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SELF-SELF-DUAL SPACES OF POLYNOMIALS

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Abstract. A space of polynomials V of dimension 7 is called self-dual if the divided Wronskian of any 6-subspace is in V. A self-dual space V has a natural inner product. The divided Wronskian of any isotropic 3-subspace of V is a square of a polynomial. We call V self-self-dual if the square root of the divided Wronskian of any isotropic 3-subspace is again in V. We show that the self-self-dual spaces have a natural non-degenerate skew-symmetric 3-form defined in terms of Wronskians.

We show that the self-self-dual spaces correspond to G_2-populations related to the Bethe Ansatz of the Gaudin model of type G_2 and prove that a G_2-population is isomorphic to the G_2 flag variety.

1. Introduction

The Bethe equation is the main equation in the Bethe Ansatz method of diagonalizing the Hamiltonians of many integrable systems of mathematical physics. Given a solution of the Bethe equation, one explicitly constructs an eigenvector of the Hamiltonian. This paper is related to the case of Gaudin model associated to G_2. In this case, the Bethe equation is the system of algebraic equations on complex variables $t_i^{(j)}$ with parameters $z_i \in \mathbb{C}, m_s^{(j)} \in \mathbb{Z}_{>0}$:

$$-\sum_{s=1}^{n} \frac{m_s^{(1)}}{t_i^{(1)} - z_s} - \sum_{k=1}^{l_2} \frac{3}{t_i^{(1)} - t_k^{(2)}} + \sum_{k=1,\ k\neq i}^{l_1} \frac{2}{t_i^{(1)} - t_k^{(1)}} = 0, \qquad i = 1, \dots, l_1,$$

$$-\sum_{s=1}^{n} \frac{3m_s^{(2)}}{t_j^{(2)} - z_s} - \sum_{k=1}^{l_1} \frac{3}{t_j^{(2)} - t_k^{(1)}} + \sum_{k=1,\ k\neq j}^{l_2} \frac{6}{t_j^{(2)} - t_k^{(2)}} = 0, \qquad j = 1, \dots, l_2.$$

A pair of polynomials $(y_1, y_2) \in (\mathbb{P}\mathbb{C}[x])^2$, is called fertile with respect to polynomials T_1, T_2 if there exist polynomials $\tilde{y}_1, \tilde{y}_2$ such that we have explicit Wronskians:

$$W(y_1, \tilde{y}_1) = T_1 y_2, \qquad W(y_2, \tilde{y}_2) = T_2 y_1^3.$$

The pair (y_1, y_2) is called generic if $y_i(x)$ have no multiple roots and no common roots.

It is shown in [MV1] that zeroes of the pair of polynomials $(\prod_{i=1}^{l_1}(x - t_i^{(1)}), \prod_{j=1}^{l_2}(x - t_j^{(2)}))$ satisfy the Bethe equation if and only if the pair is generic and fertile with respect to $T_j(x) = \prod_{i=1}^{n}(x - z_s)^{m_s^{(j)}}$. We call such a pair a Bethe pair.

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Moreover, it is shown in [MV1] that if the pair (y_1, y_2) is a Bethe pair, then the pairs $(\tilde{y}_1, y_2)$ and $(y_1, \tilde{y}_2)$ are fertile and therefore for almost all choices of $\tilde{y}_1, \tilde{y}_2$ these pairs are Bethe pairs. Thus given one Bethe pair we obtain a family of new Bethe pairs which in turn produce new Bethe pairs, etc. The Zariski closure of all Bethe pairs obtained from (y_1, y_2) is called a G_2-population originated at (y_1, y_2).

It is conjectured in [MV1] that the number of G_2-populations for generic z_i equals the multiplicity of L_{Λ_∞} in $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$. Here L_Λ denotes an irreducible finite-dimensional G_2-module with highest weight Λ, $(\Lambda_s, \alpha_i^\vee) = m_s^{(i)}$ and Λ_∞ is the unique dominant weight in the Weyl group orbit of the G_2-weight $\sum_{i=1}^n \Lambda_i - l_1\alpha_1 - l_2\alpha_2$ where α_i are simple roots of a G_2 root system.

The G_2-populations are the original motivation of this paper. It turns out that that the G_2-populations are in one-to-one correspondence with special 7-dimensional spaces of polynomials, which we call self-self-dual.

Let V be a 7-dimensional space of polynomials. Let U_i be the greatest common monic divisor of Wronskians $\{W(v_1, \dots, v_i), v_j \in V\}$. We always assume that $U_1 = 1$. Then there exist unique polynomials $T_1, \dots, T_6$ such that $U_i = T_{i-1}T_{i-2}^2 \dots T_1^{i-1}$.

The polynomial $W^\dagger(v_1, \dots, v_i) = W(v_1, \dots, v_i)/U_i$ is called the divided Wronskian.

The space V is called self-dual if the space of all divided 6-Wronskians coincides with V. Self-dual spaces were studied in [MV1]. Note that if V is self-dual then $T_i = T_{7-i}$. A self-dual space has a natural non-degenerate symmetric bilinear form given by

$$B(u, v) = W^\dagger(u, v_1, \dots, v_6), \qquad \text{if } v = W^\dagger(v_1, \dots, v_6).$$

Moreover, any divided 3-Wronskian of an isotropic 3-space is a perfect square.

We call a self-dual space *self-self-dual* if

$$\{W^\dagger(v_1, v_2, v_3), v_i \in V, B(v_i, v_j) = 0\} \ = \ \{v^2, v \in V\}.$$

Note, that if V is self-self-dual then $T_3 = T_1$, so all T_i are expressed in terms of T_1 and T_2. We prove that a self-self-dual space has a skew-symmetric non-degenerate 3-form w uniquely determined by the condition:

$$w(v_1, v_2, v_3) = B(v, v), \qquad \text{if } W^\dagger(v_1, v_2, v_3) = v^2 \text{ and } B(v_i, v_j) = 0.$$

We identify the group G_2 with the subgroup of orthogonal group $SO(V)$ which preserves w. It follows that the G_2 flag variety G_2/B is identified with the variety of G_2-isotropic flags $F = \{F_1 \subset F_2 \subset \cdots \subset F_7 = V\}$ which have the properties $B(F_i, F_{7-i}) = 0$ and $F_3 = \mathrm{Ker}(w(F_1, \cdot, \cdot))$. We supply proofs of these and some other general facts on G_2 which we failed to find in the literature.

Then we show that the first coordinates of a G_2-population span a self-self-dual space V and the G_2-population is isomorphic to the variety of G_2-isotropic flags in V. The isomorphism maps a flag F to the pair $(F_1, W^\dagger(F_2))$.

Let us say a few words about our methods. A self-dual space V is naturally a vector representation of $SO(7)$. However, we have an explicit expression for the value of the 3-form only on isotropic triples of vectors in V. The lack of general formula makes it

difficult to prove the well-definedness of the 3-form and to compute it. To overcome this problem, we consider the 8-dimensional spin representation $\hat{V}$ of $Spin(7)$. Vectors in V naturally act on $\hat{V}$. Moreover we show that the set of isotropic triples in V embeds in the projectivization of $\hat{V}$ as a non-degenerate conic. This conic defines an inner product $\hat{B}$ on $\hat{V}$ and the vectors of V act by skew-symmetric operators. It turns out that the 3-form can be defined for all triples, isotropic or not, by the same formula $w(a,b,c) = \hat{B}(abc\cdot p, p)$ where $p \in \hat{V}$ is also computed explicitly. In addition, the 2-form B is given by the formula $B(v,w) = 2\hat{B}(v\cdot p, w\cdot p)/\hat{B}(p,p)$. Those key observations allow us to finish all the proofs.

The paper is constructed as follows. In Section 2 we prepare the main facts about spinor embedding of isotropic Grassmannian of the vector representation of $SO(7)$. In Section 3 we collect and prove some general facts about G_2. In Section 4 we define and study self-self-dual spaces of polynomials. In Section 5 we show that G_2-populations are in one-to-one correspondence with self-self-dual spaces. In Section 6 we construct a special basis in a self-self-dual space so that all divided 3-Wronskians, and the 3-form have a canonical form.

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2. Spinor embedding

In this section we define two important technical tools of this paper. First we define and study the spinor embedding of isotropic Grassmannian in the vector representation of $SO(7)$ to the projectivization of the spin representation. Then we proceed to the special maps from the spin representation to the vector representation which we call invariant surjections.

2.1. **Spin representation.** We recall the basic statements about the Clifford algebra and the algebraic group $Spin(7)$. We generally follow the book of Adams, [A]. However, our exposition is adapted to the problem of embedding of the Grassmannian of isotropic 3-spaces in a seven-dimensional space with a symmetric non-degenerate form into the projectivization of the spin representation of $Spin(7)$.

Let V be a seven-dimensional complex vector space equipped with a non-degenerate quadratic form Q. Let $B(\cdot,\cdot)$ be the bilinear form given by $B(v,w) = Q(v+w) - Q(v) - Q(w)$. Note that $B(v,v) = 2Q(v)$.

By definition, the Clifford algebra $Cl(V)$ of V is the quotient of the free tensor algebra $\oplus_{n\geq 0} V^{\otimes n}$ by the two-sided ideal generated by $v\otimes v \oplus Q(v)$ for all $v \in V$. It contains the even Clifford subalgebra $Cl^+(V)$ given by the the image of the even graded components of the free tensor algebra. Dimensions of $Cl(V)$ and $Cl^+(V)$ are 2^7 and 2^6 respectively.

The spinor group $Spin(V)$ is defined as the subgroup of the invertible elements g of $Cl^+(V)$ such that $gVg^{-1} = V$. The group $Spin(V)$ is a double cover of $SO(V) \cong SO(7)$ with the map given by the above conjugation action on V, see [A].

To describe the irreducible representations of $Cl(V)$ and $Cl^+(V)$ we identify these algebras with quotients of group algebras of finite 2-groups as follows. For any orthonormal basis $v_1, \dots, v_7$ of V the images of the elements v_i in $Cl(V)$ generate a group H of order 2^8. The elements of H are of the form $\pm \prod_{i=1}^7 v_i^{r_i}$ with $r_i \in \{0,1\}$. The defining relations are $v_i^2 = -1$ and $v_i v_j = -v_j v_i$ for $i \neq j$. Let H^+ be the subgroup of H given by the condition $\sum_{i=1}^7 r_i = 0 \pmod 2$. The group H^+ has index 2 in H, and the order of H^+ is 2^7. We denote by the same letter w the central element -1 in H and H^+, we have $w^2 = 1$.

The algebra $Cl(V)$ (resp. $Cl^+(V)$) is the quotient of the group algebra of H (H^+) by $w+1$. Consequently, complex representations of $Cl(V)$ (resp. $Cl^+(V)$) are in one-to-one correspondence with representations of H (resp. H^+) for which w acts by (-1).

The quotient groups with respect to $\mathbb{Z}/2\mathbb{Z}$ subgroups generated by w, $H/\{1,w\}$ and $H^+/\{1,w\}$, are abelian 2-groups of orders 2^7 and 2^6 respectively. Therefore, the group algebra of H (resp. H^+) has 2^7 (resp. 2^6) one-dimensional representations where w acts by 1. Recall that the irreducible representations of a group algebra are in one-to-one correspondence with conjugacy classes in the group.

Since 7 is an odd number, conjugacy classes in H^+ are $\{1\}$, $\{w\}$ $\{\pm h\}$, where $h \in H^+/\{1,w\}$, $h \neq 1$. Thus, there is only one additional representation of H^+. Since the sum of squares of dimensions of all irreducible representations equals the order of the group, the dimension of this additional representation d satisfies $d^2 + 2^6 = 2^7$ which yields $d = 8$. This produces a representation of $Cl^+(V)$ and $Spin(V)$ which is called *spin representation*. We denote this representation $\hat{V}$.

The conjugacy classes of the group H are $\{1\}$, $\{w\}$, $\{w_1 := v_1 v_2 \dots v_7\}$, $\{w_2 := w w_1\}$, $\{\pm h\}$ where $h \in H/\{1,w\}$, $h \neq 1$, $h \neq \bar{w}_1$. As a result, there are two additional representations of H. On the other hand, we have two surjective group homomorphisms $H \to H^+$ given by modding out the central element w_1 or w_2. These homomorphisms produce two non-isomorphic irreducible representations of H of dimension 8.

The group H admits an outer automorphism α which sends $\prod_{i=1}^7 v_i^{r_i}$ to $w^{\sum_i r_i} \prod_{i=1}^7 v_i^{r_i}$. This automorphism permutes the two representations of H. Hence, the irreducible 8-dimensional spin representation $\hat{V}$ of $Cl^+(V)$ and $Spin(V)$ can be lifted to a representation of the full Clifford algebra $Cl(V)$ in two ways that differ by the automorphism α of $Cl(V)$ that preserves $Cl^+(V)$ and multiplies odd degree elements of $Cl(V)$ by (-1). In particular, there is an action of elements of V on the space $\hat{V}$, defined uniquely up to an overall sign.

By the construction, the action of $Spin(V)$-module V on $Spin(V)$-module $\hat{V}$ has the following equivariance property

$$g(v \cdot x) = (gv) \cdot (gx), \tag{2.1}$$

where $g \in Spin(V)$, $v \in V$ and $x \in \hat{V}$.

Now we describe the representation $\hat{V}$ explicitly. Let $\hat{V}$ be the space of polynomials in the odd variables $\hat{v}_5, \hat{v}_6, \hat{v}_7$. The space $\hat{V}$ has the basis $\{\hat{1}, \hat{v}_5, \hat{v}_6, \hat{v}_7, \hat{v}_5\hat{v}_6, \hat{v}_6\hat{v}_7, \hat{v}_5\hat{v}_7, \hat{v}_5\hat{v}_6\hat{v}_7\}$.

Choose a basis $v_1, \dots, v_7$ of V with the property $B(v_i, v_j) = (-1)^{i+1}\delta^8_{i+j}$. We will use the convention $Q(v) = \frac{1}{2}B(v,v)$. The algebra $Cl(V)$ is the associative $\mathbb{C}$-algebra with generators v_i and relations

$$v_i v_j + v_j v_i = (-1)^i \delta^8_{i+j}. \tag{2.2}$$

Define an action of $Cl(V)$ in $\hat{V}$ as follows. We let v_5, v_6, v_7 act by multiplications by $\hat{v}_5, \hat{v}_6, \hat{v}_7$ respectively, v_1, v_2, v_3 by differentiations $\partial/\partial v_7$, $-\partial/\partial v_6$,$\partial/\partial v_5$ respectively, and v_4 by $(-1)^{\deg}$, where deg is the degree of the odd polynomial.

Lemma 2.1. *The space $\hat{V}$ is an 8-dimensional irreducible representation of $Cl(V)$. In particular, $\hat{V}$ is the spin representation of $Spin(V)$.*

Proof. It is an easy check that equations (2.2) are satisfied. The irreducibility is obvious. □

We also need the decomposition of the second symmetric power of $\hat{V}$ into irreducible representations of $Spin(V)$, see [A].

Proposition 2.2. *We have the isomorphism of $Spin(V)$-modules defined uniquely up to scalars*

$$Sym^2(\hat{V}) \cong \mathbb{C} \oplus \Lambda^3(V),$$

where $\mathbb{C}$ is the trivial representation.

We also observe that while $SO(V)$ acts on $\hat{V}$ only projectively, the $Sym^2(\hat{V})$ is an honest representation of $SO(V)$, and therefore the above proposition is also a decomposition of representations of the group $SO(V)$.

2.2. **Spinor embedding of isotropic Grassmannian in $\mathbb{P}\hat{V}$.** A subspace $U \subset V$ is called *isotropic* if the the quadratic form Q vanishes on V. Let $IG(3, V)$ denote the Grassmannian of isotropic three-dimensional subspaces in V.

Let $U \subset V$ be an isotropic subspace of V of dimension 3. Let $L_U \subset \hat{V}$ be the common kernel of elements of U acting on $\hat{V}$:

$$L_U := \{x \in \hat{V} \mid v \cdot x = 0,\ v \in U\}.$$

Here $v \cdot x$ is the action of $v \in V$ on $x \in \hat{V}$. The action is defined up to a sign, therefore the space L_U is well defined.

For any complex vector space W we denote by $\mathbb{P}W$ the corresponding projective space, thought of as the space of lines in W. For any non-zero $w \in W$ we have $\mathbb{C}w \in \mathbb{P}W$.

Theorem 2.3. *For every isotropic 3-subspace $U \subset V$ the space L_U is one-dimensional. The map $\rho :\ IG(3, V) \to \mathbb{P}\hat{V}$, sending $U \mapsto L_U$ is a $Spin(V)$-equivariant embedding whose image is a smooth degree two hypersurface in $\mathbb{P}\hat{V}$.*

We will call the map ρ the *spinor embedding*.

Proof. Let v_1, v_2, v_3 be a basis of U. We extend it to a basis of V such that $B(v_i, v_j) = (-1)^{i+1}\delta^8_{i+j}$ and use the model of $\hat{V}$ as in Lemma 2.1.

It is an easy calculation to see that $L_U = \mathbb{C}\hat{1}$ and is therefore one-dimensional and the first part of theorem is proved.

Now that the map ρ is constructed, we will show that it is an injection. Let U_1 and U_2 be two isotropic 3-subspaces of V such that $L_{U_1} = L_{U_2}$. We can assume that $U_1 = \mathrm{Span}(v_1, v_2, v_3)$ and representation $\hat{V}$ is written explicitly as above. It is clear that the only elements of V that annihilate $L_{U_1} = \mathbb{C}\hat{1}$ are the ones from U_1, so $U_1 = U_2$.

To calculate the image of ρ observe that for any element z of $\hat{V}$ its annihilator in V is an isotropic subspace. Indeed, an anticommutator of two elements of annihilator must be zero on z, but anticommutators are given by the pairing. So the necessary and sufficient condition on z to come from an isotropic 3-space is to have the dimension of its annihilator at least 3 (which will then be exactly 3 by the above argument). We do this calculation explicitly. If

$$z = \alpha_\emptyset \hat{1} + \alpha_5\hat{v}_5 + \alpha_6\hat{v}_6 + \alpha_7\hat{v}_7 + \alpha_{56}\hat{v}_5\hat{v}_6 + \alpha_{67}\hat{v}_6\hat{v}_7 + \alpha_{57}\hat{v}_5\hat{v}_7 + \alpha_{567}\hat{v}_5\hat{v}_6\hat{v}_7$$

and $v = \sum_{i=1}^7 \beta_i v_i$ then $v \cdot z = 0$ if and only if

$$\begin{pmatrix} -\alpha_7 & \alpha_6 & -\alpha_5 & \frac{1}{\sqrt{2}}\alpha_\emptyset & 0 & 0 & 0 \\ \alpha_{57} & -\alpha_{56} & 0 & -\frac{1}{\sqrt{2}}\alpha_5 & \alpha_\emptyset & 0 & 0 \\ \alpha_{67} & 0 & -\alpha_{56} & -\frac{1}{\sqrt{2}}\alpha_6 & 0 & \alpha_\emptyset & 0 \\ 0 & \alpha_{67} & -\alpha_{57} & -\frac{1}{\sqrt{2}}\alpha_7 & 0 & 0 & \alpha_\emptyset \\ -\alpha_{567} & 0 & 0 & \frac{1}{\sqrt{2}}\alpha_{56} & \alpha_6 & -\alpha_5 & 0 \\ 0 & 0 & -\alpha_{567} & \frac{1}{\sqrt{2}}\alpha_{67} & 0 & \alpha_7 & -\alpha_6 \\ 0 & -\alpha_{567} & 0 & \frac{1}{\sqrt{2}}\alpha_{57} & \alpha_7 & 0 & -\alpha_5 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}}\alpha_{567} & \alpha_{67} & -\alpha_{57} & \alpha_{56} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence $\mathbb{P}\mathbb{C}z$ is in the image of ρ if and only if the rank of the above matrix is at most 4. Two 7×7 determinants of this matrix obtained by removing the first and the last row are

$$D_1 = -\frac{1}{\sqrt{2}}\alpha_{567}(\alpha_\emptyset\alpha_{567} + \alpha_6\alpha_{57} - \alpha_7\alpha_{56} - \alpha_{67}\alpha_5)^3,$$
$$D_8 = \frac{1}{\sqrt{2}}\alpha_\emptyset(\alpha_\emptyset\alpha_{567} + \alpha_6\alpha_{57} - \alpha_7\alpha_{56} - \alpha_{67}\alpha_5)^3.$$

So the image of ρ must be contained in the union of the hypersurface

$$\alpha_\emptyset\alpha_{567} + \alpha_6\alpha_{57} - \alpha_7\alpha_{56} - \alpha_{67}\alpha_5 = 0 \tag{2.3}$$

and the subspace $\alpha_\emptyset = \alpha_{567} = 0$. By an easy dimension count, the dimension of $\mathrm{IG}(3, V)$ is 6. Hence, the image of ρ is a hypersurface, so it must coincide with the smooth conic above. We also note that $IG(3, V)$ is smooth, since it is a variety with a transitive action of $SO(V)$, which implies that ρ is an embedding. Finally, this embedding is equivariant by construction and (2.1). □

Remark 2.4. In view of Proposition 2.2 it is clear that the image of ρ is given by the dual of the one-dimensional subrepresentation $\mathbb{C}$ of $Sym^2(\hat{V})$. Indeed, the image of ρ is given by an element of $Sym^2(\hat{V}^*)$ which is invariant under the action of $Spin(V)$, perhaps up to a constant. Therefore it is the dual of the trivial one-dimensional subrepresentation of $Sym^2(\hat{V})$.

2.3. **Spinor embedding and Plücker embedding.** Recall the spinor embedding ρ of the isotropic Grassmannian $IG(3,V)$ to $\mathbb{P}\hat{V}$ described in Theorem 2.3. Let $\mathcal{O}(1)$ be the tautological line bundle on $\mathbb{P}\hat{V}$. We denote by $\mathcal{O}_{IG(3,V)}(1)$ the line bundle on $IG(3,V)$ obtained as the pullback of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}\hat{V}$ under the embedding ρ. Let $\mathcal{O}_{IG(3,V)}(2)$ be the line bundle obtained as a square of $\mathcal{O}_{IG(3,V)}(1)$.

Another standard embedding of $IG(3,V)$ is the Plücker embedding $IG(3,V) \to \mathbb{P}^{34}$ given by $U \mapsto \mathbb{C}\Lambda^3(U) \in \mathbb{P}\Lambda^3 V \cong \mathbb{P}^{34}$.

Recall that we have $V \cong V^*$ as $Spin(V)$-modules and $Sym^2(V^*) = \mathbb{C} \oplus \Lambda^3(V)$, see Proposition 2.2. The following proposition connects these two embeddings.

Proposition 2.5. *The Plücker embedding is given by the full linear system of global sections of* $\mathcal{O}_{IG(3,V)}(2)$ *.*

Proof. Let $\mathcal{O}$, $\mathcal{O}(-2)$ and $\mathcal{O}_{IG(3,V)}$ be the sheaves of all functions on $\mathbb{P}\hat{V}$, functions on $\mathbb{P}\hat{V}$ vanishing on the image of ρ and functions on $IG(3,V)$ respectively. We have a short exact sequence of coherent sheaves on $\mathbb{P}\hat{V}$

$$0 \to \mathcal{O}(-2) \to \mathcal{O} \to \rho_* \mathcal{O}_{IG(3,V)} \to 0.$$

We tensor multiply it by $\mathcal{O}(2)$ and get

$$0 \to \mathcal{O} \to \mathcal{O}(2) \to \rho_* \mathcal{O}_{IG(3,V)}(2) \to 0.$$

We take the corresponding long exact sequence in cohomology. Since $H^1(\mathbb{P}\hat{V}, \mathcal{O}) = 0$ we see that the sections of $\mathcal{O}_{IG(3,V)}(2)$ are given by the quotient of $Sym^2(\hat{V}^*)$ by the equation of the image of the spinor embedding. By Remark 2.4 and Proposition 2.2, this space is isomorphic as a $Spin(V)$ representation to $\Lambda^3 V \cong \Lambda^3(V^*)$. As a result, the global sections of $\mathcal{O}_{IG(3,V)}(2)$ define a $Spin(V)$-equivariant map to $\mathbb{P}\Lambda^3 V$.

The image of $IG(3,V)$ is a six-dimensional orbit of $SO(V)$, and the stabilizer of the point which corresponds to $U \subset V$ coincides with the stabilizer of U. One can see that the only fixed point of the stabilizer of U in $\mathbb{P}\Lambda^3 V$ is $\mathbb{C}\Lambda^3 U$, which means that the map of the global sections of $\mathcal{O}(2)$ is precisely the Plücker embedding. □

2.4. **Properties of the image of the spinor embedding.** Choose a non-zero quadratic form $\hat{Q}$ on $\hat{V}$ which vanishes on $\rho(IG(3,V))$. We denote by $I \subset V$ and $\hat{I} \subset \hat{V}$, $\mathbb{C}\hat{I} = \rho(IG(3,V))$ the sets of isotropic vectors with respect to Q and $\hat{Q}$ respectively. We denote by $\hat{B}$ the bilinear form associated to $\hat{Q}$. We will also make a choice of the action of V on $\hat{V}$ (as mentioned earlier these choices differ by an overall sign only). We use calculations of Theorem 2.3 to prove some easy results about these forms and the action.

Proposition 2.6. *For any $v \in V$, $p, q \in \hat{V}$ there holds $\hat{B}(v \cdot p, q) = -\hat{B}(p, v \cdot q)$.*

Proof. By linearity, it is sufficient to show this statement for v with $Q(v) = -\frac{1}{2}$. Every such v can be given by v_4 in some basis as in Lemma 2.1 and we use the corresponding model of $\hat{V}$. It is then sufficient to check the statement for basic vectors p and q in $\hat{V}$. Since the action of v_4 on the basis elements is diagonal with entries $(-1)^{\deg}\frac{1}{\sqrt{2}}$, we have

$$\hat{B}(v \cdot p, q) = (-1)^{\deg p - \deg q}\hat{B}(p, v \cdot q).$$

It remains to observe from the explicit formula (2.3) that $\hat{B}(p, q) = 0$ unless $\deg(p) + \deg(q) = 3$. □

Corollary 2.7. *For all $v \in V$ and $p \in \hat{V}$ we have $\hat{B}(v \cdot p, p) = 0$.*

Proof. $\hat{B}(v \cdot p, p) = -\hat{B}(p, v \cdot p) = -\hat{B}(v \cdot p, p)$. □

Corollary 2.8. *For all $v \in V$ and $p, q \in \hat{V}$ we have $\hat{B}(v \cdot p, v \cdot q) = Q(v)\hat{B}(p, q)$.*

Proof. $\hat{B}(v \cdot p, v \cdot q) = -\hat{B}(v^2 \cdot p, q) = -\hat{B}(-Q(v)p, q) = Q(v)\hat{B}(p, q)$. □

Corollary 2.9. *For all $p \in \hat{V}$ and $u, v \in V$ there holds $\hat{B}(u \cdot p, v \cdot p) = B(u, v)\hat{Q}(p)$.*

Proof. We have $\hat{B}(vu \cdot p, p) = -\hat{B}(u \cdot p, v \cdot p) = -\hat{B}(v \cdot p, u \cdot p) = \hat{B}(uv \cdot p, p)$. So $\hat{B}(u \cdot p, v \cdot p) = -\frac{1}{2}\hat{B}((uv + vu) \cdot p, p) = -\frac{1}{2}\hat{B}(-B(u, v)p, p) = B(u, v)\hat{Q}(p)$. □

Proposition 2.10. *Let $u \in I$ be an isotropic vector acting on $\hat{V}$. Then we have $\mathrm{Ker}(u) = \mathrm{Im}(u) \subset \hat{I} \subset \hat{V}$, $\dim(\mathrm{Ker}(u)) = 4$. Moreover, $\mathbb{P}\,\mathrm{Ker}(u) \in \mathbb{P}\hat{V}$ coincides with $\rho(\{U \in IG(3, V) \mid u \in U\})$.*

Proof. We complete u to a basis of V as in Lemma 2.1 so that $u = v_1$. If we pick the corresponding basis in $\hat{V}$ then the $u \cdot \hat{V}$ is the span of $\hat{1}, \hat{v}_5, \hat{v}_6, \hat{v}_5\hat{v}_6$ which is easily seen to belong to $\hat{I}$. It is clearly the kernel of u.

Moreover if $U \in IG(3, V)$ and $\mathbb{C}p = \rho(U)$ then the space of elements in V which kill p is exactly U. Therefore $u \cdot p = 0$ if and only if $u \in U$. □

Proposition 2.11. *Let $p \in \hat{V}$ be a non-isotropic vector. Then $V \cdot p$ is the orthogonal complement of p in $\hat{V}$ under $\hat{Q}$.*

Proof. By Corollary 2.7 the space $V \cdot p$ is orthogonal to p. So we need to show that $V \cdot p$ has dimension 7. Suppose $v \cdot p = 0$ for some v. Then $Q(v)p = -v^2 \cdot p = 0$ so $v \in I$. By Proposition 2.10, $p \in \mathrm{Ker}(v) \subset \hat{I}$, which contradicts the condition that p is non-isotropic. □

2.5. **Invariant surjections.** A linear surjective map $\psi : \hat{V} \to V$ is called an *invariant surjection* if there exists $p \in \hat{V}$ such that $\psi(p) = 0$ and $\psi(v \cdot p) = v$.

Lemma 2.12. *If ψ be an invariant surjection then p is non-isotropic.*

Proof. If p was isotropic then we would have a non-zero $u \in V$ such that $u \cdot p = 0$ which gives a contradiction: $0 = \psi(u \cdot p) = u$. □

Let V_1, V_2 be linear spaces. We call a map of projective spaces $\mathbb{P}\pi : \mathbb{P}V_1 \to \mathbb{P}V_2$ *a projection* if it is induced from a surjective linear map of linear spaces $\pi : V_1 \to V_2$.

Let $\psi : \hat{V} \to V$ be an invariant surjection and let $\mathbb{P}\psi : \mathbb{P}\hat{V} \to \mathbb{P}V$ be the induced projection. Let $\mu_\psi : IG(3, V) \to \mathbb{P}V$ be the composition of spinor embedding $\rho : IG(3, V) \to \mathbb{P}\hat{V}$ and $\mathbb{P}\psi$.

Proposition 2.13. *For any isotropic $v \in V$ the equation $\mu_\psi(U) = \mathbb{C}v$ has a unique solution $U \in IG(3, V)$. Moreover $v \in U$.*

For any non-isotropic $v \in V$ the equation $\mu_\psi(U) = \mathbb{C}v$ has exactly two solutions $U_1, U_2 \in IG(3, V)$. Moreover $U_1 + U_2 = U_1 \oplus U_2 = v^\perp$.

Proof. Recall that the image of the spinor embedding is the set of isotropic vectors $\hat{I} \subset \hat{V}$.

Let v be a non-zero element in V. The preimage of $\mathbb{C}v$ in $\hat{V}$ is spanned by p and $v \cdot p$. We have $\hat{B}(p, v \cdot p) = 0$ by Corollary 2.7. A linear combination $\alpha p + \beta v \cdot p$ is in $\hat{I}$ if and only if (we use Corollary 2.8)

$$\hat{Q}(\alpha p + \beta v \cdot p) = \hat{Q}(p, p)(\alpha^2 + Q(v)\beta^2) = 0. \tag{2.4}$$

If v is isotropic we have $\alpha = 0$ and therefore there is a unique space $U \in IG(3, V)$ such that $\mu(U) = \mathbb{C}v$. Moreover, since $v \cdot p$ is in $Ker(v)$, we get $v \in U$ by Proposition 2.10.

If v is not isotropic, then there are exactly two linear combinations (up to a scalar), satisfying (2.4) which give two spaces $U_1, U_2 \in IG(3, V)$ such that $\mu(U_1) = \mu(U_2) = \mathbb{C}v$. Observe that $U_1 \cap U_2 = 0$. Indeed, if $u \in U_1 \cap U_2$ then u kills both $\rho(U_1)$ and $\rho(U_2)$ and therefore kills p which is a contradiction.

Finally let $u \in U_1$. Then for α, β as above, we have $u \cdot (\alpha p + \beta v \cdot p) = 0$. It follows that $(\alpha - \beta v)u \cdot p = B(u, v)p$. The left hand side is isotropic in the sense of $\hat{Q}$, because $\hat{Q}(u \cdot p) = Q(u)\hat{Q}(p) = 0$, $\hat{Q}(vu \cdot p) = Q(u)Q(v)\hat{Q}(p) = 0$ and $\hat{B}(u \cdot p, vu \cdot p) = 0$ by Corollary 2.7. Since p is non-isotropic, we obtain $B(u, v) = 0$. Therefore $v \perp U_1$. Similarly $v \perp U_2$. □

The following theorem provides a criterion for invariant surjections.

Theorem 2.14. *Let $\psi : \hat{V} \to V$ be a surjective map with kernel spanned by a non-isotropic $p \in \hat{V}$. Then ψ is an invariant surjection if and only if for a generic $u \in V$ there exists an isotropic dimension 3 subspace $U \subset V$ such that $u \perp U$ and $\mu_\psi(U) = \mathbb{C}u$.*

Proof. The only if part follows from Proposition 2.13.

Now we show the if part of the theorem. Consider the subset Z of $IG(3, V)$ which consists of U such that $\mu_\psi(U) \perp U$. Clearly, this is an algebraic subset. On the other hand, it has dimension at least 6, so the irreducibility of $IG(3, V)$ implies that $Z = IG(3, V)$ and $\mu_\psi(U) \perp U$ for all U. In particular, if $u = \mu_\psi(U)$ is isotropic then $u \in U$, since there are no isotropic 4-subspaces of V. This implies $u \cdot \rho(U) = 0$.

The preimage of u under ψ contains p and $\rho(U)$. Since $u \cdot p \neq 0$, the only solution up to scaling of $u \cdot \rho(q) = 0$ for $\mathbb{C}\psi(q) = \mathbb{C}u$ is $\rho(U)$. This implies that $\rho(U) \perp p$, since otherwise there would be two such solutions. This characterizes $\rho(U)$ as the intersection of $\psi^{-1}\mathbb{C}u$ and $V \cdot p$.

Denote by $g : V \to V$ the map $g(v) = \psi(v \cdot p)$. Surjectivity of ψ and Proposition 2.11 imply that g is invertible, and we denote its inverse by h. We have $\psi(h(u) \cdot p) = u$, which implies that $\mathbb{C}h(u) \cdot p = \rho(U)$ and $uh(u) \cdot p = 0$ for all isotropic $u \in I$.

Let $q_1, \dots, q_8$ be some basis in $\hat{V}$. Write $uh(u) \cdot p = \sum_{i=1}^{8} a_i(u) q_i$ where $a_i(u) \in \mathbb{C}$. For every linear function $r : \hat{V} \to \mathbb{C}$, we have a quadratic function $r(uh(u) \cdot p)$ on V which vanishes on I. Hence, it is proportional to Q. In particular $a_i(u) = c_i Q(u)$ for some $c_i \in \mathbb{C}$. Therefore there exists an element $p_1 = \sum_{i=1}^{8} c_i q_i \in \hat{V}$ such that for all $u \in V$

$$uh(u) \cdot p = Q(u) p_1 = -u^2 \cdot p_1.$$

Since for generic u its action on $\hat{V}$ has no kernel, we get

$$h(u) \cdot p = -u \cdot p_1 \tag{2.5}$$

for all $u \in V$. This implies that $V \cdot p = V \cdot p_1$. By passing to the orthogonal complement, we get $p_1 = cp$, which means that $h(u)$ is a multiple of the identity. We can scale p to get $h(u) = u$, which implies that ψ is an invariant surjection. □

We will need the following technical lemma later.

Lemma 2.15. *For any $c \in \mathbb{C}^*$, the set $\{v \in \hat{V} \mid \hat{Q}(v) = c\}$ is an orbit of $Spin(V)$.*

Proof. It is enough to show that any two points p_1, p_2 in $\mathbb{P}\hat{V}$ lie in the same orbit of $Spin(V)$. Indeed, then every vector can be translated to a multiple of another vector. If vectors have the same length, this multiple is ± 1, and the lemma follows since $-\mathbf{id} \in Spin(V)$.

Let p_1 and p_2 be two points in $\mathbb{P}\hat{V} - \hat{I}$. Draw generic lines l_i through p_i so that the line p_i intersects the conic $\hat{I}$ at two distinct points $q_{i,1}$ and $q_{i,2}$. These points correspond to isotropic vector spaces $U_{i,1}$ and $U_{i,2}$.

Proposition 2.13, applied to an invariant surjection corresponding to point on V which is on the line p_i, assures that $U_{i,1}$ does not intersect $U_{i,2}$.

Therefore there exists an element of $SO(V)$ which maps $U_{1,1}$ to $U_{2,1}$ and $U_{1,2}$ to $U_{2,2}$. The corresponding elements of $Spin(V)$ will map p_1 to some point p_3 on the line l_2. It remains to observe that the elements of $SO(V)$ that multiply all vectors in $U_{2,1}$ by λ, vectors in $U_{2,2}$ by λ^{-1} and fix the orthogonal complement $(U_1 \oplus U_2)^\perp$ act transitively on the non-isotropic points in l_2. Indeed, such elements scale the corresponding vectors in $\hat{V}$ with coefficients $\pm\sqrt{\lambda}$ and $\pm\sqrt{\lambda^{-1}}$ respectively. □

2.6. **A 3-form.** Let $\psi : \hat{V} \to V$ be an invariant surjection. In this section we define and study the trilinear skew-symmetric 3-form that corresponds to ψ.

Let non-isotropic $p \in \hat{V}$ span the kernel of ψ. We define a trilinear from w_ψ by:

$$w_\psi(a,b,c) = \hat{B}(abc \cdot p, p)(\hat{Q}(p))^{-1}$$

for all $a, b, c \in V$. Note that the definition of w_ψ does not depend on the choice of non-zero p in the kernel of ψ.

Proposition 2.16. *The form w_ψ is a trilinear skew-symmetric 3-form.*

Proof. We check the skew-symmetry (we use relations in Clifford algebra and Corollary 2.7):

$$\hat{B}(abc \cdot p, p) + \hat{B}(bac \cdot p, p) = -\hat{B}(B(a,b)c \cdot p, p) = 0,$$
$$\hat{B}(abc \cdot p, p) + \hat{B}(acb \cdot p, p) = -\hat{B}(B(b,c)a \cdot p, p) = 0.$$

□

Lemma 2.17. *Let $v \in V$ be an isotropic vector. Let U be the kernel of the skew-symmetric 2-form $w_\psi(v, \cdot, \cdot)$. Then U is the unique 3-dimensional isotropic space such that $\mu_\psi(U) = \mathbb{C}v$. In particular, $v \in U$.*

Proof. By Proposition 2.13 the space $U \in IG(3, V)$ such that $\rho(U) = \mathbb{C}v \cdot p$ is unique. Moreover $v \in U$.

By the definition of ρ and by the fact that the subspace of V which kills the isotropic vector $v \cdot p \in \hat{V}$ is 3-dimensional, we obtain $u \in U$ if and only if $uv \cdot p = 0$.

If $u \in U$ then for any $a \in V$, we get

$$w_\psi(a,u,v) = \hat{B}(auv \cdot p, p) = \hat{B}(0, p) = 0.$$

Therefore $U \subset \text{Ker } w_\psi(v, \cdot, \cdot)$.

If for all $a \in V$, $w_\psi(a,u,v) = 0$ then by Proposition 2.6

$$0 = w_\psi(a,u,v) = \hat{B}(auv \cdot p, p) = -\hat{B}(uv \cdot p, a \cdot p).$$

But $V \cdot p = p^\perp$ by Proposition 2.11, hence $uv \cdot p = \alpha p$ for some $\alpha \in \mathbb{C}$. But $\hat{Q}(uv \cdot p) = Q(u)Q(v)\hat{Q}(p) = 0$, therefore $\alpha = 0$, and $uv \cdot p = 0$. It follows that Ker $w_\psi(v, \cdot, \cdot) \subset U$. □

Lemma 2.18. *Let $v \in V$ be a non-isotropic vector. Then the kernel of the skew-symmetric 2-form $w_\psi(v, \cdot, \cdot)$ is one-dimensional and is spanned by v.*

Proof. Suppose there is $u \in V$ such that for all $u_1 \in V$ we have

$$\hat{B}(u_1 vu \cdot p, p) = 0.$$

We rewrite this as $\hat{B}(vu \cdot p, u_1 \cdot p) = 0$, which implies $vu \cdot p = \alpha p$ for some $\alpha \in \mathbb{C}$. Acting on both sides by v, we obtain $-Q(v)u \cdot p = \alpha v \cdot p$. Since p is non-isotropic we have $-Q(v)u = \alpha v$. □

3. Group G_2

This section collects some facts about G_2. We believe that most of these facts are standard and known. However in some cases we failed to find an adequate reference and therefore we provide the proofs.

3.1. **Definition of G_2.** Let g_2 be the semisimple complex Lie algebra with Cartan matrix

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

We have $\dim g_2 = 14$.

To describe the irreducible modules we choose a Cartan subalgebra spanned by coroots $\alpha_1^\vee, \alpha_2^\vee$. For $m, n \geq 0$, denote $L_{m,n}$ the irreducible highest module of highest weight $m\omega_1 + n\omega_2$ where ω_i are fundamental weights $\langle \omega_i, \alpha_j^\vee \rangle = \delta_i^j$. All irreducible modules have such form. The dimension of $L_{m,n}$ is given by the Weyl formula:

$$\dim L_{m,n} = \frac{1}{120}(m+1)(n+1)(m+n+2)(m+2n+3)(m+3n+4)(2m+3n+5).$$

In particular $\dim L_{0,0} = 1$, $\dim L_{1,0} = 7$, $\dim L_{0,1} = 14$, $\dim L_{2,0} = 27$. We call the representations $L_{0,0}$ and $L_{1,0}$ trivial and vector representations respectively.

Let G_2 be the connected, simply-connected complex Lie group with Lie algebra g_2.

Let V be the 7-dimensional space with a non-degenerate bilinear symmetric form B as before. Let $\hat{V}$ be the spinor representation of $Spin(V)$. The following fact is a $\mathbb{C}$-analog of [A, Theorem 5.5].

Lemma 3.1. *Let $p \in \hat{V}$ be non-isotropic. Then G_2 is isomorphic to the subgroup of $Spin(V)$ which fixes v.*

Proof. The stabilizer of p is unaffected by scaling p. Hence, by Lemma 2.15, the statement of this lemma is sufficient to check for any non-isotropic p.

In view of [A, Theorem 5.5], the Lie algebra of the compact *real* Lie group G_2 sits inside that of *real* $Spin(7)$. Consequently, the same is true for complex Lie algebras which by exponentiation implies that there is a map $G_2 \to Spin(V)$ with a finite kernel. The determinant of the Cartan matrix of g_2 is 1 and therefore the root and the weight lattices of g_2 coincide. Then by [GOV, Chapter 3, section 2.5], G_2 does not have finite subgroups and our map is injective.

Since $\hat{V}$ is a representation of $Spin(V)$, it must split into representations of G_2, which can only be one copy of a trivial and one copy of a 7-dimensional representation. Hence there is an element $p \in \hat{V}$ whose stabilizer in $Spin(V)$ contains G_2. If p was isotropic, then the Lie algebra g_2 would act non-trivially on the tangent space to $\mathbb{C}p$ inside $\mathbb{P}\hat{I}$, which is six-dimensional. This is impossible, which implies that p is non-isotropic.

It remains to show that the stabilizer G of p in $Spin(V)$ can not be bigger than G_2. In view of the dimension count this can only happen if G is disconnected. Denote by X the orbit of p, which is isomorphic to the affine conic $\hat{Q}(*) = \hat{Q}(p)$ by Lemma 2.15. The

group $Spin(V)$ is isomorphic to a fibration over X with fiber G. The exact sequence of homotopy groups gives an exact sequence $\pi_1(X) \to \pi_0(G) \to \pi_0(Spin(V))$, so it is enough to check that $\pi_1(X)$ is trivial.

The variety X is an orbit of $SO(\hat{V}) \cong SO(8)$. The stabilizer of p for the $SO(\hat{V})$ action is isomorphic to $SO(\mathbb{C}p^{\perp}) \cong SO(7)$. We have

$$\pi_1(SO(7)) \to \pi_1(SO(8)) \to \pi_1(X) \to 1.$$

The first map is an isomorphism, since it is induced by the embedding $Spin(7) \to Spin(8)$. This shows that $\pi_1(X)$ is trivial and finishes the argument. □

We also use the projective version of this statement.

Lemma 3.2. *The subgroup G of $Spin(V)$ that fixes the point $\mathbb{C}p$ in $\mathbb{P}\hat{V}$ is disconnected. The connected component of identity has index 2 and is isomorphic to the group G_2.*

Proof. Suppose that $g \in Spin(V)$ satisfies $g\mathbb{C}p = \mathbb{C}p$. This implies $gp = cp$ for some constant c. Since g preserves $\hat{Q}$, we get $c = \pm 1$. The connected component of identity is characterized by $c = 1$ and therefore consists of elements of $Spin(V)$ that fix p. On the other hand, $(-id_{\hat{V}})$ is an element of $Spin(V)$ that has $c = -1$, which implies that the index of the $c = 1$ subgroup is precisely two. □

We remark that in terms of the *projective* action of $SO(V)$ on $\mathbb{P}\hat{V}$ the isotropy subgroup of the point $\mathbb{C}p$ is isomorphic to G_2. This is because the map $Spin(V) \to SO(V)$ has kernel $\{\pm 1\}$. In particular G_2 is a subgroup of $SO(V)$ and acts on V. Therefore V is the vector representation of G_2.

3.2. G_2 **and 3-forms.** We choose an identification $\Lambda^7 V \cong \mathbb{C}$. A skew-symmetric 3-form w on V and a bilinear symmetric 2-form b on the 7-dimensional space V are called *associated* if there exists a non-zero constant $c \in \mathbb{C}$, such that for every two vectors $v_1, v_2 \in V$ we have $b(v_1, v_2) = cw(v_1, *, *) \wedge w(v_2, *, *) \wedge w(*, *, *)$.

A skew-symmetric 3-form on V is called *non-degenerate* if it is associated to a non-degenerate 2-form.

Lemma 3.3. *Let V be a vector representation of G_2. Then*

- *The space of G_2-invariant skew-symmetric three-forms on V is one-dimensional. Any such form is non-degenerate.*
- *The space of G_2-invariant symmetric bilinear 2-forms on V is one-dimensional. Any such form is non-degenerate.*

In particular, any non-zero G_2-invariant skew-symmetric three-form on V is associated to any G_2-invariant symmetric bilinear 2-form.

Proof. We have the following decompositions of G_2-modules:

$$Sym^2(V) \cong L_{2,0} \oplus L_{0,0}, \qquad \Lambda^3(V) \cong L_{2,0} \oplus L_{1,0} \oplus L_{0,0}.$$

These decompositions are easy to establish using the known dimensions of $L_{m,n}$ and the weight decompositions of each module.

It follows that the spaces of symmetric 2-forms and skew-symmetric 3-forms are one-dimensional. The bilinear form associated to a G_2-invariant 3-form is G_2-invariant.

So, we only have to show the G_2-invariant skew-symmetric form is non-degenerate. Since the bilinear form b is a multiple of the non-degenerate form B, we simply need to show that the bilinear form associated to the G_2-invariant skew-symmetric 3-form is non-zero. We use the multiple of the 3-form of Proposition 2.16 given by $w(a,b,c) = \hat{B}(abc \cdot p, p)$ where $G_2 = G_2(p)$ is realized as the subgroup of $Spin(V)$ that fixes $p \in \hat{V}$.

Consider a non-isotropic vector $v \in V$ and the skew-symmetric 6-form in $V/\mathbb{C}v$ given by $w(v,*,*) \wedge w(v,*,*) \wedge w(v,*,*)$. It is sufficient to show that this 6-form is non-zero, since this will show that $b(v,v) \neq 0$. By Lemma 2.18 the form $w(v,*,*)$ is non-degenerate, so its cube is non-zero. □

Let ψ_p be an invariant surjection from a non-isotropic vector p and let $G_2(p)$ be the group preserving p. Recall that we constructed a skew-symmetric trilinear form w_ψ on V in Section 2.6.

Proposition 3.4. *The form w_ψ is a non-zero $G_2(p)$-invariant 3-form. In particular w_ψ is associated to B.*

Proof. The form w_ψ is $G_2(p)$-invariant because p and $\hat{B}$ are $G_2(p)$-invariant. The form w_ψ is clearly non-zero in view of Proposition 2.11.

The rest follows from Lemma 3.3. □

Lemma 3.5. *For a G_2-invariant non-degenerate skew-symmetric three-form w the subgroup G of $SL(V)$ that preserves w is equal to G_2.*

Proof. Clearly, $G \supseteq G_2$. Since G preserves the associated non-degenerate symmetric bilinear form B, it lies in $SO(V)$.

We can assume that w is given by $\hat{B}(abc \cdot p, p)$. By Lemma 2.17 the action of g on $\hat{V}$ preserves the set of $\operatorname{Ker} w(v,*,*)$ for isotropic $v \in V$. Hence it preserves the linear span $V \cdot p$ of this set. It then preserves the orthogonal complement of $V \cdot p$ which is the span of p by Proposition 2.11. Lemma 3.2 then finishes the proof. □

Corollary 3.6. *The $SL(V)$ orbit of a G_2-invariant non-degenerate skew-symmetric three-form on V is a dense open subset in the projective space of all non-zero skew-symmetric three-forms on V up to scaling.*

Proof. The dimension of the $SL(V)$ orbit of w is equal to $\dim SL(V) - \dim G_2 = 48 - 14 = 34 = \dim \mathbb{P}\Lambda^3(V^*)$, which shows that the orbit is dense. □

For the sake of the completeness of exposition we prove a stronger fact, which will not be used in the rest of the paper. Namely, we now show that every non-degenerate skew-symmetric three-form on V lies in this $SL(V)$ orbit, up to scaling. This implies that every non-degenerate skew-symmetric three-form is G_2-invariant for some $G_2 \subset SL(V)$.

Proposition 3.7. *Let w be a trilinear skew-symmetric 3-form on V associated to B. Then the subgroup of $SL(V)$ which preserves w is isomorphic to G_2.*

Proof. Let w be a non-degenerate skew-symmetric three-form and let B be the associated non-degenerate bilinear symmetric form. Clearly, the group G that preserves w must preserve B as well, so it must lie in $SO(V)$.

The argument of the last paragraph of the proof of Lemma 3.3 shows that for every element $v \in V$ the bilinear form $B(v, v)$ is zero if and only if skew-symmetric bilinear form $w(v, *, *)$ on $V/\mathbb{C}v$ has zero kernel. This implies that if $w(u, v, *)$ is zero then u and v are both isotropic. Therefore, for any $u \in I$, the kernel of $w(u, *, *)$ consists of isotropic vectors of B and hence has dimension at most three. Consequently, the dimension of the kernel of $w(u, *, *)$ is three if u is isotropic and one if u is not. In both cases the kernel contains u.

The above discussion produces a map $\phi : \mathbb{P}I \to \mathbb{P}\hat{I}$ by mapping the isotropic vector to the corresponding point in $IG(3, V)$ which is identified with $\mathbb{P}\hat{I}$ via the spinor embedding.

We will now show that ϕ is an embedding. Suppose two linearly independent vectors u_1, u_2 have the same kernel U of $w(u_i, *, *)$. Then for a generic element $u \in U$ the kernel of $w(u, *, *)$ contains u, u_1 and u_2, which implies that $w(u, u', *) = 0$ for all $u, u' \in U$. Pick two vectors $v \in V$ and $u \in U$ with $B(u, v) \neq 0$. If we look at

$$B(u, v) = w(u, *, *) \wedge w(v, *, *) \wedge w(*, *, *)$$

we see that the right hand side is zero. Indeed, if we were to evaluate on the basis of V that extends the basis u_1, u_2, u_3 of U, then in each term either we have $w(u, u_i, *)$ or $w(v, u_i, u_j)$, or $w(u_i, u_j, *)$. Contradiction. We can similarly show that the map μ does not kill tangent vectors. Indeed, if we have $\mathrm{Ker}(u_1 + \epsilon u_2) = U \mod \epsilon^2$, we have $w(u_1, U, *) = w(u_2, U, *) = 0$, and the previous argument works.

As a result, the image $\phi(\mathbb{P}I)$ of ϕ is a smooth conic of dimension 5 inside a smooth conic of dimension 6 in a projective space of dimension 7. By strong Lefschetz theorem, $\phi(\mathbb{P}I)$ is a complete intersection of $\mathbb{P}\hat{I}$ and a hypersurface in $\mathbb{P}\hat{V}$. It is easy to see that this hypersurface must in fact be a hyperplane, by comparing numerical invariants of complete intersections and conics. Hence, $\phi(\mathbb{P}I)$ spans a hyperplane in $\mathbb{P}\hat{V}$. Denote by V_1 the corresponding codimension one subspace in $\hat{V}$. The restriction of $\hat{B}$ to V_1 gives a smooth conic $\phi(\mathbb{P}I)$, which implies that the orthogonal complement to V_1 is non-isotropic.

Every element $g \in G$ must preserve $\phi(\mathbb{P}I)$ as a set. Hence it preserves V_1 and its orthogonal complement, so it preserves a point in $\mathbb{P}\hat{V}$. Lemma 3.2 then shows that G is contained in G_2 so it has dimension at most 14. Then the $SL(V)$ orbit of w is dense in $\mathbb{P}\Lambda^3(V^*)$ so it must coincide with the orbit of a G_2-invariant form. This shows that G is a conjugate of some $G_2 \subset SL(V)$. □

Note that $SO(V)$ acts transitively on 3-forms associated to B (considered up to a constant). Indeed, $GL(V)$ acts transitively on the set of all non-degenerate 3-forms (because each orbit is an algebraic variety of dimension 35). And any element which maps a 3-form associated to B to a 3-form associated to B has to preserve B (all up to a constant).

In particular the dimension of the variety of 3-forms (considered up to a constant) associated to B is 7.

It follows that all subgroups of $SO(V)$ isomorphic to G_2 are conjugated. We have a bijection between the set of subgroups of $SO(V)$ isomorphic to G_2 and 3-forms associated to B (considered up to a constant). We also have a bijection between the set of subgroups of $SO(V)$ isomorphic to G_2 and $\mathbb{P}\hat{V} - \hat{I}$.

3.3. **The homogeneous space G/B.** Recall that Borel subgroup of a semisimple Lie group is a maximal connected solvable subgroup. We give explicit description of flag varieties G/B of types G_2, C_3, A_6.

A full flag F in V is the chain $F_1 \subset F_2 \subset F_3 \subset F_4 \subset F_5 \subset F_6$, where $F_i \subset V$ are subspaces of dimension i.

Let $\mathcal{F}$ be the set of all full flags in V. The group $SL(V)$ acts transitively on $\mathcal{F}$, this action identifies $\mathcal{F}$ with the quotient of $SL(V)$ by a Borel subgroup. Then $\mathcal{F}$ is an algebraic variety of dimension 21 which is called the flag variety of A_6 type.

A full flag F is called isotropic if $B(F_i, F_{7-i}) = 0$. Let $\mathcal{F}^\perp \subset \mathcal{F}$ be the set of all isotropic flags in V. The group $SO(V)$ acts transitively on $\mathcal{F}^\perp$, this action identifies $\mathcal{F}^\perp$ with the quotient of $SO(V)$ by a Borel subgroup. Then $\mathcal{F}^\perp$ is an algebraic variety of dimension 9 which is called the flag variety of B_3 type.

We choose a 3-form w on V associated to B and $G_2(w)$ the subgroup of $SO(V)$ which preserves w. An isotropic flag F is called G_2-isotropic if $F_3 = \operatorname{Ker} w(F_1, *, *)$. Let $\mathcal{F}^{\perp\perp} \subset \mathcal{F}^\perp$ be the set of all G_2-isotropic flags in V.

Proposition 3.8. *The set $\mathcal{F}^{\perp\perp}$ is a smooth algebraic variety of dimension 6.. The group $G_2(w)$ acts transitively on the space $\mathcal{F}^{\perp\perp}$, this action identifies $\mathcal{F}^{\perp\perp}$ with the quotient of $G_2(w)$ by a Borel subgroup.*

Proof. Elements of $\mathcal{F}^{\perp\perp}$ can be identified with flags $F_1 \subset F_2$ such that for bases u and (u, v) of F_1 and F_2 respectively there holds $B(u, u) = B(u, v) = B(v, v) = 0$ and $w(u, v, *) = 0$. Consequently, $\mathcal{F}^{\perp\perp}$ is a smooth variety of dimension 6 which admits a fibration $\mathcal{F}^{\perp\perp} \to \mathbb{P}I$ to a smooth conic of dimension 5. The fibers are isomorphic to $\mathbb{P}^1$. In fact, the fibration is the projectivization of the rank two vector bundle whose fiber over F_1 is given by $\operatorname{Ker} w(F_1, *, *)/F_1$.

Denote by B_1 the stabilizer of $F_1 \subset F_2$ in $\mathcal{F}^{\perp\perp}$. It is easy to see that it stabilizes a complete flag in V and so lies in the Borel subgroup of $SL(V)$. Consequently, B_1 is solvable which implies that its connected component of identity is contained in a Borel subgroup B_2 of $G_2(w)$.

The dimension of B_2 is 8, which implies that the dimension of B_1 is at most 8. Since the dimension of G_2 is 14 the dimension of $G_2(w)$ orbit of the flag is at least 6, hence it is exactly 6. Since $\mathcal{F}^{\perp\perp}$ is irreducible, this means that all $G_2(w)$ orbits on $\mathcal{F}^{\perp\perp}$ are Zariski dense hence there is only one orbit. We have thus shown the transitivity of the action.

The above argument also shows that the connected component of identity $B_1^\circ \subseteq B_1$ is equal to B_2, since their Lie algebras have the same dimension and one is contained in the other.

We now claim that B_1 is in fact connected. If B_1 was not connected, the variety $\mathcal{F}^{\perp\perp}$ would admit an unramified covering from the variety $G_2(w)/B_1^\circ$. However, the fibration description of $\mathcal{F}^{\perp\perp}$ shows that it is unirational, i.e. admits a dominant morphism from a projective space. Hence it has trivial fundamental group by [S]. □

The quotient of G_2 by a Borel subgroup is called the flag variety of G_2 type. Proposition 3.8 says that $\mathcal{F}^{\perp\perp}$ is isomorphic to the flag variety of G_2 type.

4. Self-self-dual spaces of polynomials

In this section we define the self-self-dual spaces of polynomials and show that such spaces possess a natural skew-symmetric 3-form.

4.1. **Self-dual spaces of polynomials of dimension 7.** Here we recall main facts about self-dual spaces of polynomials proved in [MV1].

Let $V \in \mathbb{C}[x]$ be a vector space of polynomials of dimension 7. The space V has *a base point* z if all polynomials in V vanish at z. We always assume that V has no base points.

Let $W(u_1, \dots, u_k)$ denote the Wronskian of functions $u_1, \dots, u_k$,

$$W(u_1, \dots, u_k) = \det(u_i^{(j-1)})_{i,j=1}^k.$$

Let U_i be the (monic) greatest common divisor of the set of all Wronskians $W(u_1, \dots, u_i)$ where $u_1, \dots, u_i \in V$. The following lemma is easy, cf. for example Lemma 4.9 in [MV2].

Lemma 4.1. *There exist unique polynomials $T_1, \dots, T_6$ such that $U_i = T_1^{i-1} T_2^{i-2} \dots T_{i-1}$.* □

We call polynomials T_i *the ramification polynomials* of V.

For $u_1, \dots, u_i \in V$, we call *the divided Wronskian* the polynomial defined by

$$W^\dagger(u_1, \dots, u_i) = \frac{W(u_1, \dots, u_i)}{U_i} = \frac{W(u_1, \dots, u_i)}{T_1^{i-1} T_2^{i-2} \dots T_{i-1}}.$$

Note that $W^\dagger(u_1, \dots, u_7) \in \mathbb{C}$.

Space V is called *self-dual* if

$$V = \{W^\dagger(u_1, \dots, u_6) \mid u_1, \dots, u_6 \in V\}.$$

If V is self-dual then $T_i = T_{7-i}$.

By Corollary 6.5 in [MV1] a self-dual space of polynomials V possesses a bilinear symmetric non-degenerate form given by

$$B(u_1, u_2) = W^\dagger(u_1, v_1, \dots, v_6), \qquad \text{if } u_2 = W^\dagger(v_1, \dots, v_6). \tag{4.1}$$

Lemma 4.2. *Let V be a self-dual space of polynomials of dimension 7. Let $u_1, u_2, u_3 \in V$ be such that $B(u_i, u_j) = 0$. Then the divided Wronskian $W^\dagger(u_1, u_2, u_3)$ is a square of a polynomial.*

Proof. By Lemma 6.16 in [MV1], the divided Wronskian is a square for generic isotropic 3-space. The condition of Lemma 6.16 in [MV1] is satisfied for a generic isotropic 3-space by Theorem 7.10 in [MV1]. Then the divided Wronskian of any isotropic 3-space is a square. □

4.2. **Definition of self-self-dual spaces and first properties.** Let V be a self-dual space of polynomials of dimension 7.

Definition 4.3. A self-dual space of polynomials V of dimension 7 is called self-self-dual if for a generic element f of V there exists a three-dimensional isotropic subspace $U \subset V$ such that $f \perp U$ and the divided Wronskian of U is (up to a constant) the square of f.

Remark 4.4. The space of polynomials of degree at most 6 is self-self-dual, see Section 5.6.

Let V be a self-self-dual space of polynomials. Then V is self-dual and all divided Wronskians of isotropic 3-spaces are squares of polynomials by Lemma 4.2. Denote by $\bar{V}$ the span of all square roots of all divided Wronskians of three-dimensional isotropic subspaces $U \subset V$. Then we have a map

$$\mu : \ IG(3, V) \to \mathbb{P}\bar{V}.$$

Proposition 4.5. *There exists a projection $P : \ \mathbb{P}\hat{V} \to \mathbb{P}\bar{V}$ such that the map μ is given by a composition of the spinor embedding $\rho : \ IG(3, V) \to \mathbb{P}\hat{V}$ and projection P.*

Proof. Let L be the tautological line bundle on $\bar{V}$. Every map from an algebraic variety M to a projective space $\bar{V}$ is given by sections of the line bundle on M which is the pull-back of L. So, we describe the pullback bundle $\mu^*(L)$. To do that we consider the pullback of L^2.

Recall that there is a bijective correspondence between line bundles and $\mathbb{C}^*$-bundles on any algebraic variety. We denote the $\mathbb{C}^*$-bundle corresponding to a line bundle M by $\tilde{M}$.

The square of $\tilde{L}^{-1}$ can be thought of as the variety of non-zero squares $f^2, f \in \bar{V}$ with the obvious map to $\mathbb{P}\bar{V}$.

Let $\tilde{R}$ be the subvariety of $\Lambda^3 V$ which corresponds to isotropic 3-spaces with a choice of a volume form:

$$\tilde{R} = \{u_1 \wedge u_2 \wedge u_3 \ | u_1 \wedge u_2 \wedge u_3 \neq 0, \ u_i \in V, \ B(u_i, u_j) = 0\} \subset \Lambda^3 V.$$

Then $\tilde{R}$ is $\mathbb{C}^*$-bundle over $IG(3, V)$ which is the dual of the $\mathbb{C}^*$-bundle associated to the Plücker bundle. By Proposition 2.5, R is $\mathcal{O}_{IG(3,V)}(-2)$.

The map μ induces a map of $\mathbb{C}^*$-bundles from $\tilde{R}$ to the pullback of $\tilde{L}^{-2}$ which maps $u_1 \wedge u_2 \wedge u_3$ to f^2 if $W^\dagger(u_1, u_2, u_3) = f^2$. Every map of $\mathbb{C}^*$ bundles which is isomorphism on the bases is an isomorphism. Therefore $\tilde{R}$ is the pullback of $\tilde{L}^{-2}$ and $\mathcal{O}_{IG(3,V)}(2)$ is the pullback of L^2.

The line bundles of any smooth hypersurface in a projective space of dimension at least 4 are integer powers of the pullback of the tautological bundle of the projective space. This

follows for example from the strong Lefschetz theorem [GH]. In particular, all bundles on $IG(3,V)$ are powers of $\mathcal{O}_{IG(3,V)}(1)$.

Consequently, the pullback of L is the spinor line bundle $\mathcal{O}_{IG(3,V)}(1)$. Thus the map μ is given by a subspace of the global sections of $\mathcal{O}_{IG(3,V)}(1)$. The global sections of this bundle are a subspace in $\hat{V}$. The lemma follows. □

Proposition 4.6. *The space $\bar{V}$ is equal to V. Moreover, there exists an invariant surjection $\psi:\ \hat{V}\to V$, such that projection P is the projectivization of ψ.*

Proof. The condition of Definition 4.3 implies that $\bar{V}$ contains V. Assume that it is bigger than V. By Proposition 4.5 the dimension of $\bar{V}$ is at most the dimension of $\hat{V}$, so it must be exactly 8 and the projection of Proposition 4.5 must in fact be an isomorphism. Then the image of μ is a non-singular conic in $\mathbb{P}\bar{V}$ and does not contain a generic element of $\mathbb{P}\bar{V}$.

Now we know that the map μ is a composition of the embedding $IG(3,V)$ as a conic in $\mathbb{P}\hat{V}$ and some projection given by a surjective map $\psi:\hat{V}\to V$. The fact that the map μ is well-defined implies that the kernel of ψ does not lie in $IG(3,V)$. The condition of Definition 4.3 implies that ψ satisfies the assumption of Theorem 2.14 and therefore ψ is an invariant surjection. □

We have an immediate corollary.

Corollary 4.7. *Every vector f in V is a constant times a square root of a divided Wronskian of a three-dimensional isotropic subspace U of V. If f is isotropic, such U is unique and $f\in U$. If f is not isotropic, there are exactly two such spaces U_1 and U_2, and $U_1+U_2=U_1\oplus U_2=f^{\perp}$.*

Proof. Follows from Proposition 2.13. □

Lemma 4.8. *If V is a self-self-dual space then $T_1=T_3=T_4=T_6$ and $T_2=T_5$.*

Proof. We only need to show $T_1=T_3$. All T_i have zeroes only at $z_1,\dots,z_n$. Let us show that T_1 and T_3 have the same order of zero at z_1.

Let T_1, T_2, T_3 have orders t_1,t_2,t_3 of zero at z_1 respectively. Let $u_1,u_2,\dots,u_7$ be a basis of V such that order of zero at z_1 of u_i is strictly smaller than the order of zero at z_1 of u_{i+1}. Then the orders of u_i at z_1 are given by

$$0,t_1+1,t_1+t_2+2,t_1+t_2+t_3+3,t_1+t_2+2t_3+4,t_1+2t_2+2t_3+5,2t_1+2t_2+2t_3+6.$$

Then $W^{\dagger}(u_5,u_6,u_7)$ has the maximal possible order of zero at z_1. Since V is self-self-dual, this implies $W(u_5,u_6,u_7)=cu_7^2T_1^2T_2$ for some $c\in\mathbb{C}^*$. But the order of zero of $W(u_5,u_6,u_7)$ is $4t_1+5t_2+6t_3+12$ and the order of zero of $u_7^2T_1^2T_2$ is $6t_1+5t_2+4t_3+12$. It follows that $t_1=t_3$.

□

4.3. **The 3-form.** Let $p \in \hat{V}$ span the kernel of the map $\bar{P}$ which corresponds to the projection P in Proposition 4.5. We denote by $G_2(p)$ the subgroup of $SO(V)$ which preserves p. The group $G_2(p)$ is isomorphic to G_2. We characterize $G_2(p)$ as the group of linear transformations of V that are compatible with the Wronskian structure.

Proposition 4.9. *The group $G_2(p)$ is the set of all elements $g \in SO(V)$ such that for any basis (a_1, a_2, a_3) of any isotropic 3-subspace of V, any $f \in V$ satisfying $W^\dagger(a_1, a_2, a_3) = f^2$ there holds*

$$W^\dagger(ga_1, ga_2, ga_3) = (gf)^2. \tag{4.2}$$

Proof. For each $g \in SO(V)$ consider a map $\mu_g : IG(3, V) \to \mathbb{P}V$ given as follows. For every isotropic 3-space U pick a basis (a_1, a_2, a_3) and define

$$\mu_g(U) := g^{-1}\Big(\sqrt{W^\dagger(g(a_1), g(a_2), g(a_3))}\Big).$$

While the square root is defined up to a sign only, the result makes sense as a point in $\mathbb{P}V$.

The proof of Proposition 4.5 is applicable to the map μ_g as well as to μ. Consequently, the map is given by a projection from a point P_{μ_g} in $\mathbb{P}\hat{V}$ which is uniquely determined by g. Moreover, the argument of Proposition 4.5 shows that the map is uniquely determined by that point, since Theorem 2.14 still applies. Our construction is $SO(V)$-equivariant, so we have $P_{\mu_g} = g\mathbb{C}p$ for the natural projective action of $SO(V)$ on $\mathbb{P}\hat{V}$. As a result, equation (4.2) implies that $g \in G_2(p)$.

On the other hand, for every $g \in G_2(p)$ we have $\mu_g = \mu$ which translates into

$$W^\dagger(g(a_1), g(a_2), g(a_3)) = c(a_1, a_2, a_3, g)(g(f))^2.$$

for some constants $c \in \mathbb{C}^*$ for all choices of a_i and f as in the statement of the proposition. The constant c depends only on g and the point $V_1 \in IG(3, V)$. Since it is homomorphic and $IG(3, V)$ is compact, it must depend on g only. Since G_2 has no non-trivial one-dimensional representations, we see that $c = 1$, so elements of $G_2(p)$ satisfy (4.2). □

Define a linear map $\nu^* : \Lambda^3 V \to Sym^2 V^*$ by the formula

$$\langle \nu^*(a \wedge b \wedge c), v \otimes v\rangle = \hat{B}(abcv \cdot p, v \cdot p),$$

where $a, b, c, v, \in V$. The argument of Proposition 2.16 shows that ν^* is well defined.

Let $\nu : \Lambda^3 V \to Sym^2 V$ be the linear map which is a composition of ν^* with the natural identification of $Sym^2 V^*$ and $Sym^2 V$ induced by B.

We remark that when one scales p or $\hat{B}$ one scales ν.

Proposition 4.10. *There exists a unique linear map $\phi : \Lambda^3 V \to Sym^2 V$ such that $\phi(a \wedge b \wedge c) = f \otimes f$ for all isotropic pairwise orthogonal a, b, c with $W^\dagger(a, b, c) = f^2$. Moreover, there is a constant C such that $\phi = C\nu$.*

Proof. We need to show that there is a constant C such that $\nu(v_1 \wedge v_2 \wedge v_3) = Cf \otimes f$ for all bases (v_1, v_2, v_3) of an isotropic subspace U and f such that $W^\dagger(v_1, v_2, v_3) = f^2$.

We extend (v_1, v_2, v_3) to a basis of V as in Theorem 2.3. To calculate $\hat{B}(v_1v_2v_3v\cdot p, v\cdot p)$, observe that $\hat{B}(v_1v_2v_3\cdot q, q)$ is proportional to α_{567}^2 in the notations of the proof of Theorem 2.3. This is also proportional to $(\hat{B}(\hat{1}, q))^2$ Since we have $\hat{1} = \gamma_1 p + \gamma_2 f \cdot p$ for some γ_1 and γ_2, we see that

$$\langle \nu^*(a \wedge b \wedge c), v \otimes v\rangle = C(a, b, c)(\hat{B}(\gamma_1 p, v \cdot p) + \hat{B}(\gamma_2 f \cdot p, v \cdot p))^2$$

$$= C_1(a, b, c)B(f, v)^2.$$

Hence $\nu(a \wedge b \wedge c) = C_1(a, b, c)f \otimes f$ for some nonzero $C_1(a, b, c)$. We now observe that C_1 in fact depends on the point of $IG(3, V)$ only. Since it is clearly holomorphic, it must be a constant.

We now show the uniqueness of ϕ. If we have ϕ_1 and ϕ_2 that satisfy the conditions of the proposition their difference ϕ_3 satisfies $\phi_3(a \wedge b \wedge c) = 0$ for all bases (a, b, c) of an isotropic subspace. Since the $SO(V)$ representation $\Lambda^3 V$ is irreducible, such $a \wedge b \wedge c$ span all of $\Lambda^3 V$, which implies that $\phi_3 = 0$. □

Let $m : \ Sym^2 V \to \mathbb{C}[x]$ be the multiplication map, sending $f \otimes g + g \otimes f \mapsto 2fg$.

Corollary 4.11. *For all $a, b, c \in V$, we have $W^\dagger(a, b, c) = m(\phi(a \wedge b \wedge c))$.*

Proof. By Proposition 4.10, the corollary is true if a, b, c span an isotropic subspace. Such triples span $\Lambda^3 V$, since they span an $SO(V)$-subrepresentation in $\Lambda^3 V$, but $\Lambda^3 V$ is an irreducible $SO(V)$-module. Then the generic case of the corollary follows from linearity. □

Now we obtain the $G_2(p)$-invariant 3-form.

Theorem 4.12. *There is a unique skew-symmetric 3-form $w \in \Lambda^3 V^*$ such that*

$$w(a_1, a_2, a_3) = B(f, f) \tag{4.3}$$

for all pairwise orthogonal isotropic a_1, a_2, a_3 with $W^\dagger(a_1, a_2, a_3) = f^2$. An element $g \in SO(V)$ preserves the form w if and only if $g \in G_2(p)$. Finally, the form w is associated to B.

Proof. We set $w(a, b, c) = B(\phi(a\wedge b\wedge c))$, where ϕ is the map in Proposition 4.10. This form clearly satisfies all the properties described in the theorem. In particular it is associated to B because all G_2-invariant forms are associated to B by Lemma 3.3. □

Corollary 4.13. *Let $v \in V$, $Q(v) = 0$. Let U be the kernel of the skew-symmetric 2-form $w(v, \cdot, \cdot)$. Then U is the unique 3-dimensional isotropic 3-space, $U \in IG(3, V)$ such that $W^\dagger(U)$ equals (up to a constant) v^2. Moreover $v \in U$.*

Proof. Follows from Lemma 2.17. □

5. Populations of critical points

Populations of critical points are defined for any Kac-Moody algebra in [MV1]. The motivation for this definition is algebraic Bethe Ansatz method. In this paper we study the populations of critical points associated with G_2 root systems. We use properties of populations of A_6 and C_3 type. We recall known facts about those cases first.

5.1. **A_6-populations.** Consider the root system of type A_6. We have

$$(\alpha_i, \alpha_i) = 2, \qquad (\alpha_i, \alpha_{i\pm 1}) = -1,$$

and other scalar products are zero. The root system of type A_6 corresponds to Lie algebra sl_7. The Weyl group of sl_7 is generated by simple reflections s_i. The shifted action of Weyl group on A_6-weights is

$$s_i \cdot \lambda = \lambda - (\lambda + \rho, \alpha_i)\alpha_i,$$

where $(\rho, \alpha_i) = 1$ for all i.

Fix polynomials $T_i(x) = \prod_{s=1}^{n}(x - z_s)^{m_s^{(i)}}$.

where $i = 1, \dots, 6$, x is a formal variable and $z_i, m_s^{(i)}$ are parameters.

A 6-tuple of polynomials $\boldsymbol{y} = (y_1, \dots, y_6)$ is called *generic* if y_i have no multiple roots and if y_i, y_{i+1} have no common roots. A generic 6-tuple of polynomials $\boldsymbol{y}$ is called *a Bethe* 6*-tuple* of type A_6 if there exist polynomials $\tilde{y}_1, \dots, \tilde{y}_6$ such that

$$W(y_i, \tilde{y}_i) = T_i y_{i-1} y_{i+1}.$$

Note that $\tilde{y}_i$ is unique up to addition of a scalar multiple of y_i.

In what follows we always consider N-tuples of polynomials up to multiplication of its coordinates by non-zero scalars.

If $\boldsymbol{y}$ is a Bethe 6-tuple and the 6-tuple $\boldsymbol{y}^{(i)} = (y_1, \dots, \tilde{y}_i, \dots, y_6)$ is generic then $y^{(i)}$ is also a Bethe 6-tuple of A_6 type, see Theorem 3.7 in [MV1]. We call $\boldsymbol{y}^{(i)}$ *an immediate descendent of* $\boldsymbol{y}$ *in the direction* i.

Let $\bar{P}$ be the minimal set of Bethe 6-tuples which contains $\boldsymbol{y}$ and such that for all $\tilde{\boldsymbol{y}} \in \bar{P}$ all immediate descendants $\tilde{\boldsymbol{y}}^{(i)}$ are also in $\bar{P}$.

The degrees of all coordinates of all tuples of $\bar{P}$ are simultaneously bounded. *The* A_6*-population of Bethe* 6*-tuples originated at* $\boldsymbol{y}$ is the closure of $\bar{P}$ in $(\mathbb{P}(\mathbb{C}[x]))^6$.

The set of populations of Bethe 6-tuples associated with polynomials T_i is in one-to-one correspondence with the set of 7-dimensional spaces of polynomials with no base points and ramification polynomials T_i. Given a population P and $\boldsymbol{y} \in P$, we define

$$\begin{aligned} D &= (\partial - \ln'(\frac{\prod_{s=1}^{6} T_s}{y_6}))(\partial - \ln'(\frac{y_6 \prod_{s=1}^{5} T_s}{y_5})) \dots (\partial - \ln'(\frac{y_2 T_1}{y_1}))(\partial - \ln'(y_1)) \\ &= \prod_{i}^{0 \to 6} (\ \partial - \ln'(\ \frac{y_{7-i} \prod_{s=1}^{6-i} T_s}{y_{6-i}})), \end{aligned} \tag{5.1}$$

The operator D does not depend on the choice of $\boldsymbol{y} \in P$. The kernel of D is the 7-dimensional space of polynomials V corresponding to P.

Let Λ_s be the unique dominant integral A_6-weight such that $(\Lambda_s, \alpha_i) = m_s^{(i)}$. Given a 6-tuple of polynomials $\boldsymbol{y}$, define the corresponding A_6-weight at infinity

$$\Lambda_\infty = \sum_{s=1}^{n} \Lambda_s - \sum_{i=1}^{6} (\deg y_i)\alpha_i.$$

The set of weights of infinity corresponding to elements of a population P form an orbit of Weyl group with respect to the shifted action.

Let $\bar{\Lambda}_\infty$ be the unique dominant weight in the orbit of Λ_∞ under the shifted action of the Weyl group. It is conjectured in [MV1] that the number of A_6-populations associated to T_i for generic z_j equals the multiplicity of $L_{\bar{\Lambda}_\infty}$ in $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$, where L_Λ is the irreducible sl_7-module of highest weight Λ.

The population P is isomorphic to the space of full flags in V, the isomorphism is given by $F \to \boldsymbol{y}$ where $y_i = W^\dagger(F_i)$. See Section 5 of [MV1] for details and proofs.

5.2. **C_3-populations.** Consider the root system of type C_3. Let α_3 be the long root and α_1, α_2 the short ones. We have

$$(\alpha_1, \alpha_1) = 2, \qquad (\alpha_2, \alpha_2) = 2, \qquad (\alpha_1, \alpha_2) = -1,$$
$$(\alpha_2, \alpha_3) = -2, \qquad (\alpha_1, \alpha_3) = 0, \qquad (\alpha_3, \alpha_3) = 4.$$

The root system C_3 corresponds to the symplectic Lie algebra sp_6. The Weyl group of sp_6 is generated by simple reflections s_i, $i = 1, 2, 3$. The shifted action of Weyl group on C_3-weights is

$$s_i \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha_i^\vee \rangle \alpha_i,$$

where $\langle \rho, \alpha_i^\vee \rangle = 1$ for all i.

Fix polynomials $T_i(x) = \prod_{s=1}^{n} (x - z_s)^{m_s^{(i)}}$ where $i = 1, 2, 3$, x is a formal variable and $z_i, m_s^{(i)}$ are parameters.

A triple of polynomials $\boldsymbol{y} = (y_1, y_2, y_3)$ is called *generic* if y_i have no multiple roots and if y_i, y_{i+1} have no common roots. A generic triple of polynomials $\boldsymbol{y}$ is called *a Bethe triple* of type C_3 if there exist polynomials $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$ such that

$$W(y_1, \tilde{y}_1) = T_1 y_2, \qquad W(y_2, \tilde{y}_2) = T_2 y_1 y_3^2, \qquad W(y_3, \tilde{y}_3) = T_3 y_2.$$

If $\boldsymbol{y}$ is a Bethe triple and the triple $\boldsymbol{y}^{(i)} = (y_1, \ldots, \tilde{y}_i, \ldots, y_3)$ is generic then $y^{(i)}$ is also a Bethe triple of C_3 type, see Theorem 3.7 in [MV1]. We call $\boldsymbol{y}^{(i)}$ *an immediate descendent of* $\boldsymbol{y}$ *in the direction* i.

Let $\bar{P}$ be the minimal set of Bethe triples which contains $\boldsymbol{y}$ and such that for all $\tilde{\boldsymbol{y}} \in \bar{P}$ all immediate descendants $\tilde{\boldsymbol{y}}^{(i)}$ are also in $\bar{P}$.

The degrees of all coordinates of all triples of $\bar{P}$ are simultaneously bounded. *The C_3-population of Bethe triples originated at* $\boldsymbol{y}$ is the closure of $\bar{P}$ in $(\mathbb{P}(\mathbb{C}[x]))^3$.

The set of population of Bethe triples associated with polynomials T_i is in one-to-one correspondence with the set of 7-dimensional self-dual spaces of polynomials with no base points and ramification polynomials $T_1, T_2, T_3, T_3, T_2, T_1$. Given a population P and $\boldsymbol{y} \in P$, we define a 6-tuple $\boldsymbol{y}^A$ by

$$\boldsymbol{y}^A = (y_1, y_2, y_3^2, y_3^2, y_2, y_1).$$

Then the self-dual space of polynomials V which corresponds to P is the kernel of $D(\boldsymbol{y}^A)$, given by (5.1).

Let Λ_s be the unique dominant integral C_3-weight such that $\langle \Lambda_s, \alpha_i^\vee \rangle = m_s^{(i)}$. Given a triple of polynomials $\boldsymbol{y}$, define the corresponding C_3-weight at infinity by

$$\Lambda_\infty = \sum_{s=1}^{n} \Lambda_s - \sum_{i=1}^{3} (\deg y_i) \alpha_i.$$

The set of weights of infinity corresponding to elements of a population P form an orbit of Weyl group with respect to the shifted action.

Let $\bar{\Lambda}_\infty$ be the unique dominant weight in the orbit of Λ_∞ under the shifted action of the Weyl group. It is conjectured in [MV1] that the number of C_3-populations associated to T_i for generic z_j equals the multiplicity of $L_{\bar{\Lambda}_\infty}$ in $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$, where L_Λ is the irreducible sp_6-module of highest weight Λ.

The population P is isomorphic to the space of isotropic flags in V, the isomorphism is given by $F \to \boldsymbol{y}$ where $y_i = W^\dagger(F_i)$, $i = 1, 2, 3$. See Sections 6 and 7 of [MV1] for details and proofs.

5.3. **G_2-populations.** Consider the root system of type G_2. Let α_1 be the long root and α_2 the short one. We have

$$(\alpha_1, \alpha_1) = 6, \qquad (\alpha_1, \alpha_2) = -3, \qquad (\alpha_2, \alpha_2) = 2.$$

The Weyl group of G_2 is generated by simple reflections s_1, s_2. The shifted action of Weyl group on G_2-weights is

$$s_i \cdot \lambda = \lambda - \langle \lambda + \rho, \alpha_i^\vee \rangle \alpha_i,$$

where $\langle \rho, \alpha_i^\vee \rangle = 1$ for all i.

Fix polynomials $T_i(x) = \prod_{s=1}^{n} (x - z_s)^{m_s^{(i)}}$, where $i = 1, 2$, x is a formal variable and $z_i, m_s^{(i)}$ are parameters.

A pair of polynomials $\boldsymbol{y} = (y_1, y_2)$ is called *generic* if y_i have no multiple roots and no common roots. A generic pair of polynomials $\boldsymbol{y}$ is called *a Bethe pair* of type G_2 if there exist polynomials $\tilde{y}_1, \tilde{y}_2$ such that

$$W(y_1, \tilde{y}_1) = T_1 y_2, \qquad W(y_2, \tilde{y}_2) = T_2 y_1^3.$$

If $\boldsymbol{y}$ is a Bethe pair and the pair $\boldsymbol{y}^{(1)} = (\tilde{y}_1, y_2)$ (resp. $\boldsymbol{y}^{(2)} = (y_1, \tilde{y}_2)$) is generic then $\boldsymbol{y}^{(1)}$ (resp. $\boldsymbol{y}^{(2)}$) is also a Bethe pair of G_2 type, see Theorem 3.7 in [MV1]. We call $\boldsymbol{y}^{(i)}$ *an immediate descendent of* $\boldsymbol{y}$ *in the direction* i.

Let $\bar{P}$ be the minimal set of Bethe triples which contains $\boldsymbol{y}$ and such that for all $\tilde{\boldsymbol{y}} \in \bar{P}$ all immediate descendants $\tilde{\boldsymbol{y}}^{(i)}$ are also in $\bar{P}$.

The degrees of all coordinates of all tuples of $\bar{P}$ are simultaneously bounded. *The G_2-population of Bethe pairs originated at $\boldsymbol{y}$* is the closure of $\bar{P}$ in $(\mathbb{P}(\mathbb{C}[x]))^2$.

Let Λ_s be the unique dominant integral G_2-weight such that $\langle \Lambda_s, \alpha_i^\vee \rangle = m_s^{(i)}$. Given a pair of polynomials $\boldsymbol{y}$, define the corresponding G_2-weight at infinity

$$\Lambda_\infty = \sum_{s=1}^{n} \Lambda_s - \sum_{i=1}^{2} (\deg y_i)\alpha_i.$$

The set of weights of infinity corresponding to elements of a population P form an orbit of Weyl group with respect to the shifted action, see [MV1], Theorem 3.12.

Let $\bar{\Lambda}_\infty$ be the unique dominant weight in the orbit of Λ_∞ under the shifted action of the Weyl group. It is conjectured in [MV1] that the number of G_2-populations associated to T_i for generic z_j equals the multiplicity of $L_{\bar{\Lambda}_\infty}$ in $L_{\Lambda_1} \otimes \cdots \otimes L_{\Lambda_n}$, where L_Λ is the irreducible G_2 module of highest weight Λ.

Given a G_2-population P and $\boldsymbol{y} \in P$, we define a 3-tuple $\boldsymbol{y}^C$ by

$$\boldsymbol{y}^C = (y_1, y_2, y_1).$$

The following lemma is obvious but very useful.

Lemma 5.1. *A pair of polynomials $\boldsymbol{y}$ is a Bethe pair of type G_2 associated to polynomials T_1, T_2 if and only if the triple of polynomials $\boldsymbol{y}^C$ is a Bethe triple of type C_3 associated to polynomials T_1, T_2, T_1. Moreover, we have an inclusion of populations $P(\boldsymbol{y}) \to P(\boldsymbol{y}^C)$ mapping $\tilde{\boldsymbol{y}} \mapsto \tilde{\boldsymbol{y}}^C$.* $\square$

Assign to a G_2-population P the space $V = \mathrm{Ker}(D(((\boldsymbol{y})^C)^A))$, where $\boldsymbol{y} \in P$ and D is defined by (5.1). Then V is a self-dual 7-dimensional space of polynomials which does not depend on the choice of $\boldsymbol{y} \in P$. Moreover for any $\boldsymbol{y} \in P$, $y_1 \in V$.

See [MV1] for details and proofs.

5.4. **The space of G_2-population.** We show that the space of the G_2-population is self-self-dual.

Let P be a G_2-population and V the corresponding self-dual space of dimension 7. We start with a description of degrees of polynomials in V first. The degrees of coordinates of pairs of polynomials in P are in one-to-one correspondence with the Weyl group orbit (with respect to the shifted action). It follows that there exists a pair $\boldsymbol{y} \in P$ such that $\deg \tilde{y}_1 > \deg y_1$ and $\deg \tilde{y}_2 > \deg y_2$. Such a pair corresponds to the dominant weight in the Weyl group orbit. Denote

$$\deg y_1 = a, \qquad \deg y_2 = b, \qquad \deg T_1 = t_1, \qquad \deg T_2 = t_2.$$

So we have $\deg(\boldsymbol{y}) = (a, b)$, $\deg(\boldsymbol{y}_1^{(2)}) = (a, 3a-b+1+t_2)$ and $\deg(\boldsymbol{y}^{(1)}) = (b-a+1+t_1, b)$. In particular our choice of $\boldsymbol{y}$ means

$$b + 1 + t_1 > 2a, \qquad 3a + 1 + t_2 > 2b. \tag{5.2}$$

In what follows we consider successive descendents of a G_2-pair. Our notation $\boldsymbol{y}^{(1)(2)(1)}$ mean $((\boldsymbol{y}^{(1)})^{(2)})^{(1)}$, etc.

Lemma 5.2. *The degrees of polynomials in V are: a, $-a+b+1+t_1$, $2a-b+2+t_1+t_2$, $3+2t_1+t_2$, $-2a+b+4+3t_1+t_2$, $a-b+5+3t_1+2t_2$, $-a+6+4t_1+2t_2$.*

Proof. These are degrees of the first coordinates of the descendents: $\boldsymbol{y}$, $\boldsymbol{y}^{(1)}$, $\boldsymbol{y}^{(2)(1)}$, $\boldsymbol{y}^{(1)(2)(1)'}$, $\boldsymbol{y}^{(1)(2)(1)}$, $\boldsymbol{y}^{(2)(1)(2)(1)}$, $\boldsymbol{y}^{(1)(2)(1)(2)(1)}$. Here by $\boldsymbol{y}^{(1)(2)(1)'}$ we mean a difference between two different descendents.

The only non-trivial part here is to find the degree of $y_1^{(1)(2)(1)'}$. Let $y_1^{(1)\pm}$ be two different immediate descendants such that $y_1^{(1)+} = y_1^{(1)-} + \alpha y_1$. Similarly, we will write $(\ )^{\pm}$ for two different descendants and denote the difference by $(\ )'$. We have the following Wronskians, with equalities up to signs.

$$W(y_1, y_1^{(1)\pm}) = T_1 y_2,\ W(y_2^{(1)(2)\pm}, y_2) = T_2 (y_1^{(1)\pm})^3,\ W(y_1^{(1)(2)(1)\pm}, y_1^{(1)\pm}) = T_1 y_2^{(1)(2)\pm}.$$

From the last equation we conclude that

$$F = W(y_1^{(1)(2)(1)'}, y_1^{(1)+}) = T_1 y_2^{(1)(2)'} - \alpha W(y_1^{(1)(2)(1)-}, y_1).$$

We compute the degree of F. We calculate

$$\begin{aligned} W(F, T_1 y_2) &= T_1^2 T_2((y_1^{(1)+})^3 - (y_1^{(1)-})^3) - \alpha W(W(y_1^{(1)(2)(1)-}, y_1), W(y_1^{(1)-}, y_1)) \\ &= \alpha^3 T_1^2 T_2 y_1^3 + 3\alpha^2 T_1^2 T_2 y_1^2 y_1^{(1)-} + 3\alpha T_1^2 T_2 y_1 (y_1^{(1)-})^2 \\ &\pm (y_1)(y_1^{(1)-})^{-1} \alpha W(W(y_1^{(1)(2)(1)-}, y_1^{(1)-}), W(y_1^{(1)-}, y_1)) \\ &= \alpha^3 T_1^2 T_2 y_1^3 + 3\alpha^2 T_1^2 T_2 y_1^2 y_1^{(1)-} + (3 \pm 1)\alpha T_1^2 T_2 y_1 (y_1^{(1)-})^2. \end{aligned}$$

Here we have used the Wronskian identity $W(W(u_1, u_2), W(u_1, u_3)) = W(u_1, u_2, u_3) u_1$ which holds for all u_1, u_2, u_3.

Using (5.2) we see that the last term has the highest degree which means that the degree of $W(F, T_1 y_2)$ equals $2t_1 + t_2 + a + 2\deg y_1^{(1)}$. This implies that the degree of F equals $1 - t_1 - b + 2t_1 + t_2 + a + 2\deg y_1^{(1)}$ which in turn implies that

$$\deg(y_1^{(1)(2)(1)'}) = 1 - \deg y_1^{(1)} + \deg F = 3 + 2t_1 + t_2.$$

Since all the degrees in the statement of the lemma are present and different (written in the increasing order) by (5.2), the lemma is proved.

□

Lemma 5.3. *Any G_2-population P is an algebraic variety of dimension 6. The set of the first coordinates of pairs in P coincides with the set of isotropic vectors in V.*

Proof. Population P is an algebraic variety by Corollary 3.13 in [MV1].

Denote $J = \{y_1 \mid (y_1, y_2) \in P\}$. Denote as before $I \subset V$ the set of isotropic vectors. Since the first coordinates of pairs in a G_2-population are also the first coordinates of triples in a C_3-population we have $J \subseteq I$. In particular $\dim J \leq \dim I = 5$.

Consider the obvious projection of algebraic varieties $P \to J$. It is well known that for any polynomial $q(x) \in \mathbb{C}[x]$ the set of planes of polynomials $H \in Gr(2, \mathbb{C}[x])$ such that the Wronskian $W(H)$ is a scalar multiple of $q(x)$, is finite. E.g. this follows from Theorem 3.18 in [MV1]. Applied to $q(x) = T_2 y_1^3$, it shows that the fibers of $P \to J$ are at most 1-dimensional and we have $\dim P \leq \dim J + 1 = 6$.

Now we show the opposite inequality. We have the following chain of descendents and their degrees:

$$\begin{aligned}
\deg(\boldsymbol{y}^{(1)}) &= (b - a + 1 + t_1, b), \\
\deg(\boldsymbol{y}^{(1)(2)}) &= (b - a + 1 + t_1, 2b - 3a + 4 + 3t_1 + t_2), \\
\deg(\boldsymbol{y}^{(1)(2)(1)}) &= (b - 2a + 4 + 3t_1 + t_2, 2b - 3a + 4 + 3t_1 + t_2), \\
\deg(\boldsymbol{y}^{(1)(2)(1)(2)}) &= (b - 2a + 4 + 3t_1 + t_2, b - 3a + 9 + 6t_1 + 3t_2), \\
\deg(\boldsymbol{y}^{(1)(2)(1)(2)(1)}) &= (-a + 6 + 4t_1 + 2t_2, b - 3a + 9 + 6t_1 + 3t_2), \\
\deg(\boldsymbol{y}^{(1)(2)(1)(2)(1)(2)}) &= (-a + 6 + 4t_1 + 2t_2, -b + 10 + 6t_1 + 4t_2).
\end{aligned}$$

We prove that the dimension of the set of the descendents in the above chain increases by one. We recall that all our pairs are considered as elements in $\mathbb{P}\mathbb{C}[x]^2$.

The point is that given a pair (y_1, y_2) on line k we can uniquely recover the pair on the line $(k-1)$ that gave rise to (y_1, y_2). Indeed, one of the coordinates is always unchanged, and the other is uniquely determined by the degree restriction. Indeed, they are obtained by doing the reverse move from (y_1, y_2) which results in a smaller degree than before. Such move is always at best unique.

On the other hand, for every (y_1, y_2) on line k there are $\mathbb{C}$ ways to extend it to line $(k+1)$. This shows that the dimension of the set of descendants on line $(k+1)$ is one bigger than that on line k.

Our chain has length 6 and therefore $\dim P \geq 6$. It follows that $\dim P = 6$ and $\dim J = 5$. Since $\mathbb{P}I$ is an irreducible algebraic variety of dimension 5 and $J \subset \mathbb{P}I$, we see that $\bar{J} = \mathbb{P}I$. On the other hand J is closed. □

Theorem 5.4. *The space V is self-self-dual.*

Proof. To check the condition of Definition 4.3, we consider the sequence of C_3 reproductions of the triple (y_1, y_2, y_1) in the directions 1,2,3. Namely we have polynomials $\tilde{y}_i$ such that

$$W(y_1, \tilde{y}_1) = T_1 y_2, \quad W(y_2, \tilde{y}_2) = T_2 \tilde{y}_1 y_1^2, \quad W(\tilde{y}_3, y_1) = T_1 \tilde{y}_2.$$

Then the triple $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ is in a C_3-population. And in particular $\tilde{y}_3$ is a divided Wronskian of an isotropic 3-space.

To show $\tilde{y}_3 \in V$ we act on $\tilde{y}_3$ by the differential operator $D(y_1, y_2, y_3^2, y_3^2, y_2, y_1)$. Applying the right factors $\tilde{y}_3$ becomes (up to a constant) $\tilde{y}_2/y_1$ then $y_1\tilde{y}_1/y_2$, then 1 and then 0. Therefore $\tilde{y}_3$ is in V which by definition is the kernel of this differential operator.

Next we show that a generic $v \in V$ is $\tilde{y}_3$ for a suitable G_2-pair (y_1, y_2). Since everything is algebraic we just have to show that the set of such v (up to a scalar multiple) has dimension 6. We have 6-dimensional G_2-population. There exists a six-dimensional subset of the variety of G_2 pairs, such that each produces a variety of $\tilde{y}_3$ which has dimension at least 3. Indeed, the dimension is exactly 3 if we start with the dominant pair (y_1, y_2) satisfying $\deg y_1 = a$, $\deg y_2 = b$ as in Lemma 5.2. To show that the dimension is at least three for a generic G_2 pair of the type $\mathbf{y}^{(1)(2)(1)(2)(1)(2)}$ of Lemma 5.3, observe that at each step of the reproduction we can keep the descendant arbitrarily close to the previous pair, in terms of the projective space, by adding a big multiple of y_i. Hence, there is an open set in the space of $\mathbf{y}^{(1)(2)(1)(2)(1)(2)}$ which is contained in a neighborhood of (y_1, y_2). Since the dimension 3 condition is clearly an open one, we see that every G_2 pair which is close enough to (y_1, y_2) satisfies it.

Now fix $\tilde{y}_3$. We need to show that the dimension of the variety of G_2-pairs (y_1, y_2) which produce $\tilde{y}_3$ is at most 3. Note, that without loss of generality we can assume that $\tilde{y}_3$ is non-isotropic. Indeed, if we start with the dominant element of the G_2-population (y_1, y_2) then $\deg \tilde{y}_3 = 3 + 2t_1 + t_2$ and $\tilde{y}_3$ is non-isotropic. Since the condition of being is non-isotropic is open, it holds for the general descendants of the G_2 pairs in the neighborhood of (y_1, y_2).

Let F be the isotropic flag which corresponds to the C_3-triple (y_1, y_2, y_1) and let $u_1, \dots, u_7$ be a Witt basis corresponding to F, such that we have $u_1 = y_1$, $W^\dagger(u_1, u_2) = y_2$ and $W^\dagger(u_1, u_2, u_3) = y_1^2$.

It follows that there exist constants C_1, C_2, C_3 such that the C_3-triple $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ corresponds to the flag $\tilde{F}$, which is related to the basis

$$\{\tilde{u}_1 = u_1 + C_1 u_2, \tilde{u}_2 = u_2 + C_2 u_3, \tilde{u}_3 = u_3 + C_3 u_4 + C_3^2 u_5/2, \tilde{u}_4, \tilde{u}_5, \tilde{u}_6, \tilde{u}_7\}$$

see Lemmas 6.14-6.16 in [MV1]. In particular, $W^\dagger(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) = \tilde{y}_3^2$.

There are only finitely many 3-spaces of polynomials with Wronskian $\tilde{y}_3^2 T_1^2 T_2$. The 3-space U spanned by $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ is one of them. Note that the intersection of F_3 and U contains $\tilde{u}_1, \tilde{u}_2$ and therefore is at least 2-dimensional.

Now we observe that for any 2-dimensional subspace $U_1 \subset U$ there are at most finitely many isotropic 3-spaces F_3 such that $U_1 \subset F_3 \cap U$ and $\sqrt{W^\dagger(F_3)} \in F_3$. Indeed, the family of isotropic 3-spaces F_3 such that $U_1 \subset F_3 \cap U$ is isomorphic to a non-degenerate conic in a $\mathbb{P}(U_1^\perp/U_1) \cong \mathbb{P}^2$. Note that the space U belongs to this family and does not contain $\sqrt{W^\dagger(U)} = \tilde{y}_3$ because y_3 is non-isotropic. Therefore our family is a proper subvariety in a non-degenerate conic, thus it is a finite set of points.

The dimension of the variety of 2-planes in U is 2 and each F_3 produces a family of G_2-pairs of dimension 1, since $y_1 = \sqrt{W^\dagger(F_3)}$ up to a scalar. It follows that the dimension

of the variety of G_2-pairs which produce a given $\tilde{y}_3$ is at most 3. Thus the variety of all possible $\tilde{y}_3$ has dimension at least 6 as needed.

Finally, we show that $\tilde{y}_3$ is orthogonal to U. Recall that $\tilde{u}_1 = \tilde{y}_1$, $W(\tilde{u}_1, \tilde{u}_2) = T_1\tilde{y}_2$. We denote $d = W(\tilde{u}_1, \tilde{u}_3)/T_1$, then $W(\tilde{y}_2, d) = T_2\tilde{y}_1\tilde{y}_3^2$.

We calculate (up to constants):

$$\begin{aligned}
W(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{y}_3) &= \tilde{y}_1^{-2}W(W(\tilde{y}_1, \tilde{u}_2), W(\tilde{y}_1, \tilde{u}_3), W(\tilde{y}_1, \tilde{y}_3)) \\
= \tilde{y}_1^{-2}W(T_1\tilde{y}_2, T_1 d, W(\tilde{y}_1, \tilde{y}_3)) &= T_1^3\tilde{y}_1^{-2}W(\tilde{y}_2, d, T_1^{-1}W(\tilde{y}_1, \tilde{y}_3)) \\
&= T_1^3\tilde{y}_1^{-2}\tilde{y}_2^{-1}W(W(\tilde{y}_2, d), W(\tilde{y}_2, T_1^{-1}W(\tilde{y}_1, \tilde{y}_3))) \\
&= T_1^3\tilde{y}_1^{-2}\tilde{y}_2^{-1}W(T_2\tilde{y}_1\tilde{y}_3^2, T_1^{-2}W(W(y_1, \tilde{y}_3), W(\tilde{y}_1, \tilde{y}_3))) \\
&= T_1^3\tilde{y}_1^{-2}\tilde{y}_2^{-1}W(T_2\tilde{y}_1\tilde{y}_3^2, T_1^{-2}\tilde{y}_3 y_1^{-1}W(W(y_1, \tilde{y}_3), W(y_1, \tilde{y}_1))) \\
&= T_1^3\tilde{y}_1^{-2}\tilde{y}_2^{-1}W(T_2\tilde{y}_1\tilde{y}_3^2, T_1^{-2}\tilde{y}_3 y_1^{-1}W(T_1\tilde{y}_2, T_1 y_2)) \\
&= T_1^3\tilde{y}_1^{-2}\tilde{y}_2^{-1}W(T_2\tilde{y}_1\tilde{y}_3^2, \tilde{y}_3 y_1^{-1}T_2\tilde{y}_1 y_1^2) = T_1^3T_2^2\tilde{y}_2^{-1}\tilde{y}_3^2W(\tilde{y}_3, y_1) \\
&= T_1^4T_2^2\tilde{y}_3^2 = T_1^4T_2^2W^\dagger(\tilde{u}_1, \tilde{u}_2, \tilde{u}_2).
\end{aligned}$$

The orthogonal complement $U^\perp$ of U is the unique 4-dimensional subspace of V such that $U \subset U^\perp$ and $W^\dagger(U^\perp) = W^\dagger(U)$. Therefore $U^\perp$ is the span of $u_1, u_2, u_3, \tilde{y}_3$. In particular $\tilde{y}_3 \perp U$. This shows that V satisfies the condition of Definition 4.3. □

Theorem 5.4 shows that V has the skew-symmetric 3-form by Theorem 4.12. Thus for each element $\boldsymbol{y}$ of population P we have a G_2-isotropic flag F in V such that

$$F_1 = W^\dagger(F_6) = y_1, \qquad W^\dagger(F_2) = W^\dagger(F_5) = y_2, \qquad W^\dagger(F_3) = W^\dagger(F_4) = y_1^2.$$

Theorem 5.5. *The population P is isomorphic to the variety $F^{\perp\!\!\perp}(V)$ of G_2-isotropic flags in V. In particular V is isomorphic to the flag variety of group G_2.*

Proof. We already know that $P \subset F^{\perp\!\!\perp}(V)$. In addition $\dim P = \dim F^{\perp\!\!\perp}(V) = 6$. The variety $F^{\perp\!\!\perp}(V)$ is a $\mathbb{P}_1$-bundle over an irreducible conic of isotropic vectors and therefore is irreducible. It follows that $P = F^{\perp\!\!\perp}(V)$.

The space $F^{\perp\!\!\perp}(V)$ is isomorphic the G_2-isotropic flag variety by Proposition 3.8. □

Let P be a G_2-population and $\boldsymbol{y} \in P$ such that Λ_∞ is dominant integral. Let F^∞ be the unique full flag of V such that the degrees of polynomials in F_i are not larger that the degrees of polynomials in F_{i+1}.

Proposition 5.6. *The closure of all elements of P with the weights at infinity equal $w\cdot\Lambda_\infty$ is the Bruhat cell $G_w^{F^\infty}$.*

Proof. Completely parallel to the proof of Corollary 5.23 in [MV1]. □

Theorem 5.7. *The set of G_2-populations associated to polynomials T_1, T_2 is in one to one correspondence with the set of self-self-dual spaces of polynomials of dimension 7 with ramification polynomials $T_1, T_2, T_1, T_1, T_2, T_1$.*

Proof. Consider a G_2-isotropic flag F. Then if the pair $(F_1, W^\dagger(F_2))$ is generic then it is a G_2 Bethe pair. Therefore, we only have to show that each self-self-dual space contains a G_2-isotropic flag F such that $(F_1, W^\dagger(F_2))$ form a generic pair. It is parallel to the proof of Theorems 7.5 and 7.10 in [MV1]. □

5.5. **Another description of self-self-dual spaces.** We show that Definition 4.3 is equivalent to a simple condition on 3-Wronskians.

Theorem 5.8. *A self-dual space of polynomials is self-self-dual if and only if*

$$\{u^2 \mid u \in V\} = \{W^\dagger(u_1, u_2, u_3) \mid u_i \in V,\ (u_i, u_j) = 0\}.$$

Proof. The only if part is Corollary 4.7.

We need to show the if part. Let V be a self-dual space. Recall that such a space corresponds to a C_3-population. Let $u_1, \dots, u_7$ be a basis of V such that $\deg u_i < \deg u_{i+1}$ and $B(u_i, u_j) = (-1)^{i+1}\delta_i^{8-j}$. Then $W^\dagger(u_1, u_2, u_3)$ has the smallest degree among all divided 3-Wronskians and therefore we get $W^\dagger(u_1, u_2, u_3) = cu_1^2$. It follows that (y_1, y_2), where $y_1 = u_1$, $y_2 = W^\dagger(u_1, u_2)$ has the reproduction properties of a G_2-pair. Namely there exist $\tilde{y}_1 = u_2$ and $\tilde{y}_2 = c_1 W^\dagger(u_1, u_3)$ such that $W(y_1, \tilde{y}_1) = y_2 T_1$ and $W(y_2, \tilde{y}_2) = y_1^3 T_2$. The pair (y_1, y_2) may be not generic and therefore it is not a G_2-pair in general.

However the triples $(y_1, \tilde{y}_2, y_1)$ and $(\tilde{y}_1, y_2, \tilde{y}_1)$ are clearly in the same C_3-population and therefore correspond to some isotropic flags. In particular, these two triples again have the reproduction properties of a G_2-pair.

It follows that the condition of Definition 4.3 is satisfied by the same argument as in Theorem 5.4. □

5.6. **Examples of self-self-dual spaces.** The simplest example of a self-self-dual space of polynomials is the space of polynomials V of degree at most 6. This space clearly corresponds to the population originated at the G_2-pair $(1, 1)$ where $T_1 = T_2 = 1$.

More generally, for every pair of integers $m < n$ a space spanned by monomials $1, x^m, x^n, x^{m+n}, x^{2m+n}, x^{2n+m}, x^{2m+2n}$ is self-self-dual. It corresponds to the population which originates at $(1, 1)$ where $T_1 = x^{m-1}$ and $T_2 = x^{n-m-1}$.

6. Standard bases of self-self-dual spaces

We recall that every self-dual space V has a basis $\{v_1, \dots, v_7\}$ such that the divided 6-Wronskians are explicitly given by

$$W^\dagger(v_1, \dots, \hat{v}_i, \dots, v_7) = v_{8-i}.$$

Such a basis is called a Witt basis. The basis $\{v_i\}$ is a Witt basis if and only if $B(v_i, v_j) = (-1)^{i+1}\delta_i^{8-j}$, see [MV1].

In this section we show that every self-self-dual space has a Witt basis $\{v_1, \dots, v_7\}$ such that the all divided 3-Wronskians are given explicitly as a certain explicit linear

combination of v_iv_j, see Table 1 below. We call such a basis a standard basis. A Witt basis is a standard basis if and only if the 3-form has the standard form, see Corollary 6.11 below.

6.1. **Standard dominant bases.** We keep the notation of Lemma 5.2. We also introduce

$$m = -2a + b + 1 + t_1,\ n = a - b + 2 + t_1 + t_2.$$

Then $m < n$ and the degree list of Lemma 5.2 can be written as

$$a, a+m, a+n, a+m+n, a+2m+n, a+m+2n, a+2m+2n.$$

First, we need we choose a Witt basis of V which is compatible with degrees.

Lemma 6.1. *The space V has a Witt basis $\{v_1, \dots, v_7\}$ of degrees*

$$a,\ a+m,\ a+n,\ a+m+n,\ a+2m+n,\ a+m+2n,\ a+2m+2n$$

and leading coefficients

$$1,\quad \frac{1}{m},\quad \frac{1}{n(n-m)},\quad \frac{1}{(m+n)nm},\quad \frac{1}{(2m+n)(m+n)(2m)m},$$
$$\frac{1}{(m+2n)(2n)(m+n)n(n-m)},\quad \frac{1}{(2m+2n)(m+2n)(2m+n)(m+n)mn}$$

respectively.

Proof. By Lemma 6.6, Lemma 6.7 in [MV1], there exists a Witt basis with the above degrees. Then we scale the basis elements so that the leading coefficients are as above. It is easy to check that the leading terms of the 6-Wronskians are equal to the leading terms of the corresponding basis elements and therefore after the scaling we again obtain a Witt basis. □

A Witt basis $\{v_1, \dots, v_7\}$ of V with the above degrees and leading coefficients is called a *standard dominant basis* if

$$W^\dagger(v_1, v_5, v_6) = \frac{1}{4}v_4^2, \qquad W^\dagger(v_2, v_3, v_7) = \frac{1}{2}v_4^2.$$

Lemma 6.2. *Let V be self-self-dual. Then there exists a standard dominant basis of V with degrees and leading terms specified by Lemma 6.1.*

Proof. By Corollary 4.7, there are two ways of writing v_4^2 as a divided Wronskian of an isotropic three-space, up to a constant. We call the corresponding spaces U_1 and U_2. For each U_i we can find a basis $\{f_{i,1}, f_{i,2}, f_{i,3}\}$ of increasing degrees. Moreover, without loss of generality we can assume that the leading coefficients and the degrees of the f_i are among those of Lemma 6.1. Since $f_{i,j}$ are isotropic their degrees are not equal to the degree of v_4.

It is easy to see that we must have

$$\deg f_{i,1} + \deg f_{i,2} + \deg f_{i,3} = 3a + 3m + 3n.$$

This implies that for each i the degrees are either $(a, a+2m+n, a+m+2n)$ or $(a+m, a+n, a+2m+2n)$. It is impossible to have degrees from the first list for both i since that would imply that $U_1 \cap U_2 \ni v_1$. Similarly, if both U_i had degrees from the second list, we would have $\dim U_i \cap \mathrm{Span}(v_1, v_2, v_3) = 2$, which again implies $U_1 \cap U_2 \neq 0$. As a result, we can assume that $f_{1,1}, f_{2,1}, f_{2,2}, v_4, f_{1,2}, f_{1,3}, f_{2,3}$ have degrees and leading terms of Lemma 6.1.

The above basis is a a Witt basis provided that $B(f_{1,i}, f_{2,j}) = 0$ if $i+j \neq 3$ and $B(f_{1,1}, f_{2,3}) = B(f_{1,2}, f_{2,2}) = -B(f_{1,3}, f_{2.1}) = 1$.

Most of these equalities hold automatically. For example, $f_{1,2} = v_5 + ...$ and $f_{2,2} = v_3 + ...$ where the dots denote linear combinations of v_i with lower i. This implies that $B(f_{1,2}, f_{2,2}) = 1$ in view of the pairing of v_i.

There are exactly three equalities that are not true for a generic choice of $f_{i,j}$. Namely, we may not have

$$B(f_{1,2}, f_{2,3}) = 0, \quad B(f_{1,3}, f_{2,2}) = 0, \quad B(f_{1,3}, f_{2,3}) = 0.$$

Then we get $B(f_{1,3}, f_{2,2}) = 0$ by adding an appropriate scalar multiple of $f_{2,1}$ to $f_{2,2}$. Note that such an addition operation does not change the leading term.

Similarly, to get $B(f_{1,2}, f_{2,3}) = 0$ and $B(f_{1,3}, f_{2,3}) = 0$ we add an appropriate linear combination of $f_{2,1}$ and $f_{2,2}$ to $f_{2,3}$. This finishes the proof. □

Example 6.3. Let V be the space of polynomials of degree at most 6. Recall that V is self-self-dual. Then the basis

$$\{1,\ x,\ \frac{x^2}{2!},\ \frac{x^3}{3!},\ \frac{x^4}{4!},\ \frac{x^5}{5!},\ \frac{x^6}{6!}\}$$

is standard dominant.

6.2. **3-Wronskians in standard bases.** Fix a standard dominant basis $\{v_1, \dots, v_7\}$ in V. We identify the spinor space $\hat{V}$ with the polynomials in odd variables $\hat{v}_5$, $\hat{v}_6$ and $\hat{v}_7$ as before. We recall that v_5, v_6, v_7 act by multiplications by $\hat{v}_5, \hat{v}_6, \hat{v}_7$ respectively, v_1, v_2, v_3 act by differentiations $-\partial/\partial\hat{v}_7$, $\partial/\partial\hat{v}_6$, $-\partial/\partial\hat{v}_5$ respectively, and v_4 acts by $\frac{1}{\sqrt{2}}(-1)^{\deg}$, where deg is the degree of the odd polynomial.

Formula (2.3) allows us to fix the explicit form of the pairing $\hat{B}$. Namely,

$$\hat{B}(\hat{1}, \hat{v}_5\hat{v}_6\hat{v}_7) = \hat{B}(\hat{v}_6, \hat{v}_5\hat{v}_7) = 1, \quad \hat{B}(\hat{v}_7, \hat{v}_5\hat{v}_6) = \hat{B}(\hat{v}_6\hat{v}_7, \hat{v}_5) = -1,$$

and all other pairings of basis elements are zero.

Recall that by Proposition 4.6 there is an invariant surjection $\psi : \hat{V} \to V$ such that if $\psi(\rho(U)) = f$ then the divided Wronskian of U is proportional to f^2.

Lemma 6.4. *The point p corresponding to the invariant surjection ψ is given by the formula $p = \hat{v}_5\hat{v}_6 + \frac{1}{\sqrt{2}}\hat{v}_7$.*

Proof. The polynomial v_4^2 is proportional to the divided Wronskians of the spans of (v_1, v_5, v_6) and (v_2, v_3, v_7). Applying the spinor embedding ρ we obtain the lines $\mathbb{C}\hat{v}_5\hat{v}_6$

and $\mathbb{C}\hat{v}_7$ in $\hat{V}$. Since the invariant surjection is a projection from a point p, we see that, up to a constant multiple,

$$p = \hat{v}_5\hat{v}_6 + \alpha\hat{v}_7$$

for some $\alpha \in \mathbb{C}$.

Consider the isotropic vector $v_1+\beta v_2$. Its square can be uniquely up to constant written as a divided Wronskian of the annihilator of $(v_1 + \beta v_2) \cdot p$. Since

$$(v_1 + \beta v_2) \cdot p = (-\partial/\partial\hat{v}_7 + \beta\partial/\partial\hat{v}_6) \cdot (\hat{v}_5\hat{v}_6 + \alpha\hat{v}_7) = -\alpha\hat{1} - \beta\hat{v}_5,$$

the corresponding isotropic 3-space is spanned by $v_1, v_2, \alpha^2 v_3+\sqrt{2}\alpha\beta v_4+\beta^2 v_5$. Therefore, we must have

$$W^\dagger(v_1, v_2, \alpha^2 v_3 + \sqrt{2}\alpha\beta v_4 + \beta^2 v_5) = c(\beta)(v_1 + \beta v_2)^2.$$

Comparing the leading coefficients of both sides, we obtain that $c(\beta)$ is a constant. Moreover, we must have

$$\alpha^2 W^\dagger(v_1, v_2, v_3) = cv_1^2, \quad \alpha W^\dagger(v_1, v_2, v_4) = \sqrt{2}cv_1v_2, \quad W^\dagger(v_1, v_2, v_5) = cv_2^2$$

for some constant c. Comparing the leading coefficients on the both sides of the last two equations, we obtain the equalities

$$\alpha\frac{m(m+n)n}{m(m+n)mn} = c\sqrt{2}\frac{1}{m}, \qquad \frac{m(2m+n)(m+n)}{m(2m+n)(m+n)(2m)m} = c\frac{1}{m^2},$$

which give $c = \frac{1}{2}$, $\alpha = \frac{1}{\sqrt{2}}$. □

The explicit knowledge of p and $\hat{B}$ allows us to calculate all divided 3-Wronskians in the standard dominant basis. The next theorem is the main result of this section.

Theorem 6.5. *The Wronskians of the basis elements of a standard dominant basis of V are given in the Table 1 below.*

Proof. The theorem essentially amounts to the calculation of the map ϕ of Proposition 4.10, since by Corollary 4.11 the divided Wronskian map to $\mathbb{C}[x]$ is a composition of the map $\frac{1}{2}\phi$ and the multiplication map $Sym^2V \to \mathbb{C}[x]$.

Since we have fixed $p = \hat{v}_5\hat{v}_6 + \frac{1}{\sqrt{2}}\hat{v}_7$ and $\hat{B}$, by Proposition 4.10 $\phi = C\nu$. From the definition of the standard dominant basis, we have $\phi(v_1 \wedge v_2 \wedge v_3) = v_1 \otimes v_1$. We compute $\nu(v_1 \wedge v_2 \wedge v_3)$:

$$\begin{aligned}
&\langle \nu^*(v_1 \wedge v_2 \wedge v_3), (\sum_{i=1}^{7} \alpha_i v_i) \otimes (\sum_{i=1}^{7} \alpha_i v_i)\rangle = \\
&\quad = \hat{B}\left(v_1v_2v_3(\sum_{i=1}^{7} \alpha_i v_i) \cdot (\hat{v}_5\hat{v}_6 + \frac{1}{\sqrt{2}}\hat{v}_7),\ (\sum_{i=1}^{7} \alpha_i v_i) \cdot (\hat{v}_5\hat{v}_6 + \frac{1}{\sqrt{2}}\hat{v}_7)\right) \\
&\quad = \hat{B}\left(\alpha_7\hat{1}, (\sum_{i=1}^{7} \alpha_i v_i) \cdot (\hat{v}_5\hat{v}_6 + \frac{1}{\sqrt{2}}\hat{v}_7)\right) = \hat{B}(\alpha_7\hat{1}, \alpha_7\hat{v}_5\hat{v}_6\hat{v}_7) = \alpha_7^2 = (B(v_1, \sum_{i=1}^{7} \alpha_i v_i))^2
\end{aligned}$$

123 v_1^2	124 v_1v_2	125 $\frac{1}{2}v_2^2$	126 $-\frac{1}{2}v_1v_4+\frac{1}{2}v_2v_3$	127 $-v_1v_5+\frac{1}{2}v_2v_4$
134 v_1v_3	135 $\frac{1}{2}v_1v_4+\frac{1}{2}v_2v_3$	136 $\frac{1}{2}v_3^2$	137 $-v_1v_6+\frac{1}{2}v_3v_4$	145 $\frac{1}{2}v_2v_4$
146 $\frac{1}{2}v_3v_4$	147 $-\frac{1}{2}v_1v_7-\frac{1}{2}v_2v_6$ $+\frac{1}{2}v_3v_5+\frac{1}{4}v_4^2$	156 $\frac{1}{4}v_4^2$	157 $-\frac{1}{2}v_2v_7+\frac{1}{2}v_4v_5$	167 $-\frac{1}{2}v_3v_7+\frac{1}{2}v_4v_6$
234 v_1v_4	235 $v_1v_5+\frac{1}{2}v_2v_4$	236 $v_1v_6+\frac{1}{2}v_3v_4$	237 $\frac{1}{2}v_4^2$	245 v_2v_5
246 $\frac{1}{2}v_1v_7+\frac{1}{2}v_2v_6$ $+\frac{1}{2}v_3v_5+\frac{1}{4}v_4^2$	247 v_4v_5	256 $\frac{1}{2}v_2v_7+\frac{1}{2}v_4v_5$	257 v_5^2	267 $-\frac{1}{2}v_4v_7+v_5v_6$
345 $-\frac{1}{2}v_1v_7+\frac{1}{2}v_2v_6$ $+\frac{1}{2}v_3v_5-\frac{1}{4}v_4^2$	346 v_3v_6	347 v_4v_6	356 $\frac{1}{2}v_3v_7+\frac{1}{2}v_4v_6$	357 $\frac{1}{2}v_4v_7+v_5v_6$
367 v_6^2	456 $\frac{1}{2}v_4v_7$	457 v_5v_7	467 v_6v_7	567 $\frac{1}{2}v_7^2$

Table 1. The entry under ijk is $W^\dagger(v_i,v_j,v_k)$.

Therefore $\nu(v_1,v_2,v_3)=v_1\otimes v_1$, so $C=1$ and $\phi=\nu$.

It is now routine to calculate all divided Wronskians. We calculate $W^\dagger(v_1,v_4,v_7)$ as an example and leave the rest to the reader.

$$\begin{aligned}
&\langle \nu^*(v_1\wedge v_4\wedge v_7),\ (\sum_{i=1}^7\alpha_iv_i)\otimes(\sum_{i=1}^7\alpha_iv_i)\rangle = \\
&\quad = \hat{B}\left(v_1v_4v_7(\sum_{i=1}^7\alpha_iv_i)\cdot(\hat{v}_5\hat{v}_6+\frac{1}{\sqrt{2}}\hat{v}_7),(\sum_{i=1}^7\alpha_iv_i)\cdot(\hat{v}_5\hat{v}_6+\frac{1}{\sqrt{2}}\hat{v}_7)\right)\\
&\quad = \hat{B}(-\alpha_1\frac{1}{2}\hat{1}+\alpha_2\frac{1}{\sqrt{2}}\hat{v}_5+\alpha_3\frac{1}{\sqrt{2}}\hat{v}_6+\alpha_4\frac{1}{2}\hat{v}_5\hat{v}_6,\\
&\quad -\alpha_1\frac{1}{\sqrt{2}}\hat{1}-\alpha_2\hat{v}_5-\alpha_3\hat{v}_6+\alpha_4(\frac{1}{\sqrt{2}}\hat{v}_5\hat{v}_6-\frac{1}{2}\hat{v}_7)+\alpha_5\frac{1}{\sqrt{2}}\hat{v}_5\hat{v}_7+\alpha_6\frac{1}{\sqrt{2}}\hat{v}_6\hat{v}_7+\alpha_7\hat{v}_5\hat{v}_6\hat{v}_7)\\
&\quad = -\frac{1}{2}\alpha_1\alpha_7-\frac{1}{2}\alpha_2\alpha_6+\frac{1}{2}\alpha_3\alpha_5+\frac{1}{4}\alpha_4^2.
\end{aligned}$$

This implies that

$$\phi(v_1\wedge v_4\wedge v_7)=\frac{1}{4}(-v_1\otimes v_7-v_7\otimes v_1-v_2\otimes v_6-v_6\otimes v_2+v_3\otimes v_5+v_5\otimes v_3+v_4\otimes v_4).$$

Hence $W^\dagger(v_1,v_4,v_7)=-\frac{1}{2}v_1v_7-\frac{1}{2}v_2v_6+\frac{1}{2}v_3v_5+\frac{1}{4}v_4^2$. □

A basis $\{v_1, \dots, v_7\}$ of V is called *standard* if the divided 3-Wronskians $W^\dagger(v_i, v_j, v_k)$ are given in Table 1. A standard dominant basis is standard.

Proposition 6.6. *Let V be a self-dual space with a standard basis. Then V is self-self-dual.*

Proof. Define $\phi_V : \Lambda^3 V \to Sym^2 V$ by the formulas in Table 1. Let U be any self-self-dual space with a standard basis $\{u_1, \dots, u_7\}$ and the map $\phi_U : \Lambda^3 U \to Sym^2 U$. Define the map $\iota : U \to V$ sending $u_i \mapsto v_i$. Then it induces the maps $\Lambda^3 U \to \Lambda^3 V$ and $Sym^2 U \to Sym^2 V$ which obviously intertwine ϕ_U and ϕ_V. The image of any isotropic 3-space in U is a tensor square of an element in U. Therefore the image of any isotropic 3-space in V is a tensor square of an element in V. It follows that divided Wronskian of any isotropic 3-space in V is a a square of an element of V. Now the proposition follows from Theorem 5.8. □

For a self-self-dual space V the knowledge of only a few 3-Wronskians is sufficient to decide whether a Witt basis $\{v_1, \dots, v_7\}$ is self-dual.

Lemma 6.7. *A Witt basis $\{v_1, \dots, v_7\}$ of a self-self-dual space is standard if and only if*

$$W^\dagger(v_2, v_3, v_7) = \frac{1}{2} v_4^2,\ W^\dagger(v_1, v_5, v_6) = \frac{1}{4} v_4^2,$$

$$W^\dagger(v_1, v_2, v_3) = v_1^2,\ W^\dagger(v_1, v_2, v_4) = v_1 v_2,\ W^\dagger(v_1, v_2, v_5) = \frac{1}{2} v_2^2.$$

Proof. The if part is a tautology. To prove the only if part, notice that the first two Wronskians assure that $p = \hat{v}_5 \hat{v}_6 + \alpha v_7$. As in the proof of Lemma 6.4 we look at

$$W^\dagger(v_1, v_2, \alpha^2 v_3 + \sqrt{2}\alpha\beta v_4 + \beta^2 v_5) = c(\beta)(v_1 + \beta v_2)^2.$$

Since $v_1^2, v_1 v_2, v_2^2$ are linearly independent, the last three Wronskians assure $\alpha = \frac{1}{\sqrt{2}}$. Then the argument of Theorem 6.5 goes through, since it only uses $W^\dagger(v_1, v_2, v_3) = v_1^2$ to fix the constant. □

6.3. **3-form in a standard basis.** Now we read off the explicit formula for the trilinear form w of Theorem 4.12 in the standard basis $\{v_i\}$.

Proposition 6.8. *For $i < j < k$ we have $w(v_i, v_j, v_k) = 0$ except for*

$$w(v_2 \wedge v_4 \wedge v_6) = w(v_1 \wedge v_4 \wedge v_7) = -w(v_3 \wedge v_4 \wedge v_5) =$$
$$= -w(v_1 \wedge v_5 \wedge v_6) = -\frac{1}{2}\ w(v_2 \wedge v_3 \wedge v_7) = \frac{1}{4}\ .$$

Proof. Since w and w_ψ are $G_2(p)$-invariant, by Lemma 3.3 w is a constant multiple of w_ψ. Therefore, there exists a constant C, such that

$$w(a \wedge b \wedge c) = C\hat{B}(abc \cdot p, p).$$

As we saw in the proof of Lemma 6.7, $p = \hat{v}_5\hat{v}_6 + \frac{1}{\sqrt{2}}\hat{v}_7$, so we need to calculate

$$\frac{1}{\sqrt{2}}(\hat{B}(abc\cdot\hat{v}_5\hat{v}_6,\hat{v}_7)+\hat{B}(abc\cdot\hat{v}_7,\hat{v}_5\hat{v}_6))+\hat{B}(abc\cdot\hat{v}_5\hat{v}_6,\hat{v}_5\hat{v}_6)+\frac{1}{2}\hat{B}(abc\cdot\hat{v}_7,\hat{v}_7).$$

Since $\hat{B}(q_1,q_2)$ is zero unless the degrees of q_i add up to three, the first of these terms is nonzero only if the degree of abc is zero, which can only happen if it is of the form $v_{\leq 3}v_4v_{\geq 5}$. It is also easy to see that the term is zero unless abc is of the form $v_iv_4v_{8-i}$, for which it equals $(-\frac{1}{2})$ for $i=1,2$ and $\frac{1}{2}$ for $i=3$.

The last two terms are nonzero only for $abc = v_2v_3v_7$ and $abc = v_1v_5v_6$ respectively, when they equal 1 and $\frac{1}{2}$ respectively. Finally, from $W^\dagger(v_1,v_5,v_6) = \frac{1}{4}v_4^2$ and $B(v_4,v_4) = -1$ we obtain the constant: $C = -\frac{1}{2}$. □

Note that the value $w(a\wedge b\wedge c)$ can be computed by applying B to the corresponding element in Table 1. For example,

$$w(v_1\wedge v_4\wedge v_7) = -\frac{1}{2}B(v_1,v_7)-\frac{1}{2}B(v_2,v_6)+\frac{1}{2}B(v_3,v_5)+\frac{1}{4}B(v_4,v_4)=\frac{1}{4}.$$

Corollary 6.9. *Let w_1 be any non-degenerate skew-symmetric 3-form in a 7-dimensional space V associated to a non-degenerate bilinear form B. Then there is a basis $\{v_1,\dots,v_7\}$ of V such that $B(v_i,v_j) = (-1)^{i+1}\delta_i^{8-j}$ and w_1 is a scalar multiple of the form described in Proposition 6.8.*

Proof. By Proposition 3.7, the group $SL(V)$ acts transitively on the set of all non-degenerate forms considered up to a constant. That implies the corollary. □

Next we describe the set of standard bases in a self-self-dual space V.

Proposition 6.10. *The group $G_2(p)$ acts transitively on the set of standard bases.*

Proof. The group $G_2(p)$ acts on the set of standard bases by Proposition 4.9. Any two standard bases can be mapped to each other by an element g of the orthogonal group $SO(V)$. But then g preserves the 3-form and therefore belongs to $G_2(p)$ by Lemma 3.5. Therefore the action is transitive. □

Corollary 6.11. *A basis $\{v_1,\dots,v_7\}$ of a self-self-dual space V is standard if and only if $B(v_i,v_j) = (-1)^{i+1}\delta_i^{8-j}$ and the 3-form has the form described in Proposition 6.8.*

Proof. Follows from Proposition 6.10. □

Corollary 6.12. *Let basis $\{v_1,\dots,v_7\}$ in V be standard and let F_i be the span of $v_1,\dots,v_i$. Then F is a G_2-isotropic flag.*

Proof. Follows from Proposition 6.10. □

Corollary 6.13. *For every G_2-isotropic flag $F\in\mathcal{F}^{\perp\!\!\perp}$, there exists a standard basis $\{v_1,\dots,v_7\}$, such that $v_1,\dots,v_i$ span F_i.*

Proof. The group G_2 acts transitively on $\mathcal{F}^{\perp\!\!\perp}$ by Proposition 3.8 and on the set of standard bases by Proposition 6.10. One standard basis exists by Lemma 6.2. It is related to a G_2-isotropic flag by Corollary 6.12. Then this basis is mapped by G_2-action to a standard basis related to any other given G_2-isotropic flag. □

We finish with a description of the reproduction procedure in a standard basis.

Let $\{v_1, \dots, v_7\}$ be a standard basis and let (y_1, y_2) be the corresponding element of the G_2-population, $y_1 = v_1$, $y_2 = W^\dagger(v_1, v_2)$.

Proposition 6.14. *For any $c \in \mathbb{C}$ the two bases of V given by*

$$\{v_1 + cv_2, v_2, v_3 + 2cv_4 + 2c^2 v_5, v_4 + 2cv_5, v_5, v_6 + cv_7, v_7\},$$
$$\{v_1, v_2 + cv_3, v_3, v_4, v_5 + cv_6, v_6, v_7\}$$

are standard. The corresponding flags are in a bijective correspondence with the set of the immediate descendents $(\tilde{y}_1, y_2)$ and $(y_1, \tilde{y}_2)$ of (y_1, y_2) in the first and in the second directions respectively.

Proof. It is straightforward to check that the new bases are standard using Table 1 and Lemma 6.7. To check that they correspond to the descendants, the only non-trivial check is $W(y_2, \tilde{y}_2) = T_2 y_1^3$ (up to a constant) for the second direction. Since $y_2 = W^\dagger(v_1, v_2)$ and $\tilde{y}_2 = W^\dagger(v_1, v_2 + cv_3)$, we have

$$W(y_2, \tilde{y}_2) = T_1^{-2} W(W(v_1, v_2), W(v_1, v_2 + cv_3)) = T_1^{-2} v_1 W(v_1, v_2, v_2 + cv_3)$$
$$= cT_1^{-2} T_1^2 T_2 v_1^3 = cT_2 v_1^3 = cT_2 y_1^3.$$

□

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