_1) - \beta_\lambda(u_2))(v_1 - v_2) + \lambda |v_1 - v_2|^2 d\Gamma$$

$$- \int_\Omega (v_1 - v_2)(u_1 - u_2)\, dx$$

$$\geq \int_\Gamma \lambda |v_1 - v_2|^2 - \frac{1}{\lambda} |u_1 - u_2| |v_1 - v_2|\, d\Gamma$$

$$- \int_\Omega |v_1 - v_2| |u_1 - u_2|\, dx$$

$$\geq -\gamma \left(\|u_1 - u_2\|^2_{H^1} + |v_1 - v_2|^2_{L^2} \right).$$

Thus (64) has a generalized solution for $\binom{u_0}{v_0} \in \overline{D(A)} = H^1(\Omega) \times L^2(\Omega)$ and $f \in L^1(0,T; L^2(\Omega))$. We multiply first (64) by $\frac{\partial u_\lambda}{\partial t}$ and integrate over Ω :

$$\frac{1}{2} \frac{d}{dt} \left| \frac{\partial u_\lambda}{\partial t} \right|^2_{L^2} + \frac{1}{2} \frac{d}{dt} |\operatorname{grad} u_\lambda|^2_{L^2} + \int_\Gamma (\beta_\lambda(u_\lambda) + \lambda \frac{\partial u_\lambda}{\partial t}) \frac{\partial u_\lambda}{\partial t}\, d\Gamma$$

$$= \int_\Omega f \cdot \frac{\partial u_\lambda}{\partial t}\, dx ,$$

which leads to a bound independent of λ for $\frac{\partial u_\lambda}{\partial t}$ in $L^\infty(0,T; L^2(\Omega))$ and for u_λ in $L^\infty(0,T; H^1(\Omega))$. Next we multiply (64) by $v_\lambda = (I + \lambda\beta)^{-1} u_\lambda$ and integrate over $\Omega \times (0,T)$:

$$\int_0^T\int_\Gamma (\beta_\lambda(u_\lambda) + \lambda \frac{\partial u_\lambda}{\partial t})v_\lambda \, d\Gamma \, dt \le \int_\Omega \left|\frac{\partial u_\lambda}{\partial t}(x,T)\right| |u_\lambda(x,T)| dx$$

$$+ \int_\Omega |v_0(x)| |u_0(x)| dx + \int_0^T\int_\Omega \left|\frac{\partial u_\lambda}{\partial t}\right|^2 dx\, dt + \int_0^T\int_\Omega |f| |u_\lambda| \, dx\, dt ,$$

which leads to a bound on $\int_0^T\int_\Gamma \beta_\lambda(u_\lambda)v_\lambda \, d\Gamma \, dt$ independent of λ.

At last we multiply (64) by a function ζ satisfying the assumptions of Theorem 34 and we integrate on $\Omega \times (0,T)$:

$$-\int_0^T\int_\Omega \frac{\partial u_\lambda}{\partial t} \cdot \frac{\partial \zeta}{\partial t} \, dx\, dt + \int_0^T\int_\Omega \operatorname{grad} u_\lambda \cdot \operatorname{grad} \zeta \, dx\, dt$$

$$-\int_0^T\int_\Gamma \beta_\lambda(u_\lambda)\zeta \, d\Gamma dt - \int_0^T\int_\Gamma \lambda \frac{\partial u_\lambda}{\partial t} \zeta \, d\Gamma \, dt$$

$$= \int_0^T\int_\Omega f \cdot \zeta \, dx\, dt + \int_\Omega v_0(x) \, \zeta(x,0) dx$$

But

$$\int_0^T\int_\Gamma \lambda \frac{\partial u_\lambda}{\partial t} \zeta \, d\Gamma \, dt = -\lambda \int_\Gamma u_0(x) \zeta(x,0) d\Gamma - \lambda \int_0^T\int_\Gamma u_\lambda \frac{\partial \zeta}{\partial t} dx\, dt.$$

Finally we let $\lambda \to 0$ as in the proof of Theorem 33. Notice that all the integrations by parts we have done can be justified by using an approximation of $\binom{u_0}{v_0}$ by $\binom{u_{0\alpha}}{v_{0\alpha}} \in D(A)$ and of f by f_α with $\frac{\partial f_\alpha}{\partial t} \in L^1(0,T;L^2(\Omega))$ so that the corresponding solution $u_{\lambda\alpha}$ of (64) is a strong solution.

<u>Comments.</u> Results related to Theorem 33 were proved by W. Strauss [31] for the case where β is singlevalued and continuous (but not necessarily monotone) and by J. L. Lions [22] for the case where

$$\beta(r) = \begin{cases} +1 & \text{if } r > 0 \\ [-1,+1] & \text{if } r = 0 \\ -1 & \text{if } r < 0 \end{cases}$$

Theorem 34 answers a question raised in [22].

Problem. More generally it would be of interest to solve the equation $\frac{d^2u}{dt^2} + \partial\varphi(u) \ni f$, $u(0) = u_0$, $\frac{du}{dt}(0) = v_0$.

In the particular case where $\varphi = I_K$ is the indicator function of a closed convex set K, the solution u represents, roughly speaking, the trajectory of an optical ray caught in K and reflecting at the boundary of K.

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