

A FINITENESS THEOREM FOR SUBGROUPS OF $\mathrm{Sp}(4, \mathbf{Z})$

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Abstract

This paper proves that there are only finitely many subgroups H of finite index in $\mathrm{Sp}(4, \mathbf{Z})$ such that the corresponding quotient \mathcal{H}/H of the Siegel upper half space of rank two is not of general type.

1 Introduction

The Siegel upper half space of rank two consists of complex symmetric two by two matrices whose imaginary part is positive definite. It will be denoted by \mathcal{H} throughout the paper. It is the moduli space of principally polarized marked abelian surfaces. The group $\mathrm{Sp}(4, \mathbf{Z})$ acts on \mathcal{H} by the automorphisms of the marking. This group consists of four by four integer matrices of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where A, B, C , and D are two by two matrices that obey $A^t B = B^t A, C^t D = D^t C, A^t D - B^t C = \mathbf{1}$. Written in coordinates, this action becomes

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot M = (AM + B)(CM + D)^{-1}.$$

It is a natural generalization of the usual upper half plane with the action of $\mathrm{Sl}(2, \mathbf{Z})$. It is related to various moduli spaces of abelian surfaces in the same way the usual upper half plane is related to moduli spaces of elliptic curves.

We shall be concerned mostly with quotients of \mathcal{H} by the action of subgroups H of finite index in $\mathrm{Sp}(4, \mathbf{Z})$. These quotients are known to be algebraic varieties of dimension 3. They have been studied extensively since the end of last century. Some of these varieties have extremely rich and beautiful geometry, see for instance [7],[11] and [6].

The goal of this paper is to prove the following statement, see proposition 6.4.

Finiteness Theorem. There are only finitely many subgroups $H \subseteq \mathrm{Sp}(4, \mathbf{Z})$ of finite index such that \mathcal{H}/H is not of general type.

The important corollary of this result is that there are only finitely many subgroups H such that the quotient \mathcal{H}/H is rational. Varieties of general type can be viewed as the generalization to higher dimension of curves of genus two or more. It is reasonable to expect that they do not have any special geometric properties, and thus all interesting quotients \mathcal{H}/H can be in principle listed. This theorem is analogous to the result of J.G. Thompson (see [14]) for the usual upper half plane. More accurate estimates for certain classes of subgroups of $\mathrm{Sp}(4, \mathbf{Z})$ have been proved in [4, 5, 9].

The method of the proof is roughly the following. It is known that H contains a principal congruence subgroup $\Gamma(n)$ of some level n . The quotient $\mathcal{H}/\Gamma(n)$ admits a well understood smooth compactification, constructed in the paper of Igusa [10]. Our aim is to construct global sections of the multicanonical line bundle on the desingularization of the compactification of $(\mathcal{H}/\Gamma(n))/(H/\Gamma(n))$ from the sections of certain line bundles on the Igusa compactification of $\mathcal{H}/\Gamma(n)$.

We will use standard facts about singular algebraic varieties, which are collected in Section 7. The results of Sections 2 and 4 are probably known to specialists in the field, although there are not many convenient references. Section 3 and 5 are the key sections of the paper. The former is a purely combinatorial calculation, and the latter is an algebra-geometrical one. In both sections we assume that n is a power of a prime, and Section 6 allows us to drop this restriction.

This paper is essentially my University of Michigan thesis. Major part of it was done when I was still in Moscow. It is influenced a lot by my advisor Vasilii Iskovskikh who taught me the basics of algebraic geometry as well as some singularity theory which comes in very handy in the paper. I would like to thank Osip Shvartsman for many stimulating discussions on the subject of this paper. My thesis advisor Igor Dolgachev has been a constant source of inspiration for my studies of algebraic geometry at the University of Michigan. I also wish to thank Gopal Prasad for several valuable conversations and Melvin Hochster for providing a useful reference.

2 Algebraic cycles on Satake and Igusa compactifications

The purpose of this section is to recall the basic facts about some special algebraic cycles on the Satake and Igusa compactifications of $\mathcal{H}/\Gamma(n)$ and to find a nice combinatorial description of their components.

We consider the principal congruence subgroup $\Gamma(n)$ of level n inside $\mathrm{Sp}(4, \mathbf{Z})$. For the rest of the section n is fixed and is greater than two. The group $\Gamma(n)$ acts on the Siegel upper half space of rank two \mathcal{H} according to the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (A\tau + B)(C\tau + D)^{-1}.$$

The quotient $\mathcal{H}/\Gamma(n)$ is a nonsingular algebraic variety. It is a Zariski open subset of the compact singular algebraic variety called the Satake compactification of

$\mathcal{H}/\Gamma(n)$. The exact references can be found in [10]. The monoidal transformation of the Satake compactification along the singular locus is nonsingular. This variety was first considered by Igusa in [10], and is called the Igusa compactification. We denote it by X_n . Points of $\mathcal{H}/\Gamma(n)$ are referred to as the *finite* part of the compactification and the rest is the part *at infinity*.

The part at infinity of the Satake compactification consists of a finite number of curves that intersect in a finite number of *cusp* points. The part at infinity of the Igusa compactification is a divisor $D = \sum_i D_i$, which has simple normal crossings. Its components are elliptic fibrations over the curves at infinity of the Satake compactification. The group $G = \Gamma(1)/\Gamma(n)$ acts on both compactifications, and the map between them is equivariant. The group G is isomorphic to $\mathrm{Sp}(4, \mathbf{Z}/n\mathbf{Z})$, and ± 1 act as the identity.

There are two more types of divisors on the Igusa compactification that will be important to our discussion. First of all, there are divisors E_i that are conjugates of the closure of the image in $\mathcal{H}/\Gamma(n)$ of the set of diagonal matrices in \mathcal{H} . They are disjoint and are isomorphic to the product of two modular curves (see [15]). We denote their sum by E . We also consider divisors that are conjugates of the closure of the image of the set of matrices $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ in \mathcal{H} . Geometrically, these matrices correspond to Jacobians of genus two curves with an extra involution, see [1]. We denote them by F_i and their sum by F . They do intersect with each other and their geometry is somewhat more complicated. We prove the necessary statements regarding these at the end of this section.

We abuse notation somewhat to denote $\mathrm{Sp}(4, \mathbf{Z})$ -conjugates of the sets $\begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}$ and $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ by E_i and F_j as well.

Let us introduce the abelian group V of column vectors of height four with coefficients in $\mathbf{Z}/n\mathbf{Z}$ provided with the skew form $\langle \ , \ \rangle$ defined by the formula $\langle {}^t(x^1, \dots, x^4), {}^t(y^1, \dots, y^4) \rangle = x^1 y^3 + x^2 y^4 - x^3 y^1 - x^4 y^2$. The group G acts naturally on V by left multiplication. Our goal here is to construct G -equivariant correspondences between components of cycles on the Satake and Igusa compactifications mentioned above and some objects defined in terms of the group V .

PROPOSITION 2.1 *The infinity divisors of the Igusa compactification (or equivalently, the curves at infinity of the Satake compactification) are in one-to-one G -equivariant correspondence with the primitive \pm vectors $\pm v$ in V . Here we call a vector v primitive iff its order is exactly n . The \pm means that we identify opposite vectors.*

Proof. It is known (see [10]) that all components of D are G -conjugate. It can be shown that the group G also acts transitively on the set of primitive \pm vectors. It remains to notice that the stabilizer of the \pm vector ${}^t(0, 1, 0, 0)$ coincides with the stabilizer of D_0 , where D_0 is the *standard* divisor that corresponds to the basis of open subsets $\left\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix}, \mathrm{Im}(z) \rightarrow +\infty \right\}$ of \mathcal{H} . The description of the stabilizer of

D_0 can be derived from [10]. It consists of matrices of the form

$$\pm \begin{pmatrix} a & 0 & b & m_3 \\ m_1 & 1 & m_2 & m_4 \\ c & 0 & d & m_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\text{mod } n),$$

$$ad - bc = 1(\text{mod } n), \quad bm_1 + m_3 = am_2(\text{mod } n), \quad dm_1 + m_5 = cm_2(\text{mod } n).$$

This allows us to construct a bijective correspondence between infinity divisors on the Igusa compactification and \pm vectors in V . We will use the notation $\pm v_\alpha$ for the \pm vector that corresponds to the divisor D_α and vice versa. \square

PROPOSITION 2.2 *Cusp points Q_i of the Satake compactification are in one-to-one G -equivariant correspondence with the following pairs $(W, \pm f)$. We consider all possible $W \subset V$ and $f : W \times W \rightarrow \mathbf{Z}/n\mathbf{Z}$ such that*

- (1) *W is a subgroup of V isomorphic to $(\mathbf{Z}/n\mathbf{Z})^2$,*
- (2) *$\langle, \rangle|_W = 0$,*
- (3) *f is a non-degenerate skew form on W with values in $\mathbf{Z}/n\mathbf{Z}$, where non-degeneracy means $f(W \times W) \ni 1(n)$.*

Proof. All cusp points are conjugates of the one described by the basis of open sets

$$\left\{ \begin{pmatrix} x & y \\ y & z \end{pmatrix}, \text{Im} \left(\begin{pmatrix} x & y \\ y & z \end{pmatrix} \right) \rightarrow +\infty \right\}$$

(see [10]). We call this point *standard*. The stabilizer of the standard point consists of matrices of the form

$$\left\{ \begin{pmatrix} A & B \\ \mathbf{0} & {}^t A^{-1} \end{pmatrix}, A^t B = B^t A, \det(A) = \pm 1(n) \right\}$$

However, this is exactly the stabilizer of the *standard* pair

$$(W, \pm f) = ({}^t(*, *, 0, 0), f({}^t(1, 0, 0, 0), {}^t(0, 1, 0, 0)) = 1(n)).$$

It can be shown that any pair (W, f) is a G -conjugate of the standard pair. As a result, we can define the required G -equivariant correspondence. We will use the notation $(W_\alpha, \pm f_\alpha)$ for the pair that corresponds to the point Q_α and vice versa. \square

PROPOSITION 2.3 *The curve at infinity of the Satake compactification that corresponds to the divisor D_α contains the cusp point Q_β iff $v_\alpha \in W_\beta$.*

Proof. Consider the action of the group that stabilizes the standard curve. It acts transitively on the set of cusp points of this curve, which are exactly the Q_i 's. Therefore, all inclusion pairs are acted upon transitively. The standard curve passes through the standard point, and ${}^t(0, 1, 0, 0) \in {}^t(*, *, 0, 0)$, which proves the only if part of the statement. On the other hand, the stabilizer of the standard point acts transitively on the \pm vectors in ${}^t(*, *, 0, 0)$, which proves the if part. \square

PROPOSITION 2.4 *Two infinity divisors D_α and D_β intersect over the point Q_δ iff $v_\alpha, v_\beta \in W_\delta$ and $f_\delta(v_\alpha, v_\beta) = \pm 1(n)$. In this case the intersection is isomorphic to \mathbf{P}^1 .*

Proof. Because of transitivity of the action, the point Q_δ may be considered standard. We follow the argument of [10] for the case where $g_0 = 0$ and $g_1 = 2$. Curves of the intersection of the two infinity divisors are conjugate to one of the curves obtained by taking the limits of the points $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$, with imaginary parts of two out of three normal coordinates $y, (-x - y), (-z - y)$ going to $-\infty$ and the remaining one being bounded. They are pairwise intersections of the divisors that correspond to the limits where exactly one of the imaginary parts goes to $-\infty$ and the other two are bounded. The divisor that corresponds to $\text{Im}(z + y) \rightarrow \infty$ is exactly the standard divisor, because $\text{Im}(y)$ is bounded. The other two divisors are obtained from the standard one by the action of $\left\{ \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & {}^t A^{-1} \end{pmatrix} \right\}$ with $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ respectively. Therefore, these divisors correspond to the \pm vectors $\pm^t(0, 1, 0, 0)$, $\pm^t(-1, 1, 0, 0)$, and $\pm^t(1, 0, 0, 0)$. This proves the "only if" part. The "if" part follows from the transitivity of the action of G on the combinatorial data on the right hand side of the statement. The fact that each irreducible component of the intersection is isomorphic to \mathbf{P}^1 is proven in [10], and the uniqueness of the irreducible component can be derived easily from the description of the divisors in terms of the above limits. \square

PROPOSITION 2.5 *Three infinity divisors $D_\alpha, D_\beta, D_\gamma$ intersect over the point Q_δ iff $v_\alpha, v_\beta, v_\gamma \in W_\delta$, the set $\{\pm v_\alpha \pm v_\beta \pm v_\gamma\}$ contains 0, and $f_\delta(v_\alpha, v_\beta) = \pm 1(n)$. In this case the intersection point is unique.*

Proof. As in the previous proposition, we prove that all points of triple intersection are conjugates of the intersection point of the divisors that correspond to $\pm^t(0, 1, 0, 0)$, $\pm^t(-1, 1, 0, 0)$, and $\pm^t(1, 0, 0, 0)$. Then we notice that any triple of \pm vectors with the above properties can be transformed to the triple $(\pm^t(0, 1, 0, 0), \pm^t(-1, 1, 0, 0), \pm^t(1, 0, 0, 0))$. \square

PROPOSITION 2.6 *Divisors E_i are in one-to-one G -equivariant correspondence with unordered pairs (W_1, W_2) such that*

- (1) W_1 and W_2 are subgroups of V isomorphic to $(\mathbf{Z}/n\mathbf{Z})^2$ each,
- (2) $W_1 \perp W_2 = V$.

Proof. All divisors E_i are conjugates of the *standard* one defined as the closure of the image of the set of diagonal matrices. The stabilizer of this standard divisor is described in [15]. It is exactly the stabilizer of the *standard* pair $({}^t(*, 0, *, 0), {}^t(0, *, 0, *))$. It is easy to show that every pair (W_1, W_2) with above properties is conjugate to this standard one, which completes the proof. For a given E_α the corresponding pair will be denoted by $(W_{\alpha 1}, W_{\alpha 2})$ and vice versa. \square

PROPOSITION 2.7 *The divisor E_α intersects the divisor D_β iff v_β lies in one of the subgroups $W_{\alpha 1}, W_{\alpha 2}$. In this case the intersection is isomorphic to the modular curve of principal level n .*

Proof. We assume that the divisor E_α is a standard one. Then the statement of the proposition follows from the description of the action of the group $\Gamma(n)$ in a neighborhood of the set of diagonal matrices (see [15]). \square

There is an alternative way to describe divisors E_i .

PROPOSITION 2.8 *Divisors E_i are in one-to-one G -equivariant correspondence with conjugates of the involution*

$$\pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

in the group $\mathrm{Sp}(4, \mathbf{Z}/n\mathbf{Z})/\{\pm 1\}$.

Proof. The action of

$$\pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

on \mathcal{H} is defined by $\begin{pmatrix} x & y \\ y & z \end{pmatrix} \rightarrow \begin{pmatrix} x & -y \\ -y & z \end{pmatrix}$, so it fixes exactly the points of the standard divisor E_0 . This gives a one-to-one correspondence between $\Gamma(1)$ conjugates of this involution and $\Gamma(1)$ conjugates of the diagonal in \mathcal{H} . This correspondence survives when we mod out by $\Gamma(n)$, and then we use surjectivity of $\Gamma(1)/\Gamma(n) \rightarrow \mathrm{Sp}(4, \mathbf{Z}/n\mathbf{Z})$. \square

This alternative description is related to the original one as follows.

PROPOSITION 2.9 *The involution φ_α that fixes all points of the divisor E_α is defined by*

- (1) $\varphi_\alpha|_{W_{\alpha 1}} = \mathrm{id}|_{W_{\alpha 1}}$,
- (2) $\varphi_\alpha|_{W_{\alpha 2}} = -\mathrm{id}|_{W_{\alpha 2}}$.

This definition makes sense, because the switch of the order of two subgroups $W_{\alpha 1}, W_{\alpha 2}$ results in the change of sign of the involution φ_α .

Proof. It is true for the standard divisor, and the rest follows from the transitivity of the action of the group G . \square

We can describe divisors F_i in the same fashion, because there is also an involution in $\mathrm{Sp}(4, \mathbf{Z})$, whose fixed points on \mathcal{H} are exactly the matrices $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ that form the standard divisor F_0 .

PROPOSITION 2.10 *Divisors F_i are in one-to-one G -equivariant correspondence with conjugates in $\mathrm{Sp}(4, \mathbf{Z}/n\mathbf{Z})/\{\pm 1\}$ of the involution*

$$\pm \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Proof. It is completely analogous to the proof of 2.8. □

Now we are going to discuss the geometry of the divisor F .

PROPOSITION 2.11 *Divisors F_i are smooth surfaces of general type if n is sufficiently big. Moreover, $\dim H^0(F_i, K_{F_i}) > 0$.*

Proof. Because F_α is an irreducible component of the set of fixed points of an involution on X , it is a smooth surface. The finite part of F_α is isomorphic to the quotient of $\mathcal{H}^1/\Gamma_1(2n) \times \mathcal{H}^1/\Gamma_1(2n)$ by the diagonal action of the group $\Gamma_1(n)/\Gamma_1(2n)$, where \mathcal{H}^1 is the usual upper half plane, and $\Gamma_1(n)$ is the principal congruence subgroup of $\mathrm{Sl}(2, \mathbf{Z})$. This can be shown by direct calculation, using an element of $\mathrm{Sp}(4, \mathbf{R})$ that maps a matrix $\begin{pmatrix} x & y \\ y & x \end{pmatrix}$ to the matrix $\begin{pmatrix} x-y & 0 \\ 0 & x+y \end{pmatrix}$. As a result, F_α admits a finite morphism to $(\mathcal{H}^1/\Gamma_1(n))^2$, which is of general type and has global 2-forms, if n is sufficiently big. □

The divisor $F + D$ does not have simple normal crossings.

PROPOSITION 2.12 *There are exactly n divisors F_γ on X_n that contain any given curve $l_{\alpha\beta} = D_\alpha \cap D_\beta$.*

Proof. We assume that the line $l_{\alpha\beta}$ is standard. Let us consider the involution that fixes all points of the divisor F_i . It fixes all points of the line $l_{\alpha\beta}$. This implies that the matrix of this involution equals

$$\pm \begin{pmatrix} 0 & 1 & 0 & b \\ 1 & 0 & -b & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We can lift these involutions to $\Gamma(1)$ so that they map $\begin{pmatrix} x & y \\ y & z \end{pmatrix}$ to $\begin{pmatrix} z+b & y \\ y & x-b \end{pmatrix}$.

The corresponding divisors F_i are $\begin{pmatrix} x & y \\ y & x-b \end{pmatrix}$. The number of $\Gamma(n)$ -inequivalent divisors of this form is equal to n . □

PROPOSITION 2.13 *If a divisor F_γ contains a line $l_{\alpha\beta}$, then $c_1(F_\gamma)l_{\alpha\beta} = -2$.*

Proof. The line $l_{\alpha\beta}$ may be assumed to be standard. Calculations in the local coordinates show that the normal bundle to $l_{\alpha\beta}$ inside X_n is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(2)$, and the normal bundle to $l_{\alpha\beta}$ inside F_γ is the subbundle of the form $(x, e^{2\pi i b/n} x)$. □

3 Upper bounds on the indices of subgroups of $\mathrm{Sp}(4, \mathbf{Z}/p^t\mathbf{Z})$

This is the key section of the paper. Its purpose is to estimate the index of the subgroup $H \subseteq \mathrm{Sp}(4, \mathbf{Z}/n\mathbf{Z})$ if H contains sufficiently many elements of a special type. We additionally assume that $n = p^t$ for some prime number p and integer t . We fix H and assume that $H \ni \pm \mathbf{1}$ throughout the rest of the section. We use the notation $[x]_p$ with $x \in \mathbf{R}_{\geq 1}$ for the maximum number of form $p^t, t \in \mathbf{N}$ that does not exceed x .

We first discuss subgroups that contain many elements that fix D_i pointwise.

DEFINITION 3.1 *For any primitive vector v we consider the subgroup $\mathrm{Ram}_G(v)$ of $G = \mathrm{Sp}(4, \mathbf{Z}/n\mathbf{Z})$ that consists of transvections, which are operators of the form*

$$r_{v,\alpha} : w \rightarrow w + \alpha \langle w, v \rangle v, \quad \alpha \in \mathbf{Z}/n\mathbf{Z}.$$

Because v is primitive, $\mathrm{Ram}_G(v) \simeq \mathbf{Z}/n\mathbf{Z}$. We denote

$$\mathrm{Ram}_H(v) = H \cap \mathrm{Ram}_G(v), \quad \mathrm{ram}_H(v) = |\mathrm{Ram}_H(v)|/n.$$

Clearly, $\mathrm{ram}_H(-v) = \mathrm{ram}_H(v)$.

REMARK 3.2 *We shall see later in Proposition 5.12 that $\mathrm{Ram}_G(v_\alpha)$ is exactly the group that fixes all points of the divisor D_α .*

PROPOSITION 3.3 *If $\sum_{\pm v} \mathrm{ram}_H(v) \geq \epsilon \cdot \sharp(\pm v)$, then $|G : H| < 2^5 \epsilon^{-2} [2^{72} \epsilon^{-42}]_p$.*

Proof. We can forget about \pm signs in the above proposition. For any set I of primitive vectors we define the *ramification mean* of I to be equal to

$$(\sum_{v \in I} \mathrm{ram}_H(v)) / |I|.$$

Clearly, the ramification mean never exceeds 1.

Among the subgroups of V that are isomorphic to $(\mathbf{Z}/n\mathbf{Z})^3$, we choose a subgroup V_3 , such that the ramification mean of the set of primitive vectors that lie in it is maximum among all such subgroups. Any two primitive vectors are contained in the same number of subgroups that are isomorphic to $(\mathbf{Z}/n\mathbf{Z})^3$, so the sum of the ramification means among these subgroups is at least ϵ times the number of subgroups. Hence, the ramification mean of V_3 is at least ϵ . Analogously, we can choose the subgroup V_2 that has the maximum ramification mean among the subgroups with the properties

- (1) $V_2 \simeq (\mathbf{Z}/n\mathbf{Z})^2$,
- (2) $V_2 \subseteq V_3$,
- (3) $\langle, \rangle|_{V_2} = 0$.

Any two primitive vectors in V_3 are conjugates with respect to the stabilizer of V_3 and therefore are contained in the same number of subgroups V_2 that satisfy the

above three properties. As a result, the ramification mean of the set of the primitive vectors that lie in V_2 is also at least ϵ . The total number of primitive vectors v in V_2 is $n^2(1 - p^{-2})$. One can easily show that at least $(\epsilon/2)n^2(1 - p^{-2})$ of them have $\text{ram}_H(v)$ bigger than $\epsilon/2$, because otherwise the ramification mean of V_2 would be less than ϵ . We call these vectors *good*.

We may additionally assume without loss of generality that $V_2 = {}^t(*, *, 0, 0)$ and $V_3 = {}^t(*, *, *, 0)$. If $v = {}^t(x, y, 0, 0)$, then $r_{v,1}$ has the matrix $\begin{pmatrix} \mathbf{1} & B \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$, where $B = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}$. Denote the group that consists of matrices $\begin{pmatrix} \mathbf{1} & B \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$ by G_{V_2} . We can prove the following statement.

LEMMA 3.4 $|G_{V_2} : (G_{V_2} \cap H)| \leq [\epsilon^{-9}2^{14}]_p$.

Proof of the lemma. We assume that $\text{ram}_H({}^t(1, 0, 0, 0)) \geq \epsilon/2$. We can do it, because there is a primitive vector in V_2 with this property and we may transform it to ${}^t(1, 0, 0, 0)$ by an element of G that stabilizes V_2 . This transformation may not stabilize V_3 , so we can not use this assumption in the proof of Proposition 3.3.

At least $\epsilon n^2(1 - p^{-2})/4$ good vectors ${}^t(x, y, 0, 0)$ satisfy $\text{g.c.d.}(y, n) \leq [4/(\epsilon(1 - p^{-2}))]_p$. Really, the number of vectors in V_2 that do not satisfy this condition is at most $\epsilon n^2(1 - p^{-2})/4$. We pick one such vector and call it ${}^t(x_1, y_1, 0, 0)$.

Consider the set of vectors $v = {}^t(x, y, 0, 0)$ that have the following properties

- (1) v is good,
- (2) $\text{g.c.d.}(y, n) \leq [4/(\epsilon(1 - p^{-2}))]_p$,
- (3) $\text{g.c.d.}(x_1y - y_1x, n) \leq [4/(\epsilon(1 - p^{-2}))]_p$.

There are at least $\epsilon n^2(1 - p^{-2})/4$ vectors that satisfy the first two conditions and there are less than $\epsilon n^2(1 - p^{-2})/4$ vectors that do not satisfy the third one. As a result, such a vector exists, and we denote it by $v = {}^t(x_2, y_2, 0, 0)$.

So H contains three elements of G_{V_2} with the matrices

$$B = \begin{pmatrix} \alpha^2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha^2 x_1^2 & \alpha^2 x_1 y_1 \\ \alpha^2 x_1 y_1 & \alpha^2 y_1^2 \end{pmatrix}, \begin{pmatrix} \alpha^2 x_2^2 & \alpha^2 x_2 y_2 \\ \alpha^2 x_2 y_2 & \alpha^2 y_2^2 \end{pmatrix},$$

where $\text{g.c.d.}(\alpha, n) \leq [2/\epsilon]_p$. They generate a subgroup of G_{V_2} of index equal to the greatest common divisor of n and the determinant of the corresponding three by three matrix. This is equal to

$$\text{g.c.d.}(n, \alpha^6 y_1 y_2 (x_1 y_2 - x_2 y_1)) \leq [(2/\epsilon)]_p^6 [4/(\epsilon(1 - p^{-2}))]^3 \leq [\epsilon^{-9}2^{14}]_p.$$

This proves the lemma. □

We now recall that the ramification mean of the set of vectors $v = {}^t(x, y, z, 0)$ is at least ϵ . It implies that there are at least $\epsilon n^3(1 - p^{-3})/2$ of them that have $\text{ram}_H(v) \geq \epsilon/2$. There are at least $\epsilon n^3(1 - p^{-3})/4$ of them that additionally satisfy $\text{g.c.d.}(z, n) \leq 4/(\epsilon(1 - p^{-3}))$. We abuse the notations and also call such vectors good. The operator $r_{v,\alpha}$ that corresponds to a vector $v \in V_3$ and a number α has

the matrix

$$\begin{pmatrix} 1 + \alpha xz & 0 & -\alpha x^2 & -\alpha xy \\ \alpha yz & 1 & -\alpha xy & -\alpha y^2 \\ \alpha z^2 & 0 & 1 - \alpha xz & -\alpha yz \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

All the elements we have described so far lie inside the group

$$G_{V_3} = \begin{pmatrix} a & 0 & b & m_3 \\ m_1 & 1 & m_2 & m_4 \\ c & 0 & d & m_5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{n},$$

$$ad - bc = 1 \pmod{n}, \quad bm_1 + m_3 = am_2 \pmod{n}, \quad dm_1 + m_5 = cm_2 \pmod{n}.$$

This group has a natural projection λ to the $\text{Sl}(2, \mathbf{Z}/n\mathbf{Z})$ defined by the entries a, b, c, d . Our next step is to show that the images of elements of H generate a subgroup of $\text{Sl}(2, \mathbf{Z}/n\mathbf{Z})$ of bounded index.

We have at our disposal the matrices $\begin{pmatrix} 1 + \alpha xz & -\alpha x^2 \\ \alpha z^2 & 1 - \alpha xz \end{pmatrix}$, as well as $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ with $\text{g.c.d.}(\beta, n) \leq [\epsilon^{-9}2^{14}]_p$. Here we use the estimate of β that comes from the result of lemma 3.4.

We fix α_0 that satisfies $\text{g.c.d.}(\alpha_0, n) = [2/\epsilon]_p$. Notice that if $(1 + \alpha_0 x_1 z_1, \alpha_0 z_1^2) \neq (1 + \alpha_0 x_2 z_2, \alpha_0 z_2^2)$, then the matrices

$$\begin{pmatrix} 1 + \alpha_0 x_1 z_1 & -\alpha_0 x_1^2 \\ \alpha_0 z_1^2 & 1 - \alpha_0 x_1 z_1 \end{pmatrix}, \begin{pmatrix} 1 + \alpha_0 x_2 z_2 & -\alpha_0 x_2^2 \\ \alpha_0 z_2^2 & 1 - \alpha_0 x_2 z_2 \end{pmatrix}$$

lie in different cosets of the subgroup $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. Therefore, we can estimate the order of the subgroup generated by the elements that lie in H simply by multiplying $n/[\epsilon^{-9}2^{14}]_p$ by the number of different pairs $(1 + \alpha_0 xz, \alpha_0 z^2)$ that we are guaranteed to have.

We have at least $\epsilon n^2(1 - p^{-3})/4$ pairs (x, z) that correspond to at least one good vector ${}^t(x, y, z, 0)$ and thus give rise to an element in H of the above form. We now need to estimate the number of pairs (x, z) that can give the same $(1 + \alpha_0 xz, \alpha_0 z^2)$. The number of different z that have the same $\alpha_0 z^2$ is at most $4 \cdot \text{g.c.d.}(\alpha_0 z^2, n)$, which does not exceed $4 \cdot [2/\epsilon]_p [4/(\epsilon(1 - p^{-3}))]_p^2$. Once we know z , the number of x that give the same $1 + \alpha_0 xz$ is at most $\text{g.c.d.}(\alpha_0 z, n)$, which is at most $[2/\epsilon]_p [4/\epsilon(1 - p^{-3})]_p$. So the total number of pairs $(1 + \alpha_0 xz, \alpha_0 z^2)$ is at least

$$(\epsilon n^2(1 - p^{-3})/4) / (4[2/\epsilon]_p^2 [4/(\epsilon(1 - p^{-3}))]_p^3) \geq \epsilon n^2 / (2^5 [2^8 \epsilon^{-5}]_p).$$

This implies that the images of elements that lie in H generate a subgroup of $\text{Sl}(2, \mathbf{Z}/n\mathbf{Z})$ of index at most

$$\frac{n^3(1 - p^{-2})(1 - p^{-1})}{(\epsilon n^2 / (2^5 [2^8 \epsilon^{-5}]_p)) \cdot (n / [\epsilon^{-9}2^{14}]_p)} \leq [2^{22} \epsilon^{-14}]_p \epsilon^{-1} 2^5.$$

On the other hand, let us estimate the index of $H \cap \text{Ker}(\lambda)$ in $\text{Ker}(\lambda)$. We use the formula

$$\begin{aligned} & \begin{pmatrix} 1 + \alpha xz & 0 & -\alpha x^2 & -\alpha xy \\ \alpha yz & 1 & -\alpha xy & -\alpha y^2 \\ \alpha z^2 & 0 & 1 - \alpha xz & -\alpha yz \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ & \begin{pmatrix} 1 + \alpha xz & 0 & -\alpha x^2 & -\alpha xy \\ \alpha yz & 1 & -\alpha xy & -\alpha y^2 \\ \alpha z^2 & 0 & 1 - \alpha xz & -\alpha yz \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 0 & 0 & -b - b\alpha xz \\ 0 & 1 & -b - b\alpha xz & b\alpha z(-2y + bz + b\alpha xz^2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ & = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -b\alpha z^2 & 1 & 0 & 0 \\ 0 & 0 & 1 & b\alpha z^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ & \text{to generate the subgroup of } \begin{pmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ of index at most } \text{g.c.d.}(\beta\alpha z^2, n), \text{ which} \end{aligned}$$

we can estimate.

$$\text{g.c.d.}(\beta\alpha z^2, n) \leq [\epsilon^{-9}2^{14}]_p [2\epsilon^{-1}]_p [4\epsilon^{-1}(1 - p^{-3})]_p^2 \leq [\epsilon^{-12}2^{20}]_p.$$

Because the kernel of λ is a semidirect product of the above group and a subgroup of G_{V_2} , we have

$$|\text{Ker}\lambda : (\text{Ker}\lambda \cap H)| \leq [\epsilon^{-12}2^{20}]_p [\epsilon^{-9}2^{14}]_p \leq [\epsilon^{-21}2^{34}]_p$$

and

$$|G_{V_3} : (G_{V_3} \cap H)| \leq [\epsilon^{-21}2^{34}]_p [\epsilon^{-14}2^{22}]_p \epsilon^{-1}2^5 \leq [\epsilon^{-35}2^{56}]_p \epsilon^{-1}2^5.$$

There is only one more step necessary to prove this proposition. Because the ramification mean of V is at least ϵ , there are at least $\epsilon n^4(1 - p^{-4})/4$ primitive vectors $v = {}^t(x, y, z, t)$ that satisfy

- (1) $\text{Ram}_H(v) \geq \epsilon/2$,
- (2) $\text{g.c.d.}(t, n) \leq [4\epsilon^{-1}(1 - p^{-4})^{-1}]_p$.

We continue to abuse the notations and call these vectors good.

We use the number α_0 defined earlier and consider elements r_{v, α_0} for all good vectors. They all lie in H , and the claim is that they lie in $\sim n^4$ different cosets of $G : G_{V_3}$. Indeed, all elements of the group G_{V_3} fix ${}^t(0, 1, 0, 0)$, and r_{v, α_0} pushes ${}^t(0, 1, 0, 0)$ to ${}^t(x\alpha_0 t, 1 + y\alpha_0 t, z\alpha_0 t, \alpha_0 t^2)$. So if

$${}^t(x\alpha_0 t, 1 + y\alpha_0 t, z\alpha_0 t, \alpha_0 t^2) \neq {}^t(x_1\alpha_0 t_1, 1 + y_1\alpha_0 t_1, z_1\alpha_0 t_1, \alpha_0 t_1^2),$$

then r_{v, α_0} and r_{v_1, α_0} lie in different cosets.

We can estimate the number of vectors that can give the same fourtuple as follows. If we know $\alpha_0 t^2$, it leaves us with at most

$$4 \cdot \text{g.c.d.}(\alpha_0 t^2, n) \leq 4[2\epsilon^{-1}]_p [4\epsilon^{-1}(1-p^{-4})^{-1}]_p^2 \leq 4[2^6\epsilon^{-3}]_p$$

options for t . Once we know t , we have at most $(\text{g.c.d.}(\alpha_0 t, n))^3$ choices for (x, y, z) . This gives us a total of at most

$$4[2^6\epsilon^{-3}]_p \cdot ([2\epsilon^{-1}]_p [4\epsilon^{-1}(1-p^{-4})^{-1}]_p)^3 \leq 4[2^{16}\epsilon^{-7}]_p$$

different good vectors v that give r_{v, α_0} from the same coset. Therefore, we can estimate the number of different cosets that have representatives in H by

$$(\epsilon n^4(1-p^{-4})/4)/(4[2^{16}\epsilon^{-7}]_p) \geq \epsilon n^4/(2^5[2^{16}\epsilon^{-7}]_p).$$

Hence, we can estimate the order of H by multiplying the estimate on the order of its intersection with G_{V_3} by the number of cosets that it has representatives in, which gives

$$|H| \geq \frac{n^6(1-p^{-2})(1-p^{-1})}{2^5\epsilon^{-1}[2^{56}\epsilon^{-35}]_p} \cdot \frac{\epsilon n^4}{(2^5[2^{16}\epsilon^{-7}]_p)} \geq n^{10} \cdot \frac{\epsilon^2 2^{-5}(1-p^{-2})(1-p^{-1})}{[2^{72}\epsilon^{-42}]_p}.$$

Therefore,

$$\begin{aligned} |G : H| &\leq n^{10}(1-p^{-4})(1-p^{-3})(1-p^{-2})(1-p^{-1}) : (n^{10} \cdot \frac{\epsilon^2 2^{-5}(1-p^{-2})(1-p^{-1})}{[2^{72}\epsilon^{-42}]_p}) \\ &< 2^5\epsilon^{-2}[2^{72}\epsilon^{-42}]_p. \end{aligned}$$

□

REMARK 3.5 *The estimate of Proposition 3.3 is probably far from optimum.*

Now we consider subgroups that contain many elements that fix E_i pointwise.

DEFINITION 3.6 *Let $(W_{\alpha 1}, W_{\alpha 2})$ be a pair of complementary isotropic subgroups that corresponds to the divisor E_α , as described in 2.6, and φ_α be the corresponding involution described in 2.9. We define $\text{ram}_H(E_\alpha)$ to equal 1 if $H \ni \varphi_\alpha$, and to equal 0 otherwise. This definition makes sense because $H \ni \pm 1$.*

REMARK 3.7 *We have shown already that φ_α fixes all points of E_α .*

PROPOSITION 3.8 *If $\sum_\alpha \text{ram}_H(E_\alpha) \geq \epsilon^\sharp(\alpha)$, then $|G : H| < 2^7\epsilon^{-2}[2^{246}\epsilon^{-130}]_p$.*

Proof. For every set of indices I we define the ramification mean of I to be $\sum_{\alpha \in I} \text{ram}_H(E_\alpha)/|I|$.

For every primitive vector v we consider the set I_v of indices α such that v is an eigenvector of φ_α . Each index α belongs to the same number of sets I_v , therefore

$$\sum_v \text{ramif.mean}(I_v) \geq \epsilon^\sharp(v).$$

Hence there are at least $(\epsilon/2) \cdot \sharp(v)$ vectors v such that the ramification mean of I_v is at least $\epsilon/2$. So now we try to estimate $\text{ram}_H(v)$ for a vector v with this property, and then use 3.3.

We assume that $v = {}^t(0, 1, 0, 0)$.

LEMMA 3.9 *Involutions $\varphi_\alpha, \alpha \in I_v$ have matrices of the form*

$$\begin{pmatrix} 1 & 0 & 0 & -2x \\ -2z & -1 & 2x & 0 \\ 0 & 0 & 1 & -2z \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The sign is chosen to satisfy $\varphi_\alpha v = -v$.

Proof of the lemma. Any involution of this kind is defined uniquely by the choice of $W_{\alpha 2}$. Because of $\langle W_{\alpha 1}, W_{\alpha 2} \rangle = 0$, the form \langle, \rangle is unimodular on $W_{\alpha 2}$. It implies that there is a basis of $W_{\alpha 2}$ that consists of v and ${}^t(x, 0, z, 1)$. The rest is just a calculation. \square

We denote the involution with the matrix

$$\begin{pmatrix} 1 & 0 & 0 & -2x \\ -2z & -1 & 2x & 0 \\ 0 & 0 & 1 & -2z \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

by $\varphi_{x,z}$. We may assume without loss of generality that $\varphi_{0,0} \in H$. There are at least $\epsilon n^2/2$ pairs (x, z) such that $\varphi_{x,z} \in H$. We call these pairs good. There are at least $\epsilon n^2/4$ good pairs that satisfy $\text{g.c.d.}(z, n) \leq [4/\epsilon]_p$. We choose one of them and denote it by (x_1, z_1) . There is at least one good pair (x, z) such that $\text{g.c.d.}(xz_1 - zx_1, n) \leq [2/\epsilon] \cdot \text{g.c.d.}(z, n)$. Then $\text{g.c.d.}(xz_1 - zx_1, n) \leq [8\epsilon^{-2}]_p$. We denote this pair by (x_2, z_2) .

It is a matter of calculation to check that

$$(\varphi_{x_1, z_1} \varphi_{0,0} \varphi_{x_2, z_2})^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 8(x_1 z_2 - x_2 z_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This element lies in H , therefore $\text{ram}_H(v) \geq 1/[8\epsilon^{-2}]_p$.

Because we can prove the same estimate for every vector v for which the ramification mean of I_v is at least $\epsilon/2$, we get

$$\sum_v \text{ram}_H(v) \geq (8\epsilon^{-2})_p^{-1} \cdot (\epsilon/2).$$

Now we use Proposition 3.3 to get

$$|G : H| < 2^7 \epsilon^{-2} [2^{246} \epsilon^{-130}]_p.$$

\square

Now let us consider subgroups that contain many elements that fix lines $D_i \cap D_j$ pointwise.

DEFINITION 3.10 Every line $l_{\alpha\beta} = D_\alpha \cap D_\beta$ is a conjugate of the standard line l_0 , which is the intersection of the divisors that correspond to the \pm vectors $\pm^t(1, 0, 0, 0)$, $\pm^t(0, 1, 0, 0)$. We define $\text{Ram}_G(l_0)$ to consist of matrices

$$\begin{pmatrix} 1 & 0 & * & 0 \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\text{mod } n).$$

We then define $\text{Ram}_G(g \cdot l_0) = g \cdot \text{Ram}_G(l_0) \cdot g^{-1}$.

It can be defined invariantly as a subgroup of all matrices that fix both v_α and v_β , and also fix a pair of the isotropic subgroups $W_1 \ni v_\alpha, W_2 \ni v_\beta$ that correspond to a divisor E_i that intersects $l_{\alpha\beta}$. It does not matter which E_i we consider.

REMARK 3.11 We shall see later in Proposition 5.14 that $\text{Ram}_G(l_{\alpha\beta})$ consists of transformations that fix all points of the line $l_{\alpha\beta}$ and do not switch the divisors D_α and D_β .

DEFINITION 3.12 We define $\text{Ram}_H(l_{\alpha\beta}) = H \cap \text{Ram}_G(l_{\alpha\beta})$. We define $\text{ram}_H(l_{\alpha\beta})$ to be the maximum order of the element of $\text{Ram}_H(l_{\alpha\beta})$ divided by n .

PROPOSITION 3.13 If $\sum_{\alpha\beta} \text{ram}_H(l_{\alpha\beta}) \geq \epsilon^\sharp(\alpha\beta)$, then

$$|G : H| \leq 2^{11} \epsilon^{-2} [2^{1020} \epsilon^{-350}]_p.$$

Proof. We will eventually use Proposition 3.3. We need another definition.

LEMMA 3.14 Let $l_{\alpha\beta}$ be the line of the intersection of the divisors D_α and D_β . Then $\text{Ram}_G(l_{\alpha\beta}) = \text{Ram}_G(v_\alpha) \oplus \text{Ram}_G(v_\beta)$.

Proof of the lemma. It is enough to consider the standard line, for which the statement follows from the explicit matrix representations of the three groups in question. \square

DEFINITION 3.15 We define

$$\text{ram}_H(l_{\alpha\beta} \subset D_\alpha) = \frac{|\text{Ram}_H(l_{\alpha\beta})|}{|\text{Ram}_H(l_{\alpha\beta}) \cap \text{Ram}_G(v_\alpha)| \cdot n}$$

If α, β are standard, then $\text{ram}_H(l_{\alpha\beta} \subset D_\alpha)$ is the inverse of the minimum g.c.d. (a, n) for

$$\begin{pmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H.$$

We notice that

$$\text{ram}_H(l_{\alpha\beta}) \leq \max\{\text{ram}_H(l_{\alpha\beta} \subset D_\alpha), \text{ram}_H(l_{\alpha\beta} \subset D_\beta)\}.$$

The usual argument shows that at least $(\epsilon/6) \cdot \#|D_\alpha|$ of divisors D_α obey the following property

(1) at least $(\epsilon/6) \cdot \#(l_{\alpha\beta} \subset D_\alpha)$ of lines $l_{\alpha\beta}$ that are contained in it have $\text{ram}_H(l_{\alpha\beta} \subset D_\alpha) \geq \epsilon/6$.

We call these divisors good. Now our goal is to prove that every good divisor D_α has sufficiently big $\text{ram}_H(v_\alpha)$.

We may assume without loss of generality that $D_\alpha = D_0$ is standard. We may also assume that the arrangement of lines in D_0 over the standard point on the Satake compactification contains at least $(\epsilon/6) \cdot n$ of the lines with $\text{ram}_H(l_{0\beta} \subset D_0) \geq \epsilon/6$. Divisors D_β that intersect D_0 over the standard point of the Satake compactification correspond to \pm vectors of the form $\pm^t(1, b, 0, 0)$, see 2.4. It implies, that there are at least $(\epsilon/6) \cdot n$ numbers b such that

$$H \ni \begin{pmatrix} 1 & 0 & a_0 & ba_0 \\ 0 & 1 & ba_0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\text{g.c.d.}(a_0, n) \leq [(\epsilon/6)^{-1}]_p$, and $*$ is an unknown number.

We can choose b_1 and b_2 that give us the above elements in H and additionally satisfy $\text{g.c.d.}(b_1 - b_2, n) \leq [6/\epsilon]_p$. Then we can divide one such element by another to get

$$H \ni \begin{pmatrix} 1 & 0 & 0 & (b_1 - b_2)a_0 \\ 0 & 1 & (b_1 - b_2)a_0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & x \\ 0 & 1 & x & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can estimate $\text{g.c.d.}(x, n) \leq [6^2\epsilon^{-2}]_p$. We denote the above element by ρ .

Now we wander away from the standard point on the Satake compactification. All other divisors D_β that intersect D_0 correspond to the \pm vectors $\pm^t(d, e, f, 0)$ with $(d, f) \neq (0, 0)(p)$. This also follows from Proposition 2.4. At least $(\epsilon/6) \cdot n^3(1 - p^{-2})$ of lines $l_{0\beta}$ satisfy $\text{ram}_H(l_{0\beta} \subset D_0) \geq \epsilon/6$. Therefore, at least one of them satisfies additionally $\text{g.c.d.}(f, n) \leq [6\epsilon^{-1}(1 - p^{-2})^{-1}]_p$. It implies, that H contains an element ρ_1 of the form

$$\begin{pmatrix} 1 + dfa_0 & 0 & -d^2a_0 & -dea_0 \\ efa_0 & 1 & -dea_0 & -ea_0^2 + c \\ f^2a_0 & 0 & 1 - dfa_0 & -efa_0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\text{g.c.d.}(f, n) \leq [6\epsilon^{-1}(1 - p^{-2})^{-1}]_p$.

One can calculate that

$$\rho_1 \rho \rho_1^{-1} \rho^{-1} \rho_1 \rho^{-1} \rho_1^{-1} \rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2x^2 f^2 a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

which implies

$$\text{ram}_H(v_0) \geq [2 \cdot (6^2 \epsilon^{-2})^2 \cdot (6\epsilon^{-1}(1-p^{-2})^{-1})^2 \cdot (\epsilon/6)^{-1}]_p^{-1} \geq [2^{19} \epsilon^{-7}]_p^{-1}.$$

As a result, $\sum_{\pm v} \text{ram}_H(v) \geq [2^{19} \epsilon^{-7}]_p^{-1} (\epsilon/6) \cdot \sharp(\pm v)$. By 3.3, it implies

$$|G : H| \leq 2^{11} \epsilon^{-2} [2^{1020} \epsilon^{-350}]_p.$$

□

We also need to deal with subgroups that contain many elements that fix F_i pointwise.

DEFINITION 3.16 *Let ψ_α be the involution that corresponds to the divisor F_α as described in 2.10. We define $\text{ram}_H(F_\alpha)$ to equal 1 if $H \ni \psi_\alpha$, and to equal 0 otherwise.*

REMARK 3.17 *We have shown already that ψ_α fixes all points of F_α .*

PROPOSITION 3.18 *If $\sum_\alpha \text{ram}_H(F_\alpha) \geq \epsilon \sharp(\alpha)$, then $|G : H| \leq 2^{13} \epsilon^{-2} [2^{1722} \epsilon^{-702}]_p$.*

Proof. There are at least $(\epsilon/2) \sharp(\alpha\beta)$ lines $l_{\alpha\beta}$ such that at least $\epsilon n/2$ of divisors F_γ that contain $l_{\alpha\beta}$ are ramification divisors. We call these lines good. Our goal is to estimate $\text{ram}_H(l_{\alpha\beta})$ for a good line $l_{\alpha\beta}$.

We may assume that $l_{\alpha\beta}$ is the standard line. If it is good, then the group H contains at least $\epsilon n/2$ elements of the form

$$\varphi_b = \begin{pmatrix} 0 & 1 & 0 & b \\ 1 & 0 & -b & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

There are two elements φ_{b_1} and φ_{b_2} in H such that $\text{g.c.d.}(n, b_1 - b_2) \leq [2\epsilon^{-1}]_p$. The matrix of the element $\varphi_{b_1} \varphi_{b_2}$ is equal to

$$\begin{pmatrix} 1 & 0 & b_1 - b_2 & 0 \\ 0 & 1 & 0 & b_2 - b_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, $\text{ram}_H(l_{\alpha\beta}) \geq [2\epsilon^{-1}]_p^{-1}$. As a result,

$$\sum_{\alpha\beta} \text{ram}_H(l_{\alpha\beta}) \geq \epsilon 2^{-1} [2\epsilon^{-1}]_p^{-1} \sharp(\alpha\beta).$$

Proposition 3.13 gives $|G : H| \leq 2^{13}\epsilon^{-2}[2^{1722}\epsilon^{-702}]_p$. \square

Finally, we will get an index estimate for subgroups such that their quotient varieties have bad singularities at the images of $D_i \cap D_j \cap D_k$. This is the most delicate calculation of the whole paper. We need some preliminary definitions.

DEFINITION 3.19 *Let $P_{\alpha\beta\gamma}$ be the point of the intersection of three infinity divisors D_α, D_β , and D_γ . Define*

$$\text{Ram}_G(P_{\alpha\beta\gamma}) = \text{Ram}_G(v_\alpha) \oplus \text{Ram}_G(v_\beta) \oplus \text{Ram}_G(v_\gamma).$$

If P is the standard point, that is the one that corresponds to $v_\alpha = {}^t(1, -1, 0, 0)$, $v_\beta = {}^t(1, 0, 0, 0)$, $v_\gamma = {}^t(0, 1, 0, 0)$, then this group consists of matrices

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{n}.$$

As usual, we define $\text{Ram}_H(P_{\alpha\beta\gamma}) = H \cap \text{Ram}_G(P_{\alpha\beta\gamma})$.

DEFINITION 3.20 *Consider the singularity at the image of $P_{\alpha\beta\gamma}$ in the quotient of a neighborhood of $P_{\alpha\beta\gamma}$ by the group $\text{Ram}_H(P_{\alpha\beta\gamma})$. We define $\text{mult}_H(P_{\alpha\beta\gamma})$ to be the multiplicity of this singular point.*

PROPOSITION 3.21 *If $\sum^* \text{mult}_H P_i \geq \epsilon^\sharp(i)$, where \sum^* means taking one point $P_{\alpha\beta\gamma}$ per orbit of the action of the group H , then $|G : H| \leq 2^{69}\epsilon^{-34}[2^{11170}\epsilon^{-5950}]_p$.*

Proof. For each point $P_{\alpha\beta\gamma}$ we define $\delta(H, P_{\alpha\beta\gamma})$ as a number δ defined in 7.13 for the group $\text{Ram}_H(P_{\alpha\beta\gamma})$ acting in the tangent space at $P_{\alpha\beta\gamma}$. Notice that there is a natural choice of coordinates (x_1, x_2, x_3) in a neighbourhood of $P_{\alpha\beta\gamma}$, such that the weights of an element $h \in \text{Ram}_H(P_{\alpha\beta\gamma})$ are determined using $\text{Ram}_G(P_{\alpha\beta\gamma}) = \text{Ram}_G(v_\alpha) \oplus \text{Ram}_G(v_\beta) \oplus \text{Ram}_G(v_\gamma)$. Then $\delta(H)$ is defined as $(1/n) \min_{l \neq 0} (l_1 + l_2 + l_3)$, where minimum is taken over all H -invariant monomials $x_1^{l_1} x_2^{l_2} x_3^{l_3}$.

First of all, we rewrite the condition of the proposition in terms of $\delta(H, P_{\alpha\beta\gamma})$. By 7.14, $\text{mult}_H P_{\alpha\beta\gamma} \leq n^3 \delta(H, P_{\alpha\beta\gamma}) / |\text{Ram}_H(P_{\alpha\beta\gamma})|$. Therefore,

$$\begin{aligned} \sum_{P_{\alpha\beta\gamma}} \delta(H, P_{\alpha\beta\gamma}) &\geq \sum_{P_{\alpha\beta\gamma}} n^{-3} |\text{Ram}_H(P_{\alpha\beta\gamma})| \text{mult}_H P_{\alpha\beta\gamma} \\ &\geq \sum^* (6n^3)^{-1} |H| \text{mult}_H P_{\alpha\beta\gamma} \geq (6n^3)^{-1} \epsilon |H| \cdot |G : H| = \epsilon^\sharp(P_{\alpha\beta\gamma}). \end{aligned}$$

For every isotropic subgroup $V_2 \simeq (\mathbf{Z}/n\mathbf{Z})^2$ in V we consider the set of the points $P_{\alpha\beta\gamma}$ with $v_\alpha, v_\beta, v_\gamma \in V_2$. Geometrically, these are the points that lie over certain cusp points of the Satake compactification, see 2.2. There are at least $(\epsilon/2)\sharp(V_2)$ of these subgroups that have

$$\sum_{v_\alpha, v_\beta, v_\gamma \in V_2} \delta(H, P_{\alpha\beta\gamma}) \geq (\epsilon/2)\sharp(v_\alpha, v_\beta, v_\gamma \in V_2).$$

We call these subgroups good. We are going to prove that if V_2 is a good isotropic subgroup, then

$$\sum_{v_\alpha, v_\beta \in V_2} \text{ram}_H(l_{\alpha\beta}) \geq \epsilon_1(\epsilon) \sharp(v_\alpha, v_\beta \in V_2),$$

and then use Proposition 3.13.

We assume without loss of generality that $V_2 = {}^t(*, *, 0, 0)$, and $\delta(H, P_0) \geq (\epsilon/2)$, where P_0 is the standard point. Notice that $\text{Ram}_G(P_{\alpha\beta\gamma})$ and $\text{Ram}_H(P_{\alpha\beta\gamma})$ do not depend on the point $P_{\alpha\beta\gamma}$, provided $v_\alpha, v_\beta, v_\gamma \in V_2$. We denote these groups by G_1 and H_1 respectively. The group G_1 is described in Definition 3.19. We are dealing with points $P_{\alpha\beta\gamma}$ obtained from the standard one by the action of elements of type $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, where $A \in \text{Gl}(2, \mathbf{Z}/n\mathbf{Z})$. Although the group H_1 is the same for all $P_{\alpha\beta\gamma}$, its action in the tangent spaces depends on $P_{\alpha\beta\gamma}$. It is the same as the action in the tangent space to the standard point P_0 of the group AH^tA , $A \in \text{Gl}(2, \mathbf{Z}/n\mathbf{Z})$, if we think of G_1 as the group of symmetric 2×2 matrices.

We define ϵ_1 by the formula

$$\sum_{v_\alpha, v_\beta \in V_2} \text{ram}_H(l_{\alpha\beta}) = \epsilon_1 \sharp(v_\alpha, v_\beta \in V_2).$$

There is a line $l_{\alpha\beta}$ such that $\text{ram}_H l_{\alpha\beta} \leq \epsilon_1$. It implies that the group $H_2 = (H_1 + [\epsilon_1^{-1}]_p G_1) / [\epsilon_1^{-1}]_p G_1$ is cyclic. When we pass from H_1 to $H_1 + [\epsilon_1^{-1}]_p G_1$, the numbers δ do not decrease. Hence,

$$\sum_{v_\alpha, v_\beta, v_\gamma \in V_2} \delta(H_1 + [\epsilon_1^{-1}]_p G_1, P_{\alpha\beta\gamma}) \geq (\epsilon/2) \sharp(v_\alpha, v_\beta, v_\gamma \in V_2).$$

This is equivalent to

$$\sum_{A \in \text{Gl}(2, \mathbf{Z}/n\mathbf{Z})} \delta(AH_1^t A + [\epsilon_1^{-1}]_p G_1, P_0) \geq (\epsilon/2) \sharp(A).$$

Let H_2 be generated by $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. One can show that $\delta(AH^t A + [\epsilon_1^{-1}]_p G_1, P_0)$ equals $\delta((\mathbf{Z}/n\mathbf{Z}) \bar{A} \bar{B}^t \bar{A}, \bar{P}_0)$, where n is replaced by $[\epsilon_1^{-1}]_p$ and bars means reduction mod $[\epsilon_1^{-1}]_p$. Therefore,

$$\sum_{C \in \text{Gl}(2, \mathbf{Z}/[\epsilon_1^{-1}]_p \mathbf{Z})} \delta((\mathbf{Z}/[\epsilon_1^{-1}]_p \mathbf{Z}) C B^t C, \bar{P}_0) \geq (\epsilon/2) \sharp(C).$$

Because of the result of Proposition 7.16, there are at most $2^{10} \epsilon^{-8} [2^{12} \epsilon^{-5}]_p$ different matrices $CB^t C$ up to proportionality that give $\delta((\mathbf{Z}/[\epsilon_1^{-1}]_p \mathbf{Z}) CB^t C, \bar{P}_0) \geq (\epsilon/4) \sharp(C)$. This implies that the orbit $CB^t C \pmod{\text{proportionality}}$ of the action of the group $\text{Gl}(2, \mathbf{Z}/[\epsilon_1^{-1}]_p \mathbf{Z})$ has length at most $2^{12} \epsilon^{-9} [2^{12} \epsilon^{-5}]_p$.

However, we can estimate this length by looking at matrices $C = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. They give $CB^t C = \begin{pmatrix} t^2 a & tb \\ tb & c \end{pmatrix}$, and so length of the orbit is at least $[\epsilon_1^{-1}]_p (1 -$

$p^{-1})/\text{g.c.d.}(bc, [\epsilon_1^{-1}]_p)$. Because we have assumed that $\delta(\mathbf{Z}/[\epsilon_1^{-1}]_p \mathbf{Z})B, \bar{P}_0) \geq (\epsilon/2)$, we have $\text{g.c.d.}(b, n) \leq [2\epsilon^{-1}]_p$ and $\text{g.c.d.}(c, n) \leq [4\epsilon^{-1}]_p$. Really, the weights of B are $(-b, a+b, c+b)$, and if $\text{g.c.d.}[c, n]$ is greater than $[4\epsilon^{-1}]_p$, then we get $\delta \geq (\epsilon/2)$ because of the invariant monomial of the form $(x_1 x_3)^{[\epsilon_1^{-1}]_p/[4\epsilon^{-1}]_p}$.

As a result, the length of the orbit is at least $[\epsilon_1^{-1}]_p(1-p^{-1})/[8\epsilon^{-2}]_p$, and we have $[\epsilon_1^{-1}]_p(1-p^{-1})/[8\epsilon^{-2}]_p \leq 2^{12}\epsilon^{-9}[2^{12}\epsilon^{-5}]_p$ and $\epsilon_1 \geq 2^{-28}\epsilon^{16}$.

We now may use the result of Proposition 3.13 with $(2^{-28}\epsilon^{16})(\epsilon/2)$ in place of ϵ . Thus, $|G : H| \leq 2^{69}\epsilon^{-34}[2^{11170}\epsilon^{-5950}]_p$. \square

4 Singularities of \mathcal{H}/H

It is easy to describe all elements of finite order in $\Gamma(2)$ by means of the following proposition.

PROPOSITION 4.1 *Any nonidentity element of finite order in $\Gamma(2)/\{\pm 1\}$ is conjugate in $\Gamma(1)/\{\pm 1\}$ to the element with the matrix*

$$\varphi_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Proof. Denote the matrix of this element by $\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Because $\Gamma(4)$ is torsion free and $\varphi^2 \in \Gamma(4)$, we obtain $\varphi^2 = 1$. Hence the following equalities hold

$$A {}^t B = B {}^t A, \quad C {}^t D = D {}^t C, \quad A {}^t D - B {}^t C = 1$$

$$A = {}^t D, \quad B = -{}^t B, \quad C = -{}^t C.$$

Really, the first three equalities hold for all symplectic matrices, and they imply $\varphi^{-1} = \begin{pmatrix} {}^t D & -{}^t B \\ -{}^t C & {}^t A \end{pmatrix}$, so $\varphi^{-1} = \varphi$ gives the last three ones. Six equalities together show that

$$\varphi = \begin{pmatrix} a_1 & a_2 & 0 & b \\ a_3 & a_4 & -b & 0 \\ 0 & c & a_1 & a_3 \\ -c & 0 & a_2 & a_4 \end{pmatrix}$$

with $(a_1 + a_4)b = (a_1 + a_4)c = (a_1 + a_4)a_2 = (a_1 + a_4)a_3 = 0$, $a_1^2 + a_2a_3 - bc = a_4^2 + a_2a_3 - bc = 1$. Hence if $\varphi \neq 1$, then $a_1 + a_4 = 0$, so

$$\varphi = \begin{pmatrix} a_1 & a_2 & 0 & b \\ a_3 & -a_1 & -b & 0 \\ 0 & c & a_1 & a_3 \\ -c & 0 & a_2 & -a_1 \end{pmatrix}$$

with $a_1^2 + a_2a_3 - bc = 1$, $(a_1 - 1), a_2, a_3, b, c \equiv 0 \pmod{2}$.

We need to prove that any matrix with these properties is conjugate to φ_0 . The vector spaces $\text{Ker}(\varphi - 1)$ and $\text{Ker}(\varphi + 1)$ are orthogonal, so we should simply find four integer vectors e_1, \dots, e_4 that obey $\varphi(e_i) = (-1)^{i+1}e_i$ and $\langle e_2, e_4 \rangle = \langle e_1, e_3 \rangle = 1$. Because of symmetry, it is enough to find e_1 and e_3 . Let us denote $d = \text{g.c.d.}(b/2, a_3/2, (a_1 - 1)/2)$. There holds $\alpha b/2 + \beta(a_1 - 1)/2 + \gamma(-a_3/2) = d$ for some integers α, β, γ . Now we simply put

$$e_1 = \begin{pmatrix} b/2d \\ 0 \\ a_3/2d \\ (1 - a_1)/2d \end{pmatrix}, \quad e_3 = \alpha \begin{pmatrix} 0 \\ -b/2 \\ (a_1 + 1)/2 \\ a_2/2 \end{pmatrix} + \beta \begin{pmatrix} a_2/2 \\ (1 - a_1)/2 \\ c/2 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} (a_1 + 1)/2 \\ a_3/2 \\ 0 \\ -c/2 \end{pmatrix}$$

and check the required conditions by direct calculation. \square

DEFINITION 4.2 *Let H be a subgroup of finite index in $\Gamma(1)$. We call E_i or F_j a ramification divisor iff H contains the involution that fixes all points of the divisor. Because of the results of 2.8 and 2.10, E_α is a ramification divisor iff $\text{ram}_H(E_\alpha) = 1$, and similarly for F_β .*

We are interested in singularities of \mathcal{H}/H . They occur at the images of the points of \mathcal{H} that have nontrivial stabilizers in H . The goal of the rest of this section is to prove the following statement.

PROPOSITION 4.3 *Singularities of the images of the points $\xi \in \mathcal{H}$ that do not lie in ramification divisors E_i or F_j are canonical. Points that do lie in ramification divisors have solvable stabilizers of order at most 72. We refer to 7.7 for the definitions of canonical and terminal singularities.*

Proof. There are two possibilities: $\xi \in \cup E_i$ and $\xi \notin \cup E_i$.

Case 1. $\xi \notin \cup E_i$. The stabilizer of ξ in $\Gamma(2)$ equals $\{\pm 1\}$ because of Proposition 4.1 and the definition of E_i . We consider the quotient of \mathcal{H} by the action of $\Gamma(2)$. It is the smooth part of the singular quartic V defined by the equation $(\sum x_i^2)^2 = 4 \sum x_i^4$ in coordinates $(x_1 : \dots : x_6, \sum x_i = 0)$ of \mathbf{P}^4 , see [7]. The group $\Gamma(1)/\Gamma(2) \simeq \Sigma_6$ acts on V by the permutations of the coordinates x_i . The stabilizer ξ in $\Gamma(1)$ equals that of the image of ξ in V in the group Σ_6 . Moreover, locally their actions are the same, so the resulting quotient singularities are isomorphic. Therefore, we need to study fixed points of Σ_6 -action on V .

LEMMA 4.4 *A point $\xi \notin \cup E_i$ with a nontrivial stabilizer in $\Gamma(1)$ either lies in $\cup F_j$ or has the image in V of type $\sigma(0 : \theta : \theta^2 : \theta^3 : \theta^4 : 1)$, $\theta = \exp(2\pi i/5)$, $\sigma \in \Sigma_6$.*

Proof of the lemma. Denote by $x = (x_1 : \dots : x_6)$ the image of ξ in V . We may assume that the stabilizer of x contains one of the permutations

$$(1, 2); (1, 2)(3, 4); (1, 2)(3, 4)(5, 6); (1, 2, 3); (1, 2, 3)(4, 5, 6); (1, 2, 3, 4, 5).$$

Let us calculate the sets of fixed points of these permutations that lie in V .

Case (1,2). We have $(x_1, x_2, \dots, x_6) = \lambda(x_2, x_1, \dots, x_6)$. If $\lambda = -1$, then $x = (-1 : 1 : 0 : 0 : 0 : 0)$, but this point does not lie in V . Hence $\lambda = 1$. The set defined by " $x_1 = x_2$ " constitutes an irreducible divisor on V , so it is the closure of the image of some submanifold of dimension two in \mathcal{H} .

Case (1,2)(3,4). We have $(x_1, x_2, x_3, x_4, x_5, x_6) = \lambda(x_2, x_1, x_4, x_3, x_5, x_6)$. If $\lambda = -1$, then $x_1 = -x_2$, $x_3 = -x_4$, $x_5 = x_6 = 0$. The equality $(\sum x_i^2)^2 = 4 \sum x_i^4$ implies that $x_1 = x_3$ or $x_1 = x_4$, so $x \in \text{Sing}(V)$, see [7]. If $\lambda = 1$, then x lies in the divisor " $x_1 = x_2$ ".

Case (1,2),(3,4),(5,6). We have $(x_1, x_2, x_3, x_4, x_5, x_6) = \lambda(x_2, x_1, x_4, x_3, x_6, x_5)$. If $\lambda = -1$, then $x_1 = -x_2$, $x_3 = -x_4$, $x_5 = -x_6$. Equality $(\sum x_i^2)^2 = 4 \sum x_i^4$ leads to $(x_1 + x_3 + x_5) \cdot (x_1 + x_3 - x_5) \cdot (x_1 - x_3 + x_5) \cdot (-x_1 + x_3 + x_5) = 0$. Each of these linear equations implies that x lies in the image of $\cup E_i$, see [7]. If $\lambda = 1$, then $x_1 = x_2$, $x_3 = x_4$, $x_5 = x_6$ so $x \in \text{Sing}(V)$.

Case (1,2,3). We have $(x_1, x_2, x_3, x_4, x_5, x_6) = \lambda(x_2, x_3, x_1, x_4, x_5, x_6)$. If $\lambda \neq 1$, then $x_1 + x_2 + x_3 = 0$, so x lies in the image of $\cup E_i$. If $\lambda = 1$, then $x_1 = x_2 = x_3$, and x lies in the divisor " $x_1 = x_2$ ".

Case (1,2,3)(4,5,6). We have $(x_1, x_2, x_3, x_4, x_5, x_6) = \lambda(x_2, x_3, x_1, x_5, x_6, x_4)$. If $\lambda = 1$, then $x = (1 : 1 : 1 : -1 : -1 : -1) \notin V$. Otherwise, $x_1 + x_2 + x_3 = 0$, and $\xi \in \cup E_i$.

Case (1,2,3,4,5). We have $(x_1, x_2, x_3, x_4, x_5, x_6) = \lambda(x_2, x_3, x_4, x_5, x_1, x_6)$. If $\lambda = 1$, then $x = (1 : 1 : 1 : 1 : 1 : -5) \notin V$. Otherwise $x = \sigma(0 : \theta : \theta^2 : \theta^3 : \theta^4 : 1)$, $\theta = \exp(2\pi i/5)$, $\sigma \in \Sigma_6$.

The above calculation shows that there is only one up to Σ_6 -action divisor on V with a nontrivial stabilizer of a generic closed point. On the other hand, the images of F_j on V obey this condition. Therefore the images of F_j are the conjugates of the divisor " $x_1 = x_2$ ", which proves the lemma. \square

REMARK 4.5 *As a corollary of this lemma, codimension one components of the ramification locus of the map from \mathcal{H}/Γ_n to \mathcal{H}/H can only be divisors E_α and F_β . Moreover, ramification occurs iff $E_\alpha(F_\beta)$ is a ramification divisor as defined above, and in this case the only nontrivial element that preserves all points of the divisor is the corresponding involution. Of course, when we consider the Igusa compactifications, we may have ramification at infinity divisors.*

Let us come back to the proof of 4.3. We try to estimate the singularity at the image of the point $\xi \notin \cup E_i$ under the quotient map $\mathcal{H} \rightarrow \mathcal{H}/H$. The group $\Gamma(2)/\{\pm 1\}$ acts freely on $\mathcal{H} - \cup E_i$, so we can work in terms of the image point $x \in V - \text{Sing}(V)$ and the group $\text{Stab}^H \xi \cdot \Gamma(2)/\Gamma(2)$, because these quotient singularities are isomorphic. There exists a useful criterion that enables one to find out whether the quotient singularity is canonical, see [12]. In particular, it is always canonical, if the image of the group in $\text{Gl}(T_x)$ lies in $\text{Sl}(T_x)$. We use these facts extensively.

First of all we consider the case $x = \sigma(0 : \theta : \dots : 1)$. Then either $\text{Stab}^H \xi \cdot \Gamma(2)/\Gamma(2) = 1$ or $\text{Stab}^H \xi \cdot \Gamma(2)/\Gamma(2) = \mathbf{Z}/5\mathbf{Z}$. A direct calculation of the weights of the generator in the tangent space and the criterion of [12] show that the quotient singularity is terminal, hence canonical.

Now let us consider other points $x = (x_1 : x_2 : x_3 : x_4 : x_5 : x_6) \in V - \text{Sing}(V)$. The group $S = \text{Stab}^H \xi \cdot \Gamma(2)/\Gamma(2)$ contains no transpositions, because ξ does not belong to any ramification divisors F_i . The proof of 4.4 shows that S does not contain permutations of types $(*,*)(*,*)(*,*)$, $(*,*,*)(*,*,*)$, and $(*,*,*,*,*)$. As a result, S consists of permutations of types $(*,*)(*,*)$, $(*,*,*)$, $(*,*,*,*)$, and $(*,*)(*,*,*,*)$ only. Calculations similar to those of 4.4 show that if the group S contains $(*,*)(*,*,*,*)$, then $\xi \in \cup E_i$. Moreover, if it contains a permutation of type $(*,*,*,*)$ and the proportionality coefficient λ does not equal 1, then $\xi \in \cup E_i$. Notice (see the proof of 4.4) that the proportionality coefficients of elements of the group S of types $(*,*)(*,*)$ and $(*,*,*)$ must also equal 1. All these restrictions on the group S imply that it consists of even permutations, and all proportionality coefficients are equal to 1. Therefore, the group S acts in the tangent space of x by matrices from Sl . The criterion of M. Reid shows that the quotient singularity is canonical.

Case 2. $\xi \in \cup E_i$. all divisors E_i are conjugates, so we may assume that ξ is represented by a diagonal matrix. Different E_i do not intersect, so the stabilizer S of ξ in $\Gamma(1)$ is a subgroup of

$$\text{Stab}(\Delta) = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & a_1 & 0 & b_1 \\ c & 0 & d & 0 \\ 0 & c_1 & 0 & d_1 \end{pmatrix} \cup \begin{pmatrix} 0 & a_1 & 0 & b_1 \\ a & 0 & b & 0 \\ 0 & c_1 & 0 & d_1 \\ c & 0 & d & 0 \end{pmatrix}, ad - bc = a_1 b_1 - c_1 d_1 = 1 \right\}.$$

Point ξ may be transformed by the group $\text{Stab}(\Delta)$ to the point $\begin{pmatrix} x_0 & 0 \\ 0 & z_0 \end{pmatrix}$ with $|\text{Re}(x_0)| \leq 1/2$, $|x_0| \geq 1$, $|\text{Re}(z_0)| \leq 1/2$, $|z_0| \geq 1$. Without any loss of generality one may consider points of this type only. The stabilizer of the general such point in $\Gamma(1)/\{\pm 1\}$ equals $\mathbf{Z}/2\mathbf{Z}$. It is generated by the involution of Proposition 4.1. If this element is in H , then Δ is a ramification divisor by our definition. The order of the stabilizer can increase in the following curves and points (we have used the symmetry between x and z)

$$\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} \rho & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & \rho \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}.$$

Let us check all these cases.

Case $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. Because Δ is not a ramification divisor, the order of $\text{Stab}^H \xi$ is at most two, so the quotient singularity is canonical.

Case $\begin{pmatrix} i & 0 \\ 0 & x \end{pmatrix}$. We get $|\text{Stab}^H \xi| = 1$ by the same argument.

Case $\begin{pmatrix} \rho & 0 \\ 0 & x \end{pmatrix}$. In this case either $|\text{Stab}^H \xi| = 1$ or $\text{Stab}^H \xi$ is generated by the element of order 3 whose action in the tangent space of ξ has determinant 1.

Case $\begin{pmatrix} i & 0 \\ 0 & \rho \end{pmatrix}$. The argument is the same as in the previous case.

Case $\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$. The stabilizer of ξ in $\Gamma(1)/\{1, -1\}$ is generated by the images of elements of $\Gamma(1)$ with matrices

$$\varphi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The relations are $\alpha\beta = \beta\alpha$, $\alpha^2 = \beta^2$, $\varphi\alpha = \beta\varphi$, $\varphi^2 = \alpha^4 = \beta^4 = 1$. The order of the group is 16.

The point ξ does not lie in the ramification divisors, so $\text{Stab}^H \xi$ does not contain any conjugates of φ . As in the above cases, $\text{Stab}^H \xi$ does not contain α^2 either. We can also employ the following simple statement: if $s^2 = 1$ for all $s \in \text{Stab}^H \xi$, then the quotient singularity is canonical. In our case it implies that if the quotient singularity is not canonical, then the group $\text{Stab}^H \xi$ contains an element of order 4. All these conditions on $\text{Stab}^H \xi$ together hold iff this group is generated by a conjugate of $\varphi\beta$. A direct calculation of the weights in the tangent space completes the argument.

Case $\begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}$. In this case it is possible to check that the condition " ξ does not belong to any ramification divisors " implies that $\text{Stab}^H \xi$ acts in the tangent space of ξ by matrices with determinant 1.

To finish the proof of Theorem 4.3, we only need to check that stabilizers of all points of \mathcal{H} are solvable groups whose orders are at most 72. It can be done using the description of $\mathcal{H}/\Gamma(2)$ as the smooth part of the singular quartic. I skip the details, because this number is clearly bounded and only slightly affects the constant in the final result. \square

5 Finiteness Theorem for subgroups $H \supseteq \Gamma(p^t)$

We assume that $n = p^t$ throughout this section. We denote the subgroup $H \supseteq \Gamma(n)$ and the quotient $H/\Gamma(n)$ by the same letter, which should not lead to a confusion. The Igusa compactifications of $\mathcal{H}/\Gamma(n)$ and \mathcal{H}/H are denoted by X and Y . The quotient map $X \rightarrow X/H = Y$ is denoted by μ .

We start by pulling the problem from Y to X .

DEFINITION 5.1 *Let $\pi : Z \rightarrow Y$ be a desingularization of Y . Denote by $-1 + \delta$ the minimum discrepancy of Y , see 7.6. Because of 7.9, δ is a positive rational number.*

DEFINITION 5.2 *Let m be a sufficiently divisible number, so that mK_Y is a Cartier divisor on Y . The vector space $H^0(Y, mK_Y - \text{mlt})$ is defined as the space of global sections s of the coherent subsheaf of $\mathcal{O}_Y(mK_Y)$ that consists of sections that lie in $m_y^{m(1-\delta)}(\mathcal{O}_Y(mK_Y))_y$ for all noncanonical singular points $y \in Y$.*

REMARK 5.3 *We assume m to be sufficiently divisible whenever it is necessary. We also omit \mathcal{O} in the notations of the space of global sections, unless it can lead to a misunderstanding.*

PROPOSITION 5.4 $\dim H^0(Z, mK_Z) \geq \dim H^0(Y, mK_Y - \text{mlt})$.

Proof. The pullbacks π^*s vanish with the multiplicity at least $m(1 - \delta)$ along exceptional divisors with negative discrepancies. Hence we can define an injective linear map from $H^0(Y, mK_Y - \text{mlt})$ to $H^0(Z, mK_Z)$. \square

DEFINITION 5.5 *Denote by $H^0(Y, mK_Y - \text{mlt}^0)$ the space of global sections $s \in H^0(Y, mK_Y)$ that satisfy $s \in m_y^{m(1-\delta)}(\mathcal{O}_Y(mK_Y))_y$ for all noncanonical singular points of Y except for the images of points $P_{\alpha\beta\gamma}$ that are triple intersections of infinity divisors on X .*

Clearly, $H^0(Y, mK_Y - \text{mlt}^0) \supseteq H^0(Y, mK_Y - \text{mlt})$.

PROPOSITION 5.6 *If $|G : H| > 2^{953}[2^{165870}]_p$, then $\dim H^0(Y, mK_Y - \text{mlt}^0) - \dim H^0(Y, mK_Y - \text{mlt}) \preceq_{m \rightarrow \infty} 2^{-8}3^{-6}5^{-1}m^3|G : H|$.*

Proof. When $m \rightarrow \infty$, the codimension we are trying to estimate grows no faster than $(\sum_{Q \in Y} \text{mult}_Q)(m^3/6)$, where $\sum_{Q \in Y} \text{mult}_Q$ is the sum over all points Q in the image of $\cup P_{\alpha\beta\gamma}$, and mult_Q is the multiplicity of the local ring of Y at Q . We want to relate it to the statement of Proposition 3.21. We need an easy lemma.

LEMMA 5.7 *Let $P_{\alpha\beta\gamma} = D_\alpha \cap D_\beta \cap D_\gamma$ be a point on X , such that $\mu(P_{\alpha\beta\gamma}) = Q$. Then $\text{mult}_Q \leq 6^3 \text{mult}_H(P_{\alpha\beta\gamma})$ with $\text{mult}_H(P_{\alpha\beta\gamma})$ defined in 3.20.*

Proof of the lemma. Every element of G that fixes $P_{\alpha\beta\gamma}$ permutes the triple of the \pm vectors $(\pm v_\alpha, \pm v_\beta, \pm v_\gamma)$. Hence the subgroup in $\text{Stab}^H(P)$ of the elements that induce trivial permutations is a normal subgroup of order at most 6. One can show that this subgroup coincides with $\text{Ram}_H(P_{\alpha\beta\gamma})$ by the explicit matrix calculation for the standard triple $v_\alpha = {}^t(0, 1, 0, 0)$, $v_\beta = {}^t(-1, 1, 0, 0)$, $v_\gamma = {}^t(1, 0, 0, 0)$.

Therefore, the singularity of Y at Q can be obtained as the quotient of the singularity of $X/\text{Ram}_H(P_{\alpha\beta\gamma})$ by the group of order at most 6. Its multiplicity can be estimated by means of Proposition 7.10. \square

As a result of this lemma, the codimension we are trying to estimate grows no faster than $m^3 6^2 \sum^* \text{mult}_H(P_{\alpha\beta\gamma})$, where one takes one point $P_{\alpha\beta\gamma}$ per orbit of H . By 3.21 with $\epsilon = 2^{-26}$, it grows no faster than $2^{-8}3^{-6}5^{-1}m^3|G : H|$, if $|G : H| > 2^{953}[2^{165870}]_p$. \square

PROPOSITION 5.8 *Let L_Y be a divisor of the modular form of weight 1 on Y . Then $\dim H^0(Y, mL_Y)$ grows as $2^{-7}3^{-6}5^{-1}m^3|G : H|$.*

Proof. It can be derived, for instance, from the formula for $\dim H^0(X, mL_X)$ and $\oplus_m H^0(Y, mL_Y) = (\oplus_m H^0(X, mL_X))^H$. \square

PROPOSITION 5.9 *If $\dim H^0(Y, m(K_Y - L_Y) - \text{mlt}^0) \neq 0$ for sufficiently big m and $|G : H| > 2^{953}[2^{165870}]_p$, then the variety Y is of general type.*

Proof. We get

$$\begin{aligned} \dim H^0(Z, mK_Z) &\geq \dim H^0(Y, mK_Y - \text{mlt}) \\ &\geq \dim H^0(Y, mK_Y - \text{mlt}^0) - 2^{-8}3^{-6}5^{-1}|G : H|m^3 \\ &\geq \dim H^0(Y, mL_Y) - 2^{-8}3^{-6}5^{-1}|G : H|m^3 \sim 2^{-8}3^{-6}5^{-1}|G : H|m^3. \end{aligned}$$

□

We shall eventually prove that if $|G : H|$ is big, then $\dim H^0(Y, m(K_Y - L_Y) - \text{mlt}^0) \neq 0$ for big m .

DEFINITION 5.10 *Let R be the ramification divisor of the morphism μ . We define $H^0(X, mK - mR - mL - \text{mlt}^0)$ to be the space of global sections of $\mathcal{O}_X(m(K_X - R - L_X))$ that satisfy certain vanishing conditions. Namely, we require their germs to lie in $m_x^{m \cdot k(\text{Stab}^H(x))} \mathcal{O}_X(m(K_X - R - L_X))_x$ for all points $x \in X$ whose images in Y have noncanonical singularities, except for $x = P_{\alpha\beta\gamma}$. Here $k(\text{Stab}^H(x))$ is defined according to remark 7.12.*

PROPOSITION 5.11 *If $|G : H| > 2^{953}[2^{165870}]_p$ and $\dim H^0(X, mK - mR - mL - \text{mlt}^0) \neq 0$ for some $m > 0$, then the variety Y is of general type.*

Proof. Because of 7.10, all H -invariant elements of $H^0(X, mK - mR - mL - \text{mlt}^0)$ can be pushed down to elements of $H^0(Y, m(K_Y - L_Y) - \text{mlt}^0)$. Notice, that $\mu^*(m(K_Y - L_Y)) = m(K_X - R - L_X)$, and $m\delta$ is dropped from the vanishing conditions to compensate for the constant N from 7.10. One can multiply the H -conjugates of any section to get an H -invariant one, so if $\dim H^0(X, mK - mR - mL - \text{mlt}^0) \neq 0$ for some $m > 0$, then $\dim H^0(Y, m(K_Y - L_Y) - \text{mlt}^0) \neq 0$ for big m , and Proposition 5.9 finishes the proof. □

Let us now describe ramification divisors and points with noncanonical images.

PROPOSITION 5.12 *The ramification divisor R equals $\sum_{\alpha}(n \cdot \text{ram}_H(v_{\alpha}) - 1)D_{\alpha} + \sum_{\alpha} \text{ram}_H(E_{\alpha})E_{\alpha} + \sum_{\alpha} \text{ram}_H(F_{\alpha})F_{\alpha}$.*

Proof. We know by 4.5 that the ramification divisor in the finite part is equal to $\sum_{\alpha} \text{ram}_H(E_{\alpha})E_{\alpha} + \sum_{\alpha} \text{ram}_H(F_{\alpha})F_{\alpha}$. We only need to show that the group of elements of G that fix all points of the divisor D_{α} is exactly $\pm \text{Ram}_G(v_{\alpha})$. It can be done explicitly in coordinates for the standard divisor D_0 . □

PROPOSITION 5.13 *If $x \in D_{\alpha}$, but $x \notin \cup(D_{\alpha} \cap D_{\beta})$, then $\text{Stab}^H(x)/(\pm \text{Ram}_H(v_{\alpha}))$ is a group of order at most 6.*

Proof. We only need to consider the standard divisor D_0 . It is the universal elliptic curve with level n structure. It can be shown that the group $\text{Stab}_{D_0}^G$ acts on it by a combination of modular transformations of the base, additions of points of order n in the fibers, and the involution $a \rightarrow -a$ of the fibers. The order 6 can be reached for the point x on the curve with complex multiplication, such that x satisfies $2n \cdot x = 0$, and all other stabilizers are even smaller. I skip the details, because a different bound here would only slightly affect the final estimate. \square

PROPOSITION 5.14 *If $x \in l_{\alpha\beta} = D_\alpha \cap D_\beta$, but $x \notin \cup P_{\alpha\beta\gamma}$, then the order of the group $\text{Stab}^H(x) / \pm \text{Ram}_H(l_{\alpha\beta})$ is at most 4.*

Proof. We may assume that $l_{\alpha,\beta} = l_0$ is the standard line. The group $\text{Stab}^G(l_0)$ contains a subgroup of index 2 of elements that preserve both $\pm^t(1, 0, 0, 0)$ and $\pm^t(0, 1, 0, 0)$. It in turn contains a subgroup of index 2 that consists of matrices $\pm \begin{pmatrix} \mathbf{1} & B \\ \mathbf{0} & \mathbf{1} \end{pmatrix}$, $B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. One can show using the explicit coordinate on l_0 , that if b is nonzero, then the action of this element has no fixed points on l_0 , except for the points of triple intersection of the infinity divisors, which finishes the proof. \square

PROPOSITION 5.15 *If $|G : H| > 2^{953}[2^{165870}]_p$ and*

$$\begin{aligned} & \dim H^0(m(K - L) - m \sum_{\alpha} n \cdot \text{ram}_H(v_{\alpha})7D_{\alpha} - m \sum_{\alpha} \text{ram}_H(E_{\alpha})73E_{\alpha} \\ & - m \sum_{\alpha} \text{ram}_H(F_{\alpha})73F_{\alpha} - m \sum_{\alpha\beta} n \cdot \text{ram}_H(l_{\alpha\beta})4l_{\alpha\beta}) \neq 0 \end{aligned}$$

for some $m > 0$, then the variety Y is of general type.

Proof. We know from Proposition 4.3 that the points in the finite part, that do not lie in the ramification divisors E_{α} or F_{β} , do not contribute to mlt^0 . Therefore,

$$m \sum_{\alpha} \text{ram}_H(E_{\alpha})73E_{\alpha} + m \sum_{\alpha} \text{ram}_H(F_{\alpha})73F_{\alpha} \geq \text{mlt} + mR$$

in the finite part. This inequality, strictly speaking, is the inclusion of the sheaves of ideals. Analogously, Propositions 5.13 and 5.14 show that

$$m \sum_{\alpha} n \cdot \text{ram}_H(v_{\alpha})7D_{\alpha} \geq mR_D + \text{mlt}^0$$

on D away from $\cup(D_{\alpha} \cap D_{\beta})$, and

$$m \sum_{\alpha\beta} n \cdot \text{ram}_H(l_{\alpha\beta})4l_{\alpha\beta} \geq \text{mlt}^0$$

on $\cup(D_{\alpha} \cap D_{\beta})$ away from points $P_{\alpha\beta\gamma}$. Then it remains to use Proposition 5.11. \square

PROPOSITION 5.16 *If the variety Y is not of general type, then at least one of the following inequalities holds true.*

(1)

$$|G : H| \leq 2^{953} [2^{165870}]_p$$

(2)

$$\begin{aligned} \dim H^0(m(K - L)) - \dim H^0(m(K - L) - m \sum_{\alpha} n \cdot \text{ram}_H(v_{\alpha}) 7D_{\alpha}) \\ \geq ((1/6)c_1(K - L)^3 m^3)/5 \end{aligned}$$

(3)

$$\begin{aligned} \dim H^0(m(K - L)) - \dim H^0(m(K - L) - m \sum_{\alpha} \text{ram}_H(E_{\alpha}) 73E_{\alpha}) \\ \geq ((1/6)c_1(K - L)^3 m^3)/5 \end{aligned}$$

(4)

$$\begin{aligned} \dim H^0(m(K - L)) - \dim H^0(m(K - L) - m \sum_{\alpha} \text{ram}_H(F_{\alpha}) 73F_{\alpha}) \\ \geq ((1/6)c_1(K - L)^3 m^3)/5 \end{aligned}$$

(5)

$$\begin{aligned} \dim H^0(m(K - L) - m \sum_{\alpha} n \cdot \text{ram}_H(v_{\alpha}) 7D_{\alpha} - m \sum_{\alpha} \text{ram}_H(E_{\alpha}) 73E_{\alpha} \\ - m \sum_{\alpha} \text{ram}_H(F_{\alpha}) 73F_{\alpha}) - \dim H^0(m(K - L) - m \sum_{\alpha} n \cdot \text{ram}_H(v_{\alpha}) 7D_{\alpha} \\ - m \sum_{\alpha} \text{ram}_H(E_{\alpha}) 73E_{\alpha} - m \sum_{\alpha} \text{ram}_H(F_{\alpha}) 73F_{\alpha}) \\ - m \sum_{\alpha\beta} n \cdot \text{ram}_H(l_{\alpha\beta}) 4l_{\alpha\beta} \geq ((1/6)c_1(K - L)^3 m^3)/5 \end{aligned}$$

Proof. If (2), (3), and (4) are all false, then

$$\begin{aligned} \dim H^0(m(K - L) - m \sum_{\alpha} n \cdot \text{ram}_H(v_{\alpha}) 7D_{\alpha} - \sum_{\alpha} \text{ram}_H(E_{\alpha}) 73E_{\alpha} \\ - \sum_{\alpha} \text{ram}_H(F_{\alpha}) 73F_{\alpha}) \succeq (2/5)(1/6)c_1(K - L)^3 m^3. \end{aligned}$$

Really, $\dim H^0(m(K - L))$ grows like $(1/6)c_1(K - L)^3 m^3$, because $K - L$ is ample for big n , and $E_{\alpha}, F_{\beta}, D_{\gamma}$ are different divisors. Hence, if (1) and (5) are also false, then Proposition 5.15 proves that the variety Y is of general type. \square

Our next goal is to show that each of the statements (2)-(5) implies that $|G : H|$ is less than some constant. We use results of Yamazaki [15] and statements of Section 3.

PROPOSITION 5.17 *If*

$$\begin{aligned} \dim H^0(m(K-L)) - \dim H^0(m(K-L) - m \sum_{\alpha} n \cdot \text{ram}_H(v_{\alpha}) 7D_{\alpha}) \\ \geq ((1/6)c_1(K-L)^3 m^3)/5, \end{aligned}$$

then $|G : H| < 2^{41}[2^{828}]_p$.

Proof. First of all, we get

$$\begin{aligned} \dim H^0(m(K-L)) - \dim H^0(m(K-L) - m \sum_{\alpha} n \cdot \text{ram}_H(v_{\alpha}) 7D_{\alpha}) \\ \leq \sum_{\alpha} (\dim H^0(m(K-L)) - \dim H^0(m(K-L) - 7mn \cdot \text{ram}_H(v_{\alpha}) D_{\alpha})). \end{aligned}$$

The standard exact sequences associated to $D_{\alpha} \subset X$ allow us to estimate that

$$\begin{aligned} \dim H^0(m(K-L)) - \dim H^0(m(K-L) - 7mn \cdot \text{ram}_H(v_{\alpha}) D_{\alpha}) \\ \leq \sum_{j=0}^{7mn \cdot \text{ram}_H(v_{\alpha}) - 1} \dim H^0(D_{\alpha}, m(K-L) - jD_{\alpha}) \\ = \sum_{j=0}^{7mn \cdot \text{ram}_H(v_{\alpha}) - 1} \dim H^0(D_{\alpha}, m(K-L) + (2j/n)(L+E)). \end{aligned}$$

The divisor $L+E$ is nef on X , because L is nef, divisors E_i are disjoint, and $(L+E)E_i = 0$. The divisor $K-L$ is ample on X , if n is sufficiently big. Therefore, we may use the Riemann-Roch formula to calculate $\dim H^0(D_{\alpha}, m(K-L) + (2j/n)(L+E))$. Because we are only interested in the coefficient of m^3 , as $m \rightarrow \infty$, we only need to take into account the term $(1/2)c_1(m(K-L) + (2j/n)(L+E))^2 c_1(D_{\alpha})$. When j grows, this intersection number grows, therefore

$$\begin{aligned} \dim H^0(m(K-L)) - \dim H^0(m(K-L) - 7mn \cdot \text{ram}_H(v_{\alpha}) D_{\alpha}) \\ \leq 7mn \cdot \text{ram}_H(v_{\alpha}) (1/2) m^2 c_1(K-L + 14 \text{ram}_H(v_{\alpha})(L+E))^2 c_1(D_{\alpha}) \\ \leq m^3 \sharp(v_{\alpha})^{-1} \text{ram}_H(v_{\alpha}) (7n/2) c_1(K-L + 14(L+E))^2 c_1(D). \end{aligned}$$

Hence, if the condition of the proposition is true, then

$$\sharp(v_{\alpha})^{-1} \sum_{\alpha} \text{ram}_H(v_{\alpha}) \geq 105^{-1} c_1(K-L)^3 / (c_1(K-L + 14(L+E))^2 c_1(nD)).$$

The right hand side can be calculated using the formulas of Yamazaki for the intersection numbers of the divisors D, L, K , and E . It is bigger than 2^{-18} if n is sufficiently big, which we may assume without loss of generality. Therefore, by the result of Proposition 3.3, $|G : H| < 2^{41}[2^{828}]_p$. \square

PROPOSITION 5.18 *If*

$$\begin{aligned} \dim H^0(m(K-L)) - \dim H^0(m(K-L) - m \sum_{\alpha} \text{ram}_H(E_{\alpha}) 73E_{\alpha}) \\ \geq ((1/6)c_1(K-L)^3 m^3)/5, \end{aligned}$$

then $|G : H| < 2^{53}[2^{3236}]_p$.

Proof. Analogously to the proof of 5.17, we estimate

$$\begin{aligned} \dim H^0(m(K-L)) - \dim H^0(m(K-L) - m \sum_{\alpha} \text{ram}_H(E_{\alpha}) 73E_{\alpha}) \\ \leq \sum_{\alpha} \text{ram}_H(E_{\alpha}) \sum_{j=0}^{73m-1} \dim H^0(E_{\alpha}, m(K-L) - jE_{\alpha}) \\ = \sum_{\alpha} \text{ram}_H(E_{\alpha}) \sum_{j=0}^{73m-1} \dim H^0(E_{\alpha}, m(K-L) + jL) \\ \leq \#(E_{\alpha})^{-1} \sum_{\alpha} \text{ram}_H(E_{\alpha}) (73/2)m^3 c_1(K+72L)^2 c_1(E). \end{aligned}$$

Therefore,

$$\#(E_{\alpha})^{-1} \sum_{\alpha} \text{ram}_H(E_{\alpha}) \geq 73^{-1} 15^{-1} c_1(K-L)^3 / (c_1(K+72L)^2 c_1(E)) > 2^{-23}.$$

Then Proposition 3.8 tells us that $|G : H| < 2^{53}[2^{3236}]_p$. □

PROPOSITION 5.19 *If*

$$\begin{aligned} \dim H^0(m(K-L)) - \dim H^0(m(K-L) - m \sum_{\alpha} \text{ram}_H(F_{\alpha}) 73F_{\alpha}) \\ \geq ((1/6)c_1(K-L)^3 m^3)/5, \end{aligned}$$

then $|G : H| < 2^{73}[2^{22782}]_p$.

Proof. As in the proof of 5.18, we estimate

$$\begin{aligned} \dim H^0(m(K-L)) - \dim H^0(m(K-L) - m \sum_{\alpha} \text{ram}_H(F_{\alpha}) 73F_{\alpha}) \\ \leq \sum_{\alpha} \text{ram}_H(F_{\alpha}) \sum_{j=0}^{73m-1} \dim H^0(F_{\alpha}, m(K-L) - jF_{\alpha}). \end{aligned}$$

Unfortunately, the geometry of F is more complicated than that of E , and we do not have a nice formula like $(L + E_{\alpha})E_{\alpha} = 0$. We can get away with it by using the adjunction formula together with the Proposition 2.11.

We can estimate

$$\sum_{\alpha} \text{ram}_H(F_{\alpha}) \sum_{j=0}^{73m-1} \dim H^0(F_{\alpha}, m(K-L) - jF_{\alpha})$$

$$\begin{aligned}
&\leq \sum_{\alpha} \text{ram}_H(F_{\alpha}) \sum_{j=0}^{73m-1} \dim H^0(F_{\alpha}, m(K-L) + jK - jK_{F_{\alpha}}) \\
&\leq \sum_{\alpha} \text{ram}_H(F_{\alpha}) \sum_{j=0}^{73m-1} \dim H^0(F_{\alpha}, m(K-L) + jK) \\
&\leq \sharp(F_{\alpha})^{-1} \sum_{\alpha} \text{ram}_H(F_{\alpha}) (73/2) m^3 c_1(74K-L)^2 c_1(F).
\end{aligned}$$

Therefore,

$$\sharp(F_{\alpha})^{-1} \sum_{\alpha} \text{ram}_H(F_{\alpha}) \geq 73^{-1} 15^{-1} c_1(K-L)^3 / (c_1(74K-L)^2 c_1(F)).$$

We need to have some upper bound on $c_1(74K-L)^2 c_1(F)$. To do this, we recall the proof of Proposition 4.3, where we have shown that the images of the divisors F_{α} on the singular quartic V have form $x_i = x_j$. The product $\prod_{i \neq j} (x_i - x_j)^2$ is invariant under the permutations of the coordinates, so it defines a modular form of weight 60, that vanishes on F . Here we use the fact that the coordinates of \mathbf{P}^4 are given by the modular forms of weight 2, see [7]. As a result, $c_1(74K-L)^2 c_1(F) \leq 60 c_1(74K-L)^2 c_1(L)$, and we can estimate $\sharp(F_{\alpha})^{-1} \sum_{\alpha} \text{ram}_H(F_{\alpha}) > 2^{-30}$.

Now Proposition 3.18 implies that $|G : H| < 2^{73} [2^{22782}]_p$. \square

PROPOSITION 5.20 *If*

$$\begin{aligned}
&\dim H^0(m(K-L) - m \sum_{\alpha} n \cdot \text{ram}_H(v_{\alpha}) 7D_{\alpha} - m \sum_{\alpha} \text{ram}_H(E_{\alpha}) 73E_{\alpha} \\
&- m \sum_{\alpha} \text{ram}_H(F_{\alpha}) 73F_{\alpha}) - \dim H^0(m(K-L) - m \sum_{\alpha} n \cdot \text{ram}_H(v_{\alpha}) 7D_{\alpha} \\
&- m \sum_{\alpha} \text{ram}_H(E_{\alpha}) 73E_{\alpha} - m \sum_{\alpha} \text{ram}_H(F_{\alpha}) 73F_{\alpha} \\
&- m \sum_{\alpha\beta} n \cdot \text{ram}_H(l_{\alpha\beta}) 4l_{\alpha\beta}) \geq ((1/6) c_1(K-L)^3 m^3) / 5,
\end{aligned}$$

then $|G : H| < 2^{65} [2^{10470}]_p$.

Proof. Denote

$$L_1 = K - L - 7 \sum_{\alpha} n \cdot \text{ram}(v_{\alpha}) D_{\alpha} - 73 \sum_{\alpha} \text{ram}_H(E_{\alpha}) E_{\alpha} - 73 \sum_{\alpha} \text{ram}_H(F_{\alpha}) F_{\alpha}.$$

Then the left hand side of the proposition does not exceed the sum over all $l_{\alpha\beta}$ of

$$\dim H^0(mL_1) - \dim H^0(mL_1 - 4mn \cdot \text{ram}_H(l_{\alpha\beta}) l_{\alpha\beta}).$$

To estimate this codimension, we consider the blow-up of the variety X along the line $l_{\alpha\beta}$, which we denote by $\pi : X_1 \rightarrow X$. The normal bundle to $l_{\alpha\beta}$ is isomorphic to $\mathcal{O}(2) \oplus \mathcal{O}(2)$. This can be checked by direct calculation. Therefore, the exceptional divisor of π is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$. We get

$$\dim H^0(mL_1) - \dim H^0(mL_1 - 4mn \cdot \text{ram}_H(l_{\alpha\beta}) l_{\alpha\beta})$$

$$\begin{aligned}
&= \dim H^0(m\pi^*L_1) - \dim H^0(m\pi^*L_1 - 4mn \cdot \text{ram}_H(l_{\alpha\beta})S) \\
&\leq \sum_{j=0}^{4mn \cdot \text{ram}(l_{\alpha\beta})-1} \dim H^0(S, m\pi^*L_1 - jS).
\end{aligned}$$

We denote the fiber and the section of $S \rightarrow l_{\alpha\beta}$ by f and s respectively and get $(m\pi^*L_1 - jS)S = m \cdot c_1(L_1)l_{\alpha\beta} \cdot f + j(2f + s)$. Hence, $H^0(S, m\pi^*L_1 - jS)$ grows when j grows, and we have

$$\begin{aligned}
&\dim H^0(mL_1) - \dim H^0(mL_1 - 4mn \cdot \text{ram}_H(l_{\alpha\beta})l_{\alpha\beta}) \\
&\leq 4mn \cdot \text{ram}_H(l_{\alpha\beta}) \dim H^0(S, mc_1(L_1)l_{\alpha\beta} \cdot 2f + 4mn \cdot \text{ram}_H(l_{\alpha\beta}) \cdot j) \\
&\leq m^3 \text{ram}_H(l_{\alpha\beta}) \cdot (8n \cdot \text{ram}_H(l_{\alpha\beta}) + c_1(L_1)l_{\alpha\beta}) \cdot 4n \cdot \text{ram}_H(l_{\alpha\beta}) \\
&\leq m^3 \text{ram}_H(l_{\alpha\beta}) (128n^3 + 16n^2 c_1(L_1)l_{\alpha\beta}) \\
&\preceq_{n \rightarrow \infty} m^3 \text{ram}_H(l_{\alpha\beta}) (128n^3 + 16n^2 \cdot (7n \cdot 2 \cdot 2 + 73 \cdot 2 \cdot n)) \\
&= m^3 \text{ram}_H(l_{\alpha\beta}) \cdot 2912n^3.
\end{aligned}$$

The number of $l_{\alpha\beta}$ is equal to $2^{-3}n^7(1-p^{-4})(1-p^{-2})$, see [15]. Therefore, if the condition of the proposition is true, then

$$(\#(l_{\alpha\beta}))^{-1} \sum_{\alpha\beta} \text{ram}_H(l_{\alpha\beta}) \geq \frac{c_1(K-L)^3}{(30 \cdot 2912 \cdot 2^{-3}n^{10}(1-p^{-4})(1-p^{-2}))} > 2^{-27}.$$

Now the result of Proposition 3.13 gives $|G : H| < 2^{65}[2^{10470}]_p$. □

We are now ready to prove the finiteness theorem for $H \supseteq \Gamma(p^t)$.

PROPOSITION 5.21 *If $|G : H| > 2^{953}[2^{165870}]_p$, then the variety Y is of general type.*

Proof. We simply combine the results of Propositions 5.17, 5.18, 5.19, 5.20, and 5.16. □

PROPOSITION 5.22 Finiteness theorem for $H \supseteq \Gamma(p^t)$. *There are only finitely many subgroups $H \subseteq \text{Sp}(4, \mathbf{Z})$ of finite index that contain $\Gamma(p^t)$ for some p and t , such that the variety \mathcal{H}/H is not of general type.*

Proof. It follows from the fact that $|G : H|$ is bounded. □

In particular, if p is sufficiently big, then for any H , $\text{Sp}(4, \mathbf{Z}) \supset H \supseteq \Gamma(p^t)$ the variety Y is of general type.

6 Finiteness Theorem, the general case

Now we no longer assume that n is a power of a prime number. Our goal is to prove that the condition $n = p^t$ can be dropped from the statement of Proposition 5.22. Our proof is the direct generalization of the argument of [14].

We first estimate prime factors of n .

PROPOSITION 6.1 *If $p > 3$, and*

$$H \cdot \Gamma(p) = \Gamma(1), \quad H \supseteq \Gamma(mp^\alpha), \quad \text{g.c.d.}(m, p) = 1,$$

then $H \supseteq \Gamma(m)$.

Proof. For any group G we denote its image modulo $\Gamma(mp^\alpha)$ by \hat{G} . We have isomorphisms

$$\hat{\Gamma}(1) \simeq \hat{\Gamma}(m) \times \hat{\Gamma}(p^\alpha), \quad \hat{\Gamma}(m) \simeq \text{Sp}(4, \mathbf{Z}/p^\alpha \mathbf{Z}), \quad \hat{\Gamma}(p^\alpha) \simeq \text{Sp}(4, \mathbf{Z}/m \mathbf{Z}).$$

The group $\text{PSp}(4, \mathbf{Z}/p \mathbf{Z})$ is simple for all $p \geq 3$. Because of $\hat{H} \cdot \hat{\Gamma}(p)/\hat{\Gamma}(p) \simeq \text{Sp}(4, \mathbf{Z}/p \mathbf{Z})$, the group \hat{H} has a section isomorphic to $\text{PSp}(4, \mathbf{Z}/p \mathbf{Z})$. Consider the following normal subgroups of $\hat{\Gamma}(1)$.

$$\hat{\Gamma}(1) \supset \hat{\Gamma}(m) \supset \hat{\Gamma}(mp) \supseteq \{e\}.$$

We easily get that $\hat{H} \cap \hat{\Gamma}(m)/\hat{H} \cap \hat{\Gamma}(mp)$ has a section isomorphic to $\text{PSp}(4, \mathbf{Z}/p \mathbf{Z})$, so there holds

$$(\hat{H} \cap \hat{\Gamma}(m)) \cdot \hat{\Gamma}(mp) = \hat{\Gamma}(m)$$

Now it will suffice to prove that the last equality implies $\hat{H} \supseteq \hat{\Gamma}(m)$. Note that $\hat{\Gamma}(m) \simeq \text{Sp}(4, \mathbf{Z}/p^\alpha \mathbf{Z})$ and $\hat{\Gamma}(mp) \simeq \text{Ker}(\text{Sp}(4, \mathbf{Z}/p^\alpha \mathbf{Z}) \rightarrow \text{Sp}(4, \mathbf{Z}/p \mathbf{Z}))$. We denote by K_i the kernels of $\text{Sp}(4, \mathbf{Z}/p^\alpha \mathbf{Z}) \rightarrow \text{Sp}(4, \mathbf{Z}/p^i \mathbf{Z})$ for $i = 1, \dots, \alpha$ and prove that $\hat{H} \supseteq K_i$ by the decreasing induction on i .

For $i = \alpha$ there is nothing to prove. Besides we already have the last step of the induction. Suppose that $\hat{H} \supseteq K_i$, $i > 1$. To prove that $\hat{H} \supseteq K_{i-1}$ consider $h \in \hat{H} \cap \hat{\Gamma}(m)$ such that

$$h \equiv \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{p}.$$

Clearly, $h^{p^i} \in K_i$. Besides, a simple calculation shows that for $p \geq 5$

$$h^{p^{i-1}} \equiv \begin{pmatrix} 1 & 0 & p^{i-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod{p^i}.$$

When the group $\hat{\Gamma}(m)$ acts on K_{i-1}/K_i by conjugation, its subgroup $\hat{\Gamma}(mp)$ acts as identity. We have already known that $(\hat{H} \cap \hat{\Gamma}(m)) \cdot \hat{\Gamma}(mp) = \hat{\Gamma}(m)$, so it is enough to show that conjugates of the element $h^{p^{i-1}}$ generate the whole group K_{i-1} modulo K_i . This can be done by a direct calculation in the abelian group K_{i-1}/K_i . \square

PROPOSITION 6.2 *There exists a natural number N such that if \mathcal{H}/H is not of general type, then*

$$H \supseteq \Gamma\left(\prod_{p_i \leq N} p_i^{n_i}\right)$$

for some natural numbers n_i .

Proof. Let n be the minimum number such that $H \supseteq \Gamma(n)$. Because of the result of 6.1, $H \cdot \Gamma(p) \neq \Gamma(1)$ for all prime factors of p of n bigger than 3. If \mathcal{H}/H is not of general type, then $\mathcal{H}/(H \cdot \Gamma_p)$ is not of general type either, see 7.8. Because of Proposition 5.22, there are only finitely many choices for p . \square

We now prove the Finiteness Theorem in full generality.

Define for any $H \subseteq \Gamma(1)$ and any prime p the p -projection of H as $H_p = \cap_1^\infty H \cdot \Gamma(p^j)$. Note that $H_p \supseteq H$ and $H_p \supseteq \Gamma(p^j)$ for some j . The following proposition allows us to work with p -projections only, after we have got an estimate on the primes.

PROPOSITION 6.3 *For any given set of subgroups $G_i \supseteq \Gamma(p_i^{n_i})$, $i = 1, \dots, k$, there are only finitely many subgroups $H \supseteq \Gamma(p_1^{\alpha_1} \cdot \dots \cdot p_k^{\alpha_k})$ with $H_{p_i} = G_i$.*

Proof. We can simply estimate the index of H if we employ the fact that $\Gamma(p_i)$ are pro- p_i -groups. \square

Now we can easily prove the Finiteness Theorem.

PROPOSITION 6.4 **Finiteness Theorem.** *There are only finitely many subgroups $H \subseteq \mathrm{Sp}(4, \mathbf{Z})$ of finite index, such that \mathcal{H}/H is not of general type.*

Proof. If \mathcal{H}/H is not of general type, then \mathcal{H}/H_p is not of general type either. Therefore, Proposition 5.22 tells us that there are only finitely many choices for H_p . By 6.2, all prime factors of n are bounded, so Proposition 6.3 finishes the proof. \square

7 Varieties of general type and singularities

We first recall some standard facts about varieties of general type and singularities.

DEFINITION 7.1 *A smooth compact algebraic variety X over \mathbf{C} is called a variety of general type if there exists some constant $c > 0$ such that $\dim H^0(X, \mathcal{O}_X(mK_X)) > cm^{\dim X}$ for all sufficiently big (equivalent condition – divisible by some integer d) positive integers m . Here K_X is the canonical divisor of X .*

REMARK 7.2 *If X and Y are birational smooth compact algebraic varieties, then $\dim H^0(X, \mathcal{O}_X(mK_X)) = \dim H^0(Y, \mathcal{O}_Y(mK_Y))$ for $m \geq 0$.*

DEFINITION 7.3 *A field $\mathcal{K} \supset \mathbf{C}$ is called a field of general type if it is a field of the rational functions of a smooth compact algebraic variety of general type.*

DEFINITION 7.4 *An algebraic variety over \mathbf{C} is called a variety of general type if its field of functions is a field of general type.*

DEFINITION 7.5 *A canonical divisor K_X of a normal variety X is a Weil divisor on X that coincides with a canonical divisor on $X - \text{Sing}(X)$. The variety X is called \mathbf{Q} -Gorenstein if mK_X is a Cartier divisor for some integer m .*

REMARK 7.6 *If the variety Y is normal \mathbf{Q} -Gorenstein but has singularities, then the condition " $\dim H^0(Y, \mathcal{O}_Y(mK_Y)) > cm^{\dim Y}$ for $m \rightarrow +\infty$ " does not imply by itself that Y is of general type. Really, if $\pi : Z \rightarrow Y$ is some desingularization, then there holds*

$$K_Z = \pi^*(K_Y) + \sum_i \alpha_i F_i, \quad \alpha_i \in \mathbf{Q}$$

in the sense of equivalence of \mathbf{Q} -Cartier divisors, where F_i are exceptional divisors of morphism π and α_i are some rational numbers called discrepancies. If some α_i are negative, then $\dim H^0(Z, \mathcal{O}_Z(mK_Z))$ may be less than $\dim H^0(Y, \mathcal{O}_Y(mK_Y))$.

DEFINITION 7.7 *A normal \mathbf{Q} -Gorenstein variety Y is said to have log-terminal singularities if for some desingularization $\pi : Z \rightarrow Y$, such that the exceptional divisor $\sum F_i$ has simple normal crossings, all discrepancies are greater than (-1) . A singular point $y \in Y$ is called canonical (resp. terminal) if the discrepancies α_i are nonnegative (resp. positive) for all i such that $\pi(F_i) \ni y$. Once satisfied for some desingularization, whose exceptional locus is a divisor with simple normal crossings, these conditions are satisfied for any desingularization (see [2]).*

PROPOSITION 7.8 *If $\mu : X \rightarrow Y$ is a finite morphism of algebraic varieties and Y is of general type, then X is also of general type.*

Proof. We find a surjective morphism $\hat{\mu} : \hat{X} \rightarrow \hat{Y}$, where \hat{X}, \hat{Y} are smooth projective birational models of X, Y , and then pull back multicanonical forms. \square

The following statement is well-known.

PROPOSITION 7.9 (see [2]) *Let X be a smooth projective algebraic variety over \mathbf{C} with an action of a finite group G . Then the quotient variety $Y = X/G$ has log-terminal singularities.*

Now we shall prove a simple but important technical result about quotient singularities. Let X be a projective algebraic variety with an action of a finite solvable group H . Let x be a (closed) point of X , such that $hx = x$ for all $h \in H$. Suppose we have $\{e\} = H_0 \subset H_1 \subset \dots \subset H_t = H$, where H_{i-1} are normal subgroups of H_i and H_i/H_{i-1} are abelian groups with exponents k_i . Denote $k = k_1 \cdot \dots \cdot k_t$. Denote the local ring of x in X by (A, \mathbf{m}) . Then $(B, \mathbf{n}) = (A^H, \mathbf{m}^H)$ is the local ring of the image of x under the quotient morphism.

PROPOSITION 7.10 *In the above setup there exists a constant N , which depends only on X and H but not on x , such that there holds $\mathbf{m}^{kl+N} \cap B \subseteq \mathbf{n}^l$ for all $l \geq 0$.*

Proof. We do not suppose X to be smooth, so it is enough to consider just the case of an abelian group H with $kH = 0$. There exists a linearized H -invariant very ample invertible sheaf \mathcal{L} on X . Consider the corresponding closed embedding $X \rightarrow \mathbf{P}^{N_0}$. Because H is abelian, there exists an open H -invariant affine neighborhood of x with the ring R equal to $\mathbf{C}[1, l_1/l_0, \dots, l_{N_0}/l_0]/I$ where $l_i \in H^0(X, \mathcal{L})$, $h(l_i) = \mu_i(h) \cdot l_i$, $\forall h \in H$ and I is some ideal. Moreover, we may assume that $f_i = l_i/l_0$ vanish at x , because of $Hx = x$. Hence the local ring (A, \mathfrak{m}) is the localization of R in $p = (f_1, \dots, f_{N_0})$. Because H is finite, one can assume that all denominators are H -invariant. Therefore, the statement of the proposition is equivalent to $p^{kl+N} \cap R^H \subseteq (p^H)^l$.

Each element of p can be represented as a polynomial in f_i with zero constant term. Therefore, each element of p^{kl+N} can be represented as a polynomial in f_i with monomials of degree no less than $kl + N$. For any given $f \in p^{kl+N} \cap R^H$ consider such a representation with the minimum possible number of monomials. Then if for some monomial g of this representation and some element $h \in H$ there holds $h(g) = w \cdot g$, $w \neq 1$, then the formula $f = f \cdot w / (w - 1) - h(f) / (w - 1)$ allows us to reduce the number of monomials. Hence every element $f \in p^{kl+N} \cap R^H$ is a sum of H -invariant monomials of degree at least $kl + N$.

Now we only need to prove that any H -invariant monomial $g = f_1^{\alpha_1} \cdot \dots \cdot f_{N_0}^{\alpha_{N_0}}$ of degree at least $kl + N$ is a product of at least l H -invariant monomials of positive degree. It is time to choose N , namely $N = k \cdot N_0$. Denote by γ_i the maximum integers that do not exceed α_i/k . Then $g = f_1^{k\gamma_1} \cdot \dots \cdot f_{N_0}^{k\gamma_{N_0}} \cdot g_1$ gives the required decomposition, because $\sum \gamma_i > \sum \alpha_i/k - N_0 \geq l$. \square

REMARK 7.11 *Due to the result of [3], the above proposition holds for scheme points which correspond to the subvarieties that are pointwise H -invariant. I wish to thank Melvin Hochster for pointing out this reference.*

REMARK 7.12 *In the rest of the paper $k(H)$ for a finite solvable group H denotes the least possible value of k that could be obtained in the above way.*

The rest of the section is devoted to multiplicities of certain toric singularities. Somewhat unnatural choice of notation is motivated by the notation of Section 3.

DEFINITION 7.13 *Let $G_1 \simeq (\mathbf{Z}/n\mathbf{Z})^3$ act on \mathbf{C}^3 according to the formula*

$$(\xi_1, \xi_2, \xi_3)(x_1, x_2, x_3) = (e^{2\pi i \xi_1/n} x_1, e^{2\pi i \xi_2/n} x_2, e^{2\pi i \xi_3/n} x_3).$$

Let H_1 be a subgroup of G_1 . Define $\delta(H_1) = (1/n) \min_{l \neq 0} (l_1 + l_2 + l_3)$, where the minimum is taken among all H_1 -invariant monomials $x_1^{l_1} x_2^{l_2} x_3^{l_3}$.

PROPOSITION 7.14 *The multiplicity of the local ring of \mathbf{C}^3/H_1 at zero is at most $n^3 \delta(H_1) / |H_1|$.*

Proof. The exponents of the H_1 -invariant monomials form a semigroup, which we denote by K . One can show that the multiplicity is equal to $\text{vol}(\mathbf{R}_{>0}^n - \text{conv}(K -$

$\{0\})/|H_1|$, where the volume is normalized to be equal one on the basic tetrahedron. This result does not seem to be stated explicitly anywhere in the literature, but its proof is completely analogous to the calculation of [13] of multiplicities of the ideals in the polynomial ring that are generated by monomials. On the other hand, this set is contained in the set

$$\begin{aligned} & \text{conv}((l_1, l_2, l_3), (0, 0, n), (0, n, 0), (0, 0, 0)) \cup \dots \\ & \dots \cup \text{conv}((l_1, l_2, l_3), (0, n, 0), (n, 0, 0), (0, 0, 0)), \end{aligned}$$

which has volume $n^3\delta(H_1)$. \square

REMARK 7.15 *Our results on the multiplicities of certain toric singularities can be generalized to arbitrary dimension, but we only need the case of dimension three.*

Now we consider in detail the case when n is a power of a prime number, and the group H_1 is cyclic.

PROPOSITION 7.16 *Let $K = K_{uvw}$ be a semigroup, defined by the conditions $\alpha u + \beta v + \gamma w = 0 \pmod{p^s}$ and $\alpha, \beta, \gamma \in \mathbf{Z}_{\geq 0}$, where u, v , and w are some natural numbers. The number δ defined in 7.13 equals $p^{-s} \min_{K - \{0\}}(\alpha + \beta + \gamma)$. Then the number of homogeneous triples $(u : v : w)$ such that $\delta(u, v, w) \geq \epsilon$ is at most $2^2\epsilon^{-8}[4\epsilon^{-5}]_p$.*

Proof. Consider the intersection of K and the coordinate plane $\alpha = 0$. It is the semigroup K_1 defined by the conditions $\beta, \gamma \in \mathbf{Z}_{\geq 0}$, $\beta v + \gamma w = 0 \pmod{p^s}$. If $\delta(u, v, w) \geq \epsilon$, then $\beta + \gamma \geq \epsilon p^s$ for all nonzero $(\beta, \gamma) \in K_1$. Therefore, the area of $\mathbf{R}_{>0}^2 - \text{conv}(K_1 - \{0\})$ is at least $\epsilon^2 p^{2s}$, if the area of the basic triangle in \mathbf{Z}^2 is equal to one. Because any triangle in \mathbf{Z}^2 with no lattice points inside and on the edges is basic, the number of points of K_1 that lie inside the positive quadrant and on the boundary of $\text{conv}(K_1 - \{0\})$ is at least $-1 + \epsilon^2 p^{2s} / |\mathbf{Z}^2 : \text{span}(K_1)| \geq -1 + \epsilon^2 p^s$.

The function $\beta - \gamma$ is monotone on the boundary of $\text{conv}(K_1 - \{0\})$, and changes by at most $2p^s$ inside the positive quadrant. Hence, there is a segment of this boundary, that is represented by the vector $(\beta_1, -\gamma_1)$ with $0 < \beta_1, \gamma_1$, $\beta_1 + \gamma_1 \leq 2\epsilon^{-2}$. Hence there holds $v\beta_1 = w\gamma_1 \pmod{p^s}$ with $0 < \beta_1, \gamma_1$, $\beta_1 + \gamma_1 \leq 2\epsilon^{-2}$.

Analogously, we have $u\alpha_2 = w\gamma_2 \pmod{p^s}$ with $0 < \alpha_2, \gamma_2$, $\alpha_2 + \gamma_2 \leq 2\epsilon^{-2}$. Besides, $\text{g.c.d.}(w, p^s) \leq [\epsilon^{-1}]_p$, because otherwise $(0, 0, p^s/\text{g.c.d.}(w, p^s))$ lies in K and gives $\delta < \epsilon$.

There are at most $[\epsilon^{-1}]_p$ choices of $w \pmod{p^s}$ up to multiplication by $(\mathbf{Z}/p^s\mathbf{Z})^*$. There are at most $2^2\epsilon^{-8}$ choices for the fourtuple $(\beta_1, \gamma_1, \alpha_2, \gamma_2)$. Once we know $(w, \beta_1, \gamma_1, \alpha_2, \gamma_2)$, there are at most $[2\epsilon^{-2}]_p$ for each of the numbers $u, v \pmod{p^s}$. This proves the proposition. \square

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