STRING COHOMOLOGY OF CALABI-YAU HYPERSURFACES VIA MIRROR SYMMETRY

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ABSTRACT. We propose a construction of string cohomology spaces for Calabi-Yau hypersurfaces that arise in Batyrev's mirror symmetry construction. The spaces are defined explicitly in terms of the corresponding reflexive polyhedra in a mirror-symmetric manner. We draw connections with other approaches to the string cohomology, in particular with the work of Chen and Ruan.

CONTENTS

1. INTRODUCTION

The notion of orbifold cohomology has appeared in physics as a result of studying the string theory on orbifold global quotients, (see [DiHVW]). In addition to the usual cohomology of the quotient, this space was supposed to include the so-called twisted sectors, whose existence was predicted by the modular invariance condition on the partition function of the theory. Since then, there have been several attempts to give a rigorous mathematical formulation of this cohomology theory. The first two, due to [BDa] and [B3], tried to define the topological invariants of certain algebraic varieties (including orbifold global quotients) that should correspond to the dimensions of the Hodge components of a conjectural string cohomology space. These invariants should have the property arising naturally from physics: they are preserved by partial crepant resolutions; moreover, they coincide with the usual Hodge numbers for smooth varieties. Also, these invariants must be the same as those defined by physicists for orbifold global quotients. In [B3, B4], Batyrev has

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successfully solved this problem for a large class of singular algebraic varieties. The first mathematical definition of the orbifold cohomology *space* was given in [ChR] for arbitrary orbifolds. Moreover, this orbifold cohomology possesses a product structure arising as a limit of a natural quantum product. It is still not entirely clear if the dimensions of the Chen-Ruan cohomology coincide with the prescription of Batyrev whenever both are defined, but they do give the same result for reduced global orbifolds.

In this paper, we propose a construction of string cohomology spaces for Calabi-Yau hypersurfaces that arise in the Batyrev mirror symmetry construction (see [B2]), with the spaces defined rather explicitly in terms of the corresponding reflexive polyhedra. A peculiar feature of our construction is that instead of a single string cohomology space we construct a finite-dimensional family of such spaces, which is consistent with the physicists' picture (see [G]). We verify that this construction is consistent with the previous definitions in [BDa], [B3] and [ChR], in the following sense. The (bigraded) dimension of our space coincides with the definitions of [BDa] and [B3]. In the case of hypersurfaces that have only orbifold singularities, we recover Chen-Ruan's orbifold cohomology as one special element of this family of string cohomology spaces. We also conjecture a partial natural ring structure on our string cohomology space, which is in correspondence with the cohomology ring of crepant resolutions. This may be used as a real test of the Chen-Ruan orbifold cohomology ring. We go further, and conjecture the B-model chiral ring on the string cohomology space. This is again consistent with the description of the B-model chiral ring of smooth Calabi-Yau hypersurfaces in [M2].

Our construction of the string cohomology space for Calabi-Yau hypersurfaces is motivated by Mirror Symmetry. Namely, the description in [M3] of the cohomology of semiample hypersurfaces in toric varieties applies to the smooth Calabi-Yau hypersurfaces in [B2]. Analysis of Mirror Symmetry on this cohomology leads to a natural construction of the string cohomology space for all semiample Calabi-Yau hypersurfaces. As already mentioned, our string cohomology space depends not only on the complex structure (the defining polynomial f), but also on some extra parameter we call ω. For special values of this parameter of an orbifold Calabi-Yau hypersurface, we get the orbifold Dolbeault cohomology of [ChR]. However, for non-orbifold Calabi-Yau hypersurfaces, there is no natural special choice of ω, which means that the general definition of the string cohomology space should depend on some mysterious extra parameter. In the situation of Calabi-Yau hypersurfaces, the parameter ω corresponds to the defining polynomial of the mirror Calabi-Yau hypersurface. In general, we expect that this parameter should be related to the "stringy complexified Kähler class", which is yet to be defined.

In an attempt to extend our definitions beyond the Calabi-Yau hypersurface case, we give a conjectural definition of string cohomology vector spaces for stratified varieties with $\mathbb{Q}$-Gorenstein toroidal singularities that satisfy certain restrictions on the types of singular strata. This definition involves intersection cohomology of the closures of strata, and we check that it produces spaces of correct bigraded dimension. It also reproduces orbifold cohomology of a $\mathbb{Q}$-Gorenstein toric variety as a special case.

Here is an outline of our paper. In Section 2, we examine the connection between the original definition of the *string-theoretic* Hodge numbers in [BDa] and the *stringy* Hodge numbers in [B3]. We point out that these do not always give the same result and argue that the latter definition is the more useful one. In

Section 3, we briefly review the mirror symmetry construction of Batyrev, mainly to fix our notations and to describe the properties we will use in the derivation of the string cohomology. Section 4 describes the cohomology of semiample hypersurfaces in toric varieties and explains how mirror symmetry provides a conjectural definition of the string cohomology of Calabi-Yau hypersurfaces. It culminates in Conjecture 4.8, where we define the stringy Hodge spaces of semiample Calabi-Yau hypersurfaces in complete toric varieties. We spend most of the remainder of the paper establishing the expected properties of the string cohomology space. Sections 5 and 6 calculate the dimensions of the building blocks of our cohomology spaces. In Section 6, we develop a theory of deformed semigroup rings which may be of independent interest. This allows us to show in Section 7 that Conjecture 4.8 is compatible with the definition of the stringy Hodge numbers from [B3]. In the non-simplicial case, this requires the use of G-polynomials of Eulerian posets, whose relevant properties are collected in the Appendix. Having established that the dimension is correct, we try to extend our construction to the non-hypersurface case. Section 8 gives another conjectural definition of the string cohomology vector space in a somewhat more general situation. It hints that the intersection cohomology and the perverse sheaves should play a prominent role in future definitions of string cohomology. In Section 9, we connect our work with that of Chen-Ruan [ChR] and Poddar [P]. Finally, in Section 10, we provide yet another description of the string cohomology of Calabi-Yau hypersurfaces, which was inspired by the vertex algebra approach to Mirror Symmetry.

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2. String-theoretic and stringy Hodge numbers

The *string-theoretic* Hodge numbers were first defined in the paper of Batyrev and Dais (see [BDa]) for varieties with Gorenstein toroidal or quotient singularities. In subsequent papers [B3, B4] Batyrev defined *stringy* Hodge numbers for arbitrary varieties with log-terminal singularities. To our knowledge, the relationship between these two concepts has never been clarified in the literature. The goal of this section is to show that the string-theoretic Hodge numbers coincide with the stringy ones under some conditions on the singular strata.

We begin with the definition of the string-theoretic Hodge numbers.

Definition 2.1. [BDa] Let $X = \bigcup_{i\in I} X_i$ be a stratified algebraic variety over $\mathbb{C}$ with at most Gorenstein toroidal singularities such that for each $i \in I$ the singularities of X along the stratum X_i of codimension k_i are defined by a k_i-dimensional finite rational polyhedral cone σ_i; i.e., X is locally isomorphic to

$$\mathbb{C}^{k-k_i} \times U_{\sigma_i}$$

at each point $x \in X_i$ where U_{σ_i} is a k_i-dimensional affine toric variety which is associated with the cone σ_i (see [D]). Then the polynomial

$$E_{\mathrm{st}}^{\mathrm{BD}}(X; u, v) := \sum_{i\in I} E(X_i; u, v) \cdot S(\sigma_i, uv)$$

is called the *string-theoretic E-polynomial* of X. Here,

$$S(\sigma_i,t) := (1-t)^{\dim\sigma_i}\sum_{n\in\sigma_i} t^{\deg n} = (t-1)^{\dim\sigma_i}\sum_{n\in\mathrm{int}\sigma_i} t^{-\deg n}$$

where deg is the linear function on σ_i that takes value 1 on the generators of one-dimensional faces of σ_i, and $\mathrm{int}\sigma_i$ is the relative interior of σ_i. If we write $E_{\mathrm{st}}(X;u,v)$ in the form

$$E^{\mathrm{BD}}_{\mathrm{st}}(X;u,v) = \sum_{p,q} a_{p,q}u^p v^q,$$

then the numbers $h^{p,q(\mathrm{BD})}_{\mathrm{st}}(X) := (-1)^{p+q}a_{p,q}$ are called the *string-theoretic Hodge numbers* of X.

Remark 2.2. The E-polynomial in the above definition is defined for an arbitrary algebraic variety X as

$$E(X;u,v) = \sum_{p,q} e^{p,q}u^p v^q,$$

where $e^{p,q} = \sum_{k\geq 0}(-1)^k h^{p,q}(H^k_c(X))$.

Stringy Hodge numbers of X are defined in terms of the resolutions of its singularities. In general, one can only define the E-function in this case, which may or not be a polynomial. We refer to [KoMo] for the definitions of log-terminal singularities and related issues.

Definition 2.3. [B3] Let X be a normal irreducible algebraic variety with at worst log-terminal singularities, $\rho : Y \to X$ a resolution of singularities such that the irreducible components $D_1,\dots,D_r$ of the exceptional locus is a divisor with simple normal crossings. Let $\alpha_j > -1$ be the discrepancy of D_j, see [KoMo]. Set $I := \{1,\dots,r\}$. For any subset $J \subset I$ we consider

$$D_J := \begin{cases} \bigcap_{j\in J} D_j & \text{if } J \neq \emptyset \\ Y & \text{if } J = \emptyset \end{cases} \qquad \text{and} \qquad D^\circ_J := D_J \setminus \bigcup_{j\in I\setminus J} D_j.$$

We define an algebraic function $E_{\mathrm{st}}(X;u,v)$ in two variables u and v as follows:

$$E_{\mathrm{st}}(X;u,v) := \sum_{J\subset I} E(D^\circ_J;u,v)\prod_{j\in J}\frac{uv-1}{(uv)^{a_j+1}-1}$$

(it is assumed $\prod_{j\in J}$ to be 1, if $J=\emptyset$). We call $E_{\mathrm{st}}(X;u,v)$ *the stringy E-function of* X. If $E_{\mathrm{st}}(X;u,v)$ is a polynomial, define the stringy Hodge numbers the same way as Definition 2.1 does.

It is not obvious at all that the above definition is independent of the choice of the resolution. The original proof of Batyrev uses a motivic integration over the spaces of arcs to relate the E-functions obtained via different resolutions. Since the work of D. Abramovich, K. Karu, K. Matsuki, J. Włodarsczyk [AKaMW], it is now possible to check the independence from the resolution by looking at the case of a single blowup with a smooth center compatible with the normal crossing condition.

Lemma 2.4. *Let X be a disjoint union of strata X_i, which are locally closed in Zariski topology, and let ρ be a resolution as in Definition 2.3. For each X_i consider*

$$E_{\mathrm{st}}(X_i\subseteq X;u,v) := \sum_{J\subset I} E(D^\circ_J\cap\rho^{-1}(X_i);u,v)\prod_{j\in J}\frac{uv-1}{(uv)^{a_j+1}-1}.$$

Then this E-function is independent of the choice of the resolution Y. The E-function of X decomposes as

$$E_{\mathrm{st}}(X;u,v) = \sum_i E_{\mathrm{st}}(X_i \subseteq X;u,v).$$

Proof. Each resolution of X induces a resolution of the complement of $\bar{X}_i$. This shows that for each X_i the sum

$$\sum_{j, X_j \subseteq \bar{X}_i} E_{\mathrm{st}}(X_j \subseteq X;u,v)$$

is independent from the choice of the resolution and is thus well-defined. Then one uses the induction on dimension of X_i. The last statement is clear. □

Remark 2.5. It is a delicate question what data are really necessary to calculate $E_{\mathrm{st}}(X_i \subseteq X;u,v)$. It is clear that the knowledge of a Zariski open set of X containing X_i is enough. However, it is not clear whether it is enough to know an analytic neighborhood of X_i.

We will use the above lemma to show that the string-theoretic Hodge numbers and the stringy Hodge numbers coincide in a wide class of examples.

Proposition 2.6. *Let $X = \bigcup_i X_i$ be a stratified algebraic variety with at worst Gorenstein toroidal singularities as in Definition 2.1. Assume in addition that for each i there is a desingularization Y of X so that its restriction to the preimage of X_i is a locally trivial fibration in Zariski topology. Moreover, for a point $x \in X_i$ the preimage in Y of an analytic neighborhood of x is complex-analytically isomorphic to a preimage of a neighborhood of $\{0\}$ in U_{σ_i} under some resolution of singularities of U_{σ_i}, times a complex disc, so that the isomorphism is compatible with the resolution morphisms. Then*

$$E_{\mathrm{st}}^{\mathrm{BD}}(X;u,v) = E_{\mathrm{st}}(X;u,v).$$

Proof. Since E-polynomials are multiplicative for Zariski locally trivial fibrations (see [DKh]), the above assumptions on the singularities show that

$$E_{\mathrm{st}}(X_i \subseteq X;u,v) = E(X_i;u,v)E_{\mathrm{st}}(\{0\} \subseteq U_{\sigma_i};u,v).$$

We have also used here the fact that since the fibers are projective, the analytic isomorphism implies the algebraic one, by GAGA. By the second statement of Lemma 2.4, it is enough to show that

$$E_{\mathrm{st}}(\{0\} \subseteq U_{\sigma_i};u,v) = S(\sigma_i, uv).$$

This result follows from the proof of [B3], Theorem 4.3 where the products

$$\prod_{j\in J} \frac{uv-1}{(uv)^{a_j+1}-1}$$

are interpreted as a geometric series and then as sums of $t^{\deg(n)}$ over points n of σ_i. □

Corollary 2.7. *String-theoretic and stringy Hodge numbers coincide for nondegenerate hypersurfaces (complete intersections) in Gorenstein toric varieties.*

Proof. Indeed, in this case, the toric desingularizations of the ambient toric variety induce the desingularizations with the required properties. □

We will keep this corollary in mind and from now on will silently transfer all the results on string-theoretic Hodge numbers of hypersurfaces and complete intersections in toric varieties in [BBo], [BDa] to their stringy counterparts.

Remark 2.8. An example of the variety where string-theoretic and stringy Hodge numbers *differ* is provided by the quotient of $\mathbb{C}^2 \times E$ by the finite group of order six generated by

$$r_1 : (x, y; z) \mapsto (x\mathrm{e}^{2\pi\mathrm{i}/3}, y\mathrm{e}^{-2\pi\mathrm{i}/3}; z),\ \ r_2 : (x, y; z) \mapsto (y, x; z + p)$$

where (x, y) are coordinates on $\mathbb{C}^2$, z is the uniformizing coordinate on the elliptic curve E and p is a point of order two on E. In its natural stratification, the quotient has a stratum of A_2 singularities, so that going around a loop in the stratum results in the non-trivial automorphism of the singularity.

Remark 2.9. We expect that the stringy Hodge numbers of algebraic varieties with abelian quotient singularities coincide with the dimensions of their orbifold cohomology, [ChR]. This is not going to be true for the string-theoretic Hodge numbers. Also, the latter numbers are not preserved by the partial crepant resolutions as required by physics, see the above example. As a result, we believe that the stringy Hodge numbers are the truly interesting invariant, and that the string-theoretic numbers is a now obsolete first attempt to define them.

3. Mirror symmetry construction of Batyrev

In this section, we review the mirror symmetry construction from [B2]. We can describe it starting with a semiample nondegenerate (transversal to the torus orbits) anticanonical hypersurface X in a complete simplicial toric variety $\mathbb{P}_\Sigma$. Such a hypersurface is Calabi-Yau. The semiampleness property produces a contraction map, the unique properties of which are characterized by the following statement.

Proposition 3.1. [M1] *Let $\mathbb{P}_\Sigma$ be a complete toric variety with a big and nef divisor class $[X] \in A_{d-1}(\mathbb{P}_\Sigma)$. Then, there exists a unique complete toric variety $\mathbb{P}_{\Sigma_X}$ with a toric birational map $\pi : \mathbb{P}_\Sigma \to \mathbb{P}_{\Sigma_X}$, such that Σ is a subdivision of Σ_X, $\pi_*[X]$ is ample and $\pi^*\pi_*[X] = [X]$. Moreover, if $X = \sum_\rho a_\rho D_\rho$ is torus-invariant, then Σ_X is the normal fan of the associated polytope*

$$\Delta_X = \{m \in M : \langle m, e_\rho\rangle \geq -a_\rho \textit{ for all } \rho\} \subset M_{\mathbb{R}}.$$

Remark 3.2. Our notation is a standard one taken from [BC, C2]: M is a lattice of rank d; $N = \mathrm{Hom}(M, \mathbb{Z})$ is the dual lattice; $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ are the $\mathbb{R}$-scalar extensions of M and N; Σ is a finite rational polyhedral fan in $N_{\mathbb{R}}$; $\mathbb{P}_\Sigma$ is a d-dimensional toric variety associated with Σ; $\Sigma(k)$ is the set of all k-dimensional cones in Σ; e_ρ is the minimal integral generator of the 1-dimensional cone $\rho \in \Sigma$ corresponding to a torus invariant irreducible divisor D_ρ.

Applying Proposition 3.1 to the semiample Calabi-Yau hypersurface, we get that the push-forward $\pi_*[X]$ is anticanonical and ample, whence, by Lemma 3.5.2 in [CKat], the toric variety $\mathbb{P}_{\Sigma_X}$ is Fano, associated with the polytope $\Delta \subset M_{\mathbb{R}}$ of the anticanonical divisor $\sum_\rho D_\rho$ on $\mathbb{P}_\Sigma$. Then, [M1, Proposition 2.4] shows that the image $Y := \pi(X)$ is an ample nondegenerate hypersurface in $\mathbb{P}_{\Sigma_X} = \mathbb{P}_\Delta$. The fact that $\mathbb{P}_\Delta$ is Fano means by Proposition 3.5.5 in [CKat] that the polytope Δ is reflexive, i.e., its dual

$$\Delta^* = \{n \in N_{\mathbb{R}} : \langle m, n\rangle \geq -1 \text{ for } m \in \Delta\}$$

has all its vertices at lattice points in N, and the only lattice point in the interior of Δ^* is the origin 0. Now, consider the toric variety $\mathbb{P}_{\Delta^*}$ associated to the polytope Δ^* (the minimal integral generators of its fan are precisely the vertices of Δ). Theorem 4.1.9 in [B2] says that an anticanonical nondegenerate hypersurface $Y^* \subset \mathbb{P}_{\Delta^*}$ is a Calabi-Yau variety with canonical singularities. The Calabi-Yau hypersurface Y^* is expected to be a mirror of Y. In particular, they pass the topological mirror symmetry test for the stringy Hodge numbers:

$$h^{p,q}_{\rm st}(Y) = h^{d-1-p,q}_{\rm st}(Y^*), 0 \le p, q \le d-1,$$

by [BBo, Theorem 4.15]. Moreover, all crepant partial resolutions X of Y have the same stringy Hodge numbers:

$$h^{p,q}_{\rm st}(X) = h^{p,q}_{\rm st}(Y).$$

Physicists predict that such resolutions of Calabi-Yau varieties have indistinguishable physical theories. Hence, all crepant partial resolutions of Y may be called the mirrors of crepant partial resolutions of Y^*. To connect this to the classical formulation of mirror symmetry, one needs to note that if there exist crepant smooth resolutions X and X^* of Y and Y^*, respectively, then

$$h^{p,q}(X) = h^{d-1-p,q}(X^*), 0 \le p, q \le d-1,$$

since the stringy Hodge numbers coincide with the usual ones for smooth Calabi-Yau varieties. The equality of Hodge numbers is expected to extend to an isomorphism (*mirror map*) of the corresponding Hodge spaces, which is compatible with the chiral ring products of A and B models (see [CKat] for more details).

4. String cohomology construction for Calabi-Yau hypersurfaces

In this section, we show how the description of cohomology of semiample hypersurfaces in [M3] leads to a construction of the string cohomology space of Calabi-Yau hypersurfaces. We first review the building blocks participating in the description of the cohomology in [M3], and then explain how these building blocks should interchange under mirror symmetry for a pair of smooth Calabi-Yau hypersurfaces in Batyrev's mirror symmetry construction. Mirror symmetry and the fact that the dimension of the string cohomology is the same for all partial crepant resolutions of ample Calabi-Yau hypersurfaces leads us to a conjectural description of string cohomology for all semiample Calabi-Yau hypersurfaces. In the next three sections, we will prove that this space has the dimension prescribed by [BDa].

The cohomology of a semiample nondegenerate hypersurface X in a complete simplicial toric variety $\mathbb{P}_\Sigma$ splits into the *toric* and *residue* parts:

$$H^*(X) = H^*_{\rm toric}(X) \oplus H^*_{\rm res}(X),$$

where the first part is the image of the cohomology of the ambient space, while the second is the residue map image of the cohomology of the complement to the hypersurface. By [M2, Theorem 5.1],

$$H^*_{\rm toric}(X) \cong H^*(\mathbb{P}_\Sigma)/Ann(X) \tag{1}$$

where $Ann(X)$ is the annihilator of the class $[X] \in H^2(\mathbb{P}_\Sigma)$. The cohomology of $\mathbb{P}_\Sigma$ is isomorphic to

$$\mathbb{C}[D_\rho : \rho \in \Sigma(1)]/(P(\Sigma) + SR(\Sigma)),$$

where

$$P(\Sigma)=\Big\langle\sum_{\rho\in\Sigma(1)}\langle m,e_\rho\rangle D_\rho : m\in M\Big\rangle$$

is the ideal of linear relations among the divisors, and

$$SR(\Sigma)=\big\langle D_{\rho_1}\cdots D_{\rho_k} : \{e_{\rho_1},\dots,e_{\rho_k}\}\not\subset\sigma \text{ for all } \sigma\in\Sigma\big\rangle$$

is the Stanley-Reisner ideal. Hence, $H^*_{\rm toric}(X)$ is isomorphic to the bigraded ring

$$T(X)_{*,*}:=\mathbb{C}[D_\rho:\rho\in\Sigma(1)]/I,$$

where $I=(P(\Sigma)+SR(\Sigma)):[X]$ is the ideal quotient, and D_ρ have the degree $(1,1)$.

The following modules over the ring $T(X)$ have appeared in the description of cohomology of semiample hypersurfaces:

Definition 4.1. Given a big and nef class $[X]\in A_{d-1}(\mathbb{P}_\Sigma)$ and $\sigma\in\Sigma_X$, let

$$U^\sigma(X)=\Big\langle\prod_{\rho\subset\gamma\in\Sigma}D_\rho : {\rm int}\gamma\subset{\rm int}\sigma\Big\rangle$$

be the bigraded ideal in $\mathbb{C}[D_\rho:\rho\in\Sigma(1)]$, where D_ρ have the degree (1,1). Define the bigraded space

$$T^\sigma(X)_{*,*}=U^\sigma(X)_{*,*}/I^\sigma,$$

where

$$I^\sigma=\{u\in U^\sigma(X)_{*,*}: uvX^{d-\dim\sigma}\in(P(\Sigma)+SR(\Sigma)) \text{ for } v\in U^\sigma(X)_{\dim\sigma-*,\dim\sigma-*}\}.$$

Next, recall from [C1] that *any* toric variety $\mathbb{P}_\Sigma$ has a homogeneous coordinate ring

$$S(\mathbb{P}_\Sigma)=\mathbb{C}[x_\rho:\rho\in\Sigma(1)]$$

with variables x_ρ corresponding to the irreducible torus invariant divisors D_ρ. This ring is graded by the Chow group $A_{d-1}(\mathbb{P}_\Sigma)$, assigning $[\sum_\rho a_\rho D_\rho]$ to $\deg(\prod_\rho x_\rho^{a_\rho})$. For a Weil divisor D on $\mathbb{P}_\Sigma$, there is an isomorphism $H^0(\mathbb{P}_\Sigma,O_{\mathbb{P}_\Sigma}(D))\cong S(\mathbb{P}_\Sigma)_\alpha$, where $\alpha=[D]\in A_{d-1}(\mathbb{P}_\Sigma)$. If D is torus invariant, the monomials in $S(\mathbb{P}_\Sigma)_\alpha$ correspond to the lattice points of the associated polyhedron Δ_D.

In [BC], the following rings have been used to describe the residue part of cohomology of ample hypersurfaces in complete simplicial toric varieties:

Definition 4.2. [BC] Given $f\in S(\mathbb{P}_\Sigma)_\beta$, set $J_0(f):=\langle x_\rho\partial f/\partial x_\rho:\rho\in\Sigma(1)\rangle$ and $J_1(f):=J_0(f):x_1\cdots x_n$. Then define the rings $R_0(f)=S(\mathbb{P}_\Sigma)/J_0(f)$ and $R_1(f)=S(\mathbb{P}_\Sigma)/J_1(f)$, which are graded by the Chow group $A_{d-1}(\mathbb{P}_\Sigma)$.

In [M3, Definition 6.5], similar rings were introduced to describe the residue part of cohomology of semiample hypersurfaces:

Definition 4.3. [M3] Given $f\in S(\mathbb{P}_\Sigma)_\beta$ of big and nef degree $\beta=[D]\in A_{d-1}(\mathbb{P}_\Sigma)$ and $\sigma\in\Sigma_D$, let $J_0^\sigma(f)$ be the ideal in $S(\mathbb{P}_\Sigma)$ generated by $x_\rho\partial f/\partial x_\rho$, $\rho\in\Sigma(1)$ and all $x_{\rho'}$ such that $\rho'\subset\sigma$, and let $J_1^\sigma(f)$ be the ideal quotient $J_0^\sigma(f):(\prod_{\rho\not\subset\sigma}x_\rho)$. Then we get the quotient rings $R_0^\sigma(f)=S(\mathbb{P}_\Sigma)/J_0^\sigma(f)$ and $R_1^\sigma(f)=S(\mathbb{P}_\Sigma)/J_1^\sigma(f)$ graded by the Chow group $A_{d-1}(\mathbb{P}_\Sigma)$.

As a special case of [M3, Theorem 2.11], we have:

Theorem 4.4. *Let X be an anticanonical semiample nondegenerate hypersurface defined by $f\in S_\beta$ in a complete simplicial toric variety $\mathbb{P}_\Sigma$. Then there is a natural isomorphism*

$$\bigoplus_{p,q} H^{p,q}(X)\cong\bigoplus_{p,q} T(X)_{p,q}\oplus\left(\bigoplus_{\sigma\in\Sigma_X} T^\sigma(X)_{s,s}\otimes R_1^\sigma(f)_{(q-s)\beta+\beta_1^\sigma}\right),$$

where $s=(p+q-d+\dim\sigma+1)/2$ and $\beta_1^\sigma=\deg(\prod_{\rho_k\subset\sigma}x_k)$.

By the next statement, we can immediately see that all the building blocks $R_1^\sigma(f)_{(q-s)\beta+\beta_1^\sigma}$ of the cohomology of partial resolutions in Theorem 4.4 are independent of the resolution and intrinsic to an ample Calabi-Yau hypersurface:

Proposition 4.5. [M3] *Let X be a big and nef nondegenerate hypersurface defined by $f\in S_\beta$ in a complete toric variety $\mathbb{P}_\Sigma$ with the associated contraction map $\pi:\mathbb{P}_\Sigma\to\mathbb{P}_{\Sigma_X}$. If $f_\sigma\in S(V(\sigma))_{\beta^\sigma}$ denotes the polynomial defining the hypersurface $\pi(X)\cap V(\sigma)$ in the toric variety $V(\sigma)\subset\mathbb{P}_{\Sigma_X}$ corresponding to $\sigma\in\Sigma_X$, then, there is a natural isomorphism induced by the pull-back:*

$$H^{d(\sigma)-*,*-1}H^{d(\sigma)-1}(\pi(X)\cap\mathbb{T}_\sigma)\cong R_1(f_\sigma)_{*\beta^\sigma-\beta_0^\sigma}\cong R_1^\sigma(f)_{*\beta-\beta_0+\beta_1^\sigma},$$

where $d(\sigma)=d-\dim\sigma$, $\mathbb{T}_\sigma\subset V(\sigma)$ is the maximal torus, and β_0 and β_0^σ denote the anticanonical degrees on $\mathbb{P}_\Sigma$ and $V(\sigma)$, respectively.

Given a mirror pair (X,X^*) of smooth Calabi-Yau hypersurfaces in Batyrev's construction, we expect that, for a pair of cones σ and σ^* over the dual faces of the reflexive polytopes Δ^* and Δ, $T^\sigma(X)_{s,s}$ with $s=(p+q-d+\dim\sigma+1)/2$, in $H^{p,q}(X)$ interchanges, by the mirror map (the isomorphism which maps the quantum cohomology of one Calabi-Yau hypersurface to the B-model chiral ring of the other one), with $R_1^{\sigma^*}(g)_{(p+q-\dim\sigma^*)\beta^*/2+\beta_1^{\sigma^*}}$ in $H^{d-1-p,q}(X^*)$ (note that $\dim\sigma^*=d-\dim\sigma+1$), where $g\in S(\mathbb{P}_{\Sigma^*})_{\beta^*}$ determines X^*. For the 0-dimensional cones σ and σ^*, the interchange goes between the *polynomial part* $R_1(g)_{*\beta^*}$ of one smooth Calabi-Yau hypersurface and the toric part of the cohomology of the other one. This correspondence was already confirmed by the construction of the generalized monomial-divisor mirror map in [M3]. On the other hand, one can deduce that the dimensions of these spaces coincide for the pair of 3-dimensional smooth Calabi-Yau hypersurfaces, by using Remark 5.3 in [M1]. The correspondence between the toric and polynomial parts was discussed in [CKat].

Now, let us turn our attention to a mirror pair of semiample singular Calabi-Yau hypersurfaces Y and Y^*. We know that their string cohomology should have the same dimension as the usual cohomology of possible crepant smooth resolutions X and X^*, respectively. Moreover, the A-model and B-model chiral rings on the string cohomology should be isomorphic for X and X^*, respectively. We also know that the polynomial g represents the complex structure of the hypersurface Y^* and its resolution X^*, and, by mirror symmetry, g should correspond to the complexified Kähler class of the mirror Calabi-Yau hypersurface. Therefore, based on the mirror correspondence of smooth Calabi-Yau hypersurfaces, we make the following prediction for the small quantum ring presentation on the string cohomology space:

$$QH^{p,q}_{\rm st}(Y)\cong\bigoplus_{(\sigma,\sigma^*)} R_1(\omega_{\sigma^*})_{(p+q-\dim\sigma^*+2)\beta^{\sigma^*}/2-\beta_0^{\sigma^*}}\otimes R_1(f_\sigma)_{(q-p+d-\dim\sigma+1)\beta^\sigma/2-\beta_0^\sigma},\tag{2}$$

where the sum is by all pairs of the cones σ and σ^* (including 0-dimensional cones) over the dual faces of the reflexive polytopes, and where $\omega_{\sigma^*}\in S(V(\sigma^*))_{\beta^{\sigma^*}}$ is a formal restriction of $\omega \in S(\mathbb{P}_{\Delta^*})_{\beta^*}$, which should be related to the complexified Kähler class of the mirror (we will discuss this in Section 9). This construction can be rewritten in simpler terms, which will help us to give a conjectural description of the usual string cohomology space for all semiample Calabi-Yau hypersurfaces.

First, recall Batyrev's presentation of the toric variety $\mathbb{P}_\Delta$ for an *arbitrary* polytope Δ in M (see [B1], [C2]). Consider the *Gorenstein* cone K over $\Delta\times\{1\} \subset M\oplus\mathbb{Z}$. Let S_Δ be the subring of $\mathbb{C}[t_0, t_1^{\pm1}, \dots, t_d^{\pm1}]$ spanned over $\mathbb{C}$ by all monomials of the form $t_0^k t^m = t_0^k t_1^{m_1} \cdots t_d^{m_d}$ where $k \geq 0$ and $m \in k\Delta$. This ring is graded by the assignment $\deg(t_0^k t^m) = k$. Since the vector $(m, k) \in K$ if and only if $m \in k\Delta$, the ring S_Δ is isomorphic to the semigroup algebra $\mathbb{C}[K]$. The toric variety $\mathbb{P}_\Delta$ can be represented as

$$\mathrm{Proj}(S_\Delta) = \mathrm{Proj}(\mathbb{C}[K]).$$

The ring S_Δ has a nice connection to the homogeneous coordinate ring $S(\mathbb{P}_\Delta) = \mathbb{C}[x_\rho : \rho \in \Sigma_\Delta(1)]$ of the toric variety $\mathbb{P}_\Delta$, corresponding to a fan Σ_Δ. If $\beta \in A_{d-1}(\mathbb{P}_\Delta)$ is the class of the ample divisor $\sum_{\rho\in\Sigma_\Delta(1)} b_\rho D_\rho$ giving rise to the polytope Δ, then there is a natural isomorphism of graded rings

$$\mathbb{C}[K] \cong S_\Delta \cong \bigoplus_{k=0}^{\infty} S(\mathbb{P}_\Delta)_{k\beta}, \tag{3}$$

sending $(m, k) \in \mathbb{C}[K]_k$ to $t_0^k t^m$ and $\prod_\rho x_\rho^{kb_\rho+\langle m,e_\rho\rangle}$, where e_ρ is the minimal integral generator of the ray ρ. Now, given $f \in S(\mathbb{P}_\Delta)_\beta$, we get the ring $R_1(f)$. The polynomial $f = \sum_{m\in\Delta} f(m) x_\rho^{b_\rho+\langle m,e_\rho\rangle}$, where $f(m)$ are the coefficients, corresponds by the isomorphisms (3) to $\sum_{m\in\Delta} f(m) t_0 t^m \in (S_\Delta)_1$ and $\sum_{m\in\Delta} f(m)[m, 1] \in \mathbb{C}[K]_1$ (the brackets [ ] are used to distinguish the lattice points from the vectors over $\mathbb{C}$), which we also denote by f. By the proof of [BC, Theorem 11.5], we have that

$$(S(\mathbb{P}_\Delta)/J_0(f))_{k\beta} \cong (S_\Delta/\langle t_i\partial f/\partial t_i : i = 0, \dots, d\rangle)_k \cong R_0(f, K)_k,$$

where $R_0(f, K)$ is the quotient of $\mathbb{C}[K]$ by the ideal generated by all "logarithmic derivatives" of f:

$$\sum_{m\in\Delta} ((m, 1)\cdot n) f(m)[m, 1]$$

for $n \in N \oplus \mathbb{Z}$. The isomorphisms (3) induce the bijections

$$S(\mathbb{P}_\Delta)_{k\beta-\beta_0} \xrightarrow{\prod_\rho x_\rho} \langle\prod_\rho x_\rho\rangle_{k\beta} \cong (I_\Delta^{(1)})_k \cong \mathbb{C}[K^\circ]_k$$

($\beta_0 = \deg(\prod_\rho x_\rho)$), where $I_\Delta^{(1)} \subset S_\Delta$ is the ideal spanned by all monomials $t_0^k t^m$ such that m is in the interior of $k\Delta$, and $\mathbb{C}[K^\circ] \subset \mathbb{C}[K]$ is the ideal spanned by all lattice points in the relative interior of K. Since the space $R_1(f)_{k\beta-\beta_0}$ is isomorphic to the image of $\langle\prod_\rho x_\rho\rangle_{k\beta}$ in $(S(\mathbb{P}_\Delta)/J_0(f))_{k\beta}$,

$$R_1(f)_{k\beta-\beta_0} \cong R_1(f, K)_k,$$

where $R_1(f, K)$ is the image of $\mathbb{C}[K^\circ]$ in the graded ring $R_0(f, K)$.

The above discussion applies well to all faces Γ in Δ. In particular, if the toric variety $V(\sigma) \subset \mathbb{P}_\Delta$ corresponds to Γ, and $\beta^\sigma \in A_{d-\dim\sigma-1}(V(\sigma))$ is the restriction of the ample class β, then

$$S(V(\sigma))_{*\beta^\sigma} \cong \mathbb{C}[C],$$

where C is the Gorenstein cone over the polytope $\Gamma \times \{1\}$. This induces an isomorphism

$$R_1(f_\sigma)_{*\beta^\sigma-\beta_0^\sigma} \cong R_1(f_C, C),$$

where $f_C = \sum_{m\in\Gamma} f(m)[m,1]$ in $\mathbb{C}[C]_1$ is the projection of f to the cone C.

Now, we can restate our conjecture (2) in terms of Gorenstein cones:

$$\bigoplus_{p,q} QH^{p,q}_{\rm st}(Y) \cong \bigoplus_{\substack{p,q\\(C,C^*)}} R_1(\omega_{C^*}, C^*)_{(p+q-d+\dim C^*+1)/2} \otimes R_1(f_C, C)_{(q-p+\dim C)/2},$$

where the sum is by all dual faces of the reflexive Gorenstein cones K and K^*. This formula is already supported by Theorem 8.2 in [BDa], which for ample Calabi-Yau hypersurfaces in weighted projective spaces gives a corresponding decomposition of the stringy Hodge numbers (see Remark 5.2 in the next section). A generalization of [BDa, Theorem 8.2] will be proved in Section 7, justifying the above conjecture in the case of ample Calabi-Yau hypersurfaces in Fano toric varieties.

It is known that the string cohomology, which should be the limit of the quantum cohomology ring, of smooth Calabi-Yau hypersurfaces should be the same as the usual cohomology. We also know the property that the quantum cohomology spaces should be isomorphic for the ample Calabi-Yau hypersurface Y and its crepant resolution X. Therefore, it makes sense to compare the above description of $QH^{p,q}_{\rm st}(Y)$ with the description of the cohomology of semiample Calabi-Yau hypersurfaces X in Theorem 4.4. We can see that the right components in the tensor products coincide, by Proposition 4.5 and the definition of $R_1(f_C, C)$. On the other hand, the left components in $QH^{p,q}_{\rm st}(Y)$ for the ample Calabi-Yau hypersurface Y do not depend on a resolution, while the left components $T^\sigma(X)$ in $H^{p,q}(X)$ for the resolution X depend on the Stanley-Reisner ideal $SR(\Sigma)$. This hints us to the following definitions:

Definition 4.6. Let C be a Gorenstein cone in a lattice L, subdivided by a fan Σ, and let $\mathbb{C}[C]$ and $\mathbb{C}[C^\circ]$, where C° is the relative interior of C, be the semigroup rings. Define "deformed" ring structures $\mathbb{C}[C]^\Sigma$ and $\mathbb{C}[C]^\Sigma$ on $\mathbb{C}[C]$ and $\mathbb{C}[C^\circ]$, respectively, by the rule: $[m_1][m_2] = [m_1+m_2]$ if $m_1, m_2 \subset \sigma \in \Sigma$, and $[m_1][m_2] = 0$, otherwise.

Given $g = \sum_{m\in C, \deg m=1} g(m)[m]$, where $g(m)$ are the coefficients, let

$$R_0(g, C)^\Sigma = \mathbb{C}[C]^\Sigma / Z \cdot \mathbb{C}[C]^\Sigma$$

be the graded ring over the graded module

$$R_0(g, C^\circ)^\Sigma = \mathbb{C}[C^\circ]^\Sigma / Z \cdot \mathbb{C}[C^\circ]^\Sigma,$$

where $Z = \{\sum_{m\in C, \deg m=1} (m\cdot n) g(m)[m] : n \in \mathrm{Hom}(L, \mathbb{Z})\}$. Then define $R_1(g, C)^\Sigma$ as the image of the natural homomorphism $R_0(g, C^\circ)^\Sigma \to R_0(g, C)^\Sigma$.

Remark 4.7. In the above definition, note that if Σ is a trivial subdivision, we recover the spaces $R_0(g, C)$ and $R_1(g, C)$ introduced earlier. Also, we should mention that the Stanley-Reisner ring of the fan Σ can be naturally embedded into the "deformed" ring $\mathbb{C}[C]^\Sigma$, and this map is an isomorphism when the fan Σ is smooth.

Here is our conjecture about the string cohomology space of semiample Calabi-Yau hypersurfaces in a complete toric variety.

Conjecture 4.8. Let $X \subset \mathbb{P}_\Sigma$ be a semiample anticanonical nondegenerate hypersurface defined by $f \in H^0(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(X)) \cong \mathbb{C}[K]_1$, and let ω be a generic element in $\mathbb{C}[K^*]_1$, where K^* is the reflexive Gorenstein cone dual to the cone K over the reflexive polytope Δ associated to X. Then there is a natural isomorphism:

$$H^{p,q}_{\mathrm{st}}(X) \cong \bigoplus_{C \subseteq K} R_1(\omega_{C^*}, C^*)^{\Sigma}_{(p+q-d+\mathrm{dim}C^*+1)/2} \otimes R_1(f_C, C)_{(q-p+\mathrm{dim}C)/2},$$

where $C^* \subseteq K^*$ is a face dual to C, and where f_C, ω_{C^*} denote the projections of f and ω to the respective cones C and C^*. (Here, the superscript Σ denotes the subdivision of K^* induced by the fan Σ.)

Since the dimension of the string cohomology for all crepant partial resolutions should remain the same and should coincide with the dimension of the quantum string cohomology space, we expect that

$$\mathrm{dim}R_1(\omega_{C^*}, C^*)^{\Sigma}_{\text{-}} = \mathrm{dim}R_1(\omega_{C^*}, C^*)_{\text{-}}, \tag{4}$$

which will be shown in Section 6 for a projective subdivision Σ. Conjecture 4.8 will be confirmed by the corresponding decomposition of the stringy Hodge numbers in Section 7. Moreover, in Section 9, we will derive the Chen-Ruan orbifold cohomology as a special case of Conjecture 4.8 for ample Calabi-Yau hypersurfaces in complete simplicial toric varieties.

5. Hodge-Deligne numbers of affine hypersurfaces

Here, we compute the dimensions of the spaces $R_1(g, C)_{\text{-}}$ from the previous section. It follows from Proposition 4.5 that these dimensions are exactly the Hodge-Deligne numbers of the minimal weight space on the middle cohomology of a hypersurface in a torus. An explicit formula in [DKh] and [BDa] for the E-polynomial of a nondegenerate affine hypersurface whose Newton polyhedra is a simplex leads us to the answer for the graded dimension of $R_1(g, C)$ when C is a simplicial Gorenstein cone. However, it was very difficult to compute the Hodge-Deligne numbers of an arbitrary nondegenerate affine hypersurface. This was a major technical problem in the proof of mirror symmetry of the stringy Hodge numbers for Calabi-Yau complete intersections in [BBo]. Here, we will present a simple formula for the Hodge-Deligne numbers of a nondegenerate affine hypersurface.

Before we start computing $\mathrm{gr.dim.}R_1(g, C)$, let us note that for a nondegenerate $g \in \mathbb{C}[C]_1$ (i.e., the corresponding hypersurface in $\mathrm{Proj}(\mathbb{C}[C])$ is nondegenerate):

$$\mathrm{gr.dim.}R_0(g, C) = S(C, t),$$

where the polynomial S is the same as in Definition 2.1 of the stringy Hodge numbers. This was shown in [B1, Theorem 4.8 and 2.11] (see also [Bo1]).

When the cone C is simplicial, we already know the formula for the graded dimension of $R_1(g, C)$:

Proposition 5.1. *Let C be a simplicial Gorenstein cone, and let $g \in \mathbb{C}[C]_1$ be nondegenerate. Then*

$$\mathrm{gr.dim.}R_1(g, C) = \tilde{S}(C, t)$$

where $\tilde{S}(C, t) = \sum_{C_1 \subseteq C} S(C_1, t)(-1)^{\mathrm{dim}C - \mathrm{dim}C_1}$.

Proof. The polynomial $\tilde{S}(C,t)$ was introduced with a slightly different notation in [BDa, Definition 8.1] for a lattice simplex. One can check that $\tilde{S}(C,t)$ in this proposition is equivalent to the one in [BDa, Corollary 6.6]. From the previous section and [B1, Proposition 9.2], we know that

$$R_1(g,C) \cong \mathrm{Gr}_F W_{\mathrm{dim} Z_g} H^{\mathrm{dim} Z_g}(Z_g),$$

where Z_g is the nondegenerate affine hypersurface determined by g in the maximal torus of $\mathrm{Proj}(\mathbb{C}[C])$. By [BDa, Proposition 8.3],

$$E(Z_g;u,v) = \frac{(uv-1)^{\mathrm{dim} C-1} + (-1)^{\mathrm{dim} C}}{uv} + (-1)^{\mathrm{dim} C} \sum_{\substack{C_1 \subseteq C \\ \mathrm{dim} C_1 > 1}} \frac{u^{\mathrm{dim} C_1}}{uv} \tilde{S}(C_1, u^{-1}v).$$

Now, note that the coefficients $e^{p,q}(Z_g)$ at the monomials $u^p v^q$ with $p+q = \mathrm{dim} Z_g$ are related to the Hodge-Deligne numbers by the calculations in [DKh]:

$$e^{p,q}(Z_g) = (-1)^{\mathrm{dim} C} h^{p,q}(H^{\mathrm{dim} Z_g}(Z_g)) + (-1)^p \delta_{pq} C^p_{\mathrm{dim} C-1},$$

where δ_{pq} is the Kronecker symbol and $C^p_{\mathrm{dim} C-1}$ is the binomial coefficient. Comparing this with the above formula for $E(Z_g;u,v)$, we deduce the result. $\square$

Remark 5.2. By the above proposition, we can see that [BDa, Theorem 8.2] gives a decomposition of the stringy Hodge numbers of ample Calabi-Yau hypersurfaces in weighted projective spaces in correspondence with Conjecture 4.8.

Next, we generalize the polynomials $\tilde{S}(C,t)$ from Proposition 5.1 to nonsimplicial Gorenstein cones in such a way that they would count the graded dimension of $R_1(g,C)$.

Definition 5.3. Let C be a Gorenstein cone in a lattice L. Then set

$$\tilde{S}(C,t) := \sum_{C_1 \subseteq C} S(C_1,t)(-1)^{\mathrm{dim} C - \mathrm{dim} C_1} G([C_1,C],t),$$

where G is a polynomial (from Definition 11.1 in the Appendix) for the partially ordered set $[C_1,C]$ of the faces of C that contain C_1.

Remark 5.4. It is not hard to show that the polynomial $\tilde{S}(C,t)$ satisfies the duality

$$\tilde{S}(C,t) = t^{\mathrm{dim} C} \tilde{S}(C,t^{-1})$$

based on the duality properties of S and the definition of G-polynomials. However, the next result and Proposition 4.5 imply this fact.

Proposition 5.5. *Let C be a Gorenstein cone, and let $g \in \mathbb{C}[C]_1$ be nondegenerate. Then*

$$\mathrm{gr.dim.} R_1(g,C) = \tilde{S}(C,t).$$

Proof. As in the proof of Proposition 5.1, we consider a nondegenerate affine hypersurface Z_g determined by g in the maximal torus of $\mathrm{Proj}(\mathbb{C}[C])$. Then [BBo, Theorem 3.18] together with the definition of S gives

$$E(Z_g;u,v) = \frac{(uv-1)^{\mathrm{dim} C-1}}{uv} + \frac{(-1)^{\mathrm{dim} C}}{uv} \sum_{C_2 \subseteq C} B([C_2,C]^*;u,v) S(C_2, vu^{-1}) u^{\mathrm{dim} C_2},$$

where the polynomials B are from Definition 11.3. We use Lemma 11.4 and Definition 5.3 to rewrite this as

$$E(Z_g;u,v) = \frac{(uv-1)^{\dim C-1}}{uv} + \frac{(-1)^{\dim C}}{uv}\times$$
$$\times \sum_{C_2\subseteq C_1\subseteq C} u^{\dim C_2} S(C_2,u^{-1}v)G([C_2,C_1],u^{-1}v)(-u)^{\dim C_1-\dim C_2}G([C_1,C]^*,uv)$$
$$= \frac{(uv-1)^{\dim C-1}}{uv} + \frac{(-1)^{\dim C}}{uv}\sum_{C_1\subseteq C} u^{\dim C_1}\tilde{S}(C_1,u^{-1}v)G([C_1,C]^*,uv).$$

The definition of G-polynomials assures that the degree of $u^{\dim C_1}G([C_1,C]^*,uv)$ is at most $\dim C$ with the equality only when $C_1 = C$. Therefore, the graded dimension of $R_1(g,C)$ can be read off the same way as in the proof of Proposition 5.1 from the coefficients at total degree $\dim C-2$ in the above sum. $\square$

6. “Deformed” rings and modules

While this section may serve as an invitation to a new theory of “deformed” rings and modules, the goal here is to prove the equality (4), by showing that the graded dimension formula of Proposition 5.5 holds for the spaces $R_1(g,C)^\Sigma$ from Definition 4.6. To prove the formula we use the recent work of Bressler and Lunts (see [BreL], and also [BaBrFK]). This requires us to first study Cohen-Macaulay modules over the deformed semigroup rings $\mathbb{C}[C]^\Sigma$.

First, we want to generalize the nondegeneracy notion:

Definition 6.1. Let C be a Gorenstein cone in a lattice L, subdivided by a fan Σ. Given $g=\sum_{m\in C,\deg m=1} g(m)[m]$, get

$$g_j = \sum_{m\in C,\deg m=1}(m\cdot n_j)g(m)[m], \qquad \text{for } j=1,\dots,\dim C$$

where $\{n_1,\dots,n_{\dim C}\}\subset \operatorname{Hom}(L,\mathbb{Z})$ descends to a basis of $\operatorname{Hom}(L,\mathbb{Z})/C^\perp$. The element g is called *Σ-regular (nondegenerate)* if $\{g_1,\dots,g_{\dim C}\}$ forms a regular sequence in the deformed semigroup ring $\mathbb{C}[C]^\Sigma$.

Remark 6.2. When Σ is a trivial subdivision, [B1, Theorem 4.8] shows that the above definition is consistent with the previous notion of nondegeneracy corresponding to the transversality of a hypersurface to torus orbits.

Theorem 6.3. (i) *The ring $\mathbb{C}[C]^\Sigma$ and its module $\mathbb{C}[C^\circ]^\Sigma$ are Cohen-Macaulay.*
(ii) *A generic element $g\in\mathbb{C}[C]_1$ is Σ-regular. Moreover, for a generic g the sequence $\{g_1,\dots,g_{\dim C}\}$ from Definition 6.1 is $\mathbb{C}[C^\circ]^\Sigma$-regular.*
(iii) *If $g\in\mathbb{C}[C]_1$ is Σ-regular, then the sequence $\{g_1,\dots,g_{\dim C}\}$ is $\mathbb{C}[C^\circ]^\Sigma$-regular.*

Proof. Part (ii) follows from the proofs of Propositions 3.1 and 3.2 in [Bo1]. The reader should notice that the proofs use degenerations defined by projective simplicial subdivisions, and any fan admits such a subdivision.

Then, part (ii) implies (i), by the definition of Cohen-Macaulay, while part (iii) follows from (i) and Proposition 21.9 in [E]. $\square$

As a corollary of Theorem 6.3, we get the following simple description of Σ-regular elements:

Lemma 6.4. *An element $g \in \mathbb{C}[C]_1$ is Σ-regular, if and only if its restriction to all maximum-dimensional cones $C' \in \Sigma(\dim C)$ is nondegenerate in $\mathbb{C}[C']$.*

Proof. Since $\mathbb{C}[C]^\Sigma$ is Cohen-Macaulay, the regularity of a sequence is equivalent to the quotient by the sequence having a finite dimension, by [Mat, Theorem 17.4].

One can check that $\mathbb{C}[C]^\Sigma$ is filtered by the modules R_k defined as the span of $[m]$ such that the minimum cone that contains m has dimension at least k. The k-th graded quotient of this filtration is the direct sum of $\mathbb{C}[C_1^\circ]$ by all k-dimensional cones C_1 of Σ. If g is nondegenerate for every cone of maximum dimension, then its projection to any cone C_1 is nondegenerate, and Theorem 6.3 shows that it is nondegenerate for each $\mathbb{C}[C_1^\circ]$. Then by decreasing induction on k one shows that $R_k/\{g_1, \ldots, g_{\dim C}\}R_k$ is finite-dimensional.

In the other direction, it is easy to see that for every $C' \in \Sigma$ the $\mathbb{C}[C]^\Sigma$-module $\mathbb{C}[C']$ is a quotient of $\mathbb{C}[C]^\Sigma$, which gives a surjection

$$\mathbb{C}[C]^\Sigma/\{g_1, \ldots, g_{\dim C}\}\mathbb{C}[C]^\Sigma \to \mathbb{C}[C']/\{g_1|_{C'}, \ldots, g_{\dim C}|_{C'}\}\mathbb{C}[C'] \to 0.$$

□

The above lemma implies that the property of Σ-regularity is preserved by the restrictions:

Lemma 6.5. *Let C be a Gorenstein cone in a lattice L, subdivided by a fan Σ. If $g \in \mathbb{C}[C]_1$ is Σ-regular, then $g \in \mathbb{C}[C_1]_1$ is Σ-regular for all faces $C_1 \subseteq C$.*

Proof. Let $g \in \mathbb{C}[C]_1$ be Σ-regular. By Lemma 6.4, the restriction $g_{C'}$ is nondegenerate in $\mathbb{C}[C']$ for all $C' \in \Sigma(\dim C)$. Since the property of nondegeneracy associated with a hypersurface is preserved by the restrictions, $g_{C_1'}$ is nondegenerate in $\mathbb{C}[C_1']$ for all $C_1' \in \Sigma(\dim C_1)$ contained in C_1. Applying Lemma 6.4 again, we deduce the result. □

The next result generalizes [B1, Proposition 9.4] and [Bo1, Proposition 3.6].

Proposition 6.6. *Let $g \in \mathbb{C}[C]_1$ be Σ-regular, then $R_0(g, C)^\Sigma$ and $R_0(g, C^\circ)^\Sigma$ have graded dimensions $S(C, t)$ and $t^k S(C, t^{-1})$, respectively, and there exists a nondegenerate pairing*

$$\langle -, - \rangle : R_0(g, C)^\Sigma_k \times R_0(g, C^\circ)^\Sigma_{\dim C - k} \to R_0(g, C^\circ)^\Sigma_{\dim C} \cong \mathbb{C},$$

induced by the multiplicative $R_0(g, C)^\Sigma$-module structure.

Proof. It is easy to see that the above statement is equivalent to saying that $\mathbb{C}[C^\circ]^\Sigma$ is the canonical module for $\mathbb{C}[C]^\Sigma$. When Σ consists of the faces of C only, this is well-known (cf. [D]). To deal with the general case, we will heavily use the results of [E], Chapter 21.

We denote $A = \mathbb{C}[C]^\Sigma$. For every cone C_1 of Σ the vector spaces $\mathbb{C}[C_1]$ and $\mathbb{C}[C_1^\circ]$ are equipped with the natural A-module structures. By Proposition 21.10 of [E], modified for the graded case, we get

$$\mathrm{Ext}^i_A(\mathbb{C}[C_1], w_A) \cong \begin{cases} \mathbb{C}[C_1^\circ], & i = \mathrm{codim}(C_1) \\ 0, & i \neq \mathrm{codim}(C_1) \end{cases}$$

where w_A is the canonical module of A.

Consider now the complex $\mathcal{F}$ of A-modules

$$0 \to F^0 \to F^1 \to \cdots \to F^d \to 0$$

where

$$F^n = \bigoplus_{C_1 \in \Sigma, \mathrm{codim}(C_1) = n} \mathbb{C}[C_1]$$

and the differential is a sum of the restriction maps with signs according to the orientations. The nontrivial cohomology of $\mathcal{F}$ is located at F^0 and equals $\mathbb{C}[C^\circ]^\Sigma$. Indeed, by looking at each graded piece separately, we see that the cohomology occurs only at F^0, and then the kernel of the map to F^1 is easy to describe. We can now use the complex $\mathcal{F}$ and the description of $\mathrm{Ext}^i_A(\mathbb{C}[C_1], w_A)$ to try to calculate $\mathrm{Hom}_A(\mathbb{C}[C^\circ]^\Sigma, w_A)$. The resulting spectral sequence degenerates immediately, and we conclude that $\mathrm{Hom}_A(\mathbb{C}[C^\circ]^\Sigma, w_A)$ has a filtration such that the associated graded module is naturally isomorphic to

$$\bigoplus_{C_1 \in \Sigma} \mathbb{C}[C_1^\circ].$$

By duality of maximal Cohen-Macaulay modules (see [E]), it suffices to show that $\mathrm{Hom}_A(\mathbb{C}[C^\circ]^\Sigma, w_A) \cong A$, but the above filtration only establishes that it has the correct graded pieces, so extra arguments are required. Let C' be a cone of Σ of maximum dimension. We observe that $\mathcal{F}$ contains a subcomplex $\mathcal{F}'$ such that

$$F'^n = \bigoplus_{C_1 \subseteq C'} \mathbb{C}[C_1].$$

Similar to the case of $\mathcal{F}$, the cohomology of $\mathcal{F}'$ occurs only at F'^0 and equals $\mathbb{C}[C'^\circ]$. By snake lemma, the cohomology of $\mathcal{F}/\mathcal{F}'$ also occurs at the zeroth spot and equals $\mathbb{C}[C^\circ]^\Sigma/\mathbb{C}[C'^\circ]$. By looking at the spectral sequences again, we see that

$$\mathrm{Ext}^{>0}(\mathbb{C}[C^\circ]^\Sigma/\mathbb{C}[C'^\circ], w_A) = 0$$

and we have a grading preserving surjection

$$\mathrm{Hom}_A(\mathbb{C}[C^\circ]^\Sigma, w_A) \to \mathrm{Hom}_A(\mathbb{C}[C'^\circ], w_A) \to 0.$$

Since $\mathrm{Hom}_A(\mathbb{C}[C'], w_A) \cong \mathbb{C}[C'^\circ]$, duality of maximal Cohen-Macaulay modules over A shows that

$$\mathrm{Hom}_A(\mathbb{C}[C'^\circ], w_A) \cong \mathbb{C}[C']$$

so for every $m \in C'$ the element $[m]$ of A does not annihilate the degree zero element of $\mathrm{Hom}_A(\mathbb{C}[C^\circ]^\Sigma, w_A)$. By looking at all C' together, this shows that

$$\mathrm{Hom}_A(\mathbb{C}[C^\circ]^\Sigma, w_A) \cong A$$

which finishes the proof. □

Proposition 6.7. *Let $g \in \mathbb{C}[C]_1$ be Σ-regular, then the pairing $\langle \_, \_ \rangle$ induces a symmetric nondegenerate pairing $\{\_, \_\}$ on $R_1(g, C)^\Sigma$, defined by*

$$\{x, y\} = \langle x, y' \rangle$$

where y' is an element of $R_0(g, C^\circ)^\Sigma$ that maps to y.

Proof. The nondegeneracy of the pairing $\{\_, \_\}$ follows from that of $\langle \_, \_ \rangle$. The pairing is symmetric, because it comes from the commutative product on $\mathbb{C}[C^\circ]^\Sigma$. □

Theorem 6.8. *Let C be a Gorenstein cone subdivided by a projective fan Σ. If $g \in \mathbb{C}[C]_1$ is Σ-regular, then the graded dimension of $R_1(g, C)^\Sigma$ is $\tilde{S}(C, t)$.*

Proof. We will use the description of Bressler and Lunts [BreL] of locally free flabby sheaves on the finite ringed topological space associated to the cone C. We recall here the basic definitions. Consider the set P of all faces of the cone C. It is equipped with the topology in which open sets are subfans, i.e. the sets of faces closed under the operation of taking a face. Bressler and Lunts define a sheaf $\mathcal{A}$ of graded commutative rings on P whose sections over each open set is the ring of continuous piecewise polynomial functions on the union of all strata of this set. The grading of linear functions will be set to 1, contrary to the convention of [BreL].

They further restrict their attention to the sheaves $\mathcal{F}$ of $\mathcal{A}$-modules on P that satisfy the following conditions.
• For every face C_1 of C, sections of $\mathcal{F}$ over the open set that corresponds to the union of all faces of C_1 is a free module over the ring of polynomial functions on C_1.
• $\mathcal{F}$ is flabby, i.e. all restriction maps are surjective.

We will use the following crucial result.

Theorem 6.9. [BreL] *Every sheaf $\mathcal{F}$ that satisfies the above two properties is isomorphic to a direct sum of indecomposable graded sheaves $\mathcal{L}_{C_1}t^i$, where C_1 is a face of C and t^i indicates a shift in grading. For each indecomposable sheaf L_{C_1} the space of global sections $\Gamma(P, \mathcal{L}_{C_1})$ is a module over the polynomial functions on C of the graded rank $G([C_1, C]^*, t)$ where $[C_1, C]$ denotes the Eulerian subposet of P that consists of all faces of C that contain C_1.*

Now let us define a sheaf $\mathcal{B}(g)$ on P whose sections over the open subset $I \in P$ are $\mathbb{C}[\cap_{i\in I} C_i]^\Sigma$. It is clearly a flabby sheaf, which can be given a grading by deg(_). Moreover, $\mathcal{B}(g)$ can be given a structure of a sheaf of $\mathcal{A}$ modules as follows. Every linear function φ on a face C_1 defines a logarithmic derivative

$$\partial_\varphi g := \sum_{m \in C_1, \deg m = 1} \phi(m) g(m)[m]$$

of g, which is an element of the degree 1 in $\mathbb{C}[C_1]^\Sigma$. Then the action of φ is given by the multiplication by $\partial_\varphi g$, and this action is extended to all polynomial functions on the cone C_1. Similar construction clearly applies to continuous piecewise polynomial functions for any open set of P.

Proposition 6.6 assures that $\mathcal{B}(g)$ satisfies the second condition of Bressler and Lunts, and can therefore be decomposed into a direct sum of $L_{C_1}t^i$ for various C_1 and i. The definition of $R_1(g, C)^\Sigma$ implies that its graded dimension is equal to the graded rank of the stalk of $\mathcal{B}(g)$ at the point $C \in P$. Since the graded rank of $\mathcal{B}$ is $S(C, t)$, we conclude that

$$S(C,t) = \sum_{C_1 \subseteq C} \text{gr.dim.} R_1(g_{C_1}, C_1)^\Sigma G([C_1, C]^*, t).$$

To finish the proof of Theorem 6.8, it remains to apply Lemma 11.2. □

7. Decomposition of stringy Hodge numbers for hypersurfaces

In this section, we prove a generalization of [BDa, Theorem 8.2] for all Calabi-Yau hypersurfaces, which gives a decomposition of the stringy Hodge numbers of the hypersurfaces. First, we recall a formula for the stringy Hodge numbers of Calabi-Yau hypersurfaces obtained in [BBo]. Then using a bit of combinatorics, we rewrite this formula precisely to the form of [BDa, Theorem 8.2] with $\tilde{S}$ defined in the previous section.

The stringy Hodge numbers of a Calabi-Yau complete intersection have been calculated in [BBo] in terms of the numbers of integer points inside multiples of various faces of the reflexive polytopes Δ and Δ^* as well as some polynomial invariants of partially ordered sets. A special case of the main result in [BBo] is the following description of the stringy E-polynomials of Calabi-Yau hypersurfaces.

Theorem 7.1. [BBo] *Let $K \subset M \oplus \mathbb{Z}$ be the Gorenstein cone over a reflexive polytope $\Delta \subset M$. For every $(m,n) \in (K, K^*)$ with $m \cdot n = 0$ denote by $x(m)$ the minimum face of K that contains m and by $x^*(n)$ the dual of the minimum face of K^* that contains n. Also, set $A_{(m,n)}(u,v)$ be*

$$\frac{(-1)^{\dim(x^*(n))}}{uv}(v-u)^{\dim(x(m))}(uv-1)^{d+1-\dim(x^*(n))}B([x(m),x^*(n)]^*;u,v)$$

where the function B is defined in Definition 11.3 in the Appendix. Then

$$E_{\rm st}(Y;u,v) = \sum_{(m,n)\in(K,K^*),m\cdot n=0} \left(\frac{u}{v}\right)^{\deg m} A_{(m,n)}(u,v) \left(\frac{1}{uv}\right)^{\deg n}$$

for an ample nondegenerate Calabi-Yau hypersurface Y in $\mathbb{P}_\Delta = \mathrm{Proj}(\mathbb{C}[K])$.

The mirror duality $E_{\rm st}(Y;u,v) = (-u)^{d-1}E_{\rm st}(Y^*;u^{-1},v)$ was proved in [BBo] as the immediate corollary of the above formula and the duality property $B(P;u,v) = (-u)^{\mathrm{rk} P}B(P^*;u^{-1},v)$. It was not noticed there that Lemma 11.2 allows one to rewrite the B-polynomials in terms of G-polynomials, which we will now use to give a formula for the $E_{\rm st}(Y;u,v)$, explicitly obeying the mirror duality. The next result is a generalization of Theorem 8.2 in [BDa] with $\tilde{S}$ from Definition 5.3.

Theorem 7.2. *Let Y be an ample nondegenerate Calabi-Yau hypersurface in $\mathbb{P}_\Delta = \mathrm{Proj}(\mathbb{C}[K])$. Then*

$$E_{\rm st}(Y;u,v) = \sum_{C\subseteq K}(uv)^{-1}(-u)^{\dim C}\tilde{S}(C,u^{-1}v)\tilde{S}(C^*,uv).$$

Proof. First, observe that the formula for $E_{\rm st}(Y;u,v)$ from Theorem 7.1 can be written as

$$\sum_{m,n,C_1,C_2}\frac{(-1)^{\dim C_2^*}}{uv}(v-u)^{\dim(C_1)}B([C_2,C_1^*];u,v)(uv-1)^{\dim C_2}\left(\frac{u}{v}\right)^{\deg m}\left(\frac{1}{uv}\right)^{\deg n}$$

where the sum is taken over all pairs of cones $C_1 \subseteq K, C_2 \subseteq K^*$ that satisfy $C_1 \cdot C_2 = 0$ and all m and n in the relative interiors of C_1 and C_2, respectively. We use the standard duality result (see Definition 2.1)

$$\sum_{n\in \mathrm{int}(C)} t^{-\deg(n)} = (t-1)^{-\dim C}S(C,t)$$

to rewrite the above formula as

$$\frac{1}{uv}\sum_{C_1\cdot C_2=0}(-1)^{\dim(C_2^*)}u^{\dim C_1}B([C_2,C_1^*];u,v)S(C_1,u^{-1}v)S(C_2,uv).$$

Then apply Lemma 11.4 to get

$$E_{\rm st}(Y;u,v) = \frac{1}{uv}\sum_{C\in K}\sum_{C_1\subseteq C, C_2\subseteq C^*}(-1)^{\dim(C_2^*)}u^{\dim C_1} \times$$

$$\times\, G([C_1,C],u^{-1}v)(-u)^{\dim C_1^*-\dim C^*}G([C_2,C^*],uv)S(C_1,u^{-1}v)S(C_2,uv).$$

It remains to use Definition 5.3. □

8. String cohomology construction via intersection cohomology

Here, we construct the string cohomology space for $\mathbb{Q}$-Gorenstein toroidal varieties, satisfying the assumption of Proposition 2.6. The motivation for this construction comes from the conjectural description of the string cohomology space for ample Calabi-Yau hypersurfaces and a look at the formula in [BDa, Theorem 6.10] for the stringy E-polynomial of a Gorenstein variety with abelian quotient singularities. This immediately leads to a decomposition of the string cohomology space as a direct sum of tensor products of the usual cohomology of a closure of a strata with the spaces $R_1(g,C)$ from Proposition 5.1. Then the property that the intersection cohomology of an orbifold is naturally isomorphic to the usual cohomology leads us to the construction of the string cohomology space for $\mathbb{Q}$-Gorenstein toroidal varieties. We show that this space has the dimension prescribed by Definition 2.1 for Gorenstein complete toric varieties and the nondegenerate complete intersections in them.

Conjectural Definition 8.1. Let $X=\bigcup_{i\in I}X_i$ be a Gorenstein complete variety with quotient abelian singularities, satisfying the assumption of Proposition 2.6. The stringy Hodge spaces of X are naturally isomorphic to

$$H^{p,q}_{\rm st}(X)\cong\bigoplus_{\substack{i\in I\\ k\geq 0}}H^{p-k,q-k}(\overline{X}_i)\otimes R_1(\omega_{\sigma_i},\sigma_i)_k,$$

where σ_i is the Gorenstein simplicial cone of the singularity along the strata X_i, and $\omega_{\sigma_i}\in\mathbb{C}[\sigma_i]_1$ are nondegenerate such that, for $\sigma_j\subset\sigma_i$, ω_{σ_i} maps to ω_{σ_j} by the natural projection $\mathbb{C}[\sigma_i]\to\mathbb{C}[\sigma_j]$.

Remark 8.2. Since $\overline{X}_i$ is a compact orbifold, the coefficient $e^{p,q}(\overline{X}_i)$ at the monomial u^pv^q in the polynomial $E(\overline{X}_i;u,v)$ is equal to $(-1)^{p+q}h^{p,q}(\overline{X}_i)$, by Remark 2.2. Therefore, Proposition 5.1 shows that the above decomposition of $H^{p,q}_{\rm st}(X)$ is in correspondence with [BDa, Theorem 6.10], and the dimensions $h^{p,q}_{\rm st}(X)$ coincide with those from Definition 2.3.

Since we expect that the usual cohomology must be replaced in Definition 8.1 by the intersection cohomology for Gorenstein toroidal varieties, the next result is a natural generalization of Theorem 6.10 in [BDa].

Theorem 8.3. *Let $X=\bigcup_{i\in I}X_i$ be a Gorenstein complete toric variety or a nondegenerate complete intersection of Cartier hypersurfaces in the toric variety, where the stratification is induced by the torus orbits. Then*

$$E_{\rm st}(X;u,v)=\sum_{i\in I}E_{\rm int}(\overline{X}_i;u,v)\cdot\tilde{S}(\sigma_i,uv),$$

where σ_i is the Gorenstein cone of the singularity along the strata X_i.

Proof. Similarly to Corollary 3.17 in [BBo], we have

$$E_{\rm int}(\overline{X}_i;u,v)=\sum_{X_j\subseteq\overline{X}_i}E(X_i;u,v)\cdot G([\sigma_i\subseteq\sigma_j]^*,uv).$$

Hence, we get

$$\sum_{i\in I} E_{\rm int}(\overline{X}_i;u,v)\cdot\tilde{S}(\sigma_i,uv)=\sum_{i\in I}\sum_{X_j\subseteq\overline{X}_i} E(X_j;u,v)G([\sigma_i\subseteq\sigma_j]^*,uv)\tilde{S}(\sigma_i,uv)$$

$$=\sum_{j\in I} E(X_j;u,v)\Big(\sum_{\sigma_i\subseteq\sigma_j} G([\sigma_i\subseteq\sigma_j]^*,uv)\tilde{S}(\sigma_i,uv)\Big)=\sum_{j\in I} E(X_j;u,v)S(\sigma_j,uv),$$

where at the last step we have used the formula for $\tilde{S}$ and Lemma 11.2. □

Based on the above theorem, we propose the following conjectural description of the stringy Hodge spaces for $\mathbb{Q}$-Gorenstein toroidal varieties.

Conjectural Definition 8.4. Let $X=\bigcup_{i\in I} X_i$ be a $\mathbb{Q}$-Gorenstein d-dimensional complete toroidal variety, satisfying the assumption of Proposition 2.6. The stringy Hodge spaces of X are defined by:

$$H^{p,q}_{\rm st}(X):=\bigoplus_{\substack{i\in I\\ k\geq 0}} H^{p-k,q-k}_{\rm int}(\overline{X}_i)\otimes R_1(\omega_{\sigma_i},\sigma_i)_k,$$

where σ_i is the Gorenstein cone of the singularity along the strata X_i, and $\omega_{\sigma_i}\in\mathbb{C}[\sigma_i]_1$ are nondegenerate such that, for $\sigma_j\subset\sigma_i$, ω_{σ_i} maps to ω_{σ_j} by the natural projection $\mathbb{C}[\sigma_i]\to\mathbb{C}[\sigma_j]$. Here, p,q are rational numbers from $[0,d]$, and we assume that $H^{p-k,q-k}_{\rm int}(\overline{X}_i)=0$ if $p-k$ or $q-k$ is not a non-negative integer.

Remark 8.5. Toric varieties and nondegenerate complete intersections of Cartier hypersurfaces have the stratification induced by the torus orbits which satisfies the assumptions in the above definition.

9. String cohomology vs. Chen-Ruan orbifold cohomology

Our next goal is to compare the two descriptions of string cohomology for Calabi-Yau hypersurfaces to the Chen-Ruan orbifold cohomology. Using the work of [P], we will show that in the case of ample orbifold Calabi-Yau hypersurfaces the three descriptions coincide. We refer the reader to [ChR] for the orbifold cohomology theory and only use [P] in order to describe the orbifold cohomology for complete simplicial toric varieties and Calabi-Yau hypersurfaces in Fano simplicial toric varieties.

From Theorem 1 in [P, Section 4] and the definition of the orbifold Dolbeault cohomology space we deduce:

Proposition 9.1. *Let $\mathbb{P}_\Sigma$ be a d-dimensional complete simplicial toric variety. Then the orbifold Dolbeault cohomology space of $\mathbb{P}_\Sigma$ is*

$$H^{p,q}_{orb}(\mathbb{P}_\Sigma;\mathbb{C})\cong\bigoplus_{\substack{\sigma\in\Sigma\\ l\in\mathbb{Q}}} H^{p-l,q-l}(V(\sigma))\otimes\bigoplus_{t\in T(\sigma)_l}\mathbb{C}t,$$

where $T(\sigma)_l=\{\sum_{\rho\subset\sigma}a_\rho[e_\rho]\in N:a_\rho\in(0,1),\sum_{\rho\subset\sigma}a_\rho=l\}$ (when $\sigma=0$, set $l=0$ and $T(\sigma)_l=\mathbb{C}$), and $V(\sigma)$ is the closure of the torus orbit corresponding to $\sigma\in\Sigma$. Here, p and q are rational numbers in $[0,d]$, and $H^{p-l,q-l}(V(\sigma))=0$ if $p-l$ or $q-l$ is not integral. (The elements of $\oplus_{0\neq\sigma\in\Sigma,l}T(\sigma)_l$ correspond to the twisted sectors.)

In order to compare this result to the description in Definition 8.4, we need to specify the ω_{σ_i} for the toric variety $\mathbb{P}_\Sigma$. The stratification of $\mathbb{P}_\Sigma$ is given by the torus orbits: $\mathbb{P}_\Sigma = \cup_{\sigma\in\Sigma}\mathbb{T}_\sigma$. The singularity of the variety $\mathbb{P}_\Sigma$ along the strata $\mathbb{T}_\sigma$ is given by the cone σ, so we need to specify a nondegenerate $\omega_\sigma \in \mathbb{C}[\sigma]_1$ for each $\sigma\in\Sigma$. If $\omega_\sigma = \sum_{\rho\subset\sigma}\omega_\rho[e_\rho]$ with $\omega_\rho \neq 0$, then one can deduce that ω_σ is nondegenerate using Remark 6.2 and the fact that the nondegeneracy of a hypersurface in a complete simplicial toric variety (in this case, it corresponds to a simplex) is equivalent to the nonvanishing of the logarithmic derivatives simultaneously. So, picking any nonzero coefficients ω_ρ for each $\rho\in\Sigma(1)$ gives a nondegenerate $\omega_\sigma\in\mathbb{C}[\sigma]_1$ satisfying the condition of Definition 8.4. For such ω_σ, note that the set $Z = \{\sum_{\rho\subset\sigma}(e_\rho\cdot m)\omega_\rho e_\rho : m\in \mathrm{Hom}(N,\mathbb{Z})\}$ is a linear span of e_ρ for $\rho\subset\sigma$. Hence,

$$R_0(\omega_\sigma,\sigma)_l = (\mathbb{C}[\sigma]/Z\cdot\mathbb{C}[\sigma])_l \cong \bigoplus_{t\in\tilde{T}(\sigma)_l}\mathbb{C}t,$$

where $\tilde{T}(\sigma)_l = \{\sum_{\rho\subset\sigma}a_\rho e_\rho\in N : a_\rho\in[0,1), \sum_{\rho\subset\sigma}a_\rho = l\}$, and

$$R_1(\omega_\sigma,\sigma)_l \cong \bigoplus_{t\in T(\sigma)_l}\mathbb{C}t.$$

This shows that the orbifold Dolbeault cohomology for complete simplicial toric varieties can be obtained as a special case of the description of string cohomology in Definition 8.4.

We will now explain how the parameter ω should be related to the complexified Kähler class. We do not have the definition of the "orbifold" Kähler cone even for simplicial toric varieties. However, we know the Kähler classes in $H^2(\mathbb{P}_\Sigma,\mathbb{R})$.

Proposition 9.2. *Let $\mathbb{P}_\Sigma$ be a projective simplicial toric variety, then $H^2(\mathbb{P}_\Sigma,\mathbb{R})\cong PL(\Sigma)/M_\mathbb{R}$, where $PL(\Sigma)$ is the set of Σ-piecewise linear functions $\varphi: N_\mathbb{R}\to\mathbb{R}$, which are linear on each $\sigma\in\Sigma$. The Kähler cone $K(\Sigma)\subset H^2(\mathbb{P}_\Sigma,\mathbb{R})$ of $\mathbb{P}_\Sigma$ consists of the classes of the upper strictly convex Σ-piecewise linear functions.*

One may call $K(\Sigma)$ the "untwisted" part of the orbifold Kähler cone. So, we can introduce the *untwisted complexified Kähler space* of the complete simplicial toric variety:

$$K_\mathbb{C}^{\mathrm{untwist}}(\mathbb{P}_\Sigma) = \{\omega\in H^2(\mathbb{P}_\Sigma,\mathbb{C}) : Im(\omega)\in K(\Sigma)\}/\mathrm{im}H^2(\mathbb{P}_\Sigma,\mathbb{Z}).$$

Its elements may be called the *untwisted complexified Kähler classes*. We can find a generic enough $\omega\in K_\mathbb{C}^{\mathrm{untwist}}(\mathbb{P}_\Sigma)$ represented by a complex valued Σ-piecewise linear function $\varphi_\omega : N_\mathbb{C}\to\mathbb{C}$ such that $\varphi_\omega(e_\rho)\neq 0$ for $\rho\in\Sigma(1)$. Setting $\omega_\rho = \exp(\varphi_\omega(e_\rho))$ produces our previous parameters ω_σ for $\sigma\in\Sigma$. This is how we believe ω_σ should relate to the complexified Kähler classes, up to perhaps some instanton corrections.

We next turn our attention to the case of an ample Calabi-Yau hypersurface Y in a complete simplicial toric variety $\mathbb{P}_\Sigma$. Section 4.2 in [P] works with a generic nondegenerate anticanonical hypersurface. However, one can avoid the use of Bertini's theorem and state the result without "generic". It is shown that the nondegenerate anticanonical hypersurface X is a suborbifold of $\mathbb{P}_\Sigma$, the twisted sectors of Y are obtained by intersecting with the closures of the torus orbits and the degree shifting numbers are the same as for the toric variety $\mathbb{P}_\Sigma$. Therefore, we conclude:

Proposition 9.3. *Let $Y \subset \mathbb{P}_\Sigma$ be an ample Calabi-Yau hypersurface in a complete simplicial toric variety. Then*

$$H^{p,q}_{orb}(Y;\mathbb{C}) \cong \bigoplus_{\substack{\sigma\in\Sigma\\ l\in\mathbb{Z}}} H^{p-l,q-l}(Y\cap V(\sigma)) \otimes \bigoplus_{t\in T(\sigma)_l} \mathbb{C}t,$$

where $T(\sigma)_l = \{\sum_{\rho\subset\sigma} a_\rho[e_\rho] \in N : a_\rho \in (0,1), \sum_{\rho\subset\sigma} a_\rho = l\}$ (when $\sigma = 0$, set $l = 0$ and $T(\sigma)_l = \mathbb{C}$).

As in the case of the toric variety, we pick $\omega_\sigma = \sum_{\rho\subset\sigma}\omega_\rho e_\rho$ with $\omega_\rho \neq 0$. Then, by the above proposition,

$$H^{p,q}_{\rm st}(Y) \cong H^{p,q}_{orb}(Y;\mathbb{C}).$$

We now want to show that the description in Proposition 9.3 is equivalent to the one in Conjecture 4.8. First, note that the proper faces C^* of the Gorenstein cone K^* in Conjecture 4.8 one to one correspond to the cones $\sigma \in \Sigma$. Moreover, the rings $\mathbb{C}[C^*] \cong \mathbb{C}[\sigma]$ are isomorphic in this correspondence. If we take $\omega \in \mathbb{C}[K^*]^\Sigma_1$ to be $[0,1] + \sum_{\rho\in\Sigma(1)}\omega_\rho[e_\rho, 1]$, then ω is Σ-regular and

$$R_1(C^*, \omega_{C^*})_l \cong R_1(\omega_\sigma, \sigma)_l \cong \oplus_{t\in T(\sigma)_l}\mathbb{C}t.$$

On the other hand, the Hodge component $H^{p-l,q-l}(Y\cap V(\sigma))$ decomposes into the direct sum

$$H^{p-l,q-l}_{\rm toric}(Y\cap V(\sigma)) \oplus H^{p-l,q-l}_{\rm res}(Y\cap V(\sigma))$$

of the toric and residue parts. Since $Y \cap V(\sigma)$ is an ample hypersurface, from [BC, Theorem 11.8] and Section 4 it follows that

$$H^{p-l,q-l}_{\rm res}(Y\cap V(\sigma)) \cong R_1(f_C, C)_{q-l+1},$$

where $C \subset K$ is the face dual to C^* which corresponds to σ, $p + q - 2l = {\rm dim}Y \cap V(\sigma) = d - {\rm dim}\sigma - 1 = d - {\rm dim}C^* - 1$. If $p + q - 2l \neq d - {\rm dim}C^* - 1$, then $H^{p-l,q-l}_{\rm res}(Y\cap V(\sigma)) = 0$. Hence, we get

$$\bigoplus_{\substack{\sigma\in\Sigma\\ l\in\mathbb{Z}}} H^{p-l,q-l}_{\rm res}(Y\cap V(\sigma)) \otimes \bigoplus_{t\in T(\sigma)_l} \mathbb{C}t \cong \bigoplus_{0\neq C\subseteq K} R_1(\omega_{C^*}, C^*)^\Sigma_a \otimes R_1(f_C, C)_b,$$

where $a = (p + q - d + {\rm dim}C^* + 1)/2$ and $b = (q - p + {\rm dim}C)/2$. We are left to show that

$$\bigoplus_{\substack{\sigma\in\Sigma\\ l\in\mathbb{Z}}} H^{p-l,p-l}_{\rm toric}(Y\cap V(\sigma)) \otimes \bigoplus_{t\in T(\sigma)_l} \mathbb{C}t \cong R_1(\omega, K^*)^\Sigma_{p+1}. \tag{5}$$

Notice that the dimensions of the spaces on both sides coincide, so it suffices to construct a surjective map between them. This will follow from the following proposition.

Proposition 9.4. *Let $\mathbb{P}_\Sigma = {\rm Proj}(\mathbb{C}[K])$ be the Gorenstein Fano simplicial toric variety, where K as above. Then there is a natural isomorphism:*

$$H^{p,p}_{\rm st}(\mathbb{P}_\Sigma) \cong \bigoplus_{\substack{\sigma\in\Sigma\\ l\in\mathbb{Z}}} H^{p-l,p-l}(V(\sigma)) \otimes \bigoplus_{t\in T(\sigma)_l} \mathbb{C}t \cong R_0(\omega, K^*)^\Sigma_p,$$

where $\omega = [0,1] + \sum_{\rho\in\Sigma(1)}\omega_\rho[e_\rho, 1]$ with $\omega_\rho \neq 0$.

Proof. First, observe that the dimensions of the spaces in the isomorphisms coincide by our definition of string cohomology, Proposition 6.6 and [BDa, Theorem 7.2]. So, it suffices to construct a surjective map between them.

We know the cohomology ring of the toric variety:

$$H^*(V(\sigma)) \cong \mathbb{C}[D_\rho : \rho \in \Sigma(1), \rho+\sigma \in \Sigma(\dim\sigma+1)]/(P(V(\sigma)) + SR(V(\sigma))),$$

where

$$SR(V(\sigma)) = \big\langle D_{\rho_1}\cdots D_{\rho_k} : \{e_{\rho_1},\dots,e_{\rho_k}\} \not\subset \tau \text{ for all } \sigma\subset\tau\in\Sigma(\dim\sigma+1)\big\rangle$$

is the Stanley-Reisner ideal, and

$$P(V(\sigma)) = \Big\langle \sum_{\rho\in\Sigma(1),\rho+\sigma\in\Sigma(\dim\sigma+1)} \langle m, e_\rho\rangle D_\rho : m \in M\cap\sigma^\perp \Big\rangle.$$

Define the maps from $H^{p-l,p-l}(V(\sigma)) \otimes \bigoplus_{t\in T(\sigma)_l} \mathbb{C}t$ to $R_0(\omega, K^*)^\Sigma$ by sending $D_{\rho_1}\cdots D_{\rho_{p-l}} \otimes t$ to $\omega_{\rho_1}[e_{\rho_1}]\cdots\omega_{\rho_{p-l}}[e_{\rho_{p-l}}]\cdot t \in \mathbb{C}[N]^\Sigma$. One can easily see that these maps are well defined. To finish the proof we need to show that the images cover $R_0(\omega, K^*)^\Sigma$. Every lattice point $[n]$ in the boundary of K^* lies in the relative interior of a face $C \subset K^*$, and can be written as a linear combination of the minimal integral generators of C:

$$[n] = \sum_{[e_\rho,1]\in C} (a_\rho + b_\rho)[e_\rho, 1],$$

where $a_\rho \in (0,1)$ and b_ρ are nonnegative integers. Let $C' \subseteq C$ be the cone spanned by those $[e_\rho, 1]$ for which $a_\rho \neq 0$. The lattice point $\sum_{[e_\rho,1]\in C'} (a_\rho)[e_\rho, 1]$ projects to one of the elements t from $T(\sigma)_l$ for some l and σ corresponding to C'. Using the relations $\sum_{\rho\in\Sigma(1)} \omega_\rho\langle m, e_\rho\rangle[e_\rho, 1]$ in the ring $R_0(\omega, K^*)^\Sigma$, we get that

$$[n] = \sum_{[e_\rho,1]\in C'} (a_\rho)[e_\rho, 1] + \sum_{\rho+\sigma\in\Sigma(\dim\sigma+1)} b'_\rho[e_\rho, 1],$$

which comes from $H^{p-l,p-l}(V(\sigma)) \otimes \mathbb{C}t$ for an appropriate p. The surjectivity now follows from the fact that the boundary points of K^* generate the ring $C[K^*]^\Sigma/\langle\omega\rangle$. □

The isomorphism (5) follows from the above proposition and the presentation:

$$H^*_{\rm toric}(Y\cap V(\sigma)) \cong H^*(V(\sigma))/\mathrm{Ann}([Y\cap V(\sigma)])$$

(see (1)). Indeed, the map constructed in the proof of Proposition 9.4 produces a well defined map between the right hand side in (5) and $R_0(\omega, K^*)^\Sigma/Ann([0,1])$ because the annihilator of $[Y\cap V(\sigma)]$ maps to the annihilator of $[0,1]$. On the other hand,

$$(R_0(\omega, K^*)^\Sigma/Ann([0,1]))_p \cong R_1(\omega, K^*)^\Sigma_{p+1},$$

which is induced by the multiplication by $[0,1]$ in $R_0(\omega, K^*)^\Sigma$.

Conjecture 9.5. We expect that the product structure on $H^*_{\rm st}(\mathbb{P}_\Sigma)$ is given by the ring structure $R_0(\omega, K^*)^\Sigma$. Also, the ring structure on $R_1(\omega, K^*)^\Sigma_{*+1}$ induced from $R_0(\omega, K^*)^\Sigma/Ann([0,1])$ should give a subring of $H^*_{\rm st}(Y)$ for a generic ω in Conjecture 4.8. Moreover,

$$\bigoplus_{p,q} R_1(\omega_{C^*}, C^*)^\Sigma_{(p+q-d+\dim C^*+1)/2} \otimes R_1(f_C, C)_{(q-p+\dim C)/2},$$

should be the module over the ring $R_0(\omega, K^*)^\Sigma/Ann([0,1])$:

$$a\cdot(b\otimes c)=\bar{a}b\otimes c,$$

for $a\in R_0(\omega,K^*)^\Sigma/Ann([0,1])$ and $(b\otimes c)$ from a component of the above direct sum, where $\bar{a}$ is the image of a induced by the projection $R_0(\omega,K^*)^\Sigma\to R_0(\omega_{C^*},C^*)^\Sigma$.

We can also say about the product structure on the B-model chiral ring. The space $R_1(f,K)\cong R_0(f,K)/Ann([0,1])$ in Conjecture 4.8, which lies in the middle cohomology $\oplus_{p+q=d-1}H^{p,q}_{\rm st}(Y)$, should be a subring of the B-model chiral ring, and

$$\bigoplus_{p,q} R_1(\omega_{C^*},C^*)^\Sigma_{(p+q-d+{\rm dim}C^*+1)/2}\otimes R_1(f_C,C)_{(q-p+{\rm dim}C)/2},$$

should be the module over the ring $R_1(f,K)$, similarly to the above description in the previous paragraph.

These ring structures are consistent with the products on the usual cohomology and the B-model chiral ring $H^*(X,\bigwedge^* T_X)$ of the smooth semiample Calabi-Yau hypersurfaces X in [M3, Theorem 2.11(a,b)] and [M2, Theorem 7.3(i,ii)].

10. Description of string cohomology inspired by vertex algebras

Here we will give yet another description of the string cohomology spaces of Calabi-Yau hypersurfaces. It will appear as cohomology of a certain complex, which was inspired by the vertex algebra approach to Mirror Symmetry.

We will state the result first in the non-deformed case, and it will be clear what needs to be done in general. Let K and K^* be dual reflexive cones of dimension $d+1$ in the lattices M and N respectively. We consider the subspace $\mathbb{C}[L]$ of $\mathbb{C}[K]\otimes\mathbb{C}[K^*]$ as the span of the monomials $[m,n]$ with $m\cdot n=0$. We also pick non-degenerate elements of degree one $f=\sum_m f_m[m]$ and $g=\sum_n g_n[n]$ in $\mathbb{C}[K]$ and $\mathbb{C}[K^*]$ respectively.

Consider the space

$$V=\Lambda^*(N_{\mathbb{C}})\otimes\mathbb{C}[L].$$

Lemma 10.1. *The space V is equipped with a differential D given by*

$$D:=\sum_m f_m \lrcorner m\otimes(\pi_L\circ[m])+\sum_n g_n(\wedge n)\otimes(\pi_L\circ[n])$$

where $[m]$ and $[n]$ means multiplication by the corresponding monomials in $\mathbb{C}[K]\otimes\mathbb{C}[K^*]$ and π_L denotes the natural projection to $\mathbb{C}[L]$.

Proof. It is straightforward to check that $D^2=0$. □

Theorem 10.2. *Cohomology H of V with respect to D is naturally isomorphic to*

$$\bigoplus_{C\subseteq K}\Lambda^{{\rm dim}C^*}C^*_{\mathbb{C}}\otimes R_1(f,C)\otimes R_1(g,C^*)$$

where $C^*_{\mathbb{C}}$ denotes the vector subspace of $N_{\mathbb{C}}$ generated by C^*.

Proof. First observe that V contains a subspace

$$\bigoplus_{C\subseteq K}\Lambda^*N_{\mathbb{C}}\otimes(\mathbb{C}[C^\circ]\otimes\mathbb{C}[C^{*\circ}])$$

which is invariant under D. It is easy to calculate the cohomology of this subspace under D, because the action commutes with the decomposition $\oplus_C$. For each C, the cohomology of D on $\Lambda^* N_{\mathbb{C}} \otimes (\mathbb{C}[C^\circ] \otimes \mathbb{C}[C^{*\circ}])$ is naturally isomorphic to

$$\Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_0(f, C^\circ) \otimes R_0(g, C^{*\circ}),$$

because $\Lambda^* N_{\mathbb{C}} \otimes (\mathbb{C}[C^\circ] \otimes \mathbb{C}[C^{*\circ}])$ is a tensor product of the Koszul complex for $\mathbb{C}[C^\circ]$ and the dual of the Koszul complex for $\mathbb{C}[C^{*\circ}]$. As a result, we have a map

$$\alpha : H_1 \to H,\ H_1 := \bigoplus_{C \subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_0(f, C^\circ) \otimes R_0(g, C^{*\circ}).$$

Next, we observe that V embeds naturally into the space

$$\bigoplus_{C \subseteq K} \Lambda^* N_{\mathbb{C}} \otimes (\mathbb{C}[C] \otimes \mathbb{C}[C^*])$$

as the subspace of the elements compatible with the restriction maps. This defines a map

$$\beta : H \to \bigoplus_{C \subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_0(f, C) \otimes R_0(g, C^*) =: H_2.$$

We observe that the composition $\beta \circ \alpha$ is precisely the map induced by embeddings $C^\circ \subseteq C$ and $C^{*\circ} \subseteq C^*$, so its image in H_2 is

$$\bigoplus_{C \subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_1(f, C) \otimes R_1(g, C^*).$$

As a result, what we need to show is that α is surjective and β is injective. We can not do this directly, instead, we will use spectral sequences associated to two natural filtrations on V.

First, consider the filtration

$$V = V^0 \supset V^1 \supset \ldots \supset V^{d+1} \supset V^{d+2} = 0$$

where V^p is defined as $\Lambda^* N_{\mathbb{C}}$ tensored with the span of all monomials $[m, n]$ for which the smallest face of K that contains m has dimension at least p. It is easy to see that the spectral sequence of this filtration starts with

$$H_3 := \bigoplus_{C \subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_0(f, C^\circ) \otimes R_0(g, C^*).$$

Analogously, we have a spectral sequence from

$$H_4 := \bigoplus_{C \subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_0(f, C) \otimes R_0(g, C^{*\circ})$$

to H, which gives us the following diagram.

$$\begin{array}{ccccc} & & H_3 & & \\ & \nearrow & \Downarrow & \searrow & \\ H_1 & \to & H & \to & H_2 \\ & \searrow & \Uparrow & \nearrow & \\ & & H_4 & & \end{array}$$

We remark that the spectral sequences mean that H is a subquotient of both H_3 and H_4, i.e. there are subspaces I_3^+ and I_3^- of H_3 such that $H \simeq I_3^+/I_3^-$, and similarly for H_4. Moreover, the above diagram induces commutative diagrams

$$\begin{tikzcd} & 0 \arrow[d] & \\ & I_3^- \arrow[d] & \\ & I_3^+ \arrow[d] \arrow[dr] & \\ H_1 \arrow[ur] \arrow[r] & H \arrow[r] \arrow[d] & H_2 \\ & 0 & \end{tikzcd} \qquad \begin{tikzcd} & 0 & \\ H_1 \arrow[r] \arrow[dr] & H \arrow[r] \arrow[u] & H_2 \\ & I_4^+ \arrow[u] \arrow[ur] & \\ & I_4^- \arrow[u] & \\ & 0 \arrow[u] & \end{tikzcd}$$

with exact vertical lines. Indeed, the filtration V^* induces a filtration on the subspace of V

$$\bigoplus_{C\subseteq K} \Lambda^* \otimes \mathbb{C}[C^\circ] \otimes \mathbb{C}[C^{*\circ}].$$

The resulting spectral sequence degenerates immediately, and the functoriality of spectral sequences assures that there are maps from H_1 as above. Similarly, the space

$$\bigoplus_{C\subseteq K} \Lambda^* \otimes \mathbb{C}[C] \otimes \mathbb{C}[C^*]$$

has a natural filtration by the dimension of C that induces the filtration on V. Functoriality then gives the maps to H_4.

We immediately get

$$Im(\beta) \subseteq Im(H_3 \to H_2) \cap Im(H_4 \to H_2)$$

which implies that

$$Im(\beta) = Im(\beta\circ\alpha) = \bigoplus_{C\subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_1(f,C) \otimes R_1(g,C^*).$$

Analogously, $Ker(\alpha) = Ker(\beta\circ\alpha)$, which shows that

$$\bigoplus_{C\subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_1(f,C) \otimes R_1(g,C^*)$$

is a direct summand of H.

The fact that

$$\begin{aligned} &Ker(\alpha) \supseteq Ker(H_1 \to H_4) \\ = \bigoplus_{C\subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} &\otimes Ker(R_0(f,C^\circ) \to R_0(f,C)) \otimes R_0(g,C^{*\circ}) \end{aligned}$$

implies that I_3^- contains the image of this space under $H_1 \to H_3$, which is equal to

$$\bigoplus_{C\subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes Ker(R_0(f,C^\circ) \to R_0(f,C)) \otimes R_1(g,C^*).$$

Similarly, I_3^+ is contained in the preimage of

$$\bigoplus_{C\subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_1(f,C) \otimes R_1(g,C^*)$$

under $H_3 \to H_2$, which is

$$\bigoplus_{C\subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_0(f,C^\circ) \otimes R_1(g,C^*).$$

As a result,

$$H = \bigoplus_{C\subseteq K} \Lambda^{\dim C^*} C^*_{\mathbb{C}} \otimes R_1(f,C) \otimes R_1(g,C^*).$$

□

Remark 10.3. If one replaces $\mathbb{C}[K^*]$ by $\mathbb{C}[K^*]^{\Sigma}$ in the definition of $\mathbb{C}[L]$, then the statement and the proof of Theorem 10.2 remain intact. In addition, one can make a similar statement after replacing $\Lambda^* N_{\mathbb{C}}$ by $\Lambda^* M_{\mathbb{C}}$ and switching contraction and exterior multiplication in the definition of D. It is easy to see that the resulting complex is basically identical, though various gradings are switched. This should correspond to a switch between A and B models.

We will now briefly outline the connection between Theorem 10.2 and the vertex algebra approach to mirror symmetry, developed in [Bo2] and further explored in [MaS]. The vertex algebra that corresponds to the N=2 superconformal field theory is expected to be the cohomology of a lattice vertex algebra $\mathrm{Fock}_{M\oplus N}$, built out of $M\oplus N$, by a certain differential $D_{f,g}$ that depends on the defining equations f and g of a mirror pair. The space $\Lambda^*(N_{\mathbb{C}})\otimes\mathbb{C}[L]$ corresponds to a certain subspace of $\mathrm{Fock}_{M\oplus N}$ such that the restriction of $D_{f,g}$ to this subspace coincides with the differential D of Theorem 10.2. We can not yet show that this is precisely the chiral ring of the vertex algebra, so the connection to vertex algebras needs to be explored further.

11. Appendix. G-polynomials

A finite graded partially ordered set is called Eulerian if every its nontrivial interval contains equal numbers of elements of even and odd rank. We often consider the poset of faces of the Gorenstein cone K over a reflexive polytope Δ with respect to inclusions. This is an Eulerian poset with the grading given by the dimension of the face. The minimum and maximum elements of a poset are commonly denoted by $\hat{0}$ and $\hat{1}$.

Definition 11.1. [S1] Let $P=[\hat{0},\hat{1}]$ be an Eulerian poset of rank d. Define two polynomials $G(P,t)$, $H(P,t)\in\mathbb{Z}[t]$ by the following recursive rules:

$$G(P,t)=H(P,t)=1 \ \text{ if } d=0;$$

$$H(P,t)=\sum_{\hat{0}<x\leq\hat{1}}(t-1)^{\rho(x)-1}G([x,\hat{1}],t) \ \ (d>0),$$

$$G(P,t)=\tau_{<d/2}\left((1-t)H(P,t)\right) \ \ (d>0),$$

where $\tau_{<r}$ denotes the truncation operator $\mathbb{Z}[t]\to\mathbb{Z}[t]$ which is defined by

$$\tau_{<r}\left(\sum_i a_i t^i\right)=\sum_{i<r}a_i t^i.$$

The following lemma will be extremely useful.

Lemma 11.2. *For every Eulerian poset $P=[\hat{0},\hat{1}]$ of positive rank there holds*

$$\sum_{\hat{0}\leq x\leq\hat{1}}(-1)^{\mathrm{rk}[\hat{0},x]}G([\hat{0},x]^*,t)G([x,\hat{1}],t)=\sum_{\hat{0}\leq x\leq\hat{1}}G([\hat{0},x],t)G([x,\hat{1}]^*,t)(-1)^{\mathrm{rk}[x,\hat{1}]}=0$$

where $()^*$ *denotes the dual poset. In other words,* $G(\_,t)$ *and* $(-1)^{\mathrm{rk}}G(\_^*,t)$ *are inverses of each other in the algebra of functions on the posets with the convolution product.*

Proof. See Corollary 8.3 of [S2]. □

The following polynomial invariants of Eulerian posets have been introduced in [BBo].

Definition 11.3. Let P be an Eulerian poset of rank d. Define the polynomial $B(P;u,v)\in\mathbb{Z}[u,v]$ by the following recursive rules:

$$B(P;u,v)=1 \text{ if } d=0,$$

$$\sum_{\hat{0}\le x\le\hat{1}} B([\hat{0},x];u,v)u^{d-\rho(x)}G([x,\hat{1}],u^{-1}v)=G(P,uv).$$

Lemma 11.4. *Let* $P=[\hat{0},\hat{1}]$ *be an Eulerian poset. Then*

$$B(P;u,v)=\sum_{\hat{0}\le x\le\hat{1}} G([x,\hat{1}]^*,u^{-1}v)(-u)^{\mathrm{rk}\hat{1}-\mathrm{rk}x}G([\hat{0},x],uv).$$

Proof. Indeed, one can sum the recursive formulas for $B([\hat{0},y])$ for all $\hat{0}\le y\le\hat{1}$ multiplied by $G([y,\hat{1}]^*,u^{-1}v)(-u)^{\mathrm{rk}\hat{1}-\mathrm{rk}y}$ and use Lemma 11.2. □

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