String Cohomology of a Toroidal Singularity

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Abstract

We construct explicitly regular sequences in the semigroup ring $R = \mathbf{C}[K]$ of lattice points of the graded cone K. We conjecture that the quotients of R by these sequences describe locally string-theoretic cohomology of a toroidal singularity associated to K. As a byproduct, we give an elementary proof of the result of Hochster that semigroup rings of rational polyhedral cones are Cohen-Macaulay.

1 Introduction

String cohomology vector space of a variety X with Gorenstein toroidal singularities is a rather mysterious object. It is supposed to be a chiral ring of no less mysterious $N = (2, 2)$ superconformal field theory constructed from X and it has known graded dimension. However, the space itself has not been identified so far in such generality. The goal of this paper is to present a candidate for the "contribution of a singular point" to this cohomology space.

The paper is organized as follows. Section 2 contains important preliminary results on the structure of lattice points of the graded cone K. Section 3 uses these results to show that some explicitly written sequences of elements of $R = \mathbf{C}[K]$ are regular in R and in R-module $R^{open} = \mathbf{C}[K^{open}]$. It also contains the proof of an analog of Poincaré duality. It is worth mentioning that we give a short elementary proof of the theorem of Hochster [13]. Finally, the last section describes the relation of these results to Mirror Symmetry and string cohomology.

The author was inspired by recent preprints of Hosono [14] and Stienstra [16] who clarified the relationship between the solutions of GKZ hypergeometric system and Mirror Symmetry. The construction of this paper belongs to the A-side of Mirror Symmetry, any B-side construction should involve solutions of GKZ systems.

One of the basic ideas of the argument has the flavor of the theory of Gröbner bases, which the author learned from [4]. It also appears that it involves the large complex structure limit, see for example [15].

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2 Decomposition of Cone Lattice Points

Let N be a free abelian group of rank r. Let K be a rational polyhedral cone inside $N\otimes\mathbf{R}$. We will assume that $K-K=N$ and $K\cap(-K)=\{0\}$. We will also assume that the cone K is graded, that is there exists an element $\deg\in M=\mathrm{Hom}(N,\mathbf{Z})$ such that the integer generators of all one-dimensional faces of the cone K have degree 1. We will denote the interior of K by K^{open}.

Another piece of data is a subset $\{e_i\}, i=1,...,d$ of the set of lattice points of degree 1 that lie in K. The only condition on the subset is that it includes the generators of all one-dimensional faces of K, that is

$$\sum\mathbf{R}_{\geq 0}e_i=K.$$

We also choose a maximum regular triangulation T based on these points e_i and denote by ψ a strictly convex function on K which is linear on the simplices of triangulation T.

Our first goal is to construct a decomposition of the sets $K\cap N$ and $K^{open}\cap N$ into the disjoint union of sets S_k of the form

$$S_k=b_k+\sum_{i\in I_k}\mathbf{Z}_{\geq 0}e_i$$

where I_k is a simplex of triangulation T of maximum dimension r and b_k is a lattice point inside $\sum_{i\in I_k}\mathbf{R}_{\geq 0}e_i$.

To carry out the construction for a given cone K we fix a generic vector $\xi\in N\otimes\mathbf{R}$ that lies in K^{open}. For every $I\in T$ of maximum dimension, we consider the coordinates of ξ in I, that is we look at $\beta_{I,i}$, such that

$$\xi=\sum_{i\in I}\beta_{I,i}e_i.$$

Because of the genericity of ξ, all β-s are non-zero. We introduce the sets $B_{I,\xi}$ and $B_{I,-\xi}$ as follows

$$B_{I,\xi}=\{b\in I\cap N,\ \text{such that}\ b=\sum_{i\in I}\gamma_i e_i\ \text{with}\ 0<\gamma_i\leq 1\ \text{if}\ \beta_{I,i}<0$$

$$\text{and}\ 0\leq\gamma_i<1\ \text{if}\ \beta_{I,i}>0\},$$

$$B_{I,-\xi}=\{b\in I\cap N,\ \text{such that}\ b=\sum_{i\in I}\gamma_i e_i\ \text{with}\ 0<\gamma_i\leq 1\ \text{if}\ \beta_{I,i}>0$$

$$\text{and}\ 0\leq\gamma_i<1\ \text{if}\ \beta_{I,i}<0\}.$$

Proposition 2.1 In the above notations the following statements hold.

(a) The set $K\cap N$ is the disjoint union of sets $b+\sum_{i\in I}\mathbf{Z}_{\geq 0}e_i$ taken over all $I\in T$ of maximum dimension and all $b\in B_{I,\xi}$.

(b) The set $K^{open}\cap N$ is the disjoint union of sets $b+\sum_{i\in I}\mathbf{Z}_{\geq 0}e_i$ taken over all $I\in T$ of maximum dimension and all $b\in B_{I,-\xi}$.

Proof. (a) If $n \in K \cap N$, consider $n + \epsilon\xi$ for very small $\epsilon > 0$. It lies in $\sum_{i\in I} \mathbf{R}_{\geq 0} e_i$ for some maximum simplex $I \in T$. Therefore, we have

$$n + \epsilon\xi = \sum_{i\in I} \alpha_i(\epsilon) e_i$$

and

$$n = \sum_{i\in I} (\alpha_i(\epsilon) - \epsilon\beta_{I,i}) e_i$$

where $\alpha_i(\epsilon) > 0$. Notice that $(\alpha_i(\epsilon) - \epsilon\beta_{I,i})$ are independent of ϵ. Therefore, they are always nonnegative. Moreover, they are positive for such i that $\beta_{I,i} < 0$. This easily implies that $n \in b + \sum_{i\in I} \mathbf{Z}_{\geq 0} e_i$ for some $b \in B_{I,\xi}$.

Conversely, if $n \in b + \sum_{i\in I} \mathbf{Z}_{\geq 0} e_i$ with $b \in B_{I,\xi}$, then for small $\epsilon > 0$, the vector $n + \epsilon\xi$ lies in $\sum_{i\in I} \mathbf{R}_{>0} e_i$, which determines I uniquely. Besides, there are clearly no intersections between $b_1 + \sum_{i\in I} \mathbf{Z}_{\geq 0} e_i$ and $b_2 + \sum_{i\in I} \mathbf{Z}_{\geq 0} e_i$ for different b_1 and b_2 from $B_{I,\xi}$. The proof of (a) could now be finished by observation that if $n + \epsilon\xi$ lies in some $\sum_{i\in I} \mathbf{R}_{>0} e_i$ for small ϵ, then n lies in K.

(b) The proof is completely analogous. We use the fact that $n - \epsilon\xi$ lies in one of $\sum_{i\in I} \mathbf{R}_{>0} e_i$ if and only if n lies in K^{open}. □

Corollary 2.2 Let us introduce polynomials

$$S(t) = (1-t)^r \sum_{n\in K\cap N} t^{\deg(n)} \quad \text{and} \quad T(t) = (1-t)^r \sum_{n\in K^{open}\cap N} t^{\deg(n)}.$$

Then

$$S(t) = \sum_{I, b\in B_{I,\xi}} t^{\deg(b)}, \quad T(t) = \sum_{I, b\in B_{I,-\xi}} t^{\deg(b)}.$$

Proof. Follows immediately from the above proposition. □

Notice that the standard duality formula

$$S(t) = t^r T(t^{-1})$$

follows immediately from this corollary together with the definitions of $B_{I,\xi}$ and $B_{I,-\xi}$.

In the next section we will use the following result. Let us fix a lattice element $n \in K$. We look for all possible ways of representing n in the form

$$n = b + \sum_{i=1}^{d} k_i e_i$$

where k_i are non-negative integers and $b \in \cup B_{I,\xi}$. The decomposition of $K \cap N$ above gives us one such representation

$$n = b_0 + \sum_{i\in I_0} l_i e_i.$$

We claim that it has special properties with respect to the convex function ψ.

Proposition 2.3 If $n = b + \sum_{i=1}^{d} k_i e_i$ then for small $\epsilon > 0$

$$\psi(n + \epsilon\xi) \geq \psi(b + \epsilon\xi) + \sum_{i=1}^{d} k_i \psi(e_i)$$

and equality holds if and only if $b = b_0,\ k_i = l_i$ for $i \in I_0,\ k_i = 0$ for $i \notin I_0$.

Proof. The inequality is the basic property of the convex function ψ. Equality holds if and only if there exists a maximum simplex I such that the cone $\sum_{i\in I} \mathbf{R}_{\geq 0} e_i$ contains $b+\epsilon\xi$ and all e_i for which k_i are non-zero. Therefore, $n+\epsilon\xi \in \sum_{i\in I} \mathbf{R}_{\geq 0} e_i$ and the proof of Proposition 2.1(a) shows that $I = I_0$. Because of $b + \epsilon\xi \in \sum_{i\in I_0} \mathbf{R}_{\geq 0} e_i$, the lattice element b lies in $b_1 + \sum_{i\in I_0} \mathbf{Z}_{\geq 0} e_i$ for some $b_1 \in B_{I_0,\xi}$. Because the union of such sets is disjoint, we have $b = b_1$. So $b \in B_{I,\xi}$ and therefore b must equal b_0. □

We will also need a similar statement for K^{open}.

Proposition 2.4 Consider a lattice element n in K^{open}. If $n = b + \sum_{i=1}^{d} k_i e_i$ for some $b \in \cup B_{I,-\xi}$ then for small $\epsilon > 0$

$$\psi(n - \epsilon\xi) \geq \psi(b - \epsilon\xi) + \sum_{i=1}^{d} k_i \psi(e_i)$$

and equality holds if and only if $b = b_0,\ k_i = l_i$ for $i \in I_0,\ k_i = 0$ for $i \notin I_0$, where $n = b_0 + \sum_{i\in I_0} l_i e_i$ is given by Proposition 2.1(b).

Proof. The proof of this proposition is completely analogous to the proof of the previous one. □

3 Regular Sequences

Let us fix a basis $m_1, ..., m_r$ of the vector space $M \otimes \mathbf{C}$ where $M = \mathrm{Hom}(N, \mathbf{Z})$. We introduce the semigroup ring $R = \mathbf{C}[K]$ and for every $n \in K$ we denote the corresponding element in R by x^n. We also introduce r elements of R by the formula

$$Z_j = \sum_{i=1}^{d} < m_j, e_i > \mathrm{e}^{2\pi \mathrm{i} a_i} x_i.$$

Here a_i are some numbers assigned to the lattice elements e_i and the elements in R that correspond to e_i are denoted by x_i. These Z_j-s act on R itself, and also on R-module $R^{open} = \mathbf{C}[K^{open}]$, which is an ideal in R.

The goal of this section is to show that for a generic choice of a_i the sequence $Z_1, Z_2, ..., Z_r$ is regular on both R and R^{open}. The following proposition is crucial.

Proposition 3.1 Denote by Z the ideal generated by $Z_1, ..., Z_r$. Then the following statements hold for generic a_i.

(a) Images of x^b for $b \in \cup B_{I,\xi}$ generate R/ZR as $\mathbf{C}$-vector space.

(b) Images of x^b for $b \in \cup B_{I,-\xi}$ generate R^{open}/ZR^{open} as $\mathbf{C}$-vector space.

Proof. (a) We introduce the ring $R_1 = \mathbf{C}[x_1, ..., x_d]$ and consider R and R/ZR as R_1-modules. Proposition 2.1 implies that these R_1-modules are generated by $x^b,\ b \in \cup B_{I,\xi}$. Therefore, for each q we have a surjective map

$$\oplus_{b\in\cup B_{I,\xi}} R_1[x^b] \to R/ZR \to 0 \tag{1}$$

of R_1-modules.

The kernel of map (1) contains generators of two types.

• *Binomial relations.* Whenever we have an identity in the lattice N

$$n = b_1 + \sum_{i=1}^{d} k_{i1}e_i = b_2 + \sum_{i=1}^{d} k_{i2}e_i$$

we have a generator of the form

$$\prod_{i=1}^{d} x_i^{k_{i1}}[x^{b_1}] - \prod_{i=1}^{d} x_i^{k_{i2}}[x^{b_2}].$$

• *Linear relations.* We have generators $Z_j r_1[x^b]$ for $j = 1, ..., d,\ b \in \cup B_{I,\xi},\ r_1 \in R_1$.

It is enough to show that $\oplus\mathbf{C}[x^b]$ maps surjectively on the part of R/ZR of degree less than some fixed big number D. Really, it is enough to show that any element of form $x_i[x^b]$ can be re-expressed as $\sum_b \alpha_b[x^b]$ modulo above relations, and degrees of x^b are less than r.

Let us pick a parameter q and choose

$$\mathrm{e}^{2\pi\mathrm{i}a_i} = q^{\psi(e_i)}.$$

We will also make the following change of variables for each non-zero q. We introduce

$$(x_i)_{new} = q^{\psi(e_i)}x_i, \quad [x^b]_{new} = q^{\psi(b+\epsilon\xi)}[x^b]$$

where ϵ is chosen to be small enough to fit in Proposition 2.3 for all n of degree less than D. Then we rewrite the generators of the kernel of map (1) in terms of new variables.

• *Binomial relations.* Whenever we have an identity in the lattice N

$$n = b_1 + \sum_{i=1}^{d} k_{i1}e_i = b_2 + \sum_{i=1}^{d} k_{i2}e_i$$

we have a generator of the form

$$q^{\psi(b_1+\epsilon\xi)+\sum_i k_{i1}\psi(e_i)-\psi(b_2+\epsilon\xi)-\sum_i k_{i2}\psi(e_i)} \prod_{i=1}^{d}(x_i)_{new}^{k_{i1}}[x^{b_1}]_{new} - \prod_{i=1}^{d}(x_i)_{new}^{k_{i2}}[x^{b_2}]_{new}.$$

• *Linear relations.* We have generators

$$Z_j r_1[x^b]_{new} = \sum_{i=1}^{d} < m_j, e_i > (x_i)_{new} r_1[x^b]_{new}.$$

Among the binomial relations, we will pick only the ones where $n = b_1 + \sum_{i=1}^{d} k_{i1} e_i$ is given by the decomposition of Proposition 2.1. Then, by Proposition 2.3, the power of q is positive, unless $b_2 = b_1,\ k_{\cdot 2} = k_{\cdot 1}$.

Pick a basis of $\oplus_b (R_1[x^b]_{new})_{\deg<D}$ that consists of the products of monomials in R_1 and $[x^b]_{new}$. For every q we can introduce a matrix $A(q)$ which describes the map to $\oplus_b (R_1[x^b]_{new})_{\deg<D}$ from the direct sum

$$\oplus_b \mathbf{C}[x^b]_{new} \ \oplus_{binomial} \mathbf{C}[binomial] \ \oplus_{j,b,r_i} \mathbf{C} Z_j r_i [x^b]_{new}$$

where the direct sum is over the binomial relations that we have just picked and r_i are chosen to be monomials in x_{new} of degree less than D.

To show that the vector space $\oplus_b \mathbf{C}[x^b]_{new}$ surjects onto $(R/ZR)_{\deg<D}$, it is enough to demonstrate that the matrix $A(q)$ has full rank. Notice, that we have picked relations in such a way that $A(q)$ has a limit $A(0)$ as $q \to 0$. Therefore, it will be enough to show that $A(0)$ has full rank.

The *binomial* relations become *monomial* in the limit $q \to 0$ and hence the image of $A(0)$ contains all basis elements of $\oplus_b (R_1[x^b]_{new})_{\deg<D}$ except, perhaps, the elements of the form $\prod_{i\in I} (x_i)_{new}^{k_i} [x^b]_{new}$ for $b \in B_{I,\xi}$. However, if we use the *linear* relations, we can express $(x_i)_{new}, i \in I$ in terms of other $(x_i)_{new}$, which shows that all the basis elements except for $[x^b]_{new}$ themselves are in the image of $A(0)$. And since $[x^b]_{new}$ are also included in the image of $A(0)$ by construction, we have the desired surjectivity of $A(0)$, which finishes the proof of (a).

The proof of (b) is completely analogous. □

From now on we assume that a_i are generic. It is easy now to prove that $Z_1, ..., Z_r$ form a regular sequence on R and R^{open}. We thus reprove for graded cones the result of Hochster [13] which states that R is Cohen-Macaulay.

Proposition 3.2 The sequence $Z_1, ..., Z_r$ is regular on R and R^{open}. Thus R^{open} is a Cohen-Macaulay module over the Cohen-Macaulay ring R.

Proof. Let us show that $Z_1, ..., Z_r$ is regular on R. For every two power series $f(t)$ and $g(t)$ we say that $f(t) > g(t)$ if the first non-zero coefficient of $f(t) - g(t)$ is positive.

For each $k = 0, ..., r$ we denote

$$f_k(t) = \sum_{l\geq 0} t^l \dim_{\mathbf{C}} (R/(Z_1, ..., Z_k)R)_{\deg=l} \ .$$

The exact sequence

$$R/(Z_1, ..., Z_k)R \to R/(Z_1, ..., Z_k)R \to R/(Z_1, ..., Z_{k+1})R \to 0$$

implies that

$$f_{k+1}(t) \geq f_k(t).$$

On the other hand, the fact that $\oplus_{b\in \cup B_{I,\xi}} \mathbf{C}[x^b]$ surjects onto R/ZR implies that the power series (in fact, it is a polynomial) $f_r(t)$ is less or equal to $\sum_{I, b\in B_{I,\xi}} t^{\deg(b)}$ which is equal to $(1-t)^r f_0(t)$ by Corollary 2.2. Therefore, all intermediate inequalities are equalities, which shows that the above sequences are exact on the left.

The same argument works for R^{open}. □

Remark 3.3 Theorem of Hochster could be proved in full generality using our methods. Really, for any cone K we can pick points e_i on one-dimensional faces that lie in the same hyperplane $\deg = 1$ for some $\deg \in M \otimes \mathbf{Q}$. Then the only difference is that $\deg(n)$ is allowed to take values in $\frac{1}{l}\mathbf{Z}$ for some l, which also requires the use of fractional powers of t. However, this does not present any problems, because the integrality of $\deg(n)$ was never used.

Corollary 3.4 Surjective maps of Proposition 3.1 are isomorphisms.

Proof. It follows from the proof of Proposition 3.2 that graded dimensions of these spaces are the same, so surjectivity implies bijectivity. □

Remark 3.5 Regularity of the sequence Z was used in a special case without proof in the paper [11]. In the later correction note [12] the result is stated explicitly, but the proof is inadequate.

Because of the duality $S(t) = t^r T(t^{-1})$, we have $\dim_{\mathbf{C}}(R^{open}/ZR^{open})_{\deg=r} = 1$. We denote by φ a surjective map $R^{open}/ZR^{open} \to \mathbf{C}$ that sends $(R^{open}/ZR^{open})_{\deg<r}$ to zero. Then we have a pairing

$$(R/ZR) \otimes_{\mathbf{C}} (R^{open}/ZR^{open}) \to \mathbf{C}$$

which maps $x \otimes y$ to $\varphi(xy)$.

Proposition 3.6 (*Poincaré Duality*) The pairing

$$(R/ZR) \otimes_{\mathbf{C}} (R^{open}/ZR^{open}) \to \mathbf{C}$$

is non-degenerate.

Proof. We need to show that for every element $x \in R^{open}/ZR^{open}$ the principal R-submodule it generates inside R^{open}/ZR^{open} is non-zero at degree r. Let us pick a homogeneous x whose principal submodule is zero in degree r, which has the highest degree (less than r) among all x with this property. Denote by $R_{>0}$ the maximum ideal in R. For every homogeneous $y \in R_{>0}$ the principal submodule of xy is zero in degree r, but xy has a higher degree, so it must be zero. This implies that there is a non-trivial homomorphism from $\mathbf{C} = R/R_{>0}$ to R^{open}/ZR^{open}, which maps 1 to x. Since the top element certainly provides us with a homomorphism $\mathbf{C} \to R^{open}/ZR^{open}$, it suffices to show that

$$\mathrm{Hom}^R(\mathbf{C}, R^{open}/ZR^{open}) \cong \mathbf{C}.$$

Now we use the well-known result (see, for example, [6]) that R^{open} is the canonical module for R. Hence

$$\mathrm{Ext}_i^R(\mathbf{C}, R^{open}) \cong 0,\ i \neq r,\ \mathrm{Ext}_r^R(\mathbf{C}, R^{open}) \cong \mathbf{C},$$

which is a standard property of canonical modules, see [5]. Now it can be easily deduced from the Koszul complex associated to Z and R^{open} that

$$\mathrm{Hom}^R(\mathbf{C}, R^{open}/ZR^{open}) \cong \mathrm{Ext}_r^R(\mathbf{C}, R^{open}) \cong \mathbf{C},$$

which completes the proof. □

4 Relation to String Cohomology

Now it is time to explain the title of the paper. String-theoretic cohomology of a variety X with toroidal Gorenstein singularities is supposed to be a generalization of the usual cohomology $H^*(X, \mathbf{C})$. However, no general construction exists at this time.

Physicists like to think of the cohomology of a smooth projective variety X as of a chiral ring of the superconformal field theory which they call A model of X (cf. [17]). Unfortunately, their definition of this object uses path integrals over spaces of all maps from a surface to X which are mathematically ill-defined. In addition, physicists introduce A models for some varieties with Gorenstein singularities that admit a crepant resolution and calculate the chiral rings in several examples (cf. [10]). This suggests that there exists a mathematical notion of string cohomology vector spaces of varieties with Gorenstein toroidal singularities which should be defined directly without any reference to path integrals. It is expected to carry a Hodge structure and to satisfy Poincare duality. In the smooth case it should coincide with the usual cohomology.

A very nice formula for the graded dimension of string-theoretic cohomology vector spaces was suggested by Batyrev and Dais in their paper [3]. It was later verified in [2] that this definition of string-theoretic Hodge numbers is compatible with mirror duality of Calabi-Yau hypersurfaces and complete intersections in Gorenstein toric Fano varieties.

Definition 4.1 ([3]) Let $X = \cup_{i \in I} X_i$ be a stratified algebraic variety over $\mathbf{C}$ with at most Gorenstein toroidal singularities such that for any $i \in I$ the singularities of X along the stratum X_i of codimension k_i are defined by a k_i-dimensional finite rational polyhedral cone K_i; that is X is locally isomorphic to

$$\mathbf{C}^{\dim(X) - k_i} \times U_{K_i}$$

at each point $x \in X_i$ where U_{K_i} is a k_i-dimensional affine toric variety which is associated with the cone K_i (see [6]). Batyrev and Dais have introduced the polynomial

$$E_{\mathrm{st}}(X; u, v) = \sum_{i \in I} E(X_i; u, v) \cdot S_{K_i}(uv)$$

where $E(X_i; u, v)$ are E-polynomials of Danilov and Khovanskiî, see [7]. It is called the *string-theoretic E-polynomial of* X. If we write $E_{\mathrm{st}}(X; u, v)$ in form

$$E_{\mathrm{st}}(X; u, v) = \sum_{p,q} a_{p,q} u^p v^q,$$

then the numbers $h^{p,q}_{\mathrm{st}}(X) = (-1)^{p+q} a_{p,q}$ are called the *string-theoretic Hodge numbers of* X.

It is known that these numbers are nonnegative and satisfy Poincare duality. Moreover, when X admits a crepant toroidal desingularization, string Hodge numbers of X coincide with the usual Hodge numbers of the desingularization (cf. [3].)

We can compare this definition with the calculation of the graded dimension of the intersection cohomology of X (cf. [8, 9]). The only difference is that for intersection cohomology polynomials S_{K_i} are replaced by polynomials G_{K_i} which depend on the combinatorics of the face poset of K_i. This suggests that there should exist a complex of sheaves $\mathcal{SC}^*(X)$ on X analogous to the intersection complex $\mathcal{IC}^*(X)$ whose hypercohomology is precisely the string cohomology of X. The cohomology sheaves of $\mathcal{SC}^*$ should be locally constant on the strata and their dimensions should be given by $S_{K_i}(t^2)$.

We suggest that locally over a point x in X_i the cohomology of $\mathcal{SC}^*(X)$ is given by $\mathbf{C}[K_i]/Z\mathbf{C}[K_i]$ where Z is the regular sequence considered in Section 3. Of course, to define such a sequence one needs to choose a generic collection of coefficients $\{a_j\}$ assigned to all points of degree one in all cones K_i. We emphasize that the complex $\mathcal{SC}^*(X)$ is not constructed yet and that its construction will require some further work. The peculiar feature of our approach is that instead of one space of string theoretic cohomology we expect to get the whole family of such spaces, depending on the choice of $\{a_j\}$.

Let us make this more precise.

Definition 4.2 Let X be a stratified variety as above and let $\{a_j\}$ be a generic collection of coefficients for all points of degree one in all cones K_i. If a stratum X_k lies in the closure of the stratum X_l then K_l is a face in K_k. We assume that the collection $\{a_j\}$ is compatible with this face inclusion, that is for any point of K_l the corresponding coefficients in K_k and K_l are the same. Then one can construct a locally constant on strata sheaf $\mathcal{A}$ on X as follows. The germ of $\mathcal{A}$ over a point x in X_i is given by $\mathbf{C}[K_i]/Z\mathbf{C}[K_i]$ where Z is the regular sequence constructed in Section 3. For a small Zariski open neighborhood U of x the sections of $\mathcal{A}$ are collections of elements of $\mathbf{C}[K_l]/Z\mathbf{C}[K_l]$ for the corresponding cones K_l (which are faces of K_i) which agree under the natural surjective maps

$$\mathbf{C}[K_i]/Z\mathbf{C}[K_i] \to \mathbf{C}[K_l]/Z\mathbf{C}[K_l]$$

induced from the projection to the face K_l.

The sheaf $\mathcal{A}$ is naturally graded by the degree in K_i. We will denote its degree d component by $\mathcal{A}_d$. For instance, $\mathcal{A}_0$ is simply the constant sheaf $\mathbf{C}(X)$.

Conjecture 4.3 For a generic choice of $\{a_j\}$ there exists a complex of coherent sheaves $\mathcal{SC}^*(X)$ on X whose odd-dimensional cohomology sheaves are zero and whose even-dimensional cohomology sheaves are

$$H^{2d}\mathcal{SC}^*(X) = \mathcal{A}_d.$$

String cohomology of X will then be defined as the hypercohomology of $\mathcal{SC}^*(X)$.

Remark 4.4 If the complex $\mathcal{SC}^*(X)$ in the above conjecture successfully constructed, one gets a spectral sequence from $\oplus H^k(\mathcal{A}_d)$ to $\oplus H^{k+2d}_{string}(X)$. We expect it to degenerate immediately in the case when X has only quotient singularities.

When X admits a crepant desingularization, this spectral sequence should be similar to the Leray spectral sequence for the constant sheaf on the desingularization. Hopefully, this could be made precise when the complex $\mathcal{SC}^*(X)$ is constructed.

Remark 4.5 The original motivation behind the idea of a family of string cohomology spaces is the following. Physicists claim that A model superconformal field theory depends on both complex and Kähler parameters when X is smooth. When X is a Calabi-Yau hypersurface in a Gorenstein toric Fano variety, the choice of the Kähler structure roughly amounts to the choice of an element of the mirror family, see [1] for precise definitions of mirror duality in the toric setting. This amounts to the choice of coefficients $\{a_i\}$ for all points of degree one in the fan defining the ambient toric variety. So it is natural no allow this data to be a part of the definition of string cohomology.

References

[1] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom., **3** (1994) 493-535.

[2] V.V. Batyrev, L.A. Borisov, *Mirror Duality and String-theoretic Hodge Numbers*, Invent. Math., **126**(1996), Fasc. 1, 183-203.

[3] V.V. Batyrev, D.I. Dais, *Strong McKay Correspondence, String-theoretic Hodge Numbers and Mirror Symmetry*, Topology, **35**(1996), 901-929.

[4] D. Bayer, D. Mumford, *What Can Be Computed in Algebraic Geometry?*, Computational algebraic geometry and commutative algebra (Cortona, 1991), 1-48, Sympos. Math., XXXIV, Cambridge Univ. Press, Cambridge, 1993.

[5] W. Bruns, J. Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1993.

[6] V.I. Danilov, *The Geometry of Toric Varieties*, Russian Math. Surveys, **33**(1978), 97-154.

[7] V.I. Danilov, A.G. Khovanskiî, *Newton Polyhedra and an Algorithm for Computing Hodge-Deligne Numbers*, Math. USSR Izvestiya, **29** (1987), 279-298.

[8] J. Denef, F. Loeser, *Weights of exponential sums, intersection cohomology, and Newton polyhedra*, Invent. Math., **106** (1991), no. 2, 275–294.

[9] K.-H. Fieseler, *Rational intersection cohomology of projective toric varieties*, J. Reine Angew. Math. **413** (1991), 88–98.

[10] B. R. Greene, *String theory on Calabi-Yau manifolds*, Fields, strings and duality (Boulder, CO, 1996), 543-726, World Sci. Publishing, River Edge, NJ, 1997.

[11] I.M. Gel'fand, A.V. Zelevinskii, M.M. Kapranov, *Hypergeometric Functions and Toral Varieties*, Funct. Analysis and its Appl., **23**(1989), 94-106.

[12] correction to [11], Funct. Analysis and its Appl., **27**(1993), 295.

[13] M. Hochster, *Rings of Invariants of Tori, Cohen-Macaulay Rings Generated by Monomials and Polytopes*, Ann. of Math., **96**(1972), 318-337.

[14] S. Hosono, *GKZ Systems, Gröbner Fans and Moduli Spaces of Calabi-Yau Hypersurfaces*, Topological field theory, primitive forms and related topics (Kyoto, 1996), 239-265, Progr. Math., 160, Birkhäuser Boston, Boston, MA, 1998.

[15] D.R. Morrison, *Making Enumerative Predictions by Means of Mirror Symmetry*, Mirror Symmetry II (B. Greene and S.-T. Yau, eds.), International Press, Cambridge, 1997, 457-482.

[16] J. Stienstra, *Resonant Hypergeometric Systems and Mirror Symmetry*, preprint alg-geom/9711002.

[17] E. Witten, *Mirror manifolds and topological field theory*, Essays on Mirror Manifolds (S.-T. Yau, ed.), International Press, Hong Kong, 1992, 120–159.