Introduction to the vertex algebra approach to mirror symmetry

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Abstract

The goal of this paper is to make the vertex operator algebra approach to mirror symmetry accessible to algebraic geometers. Compared to better-known approaches using moduli spaces of stable maps and special Lagrangian fibrations, this approach follows more closely the original line of thinking that lead to the discovery of mirror symmetry by physicists. The ultimate goal of the vertex algebra approach is to give precise mathematical definitions of N=(2,2) superconformal field theories called A and B models associated to any Calabi-Yau variety and then show that thus constructed theories are related by the mirror involution for all known examples of mirror symmetric varieties.

1 Introduction

This paper should serve as an introduction to the vertex algebra approach to mirror symmetry developed in [3]. It is thus understandable that our emphasis and selection of topics reflects the author's bias. The reader should keep in mind that other approaches to mirror symmetry exist and have independent mathematical interest. In particular, the stable maps approach allowed to state and prove mathematically the predictions for the (virtual) numbers of rational curves on a quintic threefold and other similar examples, see [11, 8, 13].

It is widely stated in physics literature that given a Calabi-Yau manifold X together with an element of its complexified Kähler cone one can construct two N=(2,2) superconformal field theories called A and B models, see for example [17]. On the other hand, all actual calculations and definitions of these theories involve Feynman type integrals over infinite-dimensional spaces of all maps from Riemann surfaces to X. While physicists have developed a good intuitive understanding of the formal properties of these integrals, they are mathematically ill-defined.

The precise axiomatic definition of $N = (2, 2)$ superconformal field theory that would include the A and B models above is still not available. Roughly speaking, this theory is a modular functor, see for example [15], but the number of labels is, perhaps, infinite. In particular, there must exist a Hilbert space H such that every Riemann surface whose oriented boundary consists of k incoming and l outcoming circles produces an operator from $H^{\otimes k}$ to $H^{\otimes l}$, perhaps defined only up to a scalar multiple. Superconformal field theory is a highly complicated object. Even when the Riemann surface is a sphere, the structure of superconformal field theory is rather non-trivial. A typical way to construct such a theory is by building it from the representation theory of vertex algebras that satisfy certain conditions, see for example [9]. In fact, mirror symmetry originated from the work of Gepner [7] who used (finite quotients of) tensor products of the so-called minimal models which are certain irreducible representations of the N=2 superconformal algebra. He was able to match the dimensions of chiral rings (see [12]) of the resulting theories with the dimensions of the cohomology spaces of the Calabi-Yau hypersurfaces in projective spaces, in particular quintic threefolds.

Vertex algebra approach to mirror symmetry attempts to define rigorously superconformal field theories associated to Calabi-Yau manifolds and then prove that the corresponding theories for mirror manifolds are related to each other. At this stage only the vertex algebra of the theory has been recovered and much work is still to be done. This review contains no new results, and no proofs are presented. It is intended as an introduction to vertex algebras for algebraic geometers, and its ultimate goal is to enable an interested reader to understand the paper [3]. In particular, only vertex algebras that appear in the context of hypersurfaces in toric varieties are discussed.

Section 2 contains basic definitions and properties of vertex algebras, and follows closely the book of Kac [10]. Section 3 provides the reader with a simplest non-trivial example of vertex algebra called one free boson. It is generalized to several bosons and several fermions in Section 4. Section 5 is devoted to the very important paper of Malikov, Schechtman and Vaintrob [14] who construct chiral de Rham complex of an arbitrary smooth variety. We are mostly interested in the case of Calabi-Yau varieties. Section 6 explains main results of [3] and the last section summarizes briefly the problems that are still to be addressed in the vertex algebra approach.

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2 Definition and basic properties of vertex algebras

The goal of this section is to state definitions of vertex algebras and to introduce important notions of normal ordered products and operator product expansions (OPE). Our treatment follows closely the book of Kac [10]. We also define N=2 superconformal structures and describe BRST cohomology construction necessary to understand [3].

Definition 2.1 ([10]) A vertex algebra is the set of data that consists of a super vector space V (over $\mathbf{C}$), a *state-field correspondence* Y and a *vacuum vector* $|0\rangle$. The fact that V is a superspace simply means that $V = V_0 \oplus V_1$. Elements of V_0 are called bosonic or even and elements from V_1 are called fermionic or odd. Vacuum vector $|0\rangle$ is a bosonic element of V. The most important structure is the state-field correspondence Y which is a parity preserving linear map from V to $\mathrm{End}V[[z,z^{-1}]]$

$$a \mapsto Y(a,z) = \sum_{n\in Z} a_{(n)} z^{-n-1}$$

such that for every two elements a and b the elements $a_{(n)}b$ are zero for all sufficiently big n. To form a vertex algebra the data $(V, Y, |0\rangle)$ must satisfy the following axioms.
•**translation covariance:** $\{T, Y(a,z)\}_- = \partial_z Y(a,z)$ where $\{,\}_-$ denotes the usual commutator and T is defined by $T(a) = a_{(-2)}|0\rangle$;
•**vacuum:** $Y(|0\rangle, z) = \mathbf{1}_V,\ Y(a,z)|0\rangle_{z=0} = a$;
•**locality:** $(z-w)^N \{Y(a,z), Y(b,w)\}_{\mp} = 0$ for all sufficiently big N, where $\mp$ is $+$ if and only if both a and b are fermionic. The equality is understood as an identity of formal power series in z, z^{-1}, w and w^{-1}. It is often expressed by saying that $Y(a,z)$ and $Y(b,z)$ are mutually local.

Let a and b be two elements of the vertex algebra V. We denote the corresponding *fields* $Y(a,z)$ and $Y(b,z)$ by $a(z)$ and $b(z)$ respectively. Locality axiom of the vertex algebra allows one to express the supercommutators of the *modes* $a_{(m)}$ and $b_{(n)}$ in a concise way in terms of *operator product expansions* (OPEs). Namely, define *normal ordered product* $:a(z)b(w): \in \mathrm{End}V[[z,z^{-1},w,w^{-1}]]$ by the formula

$$:a(z)b(w): = \sum_{m\in\mathbf{Z}_{<0}, n\in\mathbf{Z}} a_{(m)}b_{(n)} z^{-m-1}w^{-n-1} \pm \sum_{m\in\mathbf{Z}_{\geq 0}, n\in\mathbf{Z}} b_{(n)}a_{(m)} z^{-m-1}w^{-n-1}$$

where $\pm$ is $-$ if and only if both a and b are fermionic. Then it is not hard to show (see [10] for details) that locality axiom implies

$$a(z)b(w) = \sum_{j=0}^{N-1} \frac{c^j(w)}{(z-w)^{j+1}} + \ :a(z)b(w):$$

where $c^j(w)$ are some elements of $\mathrm{End}V[[w,w^{-1}]]$ and $(z-w)^{-j-1}$ is Laurent expanded in the region $|z| > |w|$. Moreover, there holds a remarkable Borcherds OPE formula, which states that $c^j(w) = Y(a_{(j)}b, w)$ so c^j are also fields in the vertex algebra. All information about supercommutators of the modes of a and b is conveniently encoded in the $\sum$ part of this OPE and the parities of a and b. The $\sum$ part is called *singular part* of the OPE.

A vertex algebra with a conformal structure is a vertex algebra $(V, Y, |0\rangle)$ with a choice of an even element v such that the corresponding field $Y(v,z) =: L(z)$ satisfies the operator product expansion

$$L(z)L(w) = \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w} + \ :L(z)L(w):$$

where c is a constant called *central charge*. In addition, one assumes that $L_{(0)}$ coincides with the operator T in the definition of vertex algebra. We also assume that $L_{(-1)}$ is diagonalizable on V, all of its eigenvalues are real numbers, and

$$\{L_{(-1)}, Y(a,z)\}_- = z\partial_z Y(a,z) + Y(L_{(-1)}a, z)$$

for all a.

Remark 2.2 The same vertex algebra can have many different conformal structures. Element v that defines a conformal structure is called Virasoro element.

Once a conformal structure is fixed, it is customary to shift the index in the definition of $a_{(n)}$ as follows. If a has eigenvalue α with respect to $L_{(-1)}$ then we introduce the notation

$$Y(a,z) = \sum_{n\in\mathbf{Z}} a_{(n)} z^{-n-1} =: \sum_{n\in\mathbf{Z}-\alpha} a[n] z^{-n-\alpha}.$$

The number α is called the *conformal weight* of a. In particular, we observe that OPE of $L(z)L(w)$ implies that conformal weight of the Virasoro element is two, and we introduce $L(z) = \sum_{n\in\mathbf{Z}} L[n]z^{-n-2}$. In these notations the endomorphisms $L[m]$ satisfy the commutator relations

$$\{L[m], L[n]\}_- = (m-n)L[m+n] + \frac{c}{12}(m^3-m)\delta^0_{m+n}$$

of the Virasoro algebra with central charge c.

When one studies Calabi-Yau manifolds, one obtains vertex algebras which have not only conformal structure, but what is called N=2 superconformal structure. It consists of the choice of conformal structure plus an even field J and two odd fields G^+ and G^- which satisfy the following OPE.

$$L(z)L(w) = \frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w} + \,:L(z)L(w):,$$

$$L(z)J(w) = \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} + \,:L(z)J(w):,$$

$$L(z)G^\pm(w) = \frac{(3/2)G^\pm(w)}{(z-w)^2} + \frac{\partial_w G^\pm(w)}{z-w} + \,:L(z)G^\pm(w):,$$

$$J(z)J(w) = \frac{c/3}{(z-w)^2} + \,:J(z)J(w):,$$

$$J(z)G^\pm(w) = \pm\frac{G^\pm(w)}{z-w} + \,:J(z)G^\pm(w):,$$

$$G^\pm(z)G^\mp(w) = \frac{2c/3}{(z-w)^3} \pm \frac{2J(w)}{(z-w)^2} + \frac{2L(w) \pm \partial_w J(w)}{z-w} + \,:G^\pm(z)G^\mp(w):,$$

$$G^\pm(z)G^\pm(w) = \,:G^\pm(z)G^\pm(w): \,.$$

It is common to introduce N=2 charge $\hat{c} = c/3$, where c is the central charge of the usual Virasoro algebra. Also one often changes the notations slightly by introducing Virasoro field $L_{top} = L(z) + (1/2)\partial_z J(z)$ of conformal charge 0. Then $G^{\pm}$, J and L_{top} form *topological structure of dimension* $\hat{c} = d$.

Notice that if one switches G^+ and G^- and changes the sign of J, then one obtains another N=2 structure. This involution is called *mirror involution* and it is expected to switch A and B models constructed from mirror symmetric Calabi-Yau manifolds. (More precisely, it is supposed to act this way on the holomorphic part and act by identity on the antiholomorphic part of N=(2,2) superconformal field theory, but this paper only deals with the holomorphic part. In fact, the absence of precise understanding of how to put together holomorphic and anti-holomorrhpic parts of the theory is a big obstacle in the vertex algebra approach to mirror symmetry.)

Given a vertex algebra $(V, Y, |0\rangle)$ one can construct other algebras by the BRST cohomology construction. If $a \in V$ is such that $a_{(0)}^2 = 0$, then one considers cohomology of V with respect to $a_{(0)}$, called *BRST cohomology.* Operator $a_{(0)}$ and field $Y(a, z)$ are called BRST operator and BRST field respectively. One can show, see for example [3], that BRST cohomology of V with respect to $a_{(0)}$ has a natural structure of vertex algebra. Moreover, if $a_{(0)}$ supercommutes with the fields of N=2 structure, then this structure descends to BRST cohomology.

3 First example of vertex algebra: one free boson

The simplest non-trivial example of the vertex algebra is called *one free (chiral) bosonic field.* The goal of this section is to describe explicitly the data $(V, Y, |0\rangle)$. Moreover, it turns out that this vertex algebra could be provided with a conformal structure, which we also describe. Most of the calculations are skipped, and the reader is referred to [10].

Consider an abstract unital associative algebra generated by elements $d[n], n \in \mathbf{Z}$ with commutator relations

$$\{d[m], d[n]\}_- = m\delta^0_{m+n}$$

In other words, $d[n]$ commutes with everything except $d[-n]$, and commutators of $d[n]$ and $d[-n]$ are proportional to the identity.

There is a standard representation of this algebra called a *Fock space.* Namely, consider a vacuum vector $|0\rangle$ such that

$$d[\mathbf{Z}_{\geq 0}]|0\rangle = 0$$

and try to see what space could be built from it. We will call operators $d[\mathbf{Z}_{>0}]$ *annihilators* and operators $d[\mathbf{Z}_{<0}]$ *creators.* Operator $d[0]$ commutes with all other operators and will equal zero on this Fock space. Its relevance will be seen later in Section 6 when we talk about vertex algebras defined by a lattice.

Notice that all creators commute with each other. If we apply all creators to the vacuum vector, assuming that the results are linearly independent, we get the space

$$V = \oplus_{n_1, n_2, \ldots} \mathbf{C} \prod_{k>0} d[-k]^{n_k} |0\rangle$$

where all n_k are non-negative integers, and only finitely many of them are nonzero. Creators obviously act on this space. The action of annihilators could be defined by means of the commutator rules and the fact that annihilators vanish on the vacuum vector. For example,

$$d[3]\,d[-1]^2 d[-3]^2 d[-5]|0\rangle = d[-1]d[-3]^2 d[-5]d[3]|0\rangle + d[-1]\{d[3], d[-3]^2\}_- d[-5]|0\rangle$$

$$= 6d[-1]d[-3]d[-5]|0\rangle.$$

Thus V is a representation of the algebra of d and it is called the Fock space of one free bosonic field. One can think of it as the space of polynomials in infinitely many variables $d[-1], d[-2], \ldots$ with creators acting by multiplication and annihilators acting by differentiation.

To describe the structure of vertex algebra on this Fock space V, we will need additional notations. We introduce the field $d(z) \in \mathrm{End}V[z, z^{-1}]$ by the formula

$$d(z) = \sum_{n \in \mathbf{Z}} d[n] z^{-n-1}.$$

Notice that $d[z]|0\rangle = \sum_{k \geq 0} d[-k-1] z^k$, and when you plug in $z = 0$ you get $d[-1]|0\rangle$. Eventually $d[z]$ will be a field that corresponds to $d[-1]|0\rangle$. To construct other fields we use the notion of normal ordering introduced in Section 2. If we try to make sense of $d(z)^2 = d(z)d(z)$ as an element of $\mathrm{End}V[[z, z^{-1}]]$, we run into infinities. On the other hand, one can plug $w = z$ into $:d(z)d(w):$ and the resulting field $:d(z)d(z):$ makes sense as an element of $\mathrm{End}V[[z, z^{-1}]]$. In terms of the modes, one can define

$$:d[m]d[n]: \quad = \quad \begin{cases} d[m]d[n] & \text{if } n \geq 0 \\ d[n]d[m] & \text{if } n < 0 \end{cases}$$

and then write

$$:d(z)d(z): \quad = \sum_{m,n \in \mathbf{Z}} :d[m]d[n]: z^{-m-n-2}.$$

We notice that this field applied to vacuum lies in $V[[z]]$ and

$$:d(z)d(z):|0\rangle_{z=0} = d[-1]^2|0\rangle.$$

Similarly, one defines fields

$$:\prod_{k \geq 0} \left(\frac{\partial^k d}{\partial z^k} \right)^{n_k}:$$

by pushing all annihilators to the right and all creators to the left. Then the claim is that these fields form the fields from the definition of the vertex algebra that correspond to the states

$$\prod_{k \geq 0} (k!)^{n_k} d[-k-1]^{n_k} |0\rangle.$$

One can also show that these fields are mutually local. This allows us to define the state-field correspondence Y which satisfies vacuum and locality axioms. One can also show that translation axiom is satisfied. Moreover, the operator T could be written in terms of $d[n]$ as

$$T = \frac{1}{2} \sum_{k \in \mathbf{Z}} d[k] d[-1-k].$$

As a result, we have constructed our first example of the vertex algebra.

We will now describe how to equip this vertex algebra $(V, Y, |0\rangle)$ with a conformal structure. Look at the field

$$L(z) = \frac{1}{2} : d(z) d(z) :$$

and introduce $L[n]$ by $L(z) = \sum_{n \in \mathbf{Z}} L[n] z^{n-2}$. Explicitly these operators could be written as $L[n] = (1/2) \sum_{k \in Z} : d[k] d[n-k] :$. Then one can check that the following commutator relations hold

$$\{L[m], L[n]\}_- = (m-n) L[m+n] + \frac{1}{12}(m^3 - m)\delta^0_{m+n}.$$

Therefore, $L[m]$ form Virasoro algebra with central charge one.

One observes that $T = L[-1]$ is the translation covariance operator T. Also, $L[0]$ is diagonalizable, because

$$L[0] d[-1]^{n_1} d[-2]^{n_2} ... |0\rangle = (\sum_i i n_i)\ d[-1]^{n_1} d[-2]^{n_2} ... |0\rangle.$$

We can conveniently rewrite the commutators of $d[n]$ in terms of the OPEs. After an easy calculation, we get

$$d(z) d(w) = \frac{1}{(z-w)^2} + \ : d(z) d(w) : \ .$$

In general, it is straightforward to calculate OPEs of products of free bosons and their derivatives using Wick's theorem, see [10]. The key point of Wick's theorem is that the commutator of products of two sets of linear operators such that the pairwise commutators are in the center can be explicitly written in terms of these pairwise commutators.

4 Further examples of vertex algebras: several free fermions and bosons

The construction of the previous section can be generalized in several directions. First of all, instead of considering one free boson, one may consider several of them. This simply means that one considers the tensor product of a number of copies of the Fock space of one free boson with the structure of vertex algebra induced in an obvious way. More generally, for every finite-dimensional vector space W of dimension r over $\mathbf{C}$ equipped with a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$, one constructs a vertex algebra. One considers a unital associative algebra generated by $w[n]$, $n \in \mathbf{Z}$, $w \in W$ with the commutator relations

$$\{w_1[m], w_2[n]\}_- = m\langle w_1, w_2\rangle \delta^0_{m+n}.$$

Then, analogously to the one-dimensional example, one defines a Fock space generated from $|0\rangle$ by applying negative modes of $w[n]$. This Fock space carries a natural structure of vertex algebra, which is isomorphic to the tensor product of $\dim W$ copies of one free boson.

We will be particularly interested in the case of the space W which is a direct sum of a space W_1 and its dual, and thus has a natural non-degenerate symmetric bilinear product denoted by $\cdot$. In terms of the OPEs, the algebra is generated by the fields $a(z)$ and $b(z)$ where $a \in W_1$, $b \in W_1^*$ and the OPEs are

$$a_1(z)a_2(w) = :a_1(z)a_2(w):, \quad b_1(z)b_2(w) = :b_1(z)b_2(w):,$$

$$a(z)b(w) = \frac{a \cdot b}{(z-w)^2} + \ :a(z)b(w): \ .$$

So far we have not introduced any fermionic elements. This is easily accomplished by changing commutators $\{\ ,\ \}_-$ to anticommutators $\{\ ,\ \}_+$ in the above formulas. While the construction could be described for a single free fermion, we will restrict our attention to $2r$ free fermions constructed from $W_1 \oplus W_1^*$ where W_1 is a vector space of dimension r. One starts with a unital associative algebra generated by $\varphi[n]$ and $\psi[n]$ with $\varphi \in W_1^*,\ \psi \in W_1,\ n \in \mathbf{Z} + \frac{1}{2}$ with the anticommutator relations

$$\{\varphi_1[m], \varphi_2[n]\}_+ = 0; \ \{\psi_1[m], \psi_2[n]\}_+ = 0;$$
$$\{\varphi[m], \psi[n]\}_+ = (\varphi \cdot \psi)\delta^0_{m+n}.$$

The Fock space is constructed by applying pairwise anticommuting creators $\varphi[(\mathbf{Z} + \frac{1}{2})_{<0}],\ \psi[(\mathbf{Z} + \frac{1}{2})_{<0}]$ to the vacuum vector $|0\rangle$ which is annihilated by the rest of the modes. The vertex algebra structure is provided by the products of various derivatives of the fields

$$\varphi(z) = \sum_{n \in \mathbf{Z}+\frac{1}{2}} \varphi[n] z^{-n-\frac{1}{2}}, \quad \psi(z) = \sum_{n \in \mathbf{Z}+\frac{1}{2}} \psi[n] z^{-n-\frac{1}{2}},$$

and the parity is defined by the total number of φ and ψ. Operator product expansions of the fields $\varphi(z)$ and $\psi(z)$ are

$$\varphi_1(z)\varphi_2(w) = :\varphi_1(z)\varphi_2(w):, \quad \psi_1(z)\psi_2(w) = :\psi_1(z)\psi_2(w):,$$

$$\varphi(z)\psi(w) = \frac{\varphi \cdot \psi}{z-w} + \ :\varphi(z)\psi(w): \ .$$

One can provide this vertex algebra with the structure of conformal vertex algebra by introducing a field

$$L(z) = \frac{1}{2}\ :\partial\psi^i(z)\varphi_i(z) - \psi^i(z)\partial\varphi_i(z):$$

where $\{\psi^i\}$ and $\{\varphi_i\}$ are dual bases of W_1 and W_1^*. We implicitly sum over all i via standard physical convention. A fermionic analog of Wick's theorem allows us to calculate that central charge of this conformal structure is $\dim W_1$.

Finally, we put together fermions and bosons. The resulting algebra is a crucial component of the constructions of [14] and [3]. Again, let W_1 and W_1^* be two dual spaces. We consider the vertex algebra which is a product of the free fermionic and free bosonic algebras constructed above. It is generated by fields $\varphi(z)$, $\psi(z)$, $a(z)$ and $b(z)$. It is equipped with a conformal structure of central charge $c = 3\dim W_1$. Moreover, one can extend this structure to an N=2 structure

$$G^+(z) := \varphi_i(z)a^i(z),\ G^-(z) := \psi^i(z)b_i(z),\ J(z) := :\varphi_i(z)\psi^i(z):,$$

$$L(z) := \frac{1}{2}\ :a^i(z)b_i(z):\ +\frac{1}{2}\ :\partial\psi^i(z)\varphi_i(z) - \psi^i(z)\partial\varphi_i(z):$$

where again $\{a^i\}$ and $\{b_i\}$ are dual bases of W_1 and W_1^*. We remark that the resulting N=2 fields are independent from the choice of these bases. The N=2 central charge is $\hat{c} = c/3 = \dim W_1$.

5 Chiral de Rham complex

In a breakthrough paper [14] Malikov, Schechtman and Vaintrob have introduced a sheaf of vertex algebras which they call chiral de Rham complex for every complex manifold. Roughly speaking, the idea is to associate to every manifold X a sheaf which locally over a neighborhood of a point $x \in X$ looks like a vertex algebra with $2\dim X$ bosons and $2\dim X$ fermions associated to the vector space $W = T_X(x) \oplus T_x^*(X)$.

There are some important details that we need to address. First of all, one needs to use a slightly different version of the definition of the vertex algebra of free bosons. Namely, instead of the commutator relations

$$\{a[m], b[n]\}_- = m(a\cdot b)\delta^0_{m+n}$$

they use the relations

$$\{a[m], b[n]\}_- = (a\cdot b)\delta^0_{m+n}.$$

The vacuum is now annihilated by $a[\mathbf{Z}_{\geq 0}]$ and $b[\mathbf{Z}_{>0}]$, but $b[0]$ are considered to be creators. The fields $a(z)$ and $b(z)$ are now defined by

$$b(z) := \sum_{n\in\mathbf{Z}} b[n]z^{-n}$$

and the basic OPE is

$$a(z)b(w) = \frac{a\cdot b}{z-w} +\ :a(z)b(w):\ .$$

Roughly speaking, one uses $\int b(w)dw$ instead of $b(w)$. The fields of the N=2 algebra are modified accordingly.

$$G^+_{\mathrm{MSV}}(z) := \varphi_i(z)a^i(z),\ G^-_{\mathrm{MSV}}(z) := \psi^i(z)\partial_z b_i(z),\ J_{\mathrm{MSV}}(z) := :\varphi_i(z)\psi^i(z):,$$

$$L_{\rm MSV}(z) := \frac{1}{2}\ :a^i(z)\partial_z b_i(z):\ + \frac{1}{2}\ :\partial\psi^i(z)\varphi_i(z) - \psi^i(z)\partial\varphi_i(z):\ .$$

The crucial observation of [14] is that the group of automorphisms of the ring of local coordinates of X at x embeds into the group of vertex algebra automorphisms of the above vertex algebra. This allows one to glue together the above spaces and construct a sheaf of vertex algebras over the variety X. Unfortunately, the fields of the $N=2$ algebra are not preserved under general automorphisms. However, if X is a Calabi-Yau variety, then the existence of the holomorphic volume form allows one to restrict the attention to the volume-preserving local changes of coordinates, and these changes do preserve the fields of $N=2$ algebra.

In [14] the resulting sheaf is called *chiral de Rham complex*, because the usual de Rham complex is naturally embedded in it. We will denote the chiral de Rham complex by $\mathcal{MSV}(X)$. An important remark here is that it is not a quasi-coherent sheaf. The multiplication map $\mathcal{O}(X)\times\mathcal{MSV}(X)\to\mathcal{MSV}(X)$ is defined but is not associative. Rather, for every open set U the space of sections of $\mathcal{MSV}$ over U forms a vertex algebra and sections of $\mathcal{O}$ over U are mapped to the set of pairwise commuting bosonic fields in $\Gamma(U,\mathcal{MSV})$. This type of sheaf was called a quasi-loop-coherent sheaf of vertex algebras in [3].

One then observes that cohomology $\mathrm{H}^*(X,\mathcal{MSV})$ of the chiral de Rham complex is provided with a natural structure of vertex algebra, essentially via a cup product, see [3] for details. This vertex algebra has a natural N=2 structure, if X is Calabi-Yau. The corresponding structures of topological algebras, see Section 2, should correspond to the A and B models. In fact, it was the topological twist of the above algebra that was considered in [14], so their definition of L is slightly different. As a result, the (half-integer) notations for the fermionic modes of the operators φ and ψ that we have used above are different from the (integer) notations used in [14]. However, in terms of the natural modes $\varphi_{(n)}$ and $\psi_{(n)}$ our notations coincide.

It is worth mentioning that $\mathcal{MSV}(X)$ possesses a natural filtration, such that the graded object is a quasi-coherent sheaf isomorphic to a tensor product of an infinite number of copies of symmetric and exterior algebras of the tangent and cotangent sheaves on X. This remark allows one to show that the elliptic genus of the variety X can be naturally formulated in terms of the supertrace over the cohomology of $\mathcal{MSV}(X)$ of the operator $y^{J[0]}q^{L[0]}$. The reader is referred to [4] for details.

It would be very interesting to compare the approach of [14] with that of the monograph of Tamanoi [16].

6 Vertex algebras of Calabi-Yau hypersurfaces in toric varieties

We will now talk about the contents of the paper [3]. Its major achievement is an explicit calculation of the cohomology of the chiral de Rham complex for a generic Calabi-Yau hypersurface X in a smooth toric nef-Fano variety $\mathbf{P}$. There is also some progress made in the problem of defining chiral de Rham complex for varieties with

Gorenstein singularities. The description uses certain vertex algebra constructed from a lattice, whose definition will be provided below.

Before we can describe the cohomology of the chiral de Rham complex of a generic Calabi-Yau toric hypersurface, we must recall the combinatorial data which define it that were discovered by Batyrev in [1]. Let M_1 and N_1 be two dual lattices of rank $d+1$ that contain dual reflexive polyhedra Δ_1 and Δ_1^*. One defines dual lattices $M = M_1 \oplus \mathbf{Z}$ and $N = N_1 \oplus \mathbf{Z}$ of rank $d+2$ and cones $K = \{(t\Delta_1, t),\, t \geq 0\}$ and $K^* = \{(t\Delta_1^*, t),\, t \geq 0\}$ in M and N respectively. The conditions on Δ_1 and Δ_1^* to form a dual pair of reflexive polytopes is that all their vertices are lattice points, and that K and K^* are dual to each other, as the notation suggests. To specify a nef-Fano toric variety $\mathbf{P}$ one also chooses a fan Σ in N_1 which subdivides the minimum fan defined by the faces of Δ_1^*. Toric variety $\mathbf{P}$ is smooth if and only if all cones of Σ are generated by a part of the basis of the lattice N_1.

We denote the bilinear form on $M \oplus N$ by $\cdot$. We also denote $(0,1) \in M$ by deg and $(0,1) \in N$ by $\deg^*$. We call $\deg \cdot n$ and $\deg^* \cdot m$ the degree of $n \in N$ and $m \in M$ respectively. Codimension one polytopes $\Delta_1 + \deg$ and $\Delta_1^* + \deg^*$ are denoted by Δ and Δ^* respectively. A generic hypersurface X in $\mathbf{P}$ is defined by a generic collection of coefficients f_m for all lattice points $m \in \Delta$. It is (in general only partial) desingularization of $\mathrm{Proj}(\mathbf{C}[K]/f)$. Here f is an element of degree one defined by $\sum_{m \in \Delta} f_m x^m$ where x is a dummy variable used to write $\mathbf{C}[K]$ in a multiplicative form.

To explain the results of [3], we need to introduce a vertex algebra associated to the lattice $M \oplus N$. As a vector space, it is isomorphic to the vector space of the vertex algebra of $2(d+2)$ free fermions and $2(d+2)$ free bosons constructed in Section 4 tensored with $\mathbf{C}[M \oplus N]$. For any subset I of $M \oplus N$ we denote by Fock_I the space obtained by tensoring (over $\mathbf{C}$) of the vertex algebra of Section 4 and vector space $\mathbf{C}[I]$.

First of all, we need to define how the fields of $\mathrm{Fock}_{0 \oplus 0}$ act on $\mathrm{Fock}_{M \oplus N}$. We use the notations of [3] and denote the bosonic fields of $\mathrm{Fock}_{0 \oplus 0}$ by $m \cdot B(z)$ and $n \cdot A(z)$ and fermionic fields by $m \cdot \Phi(z)$ and $n \cdot \Psi(z)$. Here A, B, Φ, Ψ are vector valued, and m and n are in $M \otimes \mathbf{C}$ and $n \otimes \mathbf{C}$ respectively. For a fixed pair of lattice elements (m, n), the action of fermionic fields Φ and Ψ on $Fock_{m \oplus n}$ is simply induced by their action on $\mathrm{Fock}_{0 \oplus 0}$. The action of the bosonic fields is modified so that the zero modes $(m_1 \cdot B)[0]$ and $(n_1 \cdot A)[0]$ do not annihilate $\mathrm{Fock}_{m \oplus n}$. Rather, they act by a scalar multiplication by $m_1 \cdot n$ and $n_1 \cdot m$ respectively.

To define state-field correspondence we still need to specify which fields of $\mathrm{Fock}_{M \oplus N}$ correspond to the elements of $\mathrm{Fock}_{m \oplus n}$ with non-zero (m, n). We denote the element $(|0\rangle, m \oplus n) \in \mathrm{Fock}_{M \oplus N}$ by $|m, n\rangle$, and our first goal is to describe the field $Y(|m, n\rangle, z)$. For all $(m_1, n_1) \in M \oplus N$ all modes of the field $Y(|m, n\rangle, z)$ map $\mathrm{Fock}_{m_1 \oplus n_1}$ to $\mathrm{Fock}_{(m+m_1) \oplus (n+n_1)}$. We denote the endomorphism of $\mathrm{Fock}_{M \oplus N}$ that commutes with all non-zero modes of A and B and sends $|m_1, n_1\rangle$ to $|m+m_1, n+n_1\rangle$ by $\gamma_{m,n}$. Then

$$Y(|m, n\rangle, z) := \gamma_{m,n} c_{m,n} \; :\mathrm{e}^{\sum_{0 \neq k \in \mathbf{Z}} (-z^{-k}/k)((m \cdot B)[k] + (k \cdot A)[k])}: \; z^{m \cdot B[0]} z^{n \cdot A[0]}$$

where $c_{m,n}$ acts on Fock_{m_1, n_1} by multiplication by $(-1)^{m \cdot n_1}$. One defines $Y(a, z)$ for other elements of $\mathrm{Fock}_{m\oplus n}$ by inserting appropriate free fields and their derivatives inside the normal ordering. It is not hard to see that these operators are well defined, moreover one can show that they are mutually local and satisfy some nice OPEs. Instead of the complicated notation above we use

$$Y(|m,n\rangle,z) := :\mathrm{e}^{\int (m\cdot B+n\cdot A)(z)\,dz}:$$

and similarly for other elements of $\mathrm{Fock}_{m\oplus n}$. All $|m,n\rangle$ are bosonic and satisfy

$$Y(|m,n\rangle,z)Y(|m_1,n_1\rangle,w) = \frac{:\mathrm{e}^{\int (m\cdot B+n\cdot A)(z)\,dz}\mathrm{e}^{\int (m_1\cdot B+n_1\cdot A)(w)\,dw}:}{(z-w)^{m\cdot n_1+m_1\cdot n}}$$

which can be Taylor expanded around $z=w$ to give the OPEs. The details of this calculation could be found in [10]. In general, it is straightforward to calculate OPEs in the lattice algebra, but the resulting expressions could be quite complicated.

The vertex algebra $\mathrm{Fock}_{M\oplus N}$ can be equipped with the following N=2 structure of central charge $\hat{c}=d$. Notice that the rank of the lattice M is $d+2$. We will call it Calabi-Yau N=2 structure, because we will see in a second that it is related to the N=2 structure of the cohomology of the chiral de Rham complex of Calabi-Yau hypersurfaces in toric varieties.

$$G^+_{CY}(z) := (A\cdot\Phi)(z) - \deg\cdot\partial_z\Phi(z)$$

$$G^-_{CY}(z) := (B\cdot\Psi)(z) - \deg^*\cdot\partial_z\Psi(z)$$

$$J_{CY}(z) := :(\Phi\cdot\Psi)(z): + \deg\cdot B(z) - \deg^*\cdot A(z)$$

$$L_{CY}(z) := :(B\cdot A)(z): + \frac{1}{2}\ :(\partial_z\Phi\cdot\Psi - \Phi\cdot\partial_z\Psi)(z): - \frac{1}{2}\deg^*\cdot\partial_z A(z) - \frac{1}{2}\deg\cdot\partial_z B(z).$$

We will also need to describe a certain deformation of the vertex algebra structure on $\mathrm{Fock}_{M\oplus N}$ defined by the fan Σ_1 which is used to define the ambient toric variety $\mathbf{P}$. Fan Σ_1 naturally gives rise to a (generalized) fan Σ in N by simply extending all of the cones of Σ_1 in the direction of $\deg^*$. One can then define a vertex algebra structure on $\mathrm{Fock}_{M\oplus N}$ by changing $\gamma_{m,n}$ as follows.

$$\gamma_{m,n}|m_1,n_1\rangle = \begin{cases} |m+m_1,n+n_1\rangle & \text{if there exists } C\in\Sigma, \text{ such that } n_1,n\in C \\ 0 & \text{otherwise} \end{cases}$$

This gives a different structure of the vertex algebra, and we denote it by $\mathrm{Fock}^{\Sigma}_{M\oplus N}$ to distinguish it from the usual structure on $\mathrm{Fock}_{M\oplus N}$. Operator product expansions of the new fields are either identical to the OPEs in $\mathrm{Fock}_{M\oplus N}$ or vanish. The N=2 superconformal structure on this new algebra is given by the same formulas.

The following theorem is the main result of [3] in the case of smooth ambient toric variety $\mathbf{P}$.

Theorem 6.1 Let Δ and Δ^* be as above, and let Σ_1 define a non-singular toric variety $\mathbf{P}$. Let $f:(\Delta\cap M)\to\mathbf{C}$ define a generic hypersurface $X\subset\mathbf{P}$. We pick a generic collection of coefficients $\{g_n, n\in\Delta^*\cap N\}$. Then the cohomology of the chiral de Rham complex of X is isomorphic as a vertex algebra to the BRST quotient of $\mathrm{Fock}^{\Sigma}_{M\oplus N}$ by the BRST operator

$$\mathcal{BRST}_{f,g} := \oint (\sum_{m\in\Delta} f_m (m\cdot\Phi)(z)\ :\mathrm{e}^{\int m\cdot B(z)}:\ + \sum_{n\in\Delta^*} g_n (n\cdot\Psi)(z)\ :\mathrm{e}^{\int n\cdot A(z)}:)dz.$$

Moreover, the N=2 superconformal structure on $\mathrm{H}^*(\mathcal{MSV}(X))$ coincides with the structure induced by $G^{\pm}_{CY}, J_{CY}, L_{CY}$ introduced above.

We remark that BRST operator above supercommutes with the fields of N=2 algebra which allows one to induce these fields on the BRST quotient. Moreover, it was shown in [3] that eigenvalues of $L[0]$ are non-negative on the BRST cohomology, even though they can definitely be negative on elements of $\mathrm{Fock}^{\Sigma}_{M\oplus N}$. In addition, all eigenspaces of $L[0]$ have finite dimension.

Notice that a nice feature of the above result is that the cohomology of the chiral de Rham complexes for mirror symmetric toric hypersurfaces are obviously related to each other as deformations of a single family of the vertex algebras where one uses $\mathrm{Fock}_{M\oplus N}$ instead of $\mathrm{Fock}^{\Sigma}_{M\oplus N}$. Then the only difference between mirror pictures is that one switches the roles of M and N which amounts exactly to the mirror involution $G^{\pm}\to G^{\mp}, J\to -J, L\to L$. Besides, when $\mathbf{P}=\mathbf{P}^4$ and one picks f_m and g_n to be non-zero at the vertices of the simplices only, one appears to recover the (finite quotient of) tensor product of vertex algebras with fractional charges that corresponds to the Gepner model [7]. Some details of this correspondence are still to be worked out, but note the paper [5].

One also observes that combinatorial description of the N=2 superconformal vertex algebra as a BRST quotient of a certain lattice vertex algebra makes perfect sense whether or not the ambient toric variety $\mathbf{P}$ is non-singular. This prompts one to try to define chiral de Rham complex for a singular Calabi-Yau hypersurface $X=\mathrm{Proj}(\mathbf{C}[K]/f)$ according to the general philosophy that one should be able to understand mirror symmetry without using partial crepant desingularizations, see for example [2]. On the other hand it suggests that this chiral de Rham complex will depend not only on the scheme structure of X but also on some mysterious coefficients g_n. In the smooth case these coefficients are rather irrelevant (all one needs is for them to be non-zero) but in general they seem to be of importance.

Paper [3] contains a definition of a sheaf of vertex algebras $\mathcal{MSV}(X)$ for a Calabi-Yau hypersurface in a toric variety. Namely, for a toric affine chart $\mathbf{A}_C$ of $\mathbf{P}$ that corresponds to a cone $C_1\in\Sigma_1$ one considers a subcone $C\subset K$ defined by $\{(n_1,t)\in K,\ \text{s.t. } n_1\in C\}$. Then sections of $\mathcal{MSV}(X)$ over the intersection of X with $\mathbf{A}_C$ are defined as BRST quotient of $\mathrm{Fock}_{M\oplus C}$ by $\mathrm{BRST}_{f,g}$. Here one induces the vertex algebra structure from $\mathrm{Fock}_{M\oplus N}$ to $\mathrm{Fock}_{M\oplus C}$ and ignores $n\notin C$ in the definition of $\mathrm{BRST}_{f,g}$. Some progress is made in the singular case, but the exact analog of 6.1 is still an open problem. However, we were able to use the results of this analysis in the singular case to define the elliptic genus of a singular Calabi-Yau hypersurface in a toric variety and prove that it satisfies the expected mirror duality, see [4].

7 Open questions

In this short section we briefly describe major open problems and minor technical obstacles that are still to be faced in the vertex algebra approach to mirror symmetry.

The most important problem is that it is not clear how to see instanton corrections in terms of the cohomology of the chiral de Rham complex of a variety X. Geometrically, chiral de Rham complex seems to deal with the neighborhood of the constant loops in the loop space of X, while the instanton corrections are more global in nature. One may have to introduce some modules over the vertex algebra $\mathrm{H}^*(\mathcal{MSV}(X))$ to deal with this difficulty. This is also related to the problem of putting together holomorphic and antiholomorpic parts of the N=(2,2) superconformal field theory associated to a Calabi-Yau manifold. One distinct possibility, which the author plans to explore, is that the true vertex algebra of the superconformal theory associated to a generic Calabi-Yau hypersurface in a toric variety is BRST quotient of $\mathrm{Fock}_{M\oplus N}$, and that $\mathrm{Fock}_{M\oplus N}^{\Sigma}$ appears when one expands the correlators around the limiting point that corresponds to the degeneration of $\mathbf{C}[M\oplus N]$ into $\mathbf{C}[M\oplus N]^{\Sigma}$. Then one hopes to recover instanton corrections as higher terms in this expansion.

Another big issue is a possible extension of these definitions to curves of higher genus. This is always a highly non-trivial problem in conformal field theory, and it is interesting to see if an explicit description could be obtained in the case of hypersurfaces in toric varieties.

Less formidable problems include the extension of all definitions and results to the case of singular varieties with some mild singularities. One also wants to show that the families of vertex algebras given by Theorem 6.1 are flat in the appropriate sense, that is the dimensions of the $L[0]$ graded components are generically constant. One should somehow see how GKZ hypergeometric system of differential equations appears in the context of the above vertex algebra. Also, it was suggested by Yi-Zhi Huang that one should expect this vertex algebra to have an invariant bilinear form in the sense of [6].

It is our hope that the interplay of vertex algebras and algebraic geometry will enrich both fields and will provide deep mathematical understanding of conformal field theories that are currently only defined in terms of path integrals.

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