ON THE CONJECTURE OF KING FOR SMOOTH TORIC DELIGNE-MUMFORD STACKS

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Abstract. We construct full strong exceptional collections of line bundles on smooth toric Fano Deligne-Mumford stacks of Picard number at most two and of any Picard number in dimension two. It is hoped that the approach of this paper will eventually lead to the proof of the existence of such collections on all smooth toric nef-Fano Deligne-Mumford stacks.

1. Introduction

It has been suggested by Alastair King in [Ki] that every smooth toric variety has a full strong exceptional collection of line bundles. While this turned out to be false, see [HP], it is still natural to conjecture that every smooth nef-Fano toric variety possesses such a collection, and there is some numerical evidence towards it. Here a variety is called nef-Fano (also often referred to as weak Fano) if its anticanonical class is nef and big, though not necessarily ample. We refer the readers to the introduction section of [CS] for the more detailed exposition of this area. In this paper we propose to extend the conjecture of King to smooth toric Deligne-Mumfod stacks, which were defined in [BCS].

Conjecture 3.14. *Every smooth nef-Fano toric DM stack possesses a full strong exceptional collection of line bundles.*

There are multiple advantages to working with stacks rather than varieties in the context of this conjecture. Smooth toric DM stacks behave like smooth toric varieities in many ways, so it is plausible that if Conjecture 3.14 holds in the case of varieties, then it holds in this more general setting, at least when the stacks are generically schemes. On the other hand, while there are only finitely many smooth toric nef-Fano varieties in any given dimension, there are infinitely many smooth toric Fano stacks, and they correspond to nice combinatorial data of simplicial convex lattice polytopes. Consequently, working with stacks allows one to test the conjecture on numerous families of examples, and

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to concentrate on the more essential features of the Fano condition. Last, but not least, stacks appear naturally from the point of view of homological mirror symmetry. For example, it is natural to try to extend the work of [A] to this generality,

We have been able to construct full strong exceptional collections of line bundles for all smooth toric Fano DM stacks $\mathbb{P}_{\mathbf{\Sigma}}$ of Picard number at most two or, significantly, of dimension at most two. This dimension two case is of special importance since it is related to (noncompact) toric Calabi-Yau stacks of dimension three.

The main ingredient of the argument is a convex polytope P in $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}) \otimes \mathbb{R}$ which is to be thought of as a window into $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$. For a generic point $p \in \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}) \otimes \mathbb{R}$, we define the strong exceptional collection S as the set of line bundles such that the corresponding points in $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}) \otimes \mathbb{R}$ lie in $p + P$. In other words, S is the set of line bundles that we can see through the P window, when it is shifted by p. We then move p and $p + P$, and as new line bundles appear in the window, we use Koszul complexes to generate them from the line bundles that we have already seen. In the Picard number one case, P is a segment, and in the Picard number two case it is a parallelogram, irrespective of the dimension of $\mathbb{P}_{\mathbf{\Sigma}}$. In the case of Picard number three and dimension two, the polytope P is a 10-gonal prism, and careful arguments of convex geometry are needed to establish its various properties. For arbitrary Picard numbers and dimension two, P is a zonotope, i.e. a Minkowski sum of line segments.

One key property of the polytope P is that the differences between all pairs of its interior points give acyclic line bunldles. To prove this, we introduce the notions of strong acyclicity and forbidden cones, see Definition 4.4. This approach follows the calculations of Danilov [D] and is similar to the recent work of Perling [P] in the scheme setting. The notion of strong acyclicity allows one to reduce the calculations to questions of convex geometry.

The paper is organized as follows. In Section 2 we briefly review the definition of smooth toric Deligne-Mumford stacks. In Section 3 we describe line bundles on these stacks and state the main Conjecture 3.14. In Section 4 we give a combinatorial description of cohomology of a line bundle on a smooth toric Deligne-Mumford stack and introduce the notions of strong acyclicity and forbidden cones. In Section 5 we describe the construction in the cases of Picard number one and two. Sections 6 and 7 treat the case of smooth toric del Pezzo DM stacks. The former section contains the calculations in the quotient of $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ by the span of the canonical class $\mathbb{R}K$, and the latter finishes the argument. Section 8 describes our construction in the case of dimension two and Picard number three. Finally, in Section 9 we briefly describe the difficulties one encounters when one tries to extend the method to higher dimension.

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2. An overview of toric DM stacks

Let N be a finitely generated abelian group and let Σ be a complete simplicial fan in N (which is simply a pullback of a simplicial fan Σ_{free} in $N_{\text{free}} = N/\text{torsion}(N)$). If one chooses a non-torsion element v in each of the one-dimensional cones of Σ, one gets the data of a complete stacky fan $\mathbf{\Sigma} = (\Sigma, \{v_i\})$, see [BCS]. To these data one can then associate a smooth toric Deligne-Mumford stack $\mathbb{P}_{\mathbf{\Sigma}}$ whose coarse moduli space is the proper simplicial toric variety given by Σ_{free}.

We will assume from now on that N has no torsion, to simplify the discussion, although it appears that the general case is not very different. This assumption will allow us to avoid the technicalities of the derived Gale duality of [BCS]. The toric Deligne-Mumford stack $\mathbb{P}_{\mathbf{\Sigma}}$ is obtained by a stacky version of the Cox's homogeneous coordinate ring construction of [C]. More specifically, if Σ has n one-dimensional cones, then we have a map

$$\rho\colon \mathbb{Z}^n \to N$$

defined by $(l_1, \ldots, l_n) \mapsto \sum_i l_i v_i$ where v_i are the chosen elements of $\mathbf{\Sigma}$. We dualize to get an injection

$$\rho^*\colon N^* \to \mathbb{Z}^n$$

and we denote the cokernel of ρ^* by $\text{Gale}(N)$. The group $\text{Gale}(N)$ is a finitely generated abelian group of rank $n - \text{rk}(N)$ and it has torsion if and only if ρ is not surjective. We define the abelian complex algebraic group G by

$$G := \text{Hom}(\text{Gale}(N), \mathbb{C}^*).$$

The group G is (non-canonically) isomorphic to a product of $(\mathbb{C}^*)^{n-\text{rk}(N)}$ and a finite abelian group.

The map ρ^* induces an injection

$$G \subseteq (\mathbb{C}^*)^n$$

and an element $(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ lies in G if and only if

$$\prod_{i=1}^{n} \lambda_i^{w\cdot v_i} = 1$$

for all $w \in N^*$, where $\cdot$ denotes the natural pairing.

Consider the open set U in $\mathbb{C}^n$ defined as follows. A point $(z_1, \ldots, z_n)$ lies in U if and only if there exists a cone in Σ which contains all v_i for which $z_i = 0$. It turns out that the action of G has only finite isotropy subgroups on U, and $\mathbb{P}_{\mathbf{\Sigma}}$ is then defined as the stack quotient $[U/G]$, see [BCS].

3. Derived category of toric stacks and King's conjecture

We keep the notations from the previous section. In this section we will describe some of the known results about the derived category of coherent sheaves on $\mathbb{P}_{\mathbf{\Sigma}}$ and will formulate the conjecture, whose original version is due to Alastair King, [Ki]. See [CS] for a short review of the related results.

The category of coherent sheaves on $\mathbb{P}_{\mathbf{\Sigma}}$ is equivalent to the category of G-equivariant sheaves on U, see [V, 7.12]. In particular, the line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ have the following explicit description.

Definition 3.1. For each $(r_1, \ldots, r_n) \in \mathbb{Z}^n$ consider the trivial line bundle $\mathbb{C} \times U \to U$ with the G-linearization $G \times \mathbb{C} \times U \to \mathbb{C} \times U$ given by

$$((\lambda_1, \ldots, \lambda_n), t, (z_1, \ldots, z_n)) \mapsto (t \prod_{i=1}^{n} \lambda_i^{r_i}, (\lambda_1 z_1, \ldots, \lambda_n z_n)).$$

By [V], this gives a line bundle on $\mathbb{P}_{\mathbf{\Sigma}}$. We will denote it by $\mathcal{O}(\sum_i r_i E_i)$.

Remark 3.2. We will implicitly identify line bundles and invertible sheaves of their regular sections throughout the paper.

Proposition 3.3. *All line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ are given by the construction of Definition 3.1. The Picard group of $\mathbb{P}_{\mathbf{\Sigma}}$ is isomorphic to the quotient of $\mathbb{Z}^n$ with basis (E_i) by the subgroup of elements of the form*

$$\sum_{i=1}^{n} (w \cdot v_i) E_i$$

for all $w \in N^*$.

Proof. The line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ correspond to G-equivariant line bundles on U. The open set U is a smooth toric variety, so its Picard group is generated by invariant divisors $z_i = 0$, which are clearly trivial. Consequently, every line bundle on U can be trivialized. To classify line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ one thus needs to classify the G-linearizations of the trivial line bundle $\mathbb{C} \times U \to U$.

For every $g \in G$, we have

$$g\colon (t, \mathbf{z}) \mapsto (t\, r_g(\mathbf{z}), g\mathbf{z}).$$

The function r_g is an invertible regular function on U. Since U is obtained from $\mathbb{C}^n$ by removing subspaces of codimension at least two, the ring of regular functions on U is $\mathbb{C}[z_1, \ldots, z_n]$, and any invertible regular function on U is a nonzero constant. Then the definition of G-linearization shows that the map $G \to \mathbb{C}^*$ given by $g \mapsto r_g$ gives a line bundle if and only if it is a character of G. The characters of G are given by $\mathrm{Gale}(N)$, which has the desired description in terms of E_i. □

The following result has been shown in [BH1].

Theorem 3.4. *The derived category of $\mathbb{P}_{\mathbf{\Sigma}}$ is generated by line bundles.*

Proof. See Corollary 4.8 of [BH1]. □

The focus of this paper is on constructing, in some special cases, collections of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ which satisfy certain cohomological properties.

Definition 3.5. A sequence of line bundles $(\mathcal{L}_1, \ldots, \mathcal{L}_r)$ on $\mathbb{P}_{\mathbf{\Sigma}}$ is called a strong exceptional collection if

$$\mathrm{Ext}^i_{\mathbb{P}_{\mathbf{\Sigma}}}(\mathcal{L}_{j_1}, \mathcal{L}_{j_2}) = 0$$

unless $i = 0$ and $j_1 \leq j_2$.

Remark 3.6. A subset S of $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ can be indexed to form a strong exceptional collection, as long as $\mathrm{Ext}^i_{\mathbb{P}_{\mathbf{\Sigma}}}(\mathcal{L}_1, \mathcal{L}_2) = 0$ for all $i > 0$ and all $\mathcal{L}_1$ and $\mathcal{L}_2$ in S. Indeed, the existence of nonzero $\mathrm{Hom}_{\mathbb{P}_{\mathbf{\Sigma}}}(\mathcal{L}_1, \mathcal{L}_2)$ induces a partial order on the set S, which can then be extended to a linear order.

Definition 3.7. A finite set S of line bundles on $\mathbb{P}_\Sigma$ is called a full strong exceptional collection if

$$\mathrm{Ext}^i_{\mathbb{P}_{\mathbf{\Sigma}}}(\mathcal{L}_1, \mathcal{L}_2) = 0 \text{ for all } i > 0 \text{ and all } \mathcal{L}_1, \mathcal{L}_2 \in S,$$

and the line bundles in S generate the derived category of $\mathbb{P}_{\mathbf{\Sigma}}$.

It is only natural to ask the following question.

Question 3.8. Does $\mathbb{P}_{\mathbf{\Sigma}}$ possess a full strong exceptional collection of line bundles?

Remark 3.9. Kawamata has shown that the derived category of $\mathbb{P}_{\mathbf{\Sigma}}$ possesses a full exceptional collection of objects, see [Ka1]. In his construction, the objects are typically sheaves rather than line bundles, and the collection is only exceptional, rather than strong exceptional (some nontrivial higher Ext spaces are allowed).

Remark 3.10. There is an example of a smooth toric surface which does not admit a full strong exceptional collection of line bundles, see [HP]. A quick review of the related results can be found in [CS]. It has been subsequently suggested, that in the case of varieties a sufficient condition for the positive answer to Question 3.8 is that $\mathbb{P}_\Sigma$ is a Fano variety. We will argue in this paper that it is reasonable to expect that Question 3.8 has a positive answer for all nef-Fano Deligne-Mumford stacks, to be defined below.

Definition 3.11. A toric Deligne-Mumford stack $\mathbb{P}_{\mathbf{\Sigma}}$ is called Fano if the chosen points v_i are precisely the vertices of a simplicial convex polytope in $N_\mathbb{R}$. More generally, it is called nef-Fano if all v_i lie on the boundary of

$$\Delta = \mathrm{ConvexHull}(v_1, \ldots, v_n)$$

but are not necessarily its vertices, nor is Δ assumed to be simplicial.

Remark 3.12. The terminology of Definition 3.11 is justified as follows. A positive power of the anticanonical line bundle on $\mathbb{P}_{\mathbf{\Sigma}}$ is a pullback of a line bundle on the coarse moduli space. The stack $\mathbb{P}_{\mathbf{\Sigma}}$ is Fano (resp. nef-Fano) if the corresponding Cartier divisor is ample (resp. nef and big). Since we do not use this interpretation of the definition, we leave the verification of the above statement to the reader.

Remark 3.13. In dimension two case, we call the Fano stacks del Pezzo, in accordance with the common terminology for varieties.

We are now ready to formulate the stack version of King's conjecture.

Conjecture 3.14. *Every smooth nef-Fano toric DM stack possesses a full strong exceptional collection of line bundles.*

Remark 3.15. From the general theory of exceptional collections, the number of elements in a strong exceptional collection of line bundles equals the rank of K-theory. For a smooth toric nef-Fano DM stack this rank in turn equals $\mathrm{rk}(N)!\mathrm{Vol}(\Delta)$, where the volume is normalized so that the volume of $N_\mathbb{R}/N$ is one, see for example [BH2].

4. Strongly acyclic line bundles

The following rather standard calculation provides a description of cohomology of a line bundle $\mathcal{L}$ on $\mathbb{P}_{\mathbf{\Sigma}}$. For every $\mathbf{r} = (r_i)_{i=1}^n \in \mathbb{Z}^n$ we denote by $\mathrm{Supp}(\mathbf{r})$ the simplicial complex on n vertices $\{1, \ldots, n\}$

which consists of all subsets $J \subseteq \{1, \ldots, n\}$ such that $r_i \geq 0$ for all $i \in J$ and there exists a cone of Σ that contains all $v_i, i \in J$. We will abuse notation to also denote by $\mathrm{Supp}(\mathbf{r})$ the subfan of Σ whose cones are the minimum cones of Σ that contain all $v_i, i \in J$ for all subsets J as above. It should be clear from the context whether $\mathrm{Supp}(\mathbf{r})$ refers to the simplicial complex or to its geometric realization as a subfan of Σ. For example, if all coordinates r_i are negative then the simplicial complex $\mathrm{Supp}(\mathbf{r})$ consists of the empty set only, and its geometric realization is the zero cone of Σ. In the other extreme case, if all r_i are nonnegative then the simplicial complex $\mathrm{Supp}(\mathbf{r})$ encodes the fan Σ, which is its geometric realization.

Proposition 4.1. *The cohomology* $\mathrm{H}^p(\mathbb{P}_{\mathbf{\Sigma}}, \mathcal{L})$ *is isomorphic to the direct sum over all* $\mathbf{r} = (r_i)_{i=1}^n$ *such that* $\mathcal{O}(\sum_{i=1}^n r_i E_i) \cong \mathcal{L}$ *of the* $(\mathrm{rk}(N) - p)$*-th reduced homology of the simplicial complex* $\mathrm{Supp}(\mathbf{r})$*.*

Proof. Consider the left exact functor $\mathrm{H}^0(\mathbb{P}_{\mathbf{\Sigma}}, \bullet)$ on the category of G-equivariant quasi-coherent sheaves on U. It sends a G-equivariant sheaf on U to the space of its G-invariant global sections. Hence, it is the composition of the functor of global sections and the functor of taking G-invariants. Since G is reductive, the latter is exact, consequently,

$$\mathrm{H}^p(\mathbb{P}_{\mathbf{\Sigma}}, \mathcal{L}) = (\mathrm{H}^p(U, \mathcal{L}))^G.$$

Recall that $\mathcal{L} \cong \mathcal{O}_U$ if one ignores the action of G. We can calculate $H^p(U, \mathcal{O})$ by resolving $\mathcal{O}$ via a toric Čech complex. Specifically, U is a toric variety whose toric affine charts U_σ are indexed by $\sigma \in \Sigma$. A point $(z_1, \ldots, z_n)$ lies in U_σ if and only if all v_i for which $z_i = 0$ lie in σ. Consequently, $\Gamma(U_\sigma, \mathcal{O})$ has a monomial basis of $\prod_i z_i^{a_i}$ with $a_i \geq 0$ for all $v_i \in \sigma$ and $a_i \in \mathbb{Z}$ otherwise. The cohomology of $\mathcal{O}$ on U is naturally isomorphic to the cohomology of the toric Čech complex

(4.1)
$$0 \to \bigoplus_{\substack{\sigma\in\Sigma, \\ \dim\sigma=\mathrm{rk}(N)}} \Gamma(U_\sigma, \mathcal{O}) \to \bigoplus_{\substack{\sigma\in\Sigma, \\ \dim\sigma=\mathrm{rk}(N)-1}} \Gamma(U_\sigma, \mathcal{O}) \to \cdots \to \Gamma(U_{\{0\}}, \mathcal{O}) \to 0.$$

The maps in this complex are direct sums of the maps from $\Gamma(U_\sigma, \mathcal{O})$ to $\Gamma(U_{\sigma'}, \mathcal{O})$ which are zero unless σ' is a codimension one face of σ. In this case the map is, up to a sign, the restriction map with the sign determined as follows. If

$$\mathbb{R}_{\geq 0}\sigma = \bigoplus_{j=1}^{\dim\sigma} \mathbb{R}_{\geq 0} v_{i_j}, \ \mathbb{R}_{\geq 0}\sigma' = \bigoplus_{j=1, j\neq k}^{\dim\sigma} \mathbb{R}_{\geq 0} v_{i_j}$$

with $i_1 < \ldots < i_{\dim\sigma}$, then the sign is $(-1)^k$.

This complex is graded by the characters of $(\mathbb{C}^*)^n$, i.e. by multidegree of the monomials. For any given collection $\mathbf{r} = (r_i)_{i=1}^n \in \mathbb{Z}^n$, the graded piece of the complex (4.1) at multidegree $\mathbf{r}$ is precisely the reduced homology complex of $\mathrm{Supp}(\mathbf{r})$. Indeed, the space of sections of $\mathcal{O}$ on U_σ contains a one-dimensional graded piece $\mathbb{C}\prod_i z_i^{r_i}$ if and only if σ contains no v_i for which $r_i < 0$, i.e. the set J of i such that $v_i \in \sigma$ is a subset of the simplicial complex $\mathrm{Supp}(\mathbf{r})$. Moreover, the maps in (4.1) are the same as in the reduced homology complex of $\mathrm{Supp}(\mathbf{r})$.

It remains to show that taking G-invariants amounts to only picking $\mathbf{r}$ with $\mathcal{O}(\sum_{i=1}^n r_i E_i) \cong \mathcal{L}$, which follows from Definition 3.1 and the description of G in Section 2. □

Remark 4.2. For example, $\mathrm{H}^0(\mathcal{L})$ only comes from $\mathbf{r}$ for which $\mathrm{Supp}(\mathbf{r})$ is the entire fan Σ, i.e. $\mathcal{O}(\sum_{i=1}^n r_i E_i) \cong \mathcal{L}$ with $\mathbf{r} \in \mathbb{Z}^n_{\geq 0}$. In the other extreme case $\mathrm{H}^{\mathrm{rk}(N)}(\mathcal{L})$ only appears when the simplicial complex $\mathrm{Supp}(\mathbf{r}) = \{\emptyset\}$, i.e. when $\mathcal{O}(\sum_{i=1}^n r_i E_i) \cong \mathcal{L}$ with all $r_i \leq -1$.

As usual, we will call a line bundle acyclic if all of its higher cohomology groups vanish. Based on Proposition 4.1 we can describe all acyclic line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ as follows. For every subset $I \subseteq \{1, \ldots, n\}$ consider the simplicial complex C_I which encodes the cones of Σ, such that the indices of all rays of the cone lie in I. In other words, this complex is $\mathrm{Supp}(\mathbf{r})$ where $r_i = -1$ for $i \notin I$ and $r_i = 0$ for $i \in I$.

Proposition 4.3. *Consider all proper subsets $I \subset \{1, \ldots, n\}$ such that C_I has nontrivial reduced homology. For each such subset consider the set of line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ of the form*

$$\mathcal{O}(-\sum_{i \notin I} E_i + \sum_{i \in I} r_i E_i - \sum_{i \notin I} r_i E_i)$$

where $r_i \in \mathbb{Z}_{\geq 0}$ for all i. Then a bundle $\mathcal{L}$ is acyclic if and only if it does not lie in any of the above sets.

Proof. This is an immediate corollary of Proposition 4.1. □

It is in principle rather difficult to apply the above criterion. We can provide a more manageable sufficient condition of acyclicity as follows. Consider $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}}) := \mathrm{Pic}_{\mathbb{Z}}(\mathbb{P}_{\mathbf{\Sigma}}) \otimes \mathbb{R}$. We can think of it as a quotient of $\mathbb{R}^n$ with basis elements given by E_i.

Definition 4.4. For each proper subset $I \subset \{1, \ldots, n\}$ such that C_I has nontrivial reduced homology define the *forbidden point*

$$q_I = -\sum_{i \notin I} E_i \in \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$$

Define the *forbidden cone* $F_I \subseteq \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ by

$$F_I = q_I + \sum_{i\in I} \mathbb{R}_{\geq 0} E_i - \sum_{i\notin I} \mathbb{R}_{\geq 0} E_i.$$

A line bundle $\mathcal{L}$ is called *strongly acyclic* if its image in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ does not lie in any of the forbidden cones.

Proposition 4.5. *Every strongly acyclic line bundle is acyclic.*

Proof. This statement follows immediately from Proposition 4.3. □

Remark 4.6. The concept of strong acyclicity has several advantages over the usual acyclicity. For example, it can be checked for by looking at a finite set of inequalities. It would be interesting to figure out the geometric meaning of strong acyclicity and to see if this notion can be defined beyond the toric case.

Remark 4.7. An example of a line bundle which is acyclic but not strongly acyclic is $\mathcal{O}(-6)$ on the weighted projective line with weights 2 and 3. Here the Picard group is isomorphic to $\mathbb{Z}$ with images of $\mathcal{O}(E_1)$ and $\mathcal{O}(E_2)$ given by $\mathcal{O}(2)$ and $\mathcal{O}(3)$ respectively. It is impossible to write $\mathcal{O}(-6) = \mathcal{O}(r_1E_1 + r_2E_2)$ with negative integer r_i, which means that $\mathcal{O}(-6)$ is acyclic. On the other hand the forbidden cone $F_\emptyset$ contains the images of all $\mathcal{O}(k)$ with $k \leq -5$, so $\mathcal{O}(-6)$ is not strongly acyclic.

5. The case of $\mathrm{rk}(\mathrm{Pic}) \leq 2$

In this section we will argue that Conjecture 3.14 is true for toric Fano Deligne-Mumford stacks $\mathbb{P}_{\mathbf{\Sigma}}$ with $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) \leq 2$.

We first consider the case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 1$. In this case Δ is a simplex in the lattice N of rank $(n-1)$. The only forbidden cone occurs for $I = \emptyset$, with the corresponding forbidden point $-\sum_{i=1}^n E_i$. Denote by

$$\deg\colon \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}}) \to \mathbb{Z}$$

the linear function that takes value 1 on the positive generator of $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$. Then the forbidden cone is given by $x \in \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ such that

$$\deg(x) \leq -\sum_{i=1}^n \deg(E_i) = \deg(K)$$

where K is the canonical divisor.

Proposition 5.1. *Consider the set S of line bundles $\mathcal{L}$ with* $\deg \mathcal{L} \in [\deg(K)+1, 0]$*. Then the set S forms a full strong exceptional collection on $\mathbb{P}_{\mathbf{\Sigma}}$.*

Proof. It is clear that for any two $\mathcal{L}_1$ and $\mathcal{L}_2$ in S, the line bundle $\mathcal{L}_2 \otimes \mathcal{L}_1^{-1}$ has degree bigger than $\deg(K)$ and is therefore acyclic by Proposition 4.5.

Consider the subcategory D of the derived category of $\mathbb{P}_{\mathbf{\Sigma}}$ which is generated by $\mathcal{L}$ in S. In view of Theorem 3.4, it suffices to show that all line bundles on $\mathbb{P}_{\mathbf{\Sigma}}$ lie in D.

Let us first prove this for all line bundles of nonnegative degree by induction on $\deg(\mathcal{L})$. The base of induction $\deg(\mathcal{L}) = 0$ is clear. Suppose now that we have shown this for all $\mathcal{L}$ of $0 \leq \deg(\mathcal{L}) \leq k$. Then if $\mathcal{L} = \mathcal{O}(E)$ has degree $(k+1)$, consider the Koszul complex

$$0 \to \mathcal{O}(E - \sum_{i=1}^{n} E_i) \to \ldots \to \oplus_{i=1}^{n} \mathcal{O}(E - E_i) \to \mathcal{L} \to 0.$$

This comes from a Koszul complex on $\mathbb{C}^n$ which resolves the point $(0, \ldots, 0) \notin U$. As a result, it leads to an exact complex on $\mathbb{P}_\Sigma$, see [BH1]. All but the last term of the complex are in D, hence so is $\mathcal{L}$, which proves the induction step.

A similar, decreasing, induction on the degree allows us to handle the case of $\deg(\mathcal{L}) \leq \deg(K)$, which finishes the proof. ☐

Remark 5.2. The number of elements of S equals $(-\deg(K))d$ where d is the order of the torsion subgroup of $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$. This coincides with the rank of the Grothendieck group of $\mathbb{P}_{\mathbf{\Sigma}}$, which is not a coincidence, but rather is expected by Remark 3.15.

Remark 5.3. The case of Picard number one has already been settled in [Ka1], but we have treated it here nonetheless, to give a unified picture of our approach.

We will now move to the more challenging case of $\mathrm{rk}(\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})) = 2$. We have n elements v_i of the lattice N of rank $n-2$, which form the set of vertices of a simplicial polytope Δ.

Proposition 5.4. *There exists a unique up to scaling collection of rational numbers α_i, such that $\sum_{i=1}^n \alpha_i = 0$ and $\sum_{i=1}^n \alpha_i v_i = 0$. Moreover, all of the α_i in this relation are nonzero.*

Proof. Since Σ is a complete fan, the vertices v_i generate $N \otimes \mathbb{Q}$, so the space of linear relations on v_i is of dimension two. Since 0 is in the convex hull of v_i, it can be written as a sum of v_i with nonnegative coefficients. Hence, there is a relation on v_i with $\sum_{i=1}^n \alpha_i > 0$. Consequently, the condition $\sum_{i=1}^n \alpha_i = 0$ cuts out a dimension one subspace of relations.

Suppose some α_i is zero. It means that $v_j, j \neq i$ lie in a proper affine subspace of $N \otimes \mathbb{Q}$. It then gives a supporting hyperplane of Δ which has $(n-1)$ points in it, in contradiction with simpliciality of Δ. $\square$

We will pick one such relation $\sum_{i=1}^n \alpha_i v_i = 0$. We will denote by I_+ (resp. I_-) the sets of i with positive α_i (resp. negative α_i).

Proposition 5.5. *The facets of Δ are precisely convex hulls of $(n-2)$ of the v_i-s, such that one of the remaining two indices lies in I_+ and the other lies in I_-.*

Proof. Consider a subset $I \subset \{1, \ldots, n\}$ of cardinality $(n-2)$. The convex hull of $v_i, i \in I$ does *not* form a face of Δ if and only if the segment through remaining two vertices intersects the affine span of this set. This is equivalent to the existence of a relation

$$\sum_{i \in I} \beta_i v_i = \sum_{j \notin I} \beta_j v_j$$

with $\sum_{i \in I} \beta_i = 1 = \sum_{j \notin I} \beta_j$ and with the two β_j both positive. By comparing with the result of Proposition 5.4, this implies that the complement of I is a subset of I_+ or of I_-. Conversely, for any two indices j_1, j_2 in I_+ or I_-, one can move $\alpha_{j_1} v_{j_1} + \alpha_{j_2} v_{j_2}$ to the right hand side in the equation $\sum_{i=1}^n \alpha_i v_i = 0$ and then divide by $-(\alpha_{j_1} + \alpha_{j_2})$ to get

$$\sum_{i \neq j_1, j_2} \beta_i v_i = \beta_{j_1} v_{j_1} + \beta_{j_2} v_{j_2}$$

with $\sum_{i \neq j_1, j_2} \beta_i = 1 = \beta_{j_1} + \beta_{j_2}$ and positive β_{j_1} and β_{j_2}. $\square$

Corollary 5.6. *A subset I of $\{1, \ldots, n\}$ corresponds to a face of Δ if and only if the complement of I is not contained in I_+ or I_-. In addition, the sets of I_+ and I_- have at least two elements each.*

Proof. The first statement follows immediately from Proposition 5.5. If I_+ or I_- has only one element, then the corresponding v_i does not lie in any face of Δ. $\square$

The following proposition classifies the forbidden cones in this case.

Proposition 5.7. *There are precisely three forbidden cones, which correspond to the subsets $\emptyset$, I_+ and I_- of $\{1, \ldots, n\}$.*

Proof. Suppose that both I and its complement $\bar{I}$ intersect I_+ nontrivially. Pick $i \in I \cap I_+$. By Corollary 5.6 the simplicial complex C_I is a cone over i (i.e. i can be added to any of its subsets) and is thus acyclic. Similarly, if I and $\bar{I}$ intersect I_- nontrivially, then C_I is acyclic.

It remains to observe that for I that are equal to $I_\pm$ the corresponding simplicial complex C_I has a geometric realization of the sphere and consequently has nontrivial reduced homology. $\square$

For what follows we pick and fix a collection of positive numbers $r_i, i = 1, \ldots, n$, such that $\sum_i r_i = 1$ and $\sum_i r_i v_i = 0$. This collection gives a linear function f on $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ with $f(E_i) = r_i$. Similarly, we define a linear function α with $\alpha(E_i) = \alpha_i$ from Proposition 5.4. Consider the parallelogram P in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ which is given by the inequalities

$$|f(x)| \leq \frac{1}{2}, \qquad |\alpha(x)| \leq \frac{1}{2} \sum_{i \in I_+} \alpha_i.$$

Proposition 5.8. *The interior of the parallelogram $2P$ contains no points from the forbidden cones. The only points on the boundary of $2P$ that lie in the forbidden cones are $-\sum_{i=1}^n E_i$, $-\sum_{i\in I_-} E_i$ and $-\sum_{i \in I_+} E_i$, see Figure 1.*

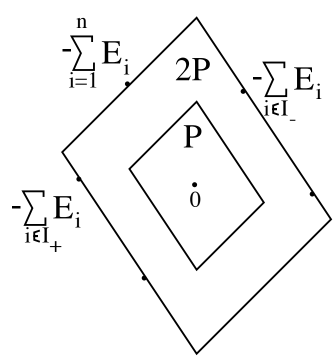

Figure 1.

Proof. There are three forbidden cones, described in Proposition 5.7. We will show that for each of these cones the corresponding forbidden point lies on the side of $2P$ with the side giving a supporting hyperplane of the cone. For x in the cone

$$-\sum_{i\in I_-} E_i + \sum_{i \in I_+} \mathbb{R}_{\geq 0} E_i - \sum_{j \in I_-} \mathbb{R}_{\geq 0} E_j,$$

we have

$$\alpha(x) = -\sum_{i \in I_-} \alpha_i + \sum_{i\in I_+} t_i \alpha_i - \sum_{j \in I_-} t_j \alpha_j \geq -\sum_{i\in I_-} \alpha_i = \sum_{i \in I_+} \alpha_i,$$

with the equality if and only if all t_i and t_j are zero. The other two cones are handled similarly. $\square$

Proposition 5.9. *Consider the four points*

$$\pm\frac{1}{2}\sum_{i\in I_+} E_i, \pm\frac{1}{2}\sum_{i\in I_-} E_i.$$

They lie on two opposite sides of P. Moreover, each of the opposite sides of P can be subdivided into a pair of segments with these points as centers, as in Figure 2.

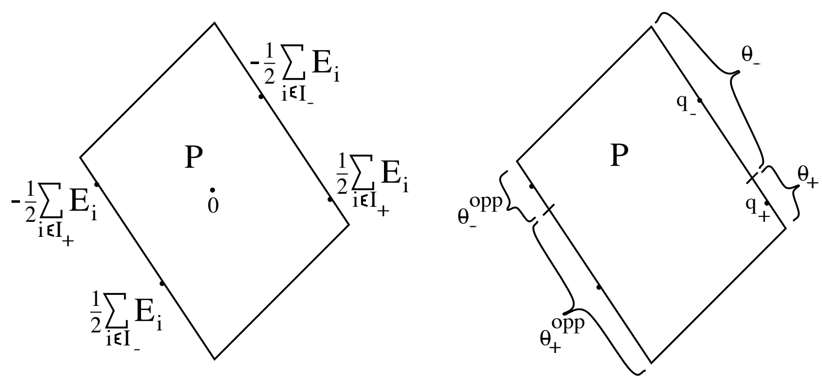

FIGURE 2.

Proof. In view of central symmetry of P it suffices to show that $q_+ = \frac{1}{2}\sum_{i\in I_+} E_i$ and $q_- = -\frac{1}{2}\sum_{i\in I_-} E_i$ lie on its sides. It is clear that

$$\alpha(q_\pm) = \frac{1}{2}\sum_{i\in I_+} \alpha_i,$$

so it remains to check $f(q_\pm)$. We have

$$-\frac{1}{2} = -\frac{1}{2}\sum_{i=1}^{n} r_i < -\frac{1}{2}\sum_{i\in I_-} r_i = f(q_-) < 0 < \frac{1}{2}\sum_{i\in I_+} r_i = f(q_+) < \frac{1}{2}.$$

To show the last statement, observe that $f(q_+) - f(-q_-) = -\frac{1}{2}$, so the distance between the two points on the side of P is half the length of the side of P. $\square$

We will denote the four segments on the sides of P by $\theta_\pm$ and $\theta_\pm^{\mathrm{opp}}$, see Figure 2. The following proposition is crucial.

Proposition 5.10. *Let q be a point in the interior of the segment $\theta_\pm$. Then $q \mp \sum_{i\in I_\pm} E_i$ lies in the interior of the segment $\theta_\mp^{\rm opp}$, and for any proper nonempty subset $J \subset I_\pm$ the point $q \mp \sum_{i\in J} E_i$ lies in the interior of P.*

Proof. Since $2q_\pm = \pm\sum_{i\in I_\pm} E_i$, and $\theta_\pm$ has the same length as $\theta_\mp^{\rm opp}$, the translate of the interior of $\theta_\pm$ by $\mp\sum_{i\in I_\pm} E_i$ is the interior of $\theta_\mp^{\rm opp}$. For each $J \subset I_\pm$ the values of $f(q \mp \sum_{i\in J} E_i)$ and $\alpha(q \mp \sum_{i\in J} E_i)$ are in between those for $J = \emptyset$ and $J = I_\pm$, in view of the signs of r_i and α_i. This shows that $q \mp \sum_{i\in J} E_i$ is in the interior of P. $\square$

We are now ready to construct a strong exceptional collection S in $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$. Pick a generic point $p \in \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ so that the lines along the sides of the parallelogram $p + P$ do not contain any points from $\mathrm{Pic}_{\mathbb{Q}}(\mathbb{P}_{\mathbf{\Sigma}})$.

Theorem 5.11. *The set S of line bundles $\mathcal{L}$ such that their image in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ lies in $P+p$ forms a full strong exceptional collection on $\mathbb{P}_{\mathbf{\Sigma}}$.*

Proof. First of all, we will show that this set forms a strong exceptional collection. For this it suffices to show that the difference of any two points in the interior of $p+P$ lies outside of the forbidden cones. Since $p + P - (p + P) = 2P$, this statement follows from Proposition 5.8.

In view of Theorem 3.4, we now need to show that the category D generated by the line bundles from S contains all line bundles. At the first step of the construction we will move the polytope $p+P$ by moving the point p in the line with constant $f(p)$. We claim that the newly appearing line bundles lie in D. Let us first show it for the direction indicated by the arrow in Figure 3.

Every time that the image in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ of a line bundle $\mathcal{L} = \mathcal{O}(E)$ fits into $p + P$, this image will be in the interior of $p + \theta_\pm$, since we can assure that the intersection point of $p + \theta_+$ and $p + \theta_-$ is moving along a non-rational line by a generic choice of (r_i) and p. Suppose that the image of $\mathcal{L}$ lies in θ_+. Proposition 5.10 then implies that for any nonempty $J \subset I_+$ the line bundle $\mathcal{O}(E - \sum_{i\in J} E_i)$ lies in D. Consider the Koszul complex on $\mathbb{C}^n$ given $z_i, i \in I_+$. It resolves the structure sheaf of a coordinate subspace which is outside of U. This yields a long exact sequence of sheaves on $\mathbb{P}_{\mathbf{\Sigma}}$ (see also [BH1]), which after twisting by $\mathcal{L}$ becomes

$$0 \to \mathcal{O}(E - \sum_{i\in I_+} E_i) \to \ldots \to \oplus_{i\in I_+}\mathcal{O}(E - E_i) \to \mathcal{L} \to 0.$$

All but the last terms of this sequence lie in D, hence $\mathcal{L}$ lies in D.

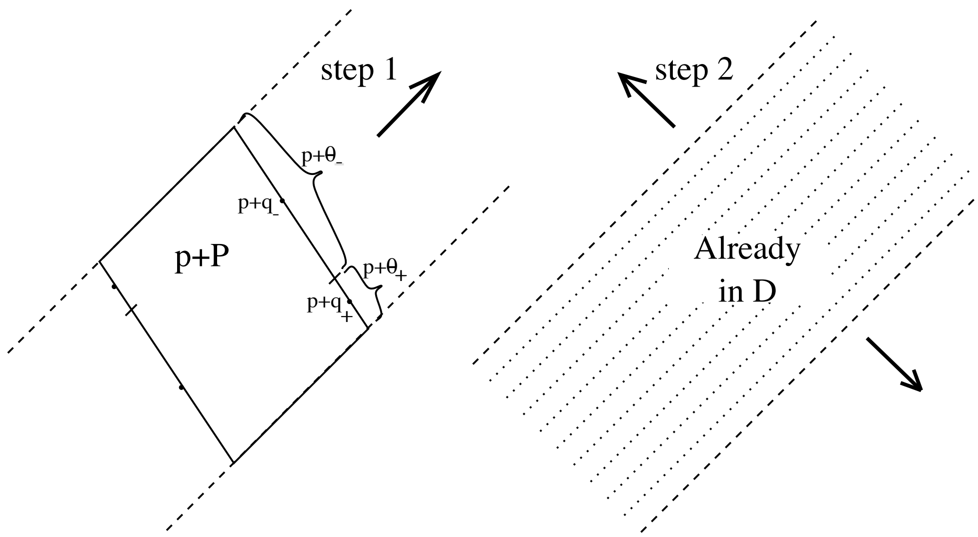

FIGURE 3.

The calculation for the case when the image of $\mathcal{L}$ is in $p + \theta_-$ is completely analogous, as are the calculations for when the point p is moving in the opposite direction. As such, they are left to the reader.

So now we have shown that all $\mathcal{L}$ with the property that their image q in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ satisfies $|f(q-p)| \leq \frac{1}{2}$ lie in D. The second step of the construction is to move this whole slab in both directions by making similar use of the Koszul relation for I_+ (or I_-, at this stage either one of the two suffices), see Figure 3. We make use of the inequalities $-1 < f(-\sum_{i\in J} E_i) < 0$ for any $\emptyset \neq J \subseteq I_+$. This finishes the proof. $\square$

Remark 5.12. Similar to Remark 5.2, it can be shown that, with the area form that makes the volume of $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})/\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ equal the order of the torsion subgroup of $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$, the area of the parallelogram P is $(n-2)!\mathrm{Vol}(\Delta)$. It can also be shown that the number of elements of S equals $(n-2)!\mathrm{Vol}(\Delta)$. This is, again, expected, since the number of elements of S needs to coincide with the rank of the Grothendieck group of $\mathbb{P}_{\mathbf{\Sigma}}$.

Remark 5.13. The case of toric varieties of Picard number at most two has been settled in [CM] by a different method. Notice that [CM] does not assume that the variety is Fano. We thank Rosa Miró-Roig for bringing this article to our attention.

6. The case of del Pezzo toric stacks, the preliminaries

In this section we will describe a construction in convex geometry which will eventually allow us to prove Conjecture 3.14 in the case of del Pezzo toric Deligne-Mumford stacks. The reader may refer to Section 8 for the example of this construction for the case $n = 5$.

Let $\Delta = A_1A_2\ldots A_n$ be a convex n-gon in $N = \mathbb{Z}^2$, with the vertices counted clockwise, which contains 0 in its interior. Let $\mathbf{\Sigma}$ be the corresponding stacky fan and $\mathbb{P}_{\mathbf{\Sigma}}$ the corresponding del Pezzo DM stack. As before, we denote by v_i the vector from 0 to A_i and by E_i the corresponding elements of the Picard group of $\mathbb{P}_{\mathbf{\Sigma}}$.

We will first introduce additional notation. Recall that by Proposition 3.3 the Picard group $\mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})$ is the quotient of $\mathbb{Z}^n$ with basis E_i by the linear relations

$$\sum_{i=1}^n (w\cdot v_i)E_i$$

for $w \in N^*$.

Definition 6.1. We mod out by the span of the canonical divisor to define the group $\widehat{\mathrm{Pic}}(\mathbb{P}_{\mathbf{\Sigma}})$ by

$$\widehat{\mathrm{Pic}}(\mathbb{P}_{\mathbf{\Sigma}}) = \mathrm{Pic}(\mathbb{P}_{\mathbf{\Sigma}})/\mathbb{Z}(\sum_{i=1}^n E_i).$$

We denote by $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ its tensor product with $\mathbb{R}$. We denote by $\widehat{E}_i$ the images of E_i in $\widehat{\mathrm{Pic}}(\mathbb{P}_{\mathbf{\Sigma}})$ and $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$.

Definition 6.2. We denote by Q the convex polytope in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ which is the convex hull of the points $\widehat{E}_I = \sum_{i\in I}\widehat{E}_i$ for all subsets $I \subseteq \{1,\ldots,n\}$.

Remark 6.3. Polytope Q is the Minkowski sum of line segments $[0,\widehat{E}_i]$ and is thus a zonotope. The condition $\sum_{i=1}^n \widehat{E}_i = 0$ ensures that the center of symmetry of Q is 0.

The following proposition describes the vertices of Q.

Proposition 6.4. *The point $\widehat{E}_I$ is a vertex of Q if and only if I is a nonempty proper subset and $q_I = -\sum_{i\notin I} E_i$ is a forbidden point. Equivalently, $\widehat{E}_I$ is a vertex of Q if and only if the simplicial complex C_I contains more than one connected component.*

Proof. It is clear that the set of vertices of Q is a subset of the set of $\widehat{E}_I$. For $\widehat{E}_I$ to be a vertex of Q there has to exist a linear function on

$\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ which is maximized on it among other vertices of Q. In other words, this linear function should take positive values on $\widehat{E}_i$ for $i \in I$ and negative values on $\widehat{E}_i$ for $i \notin I$. Linear functions f on $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ are collections of n real numbers $r_i = f(\widehat{E}_i)$ that satisfy $\sum_{i=1}^n r_i = 0$ and $\sum_{i=1}^n (w \cdot v_i) r_i = 0$ for all $w \in N^*$. In other words, $\sum_i r_i = 0$ and

$$\sum_{i=1}^{n} r_i v_i = 0. \tag{6.1}$$

Since $\sum_i r_i = 0$, it means that I and its complement are nonempty. We can then write (6.1) as

$$\frac{1}{\sum_{i\in I} r_i} \sum_{i\in I} r_i v_i = \frac{1}{\sum_{i\notin I}(-r_i)} \sum_{i\notin I} (-r_i) v_i. \tag{6.2}$$

So the existence of the linear function with the required property is equivalent to the condition that I is proper and nonempty and the relative interiors of the convex hulls $\mathrm{conv}(\{v_i, i \in I\})$ and $\mathrm{conv}(\{v_i, i \notin I\})$ intersect. It is straightforward to see that in a convex polygon Δ the latter condition is equivalent to C_I having more than one connected component. $\square$

Remark 6.5. Already in dimension three, the condition that relative interiors of $\mathrm{conv}(\{v_i, i \in I\})$ and $\mathrm{conv}(\{v_i, i \notin I\})$ intersect is only necessary, but not sufficient to assure that C_I is not acyclic. Consequently, if dimension of Δ is bigger than two, then some vertices of Q may not be images of forbidden points.

Proposition 6.6. *For a vertex $\widehat{E}_I$ of Q the image of the corresponding forbidden cone F_I under the projection $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}}) \to \widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ is the opposite of the angle cone. In other words, the image of F_I is*

$$\widehat{E}_I - \mathbb{R}_{\geq 0}(Q - \widehat{E}_I).$$

Proof. This follows immediately from the definition of Q and the description of F_I in Definition 4.4. $\square$

The argument of Proposition 6.4 can be generalized to describe all faces of Q, and in particular its facets.

Proposition 6.7. *Faces of Q correspond to ordered pairs of disjoint subsets I and J of $\{1, \ldots, n\}$, such that the relative interiors of the convex hulls $\mathrm{conv}(\{v_i, i \in I\})$ and $\mathrm{conv}(\{v_i, i \in J\})$ intersect. In particular, facets of Q are in one-to-one correspondence with ordered pairs* *of intersecting diagonals in* Δ. *Specifically, for a pair* (I, J), *the corresponding face is given by*

$$\theta_{I,J} = \widehat{E}_I + \sum_{i \notin I \cup J} [0, \widehat{E}_i].$$

Proof. Faces of Q are maximum sets on Q of the linear functions on $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$. If a linear function f is given by $r_i = f(\widehat{E}_i)$, then consider the set I of indices i for which $r_i > 0$ and the set J of indices j for which $r_j < 0$. As before, we see that the relative interiors of the convex hulls $\mathrm{conv}(\{v_i, i \in I\})$ and $\mathrm{conv}(\{v_i, i \in J\})$ intersect. Vice versa, any such intersection allows us to define r_i that give a linear function on $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$. The maximum set of a linear function on the Minkowski sum of polytopes is the Minkowski sum of its maximum sets on the individual polytopes. Hence, we get the formula for $\theta_{I,J}$. Finally, the pairs (I, J) with the above property are partially ordered by inclusion. This partial order is the reverse of the inclusion order of the faces. So the facets correspond to the minimum pairs (I, J) with $\mathrm{conv}(\{v_i, i \in I\}) \cap \mathrm{conv}(\{v_i, i \in J\}) \neq \emptyset$, and these are precisely the pairs of intersecting diagonals. $\square$

Remark 6.8. The condition $\mathrm{conv}(\{v_i, i \in I\}) \cap \mathrm{conv}(\{v_i, i \in J\}) \neq \emptyset$, on I and J above is purely combinatorial, in the sense that it does not depend on the geometry of Δ, provided that Δ is convex. Specifically, it is equivalent to the existence of indices $i_1, i_2 \in I$ and $j_1, j_2 \in J$, such that i_1, j_1, i_2, j_2 are counted clockwise, if one sets $\{1, \ldots, n\}$ in a clockwise circle.

Our next goal is to construct a polytope $\widehat{P}$ in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ with centrally symmetric faces which has the peculiar property that the midpoints of all facets of $\widehat{P}$ are vertices of Q and all vertices of Q are midpoints of some faces of $\widehat{P}$. This is the key ingredient of the argument of this paper. This polytope $\widehat{P}$ will also be a zonotope and it will have a combinatorial structure that is identical to that of Q.

Consider the stacky fan $\mathbf{\Sigma}_1$ in N given by the rays

$$(t_1, t_2, \ldots, t_n) = (v_1 - v_n, v_2 - v_1, \ldots, v_n - v_{n-1}).$$

It will be convenient for us to consider our subscripts to be elements of $\mathbb{Z}/n\mathbb{Z}$, so that we can write the above equation simply as $t_i = v_i - v_{i-1}$. Note that the convexity of Δ assures that t_i are counted clockwise, although we can no longer assume that they are vertices of a convex polytope. Consider a collection of positive numbers ϕ_i such that $\frac{1}{\phi_i} t_i$

are vertices of a convex polytope. By scaling ϕ_i we may arrange so that

$$\sum_{i=1}^{n} \phi_i = 1. \tag{6.3}$$

Remark 6.9. There is a fairly natural choice of ϕ given by $\phi_i = v_i \wedge v_{i-1}$ for some area form $N_{\mathbb{R}} \wedge N_{\mathbb{R}} \to \mathbb{R}$. This corresponds to considering the dual polytope Δ^* and placing it in N via the identification of N and N^* via the above form. It can consequently be scaled to get $\sum_{i=1}^{n} \phi_i = 1$. On the other hand, the arguments of the paper go through for any convex ϕ, even if some ϕ_i are negative.

Definition 6.10. We define the zonotope $\widehat{P}$ in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ which is the Minkowski sum of segments $[\widehat{t}_i, -\widehat{t}_i]$ where $\widehat{t}_i$ are given by

$$\widehat{t}_{i+1} - \widehat{t}_i = \widehat{E}_i, \text{ for all } i \in \mathbb{Z}/n\mathbb{Z}$$

and

$$\sum_{i=1}^{n} \phi_i \widehat{t}_i = 0.$$

Remark 6.11. It is easy to see that Definition 6.10 determines $\widehat{t}_i$ and hence $\widehat{P}$ uniquely. Indeed, the first set of equations can be solved because $\sum_{i=1}^{n} \widehat{E}_i = 0$. It determines $\widehat{t}_i$ uniquely up to an addition of an element of $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ and then the last relation removes the remaining ambiguity uniquely in view of (6.3). Specifically, we get

$$\widehat{t}_i = \frac{1}{n}\Big(\sum_{j=0}^{n-1} j\widehat{E}_{i+j} - \sum_{k=1}^{n}\sum_{j=0}^{n-1} j\phi_k \widehat{E}_{k+j}\Big)$$

but we will not need this form of the solution.

We can explicitly describe the face structure of $\widehat{P}$.

Proposition 6.12. *Faces of $\widehat{P}$ correspond to ordered pairs of disjoint subsets I and J of $\mathbb{Z}/n\mathbb{Z}$, such that the relative interiors of the convex hulls $\mathrm{conv}(\{v_i, i \in I\})$ and $\mathrm{conv}(\{v_i, i \in J\})$ intersect. In particular, facets of $\widehat{P}$ are in one-to-one correspondence with ordered pairs of intersecting diagonals in Δ. Specifically, for a pair (I, J), the corresponding face is given by*

$$\theta_{I,J} = \sum_{i\in I} \widehat{t}_i - \sum_{i \in J} \widehat{t}_i + \sum_{i \notin I\cup J} [-\widehat{t}_i, \widehat{t}_i].$$

Proof. Consider a supporting function f with $r_i = f(\widehat{t_i})$. In view of the definition of $\widehat{t_i}$, these r_i satisfy a three-dimensional space of linear relations

$$\sum_{i=1}^{n} \phi_i r_i = 0$$

and

$$\sum_{i=1}^{n} (w \cdot v_i)(r_{i+1} - r_i) = 0$$

for all $w \in N^*$. The latter relation can be rewritten as

$$\sum_{i=1}^{n} r_i t_i = 0.$$

This can be then thought of as a linear relation

$$\sum_{i=1}^{n} (\phi_i r_i)(\frac{1}{\phi_i} t_i) = 0 \tag{6.4}$$

on points $\frac{1}{\phi_i} t_i$ in $N_{\mathbb{R}}$ with $\sum_i \phi_i r_i = 0$. If I is the set of i with $r_i > 0$ (and hence $\phi_i r_i > 0$) and J is the set of i with $r_i < 0$, then, similarly to (6.2), we can see (6.4) as a statement that the relative interiors of the convex hulls of $\frac{1}{\phi_i} t_i, i \in I$ and $\frac{1}{\phi_i} t_i, i \in J$ intersect. In view of Remark 6.8 we can replace $\frac{1}{\phi_i} t_i$ by v_i.

The calculation of the maximum set of f on $\widehat{P}$ is then straightforward. The statement about facets is also clear. □

Remark 6.13. A reader familiar with Gale duality will notice that the proofs of Propositions 6.7 and 6.12 can be stated naturally in its terms, since the facet structure of a zonotope is encoded by the linear combinations in the Gale dual picture. However, we preferred to give a direct argument to avoid introducing additional terminology.

The main properties of $\widehat{P}$ are summarized in the following proposition.

Proposition 6.14. *The polytope $\widehat{P}$ is centrally symmetric. All vertices of Q are midpoints of some faces of $\widehat{P}$. A vertex $\widehat{E}_I$ of Q is a midpoint of a facet of $\widehat{P}$ if and only if C_I has exactly two connected components. The midpoint of every facet of $\widehat{P}$ is a vertex of Q.*

Proof. Denote by $[i, j)$ the set of indices in $\mathbb{Z}/n\mathbb{Z}$ starting from i (included) and ending with j (excluded), counted clockwise. Then

$$\widehat{E}_{[i,j)} = \widehat{t_j} - \widehat{t_i}. \tag{6.5}$$

Every proper subset I of $\mathbb{Z}/n\mathbb{Z}$ such that simplicial complex C_I has nontrivial reduced homology can be uniquely written as a disjoint union of l intervals $[i_k, j_k)$, $k = 1, \dots, l$ with $l \geq 2$. Equation (6.5) then gives

$$\widehat{E}_I = \sum_{k=1}^{l} \widehat{t}_{j_k} - \sum_{k=1}^{l} \widehat{t}_{i_k},$$

which is the midpoint of a face of $\widehat{P}$ by Proposition 6.12. In particular, this face is a facet of $\widehat{P}$ if and only if $l = 2$, which is equivalent to C_I having two connected components. Finally, all facets of $\widehat{P}$ are obtained by this procedure. $\square$

Corollary 6.15. *The interior of $\widehat{P}$ lies outside of the images of the forbidden cones F_I for all proper subsets I of $\{1, \dots, n\}$ with non-acyclic C_I.*

Proof. For each forbidden cone F_I consider the corresponding point $\widehat{E}_I$ on the boundary of $\widehat{P}$. By Proposition 6.14 the polytope Q is contained in $\widehat{P}$, so any supporting hyperplane of $\widehat{E}_I$ for $\widehat{P}$ is also a supporting hyperplane for it for Q. It remains to observe that the image of the forbidden cone for Q lies on the side of this hyperplane away from Q and hence away from the interior of $\widehat{P}$ by Proposition 6.4. $\square$

Proposition 6.16. *Let $I = [i_1, j_1) \sqcup [i_2, j_2)$ with i_1, j_1, i_2, j_2 indexed clockwise and let*

$$\theta_I = \widehat{t}_{j_1} + \widehat{t}_{j_2} - \widehat{t}_{i_1} - \widehat{t}_{i_2} + \sum_{k \neq i_1, i_2, j_1, j_2} [-\widehat{t}_k, \widehat{t}_k]$$

be the facet of $\widehat{P}$ that contains $\widehat{E}_I$ as a midpoint. Then the shifts of the relative interiors of θ_I by $-2\widehat{E}_{[i_1,j_1)}$, $-2\widehat{E}_{[i_2,j_2)}$, $2\widehat{E}_{[j_1,i_2)}$ and $2\widehat{E}_{[j_2,i_1)}$ are contained in the interior of $\widehat{P}$. In addition, the shift of the relative interior of θ_I by $-2\widehat{E}_I = 2\widehat{E}_{\bar{I}}$ is the opposite face $-\theta_I = \theta_{\bar{I}}$ of $\widehat{P}$.

Proof. We will prove the last of the four statements. The proof of the other three is completely analogous and is left to the reader. In view of the equation (6.5), the shift of the relative interior of θ_I by $-2\widehat{E}_{[j_2,i_1)}$ is given by

$$\widehat{t}_{i_1} - \widehat{t}_{j_2} + \widehat{t}_{j_1} - \widehat{t}_{i_2} + \sum_{k \neq i_1, i_2, j_1, j_2} (-\widehat{t}_k, \widehat{t}_k).$$

Every point p of this set clearly lies in $\widehat{P}$, so it remains to show that it does not lie on the boundary of $\widehat{P}$. Assume the contrary. For any index $k \neq i_1, i_2, j_1, j_2$ the point p can be moved by $\epsilon\widehat{t}_k$ for small $|\epsilon|$, so that the result is still in $\widehat{P}$. Consequently, any supporting hyperplane at p should be given by a linear function f with $f(\widehat{t_k}) = 0$ for $k \neq i_1, i_2, j_1, j_2$. This leads to a statement that the interiors of the segments $[\frac{1}{\phi_{i_1}} t_{i_1}, \frac{1}{\phi_{j_1}} t_{j_1}]$ and $[\frac{1}{\phi_{i_2}} t_{i_2}, \frac{1}{\phi_{j_2}} t_{j_2}]$ intersect, which is false.

The statement about the shift by $-2\widehat{E}_I$ is an easy calculation which we leave to the reader. $\square$

7. The case of del Pezzo toric stacks, the full strong exceptional collection

In this section we will prove Conjecture 3.14 for toric del Pezzo Deligne-Mumford stacks.

We will be using the notations of the preceding section. First, we will define a polytope P in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ as follows. Fix a collection of positive numbers r_i such that $\sum_{i=1}^n r_i = 1$ and $\sum_{i=1}^n r_i v_i = 0$. This collection defines a linear function f on $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ with $f(E_i) = r_i$.

Definition 7.1. We define a convex polytope P in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ which consists of points of x with $|f(x)| \leq \frac{1}{2}$ such that the image of x in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ lies in $\frac{1}{2}\widehat{P}$.

We pick a generic $p \in \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$. As in Theorem 5.11 we consider the set S of line bundles $\mathcal{L}$ such that their image in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ lies in $P + p$.

Proposition 7.2. *The set S forms a strong exceptional collection.*

Proof. We simply need to check that the differences of any two line bundles in $\mathcal{L}$ lie outside of all forbidden cones. Since p is generic, $p+P$ has no lattice points on the boundary, consequently, the differences of line bundles in S map to the interior of $p + P + (-p - P)$. Because P is centrally symmetric, these differences are then in the interior of $2P$. Points $x = \sum_{i=1}^n x_i E_i$ in the interior of $2P$ satisfy $\sum_{i=1} r_i x_i > -1 = f(-\sum_{i=1}^n E_i)$, which shows that they lie outside the forbidden cone for $I = \emptyset$. To show that the other forbidden cones F_I do not intersect the interior of $2P$ consider their projections to $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$. By Corollary 6.15, they do not intersect the interior of $\widehat{P}$, which is precisely the projection of the interior of $2P$. $\square$

We are now ready to prove the main result of this paper, which is to show that S is a full strong exceptional collection.

Theorem 7.3. *For a generic $p \in \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ the set S of line bundles $\mathcal{L}$ that map inside $P + p$ forms a full strong exceptional collection on $\mathbb{P}_{\mathbf{\Sigma}}$.*

Proof. Denote by D the subcategory of the derived category of the coherent sheaves on $\mathbb{P}_{\mathbf{\Sigma}}$ generated by the elements of S.

We will first show that D contains all line bundles $\mathcal{L}$ whose image x in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ satisfies $|f(x-p)| \leq \frac{1}{2}$. Fix one such $\mathcal{L}$ and the corresponding x. Since p is chosen generic, we may safely assume that no lattice points y satisfy $|f(y-p)| = \frac{1}{2}$. Consider the set of elements p_1 of $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ such that $f(p_1) = 0$ and $p + p_1 + P$ contains x in its interior. This is a relatively open subset in the codimension one subspace of $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ characterized by $f(p_1) = 0$. Pick a generic such p_1 and consider for all t from 0 to 1 the collection $S(t)$ of line bundles $\mathcal{L}$ whose images in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ lie in $p + tp_1 + P$.

The assumption that p and p_1 are generic implies that for all t there are no lattice points on the codimension two or higher faces of $p+tp_1+P$. Indeed, since there are no lattice points that satisfy $|f(y-p)| = \frac{1}{2}$ we may assume that the face in question is a shift of the preimage of a face in $\frac{1}{2}\widehat{P}$ of codimension two or more. Since for a given x the union of all possible $p + tp_1 + P$ (as p_1 and t vary) is a bounded subset of $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$, there are only finitely many lattice points that could be on the sides of some $p + tp_1 + P$. For each such lattice point y and each face θ of P the condition $y \in p + tp_1 + \theta$ cuts out a space of codimension at least two in the space of all possible tp_1. This in turn gives a codimension at least one space of p_1 such that for some $t \in [0,1]$ there holds $y \in p + tp_1 + \theta$.

Similarly we may also assume that the boundaries of $p + P$ and $p + p_1 + P$ contain no lattice points. Assume that the line bundle $\mathcal{L}$ does not lie in D. Since there are only finitely many lattice points in $p + [0, p_1] + P$, the segment $[0,1]$ is subdivided into a finite number of segments on which the set of lattice point in $p + tp_1 + P$ is constant. Consequently, we may consider the smallest value t_1 of t such that there is a lattice point x_1 in $p + tp_1 + P$ such that there is a line bundle $\mathcal{L}_1$ which maps to it and does not lie in D.

By the above argument, x_1 lies in a face θ of codimension one of $p + t_1p_1 + P$ which is the preimage of the shift of a facet $\widehat{p} + t_1\widehat{p}_1 + \frac{1}{2}\widehat{\theta}$ of $\widehat{p} + t_1\widehat{p}_1 + \frac{1}{2}\widehat{P}$ under the projection map. The corresponding set I is the union of two intervals

$$I = [i_1, j_1) \sqcup [i_2, j_2).$$

The facet θ contains x_1 and the points $p+t_1p_1-\frac{1}{2}\sum_{i\notin I}E_i$, $p+t_1p_1+\frac{1}{2}\sum_{i\in I}E_i$. We have

$$f(p+t_1p_1+\frac{1}{2}\sum_{i\in I}E_i)-f(p+t_1p_1-\frac{1}{2}\sum_{i\notin I}E_i)=\frac{1}{2}\sum_{i=1}^{n}r_i=\frac{1}{2}.$$

This allows us to decompose $p+t_1p_1+\theta$ into the union of two centrally symmetric polytopes, with centers of symmetry $p+t_1p_1+\frac{1}{2}\sum_{i\in I}E_i$ and $p+t_1p_1-\frac{1}{2}\sum_{i\notin I}E_i$, as in Figure 2. To determine which polytope x_1 belongs to, we need to compare $f(x_1-p+\sum_{i\notin I}E_i)$ to $\frac{1}{2}$. We can safely assume that it is not equal to $\frac{1}{2}$ since p is picked to be generic.

Case $f(x_1+\sum_{i\notin I}E_i-p)<\frac{1}{2}$. In this case, the points

$$x_1+\sum_{i\in[j_1,i_2)}E_i,\ x_1+\sum_{i\in[j_2,i_1)}E_i$$

lie in the interior of $p+t_1p_1+P$ and the point

$$x_1+\sum_{i\notin I}E_i$$

lies in the interior of the opposite facet $2p+2t_1p_1-\theta$ of $p+t_1p_1+P$. Indeed, the projections of the first two points to $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ lie in the interior of and the projection of the third point lies on the opposite facet $\widehat{p}+t_1\widehat{p}_1-\theta_I$ of $\widehat{p}+t_1\widehat{p}_1+\frac{1}{2}\widehat{P}$ by Proposition 6.16. It remains to check the property $|f(y-p)|<\frac{1}{2}$ for each of these points. Since $f(x_1-p)>-\frac{1}{2}$ and $f(E_i)=r_i>0$, we only need to check $f(y-p)<\frac{1}{2}$. The largest of these values occurs for $f(x_1+\sum_{i\notin I}E_i)$, which is less than $\frac{1}{2}$, by our assumption.

Observe that for small $\epsilon>0$ the three points of interest lie in the interior of $p+(t_1-\epsilon)p_1+P$. Indeed, by our assumption of minimality of t_1 the point x_1 does not lie in $p+(t_1-\epsilon)p_1+P$ for small positive ϵ, which means that the value of the supporting function of θ is negative on p_1. As a result, every point in the interior of the opposite face will lie in the interior of $p+(t_1-\epsilon)p_1+P$ for small $\epsilon>0$. By the minimality assumption we conclude that the line bundles $\mathcal{L}_1(\sum_{i\in[j_1,i_2)}E_i)$, $\mathcal{L}_1(\sum_{i\in[j_2,i_1)}E_i)$ and $\mathcal{L}_1(\sum_{i\notin I}E_i)$ lie in the category D. We can then write a Koszul sequence on $\mathbb{P}_{\mathbf{\Sigma}}$

$$0\to\mathcal{L}_1\to\mathcal{L}_1(\sum_{i\in[j_1,i_2)}E_i)\oplus\mathcal{L}_1(\sum_{i\in[j_2,i_1)}E_i)\to\mathcal{L}_1(\sum_{i\notin I}E_i)\to 0$$

which is exact, since the divisors $\sum_{i\in[j_1,i_2)}E_i$ and $\sum_{i\in[j_2,i_1)}E_i$ have no common zeroes in $\mathbb{P}_{\mathbf{\Sigma}}$. This shows that $\mathcal{L}_1$ is in D, contradiction.

Case $f(x_1 + \sum_{i \notin I} E_i - p) > \frac{1}{2}$. Observe that

$$f(x_1 - \sum_{i \in I} E_i - p) = f(x_1 + \sum_{i \notin I} E_i - p) - \sum_{i=1}^{n} f(E_i) > \frac{1}{2} - 1 = -\frac{1}{2}.$$

Similarly to the previous case we can show, using Proposition 6.16, that $\mathcal{L}_1(-\sum_{i \in [i_1, j_1)} E_i)$, $\mathcal{L}_1(-\sum_{i \in [i_2, j_2)} E_i)$ and $\mathcal{L}_1(-\sum_{i \in I} E_i)$ are in D. The Koszul short exact sequence

$$0 \to \mathcal{L}_1(-\sum_{i \in I} E_i) \to \mathcal{L}_1(-\sum_{i \in [i_1, j_1)} E_i) \oplus \mathcal{L}_1(-\sum_{i \in [i_2, j_1)} E_i) \to \mathcal{L}_1 \to 0$$

then finishes the argument.

So we have shown that all $\mathcal{L}$ which map to x with $|f(x-p)| \leq \frac{1}{2}$ lie in D. By looking at shifts of any short exact Koszul complex we can then extend the range of values of $f(x)$ in both directions to finish the argument. □

Remark 7.4. In our desire to focus on the basic features of the problem, we have restricted our attention to the del Pezzo case, as opposed to the nef del Pezzo case. It is likely that a slight modification of our argument would allow one to handle the nef del Pezzo case as well. Indeed, our construction of polytope $\widehat{P}$ is continuous in vertices of Δ, and we should be able to take a limit as a side of Δ flattens. We might no longer be able to guarantee in Proposition 6.16 that the shifts of the interior of the face lie in the interior of $\widehat{P}$, but it is still likely that in the proof of Theorem 7.3 the new points in $p + tp_1 + P$ could be expressed in terms of the old ones as t increases. Another reason not to consider the nef case in this article is that the work of Kawamata [Ka2] assures that the derived category of a nef del Pezzo toric stack is equivalent to that of the corresponding K-equivalent del Pezzo toric stack obtained by only keeping the vertices of Δ. Hence our results already guarantee the existence of a strong exceptional collection of objects in the derived category of a nef del Pezzo toric stack, although these objects might not be line bundles.

8. The case of $\mathrm{rk}(\mathrm{Pic}) = 3$ and $\dim = 2$

In this section we will illustrate the result and construction of Sections 6 and 7 in the case of $n = 5$.

Let $\Delta = A_1A_2A_3A_4A_5$ be a convex pentagon in $N = \mathbb{Z}^2$, with the vertices counted clockwise, which contains 0 in its interior. Let $\mathbf{\Sigma}$ be the corresponding stacky fan and $\mathbb{P}_{\mathbf{\Sigma}}$ the corresponding del Pezzo DM stack. As before, we denote by v_i the vector from 0 to A_i and by E_i

the corresponding elements of the Picard group. We will introduce the notation
$$\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}}) = \mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})/\mathbb{R}K$$
where K is the canonical class. We will abuse the notation and denote by E_i the image of $\mathcal{O}(E_i)$ in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$. We will use the notation $\widehat{E}_i$ for the image of E_i in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$.

The polytope Q in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ is given in Figure 4. It is a convex centrally symmetric 10-gon.

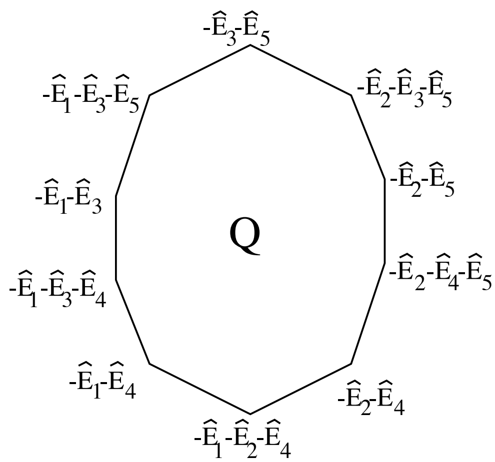

FIGURE 4.

The projections of the forbidden cones for proper subsets $I \subset \{1, \ldots, 5\}$ are given in Figure 5. The complement of it is the acyclic region, in the sense that any line bundle $\mathcal{L}$ which projects to it has no middle cohomology.

For any point V in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$, we can consider the points obtained from it by flipping it across the vertices of Q. A flip of a point A across a point B is $2B - A$. It is easy to see that $\sum_{i=1}^{5} \widehat{E}_i = 0$ implies that if one starts with a point V and flips it across $-\widehat{E}_1 - \widehat{E}_3$, then the 10-th vertex is again V, and the ten vertices are
$$V, -V - 2\widehat{E}_3 - 2\widehat{E}_5, V - 2\widehat{E}_2, -V - 2\widehat{E}_5, V - 2\widehat{E}_2 - 2\widehat{E}_4,$$
$$-V, V + 2\widehat{E}_3 + 2\widehat{E}_5, -V + 2\widehat{E}_2, V + 2\widehat{E}_5, -V + 2\widehat{E}_2 + 2\widehat{E}_4.$$

It is a priori not obvious that one can pick V in such a way that the resulting ten points form vertices of a convex polytope that contains Q. However, by Proposition 6.14 there exists a convex polygon $\widehat{P}$ such that the midpoints of its edges are the vertices of Q. Hence one can

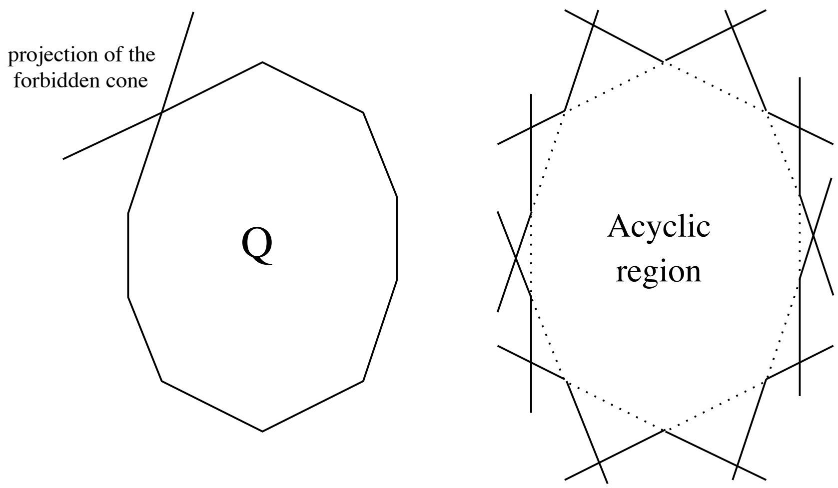

FIGURE 5.

pick V so that the above 10 points are vertices of $\widehat{P}$. In particular, the interior of $\widehat{P}$ lies in the acyclic region, see Figure 6.

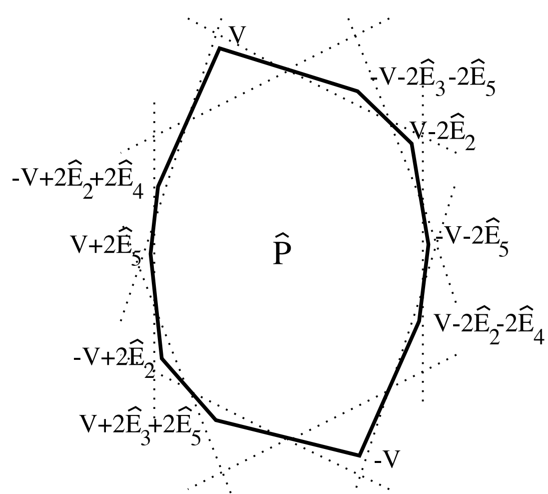

FIGURE 6.

Proposition 6.16 can be stated as follows. It will be convenient to consider the indices i of A_i to lie in $\mathbb{Z}/5\mathbb{Z}$.

Proposition 8.1. *For an edge of $\widehat{P}$ that contains $-\widehat{E}_{i-1} - \widehat{E}_{i+1}$, the translates of its interior by $2\widehat{E}_{i-1}$, $2\widehat{E}_{i+1}$, $-2\widehat{E}_i$ and $-2(\widehat{E}_{i-2}+\widehat{E}_{i+2})$ lie* *in the interior of $\widehat{P}$. For an edge of $\widehat{P}$ that contains $-\widehat{E}_i-\widehat{E}_{i-2}-\widehat{E}_{i+2}$, the translates of its interior by $2\widehat{E}_i$, $2(\widehat{E}_{i-2}+\widehat{E}_{i+2})$, $-2\widehat{E}_{i-1}$ and $-2\widehat{E}_{i+1}$ lie in the interior of $\widehat{P}$.*

For what follows we pick and fix a generic collection of positive numbers $r_i, i = 1, \ldots, 5$, such that $\sum_i r_i = 1$ and $\sum_i r_i v_i = 0$. This collection gives a linear function f on $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ by $f(E_i) = r_i$.

The convex polytope P in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ given by the inequalities $|f(x)| \leq \frac{1}{2}$ and the condition that the image of x in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ lies in $\frac{1}{2}\widehat{P}$ is depicted in Figure 7. The polytope P has two 10-gonal faces and 10 parallelogram faces that are preimages of the sides of the pentagon $\frac{1}{2}\widehat{P}$, see Figure 7.

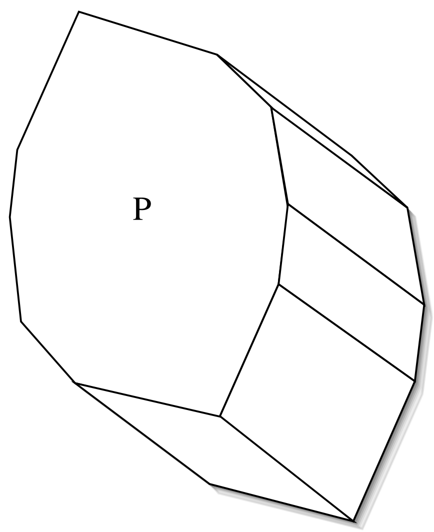

FIGURE 7.

The proof of Theorem 7.3 can be visualized as follows. We place the polytope P somewhere generically in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ by considering its shift $p + P$. The set of line bundles whose images in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ lie in $p + P$ form a strong exceptional collection. Indeed, the differences avoid the forbidden cone $F_\emptyset$ with the vertex $q_\emptyset = -\sum_{i=1}^{5} E_i$ because they have $f(\cdot) > -1$, and they avoid the other forbidden cones because their image in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ is contained in $\widehat{P}$ and hence in the acyclic region. We then define the category D generated by the line bundles in this strong exceptional collection. We first move the polytope $p + P$ in generic directions that are parallel to its 10-gonal facets. As the polytope moves, any new points can be connected to already covered points by means of Proposition 8.1. This in turn leads to Koszul complexes which allow one to show that the corresponding line bundles lie in D. After we have guaranteed that all points between the supporting planes of the 10-gonal facets of $p+P$ correspond to line bundles in D, we use Koszul complexes to extend in the orthogonal direction, as in the second panel of Figure 3.

9. Comments

It is natural to try to apply the techniques of this paper to the general case of King's conjecture. For an arbitrary rank of the Picard group, and arbitrary dimension, one can still define the polytope Q in $\widehat{\mathrm{Pic}}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ as the Minkowski sum of $[0, \widehat{E}_i]$. One then wants to construct a polytope $\widehat{P} \supseteq Q$ with the property that all vertices of Q that correspond to forbidden cones are midpoints of some of the faces of $\widehat{P}$ and that midpoints of all facets of $\widehat{P}$ are images of the vertices of the forbidden cones.

It is not a priori clear that $\widehat{P}$ should be a zonotope, although this is a plausible assumption. However even assuming that it is, the combinatorics of it is generally unclear and remains the key challenge. Once the polytope $\widehat{P}$ is constructed, we can define the polytope P in $\mathrm{Pic}_{\mathbb{R}}(\mathbb{P}_{\mathbf{\Sigma}})$ as in Section 7. It remains to be seen whether this approach will lead to a proof of Conjecture 3.14, but it appears promising.

Even in its current state the paper can be applied to the study of (noncompact) toric Calabi-Yau threefolds, which are defined by triangulations of the polygon $(\Delta, 1)$ in $N \oplus \mathbb{Z}$.

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