ON WRONSKIANS OF WEIGHT ONE EISENSTEIN SERIES

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Abstract. We describe the span of Hecke eigenforms of weight four with nonzero central value of L-function in terms of Wronskians of certain weight one Eisenstein series.

1. Introduction

For any positive integer l we consider the congruence subgroup $\Gamma_1(l) \subseteq Sl_2(\mathbb{Z})$. The space of cusp forms for $\Gamma_1(l)$ of a given weight k splits according to the eigenvalues of Hecke operators. We say that a Hecke eigenform has analytic rank zero, if the central value of the corresponding L-function is nonzero.

It has been shown in [BG1] that the span of Hecke eigenforms of weight two coincides with the span of the cuspidal parts of products of certain weight 1 Eisenstein series for the group $\Gamma_1(l)$. These series are the logarithmic derivatives in the z direction of the standard θ-function, evaluated at $\frac{a}{l}$ for $a = 1, \ldots, l-1$. It is convenient to look at the Fricke involutions of these Eisenstein series. These are linear combinations of the original series and are given by

$$s_a(q) = (\frac{1}{2} - \{\frac{a}{l}\}) + \sum_{n>0} q^n \sum_{d|n} (\delta_d^{a \bmod l} - \delta_d^{-a \bmod l}),$$

where $q = \exp(2\pi i \tau)$ and δ is a version of Kronecker symbol. In this paper we look at the Wronskians $W(s_a, s_b)$ defined as usual by $W(s_a, s_b) = (\frac{d}{d\tau} s_a) s_b - (\frac{d}{d\tau} s_b) s_a$. It is easy to see that $W(s_a, s_b)$ is always a cusp form of weight 4, and the main result of this paper relates the span of such forms with the span of Hecke eigenforms of analytic rank zero.

Theorem 6.5. For arbitrary $l > 1$ the span of Hecke eigenforms of weight four and analytic rank zero is equal to the span of the Wronskians $W(s_a(\tau), s_b(\tau))$ for all $a, b \in \mathbb{Z}/l\mathbb{Z}$.

Before we explain the idea of the proof of this paper, we remark that it should be possible to prove Theorem 6.5 using Rankin-Selberg method, by combining the formulas [Z, 4.3, equation (4)] and [Sc, Theorem 4.6.3]. However, we chose to use the technique of [BG1] and [BG2] that emphasizes the map from modular symbols to modular forms.

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The space $M_4(l)$ of modular symbols of weight four can be thought of as a combinatorial counterpart to the space of modular forms. It is a vector space of roughly twice the dimension, and it contains subspaces $S_4(l)_+$ and $S_4(l)_-$ which are naturally dual to the space $\mathcal{S}_4(l)$ of cusp forms of weight four. Moreover, the action of Hecke operators on the space of modular symbols is given explicitly, see [M1]. Ignoring minor complications due to old forms, the span of Hecke eigenforms of weight four and analytic rank zero can be seen as the image of the endomorphism $\rho : \mathcal{S}_4(l) \to \mathcal{S}_4(l)$ given by

$$\rho(f) = \sum_{n>0} L(T_n f, 2) q^n$$

where T_n denote the Hecke operator. We observe that $L(f,2)q^n$ is the result of the pairing $\langle f, xy(0,1)_- \rangle$ of f a certain element of $S_4(l)_-$ to calculate ρ in terms of modular symbols as a composition of maps

$$\mathcal{S}_4(l) \stackrel{Int}{\to} (S_4(l)_-)^* \stackrel{PD}{\to} S_4(l)_+ \stackrel{\mu}{\to} \mathcal{S}_4(l)$$

where Int is induced by the integration pairing of $\mathcal{S}_4(l)$ and $S_4(l)_-$, the PD is the Poincaré duality map which we define in Section 3, and μ is the Wronskian map, defined in Section 4.

The map PD is a weight four analog of the intersection pairing on weight 2 symbols considered in [BG1]. It is shown to be nondegenerate in Section 3 as a consequence of a modular symbol formula for Petersson inner product. The map μ is the main novelty of this paper. It is a map from the space of modular symbols to the space of modular forms, which in particular maps $xy(a,b)$ to the Wronskian $W(s_a, s_b)$. Our calculations are purely elementary and rely on properties of the Euclid algorithm and some explicit calculations with modular symbols.

There are several directions in which one can try to extend the results of this paper. For example, one can look at the subspaces in the spaces of modular forms of higher weight that are spanned by Wronskians of Eisenstein series of higher weight. Intuition derived from [BG2] and [Z] suggests that these would be related to values of the L-function at 2. Consequently, we expect the Wronskians to span the whole space in the higher weight setting.

It is worth mentioning that the product and the Wronskian are the first two cases of Cohen operators (see [Z]). One can wonder if the forms of higher weight of analytic rank zero can be described in terms of higher Cohen operators of s_a. Clearly, for a high enough weight this seems impossible for dimension reasons. On the other hand, one could perhaps apply Cohen operators to the theta function itself, rather than its logarithmic derivatives, similar to the definition of μ on the noncuspidal symbols of weight four. But this is all but a speculation at this point.

One might hope to use the construction of this paper to give upper bounds on the number of Hecke eigenforms of higher analytic rank. However, analogous statements for weight two, at least so far, has not lead to such results.

It can also be argued that there may be some deeper reason behind the results of this paper and [BG1] which is yet to be uncovered. From this point of view, it would be tempting to try to see the sums along the runs of Euclid algorithm as a calculation of an Euler characteristics of some complex, whose cohomology is located at top and bottom location only. But at the moment we do not have a suitable candidate for it. Finally, one can wonder whether derivatives of L-function at the central value can be somehow seen in terms of Eisenstein series and Cohen operators.

Notations. We denote by $\mathcal{H}$ the upper half-plane and denote by τ, $\Im(\tau) > 0$ the complex coordinate on it. We use the notation $q = \exp(2\pi i\tau)$ when writing Fourier expansion of modular forms. Throughout the paper l denotes the level, and it is generally fixed, except for the proof of Theorem 6.5 that requires induction on the level. We use a slightly modified Kronecker δ notation $\delta_u^{v \bmod w}$ which gives 1 when $u = v \bmod w$ and 0 otherwise.

Acknowledgments. This paper grew out of a search of a (weight two) skew-symmetric analog of [BG1] which the author talked about on and off for a few years with Paul Gunnells. The author also thanks Loïc Merel for helpful remarks regarding the Poincaré duality map.

2. Modular symbols of weight four

Our main reference for modular symbols is the paper [M1] by Merel, which in turn builds on the work of Manin and Shokurov. In this section we recall the purely combinatorial description of modular (Manin, in the terminology of [M1]) symbols of weight four for the group $\Gamma_1(l)$.

The modular symbols of weight four and level l is a quotient of the vector space with basis $x^2(u,v)$, $xy(u,v)$, $y^2(u,v)$, with $(u,v) \in (\mathbb{Z}/l\mathbb{Z})^2$, $gcd(u,v,l) = 1$ by the span of the relations

(2.1)
$$x^2(u,v) + y^2(v,-u),\ xy(u,v) - xy(v,-u),\ y^2(u,v) + x^2(v,-u)$$
$$xy(v,-u-v) - xy(-u-v,u) + y^2(-u-v,u) + x^2(u,v) - xy(u,v)$$

for all u, v with $gcd(u,v,l) = 1$.

Remark 2.1. Our set of relations looks somewhat smaller than that of [M1], where the relations are

$$P(x,y)(u,v) + P(y,-x)(u,v)$$
$$P(x,y)(u,v) + P(y-x,-x)(-u-v,u) + P(-y,x-y)(v,-u-v)$$

for an arbitrary degree two homogeneous polynomial $P(x,y)$. The "missing" relations are obtained by cyclic permutations of $(u,v,-u-v)$ in the last line of (2.1), so the two definitions of modular symbols are equivalent.

Recall that the subspace $S_4(l) \subset M_4(l)$ of cuspidal modular symbols is characterized as follows. The cusps of the modular curve $X_1(l) = \overline{\mathcal{H}/\Gamma_1(l)}$ are in one-to-one correspondence with elements of the set $I = \{(a,b),\ a \in \mathbb{Z}/l\mathbb{Z},\ b \in (\mathbb{Z}/(a,l)\mathbb{Z})^*\}/\pm$. This correspondence maps (a,b) to $\frac{b^*}{a} \in \mathbb{Q} \cup i\infty$

where b^* is the inverse of $b \operatorname{mod}(a,l)$. For every element of I there is a map $M_4(l) \to \mathbb{C}$ defined by

$$
\begin{array}{lll}
x^2(u,v) & \mapsto & \delta_u^{a \operatorname{mod} l} \delta_v^{b \operatorname{mod}(a,l)} + \delta_u^{-a \operatorname{mod} l} \delta_v^{-b \operatorname{mod}(a,l)} \\
y^2(u,v) & \mapsto & -\delta_v^{a \operatorname{mod} l} \delta_u^{-b \operatorname{mod}(a,l)} - \delta_v^{-a \operatorname{mod} l} \delta_u^{b \operatorname{mod}(a,l)} \\
xy(u,v) & \mapsto & 0
\end{array}
\tag{2.2}
$$

Then the space of cuspidal symbols $S_4(l)$ is defined as the intersection of the kernels of all these maps.

We are now ready to formulate the main result of this section.

Proposition 2.2. *The space of cuspidal symbols $S_4(l)$ is spanned by the modular symbols of the form $xy(u,v)$.*

Proof. We can use the first set of equations to solve for $y^2(u,v)$. Then we can think of modular symbols of weight four as being spanned by $x^2(u,v)$ and $xy(u,v)$, subject to conditions

(2.3)

$$
\begin{array}{l}
x^2(u,v) - x^2(-u,-v) = 0,\ xy(u,v) - xy(v,-u) = 0, \\
x^2(u,u+v) - x^2(u,v) = xy(v,-u-v) - xy(-u-v,u) - xy(u,v).
\end{array}
$$

Clearly, $xy(u,v)$ are cuspidal. On the other hand, if a linear combination of $w = \sum_{u,v} \alpha_{u,v} x^2(u,v)$ is cuspidal, then for each $a \operatorname{mod} l$ and each $b \operatorname{mod}(a,l)$

$$
\sum_{v=b \operatorname{mod}(a,l)} (\alpha_{u,v} + \alpha_{-u,-v}) = 0
$$

By using relations $x^2(u,v) = x^2(-u,-v)$ we can write w as a linear combination of $x^2(u,v+ku) - x^2(u,v+(k-1)u)$ which is then written as a linear combination of $xy(u',v')$. □

Remark 2.3. In addition to the obvious symmetry relations $xy(u,v) = xy(v,-u)$ there are still some other linear relations among the symbols $xy(u,v)$ in $S_4(l)$. In fact, one can show that for $l \geq 5$ the linear relations on $xy(u,v)$ in $M_4(l)$ (or $S_4(l)$) are spanned by the symmetry relations and

$$
\sum_{k=0}^{l-1} \Big(xy(v+ku, -(k+1)u - v) - xy(-(k+1)u - v, u) - xy(u, v+ku) \Big) = 0
$$

for all u and v with $gcd(u,v,l) = 1$. We leave the proof of this claim to the reader, as it will not be used elsewhere in the paper.

We now recall that $M_4(l)$ and $S_4(l)$ naturally split according to the eigenvalue of the involution i given by

$$
x^2(u,v) \mapsto x^2(-u,v),\ xy(u,v) \mapsto -xy(-u,v),\ y^2(u,v) \mapsto y^2(-u,v).
$$

We define the corresponding eigenspaces by $M_4(l)_+$, $M_4(l)_-$, $S_4(l)_+$ and $S_4(l)_-$. There are symmetrization maps $M_4(l) \to M_4(l)_\pm$ given by $t \to \frac{1}{2}(t \pm i(t))$, and similarly for $S_4(l) \to S_4(l)_\pm$. We use a subscript to indicate the symmetrization of a symbol. We can now apply Proposition 2.2 to $S_4(l)_\pm$.

Corollary 2.4. *The space $S_4(l)_\pm$ is a linear span of the symbols $xy(u,v)_\pm$ with $(u,v) \in (\mathbb{Z}/l\mathbb{Z})^2$ and $gcd(u,v,l) = 1$.*

Remark 2.5. It is amusing to observe that for prime $l \geq 3$ the space $S_4(l)_+$ is a linear span of symbols $xy(u,v)_+$ with $(u,v) \in (\mathbb{Z}/l\mathbb{Z})^2 - (0,0)$ with linear relations among these symbols generated by

$$xy(u,v)_+ = -xy(-u,v)_+ = -xy(v,u)_+.$$

Clearly, these relations hold in $S_4(l)_+$ and, by themselves, they cut its dimension down to at most $\frac{1}{8}(l-1)(l-3)$. On the other hand, by [M1], $S_4(l)_+$ is dual to the space $\mathcal{S}_4(l)$ of cusp forms of weight four, which has dimension $\frac{1}{8}(l-1)(l-3)$ by the usual Riemann-Roch calculation. This shows that all other relations on $xy(u,v)_+$ follow from the above symmetry relations (which can also be checked directly along the lines of Remark 2.3). One can thus identify $S_4(l)_+$ with the second exterior power of the vector space of dimension $(l-1)/2$ which is generated by the symbols r_a for $a \bmod l$ with $r_{-a} = -r_a$.

3. Poincaré Duality for Modular Symbols

The goal of this section is to explicitly describe a certain map $PD : M_4(l)^* \to M_4(l)$ which is a weight four analog of the Poincaré duality for the weight two cuspidal symbols. It is rather easy to show that $PD(M_4(l)) \subseteq S_4(l)$. In fact, we will see that $PD(M_4(l)) = S_4(l)$, which is crucial for the argument of this paper. This is proved by comparison of PD and the expression of the Petersson inner product of cusp forms of weight four in terms of their period integrals.

Definition 3.1. The linear map $PD : M_4(l)^* \to M_4(l)$ is defined by sending any linear function $\phi : M_4(l) \to \mathbb{C}$ to the element of $M_4(l)$ given by

$$\begin{aligned}\tfrac{1}{24}\textstyle\sum_{u,v\in\mathbb{Z}/l\mathbb{Z}}(&\phi((y-x)^2(-v,u+v) - (y+x)^2(v,v-u))x^2(u,v)\\ &-2\phi(y(y-x)(-v,u+v) - (-y)(y+x)(v,v-u))xy(u,v)\\ &+\phi(y^2(-v,u+v) - (-y)^2(v,v-u))y^2(u,v))\end{aligned}$$

where we adopt the convention $\phi(P(x,y)(u,v)) = 0$ for any $P(x,y)$ if $gcd(u,v,l) > 1$.

Proposition 3.2. *The bilinear form on $M_4(l)$ induced by PD is skew-symmetric. Namely, for any $\phi, \lambda \in M_4(l)^*$ one has*

$$\lambda(PD(\phi)) = -\phi(PD(\lambda)).$$

Proof. We can express $\lambda(PD(\phi))$ as

$$\begin{aligned}\tfrac{1}{24}\textstyle\sum_{u,v\in\mathbb{Z}/l\mathbb{Z}}(&\phi((y-x)^2(-v,u+v) - (y+x)^2(v,v-u))\lambda(x^2(u,v))\\ &-2\phi(y(y-x)(-v,u+v) - (-y)(y+x)(v,v-u))\lambda(xy(u,v))\\ &+\phi(y^2(-v,u+v) - (-y)^2(v,v-u))\lambda(y^2(u,v))).\end{aligned}$$

We then use the relations $P(x,y)(c,d) = -P(y,-x)(-d,c)$ to rewrite the terms with $(v, v-u)$ in terms of $(u-v, v)$. Afterwards, we use the relation $P(x,y)(c,d) = -P(y-x,-x)(-c-d,c) - P(-y,x-y)(d,-c-d)$ to further rewrite them in terms of $(v,-u)$ and $(-u,u-v)$. Finally, we use the relations $P(x,y)(c,d) = -P(y,-x)(d,-c)$ to write the result in terms of (u,v) and $(u,u-v)$. We also rewrite the terms with $(-v,u+v)$ in terms of (u,v) and $(-u-v,u)$. After simplifications, this gives

$$\begin{aligned}
(y-x)^2(-v,u+v) - (y+x)^2(v,v-u) = {} & x^2(u-v,u) \\
& -x^2(-u-v,u) \\
y(y-x)(-v,u+v) - (-y)(y+x)(v,v-u) = {} & -x(y+x)(u-v,u) \\
& -x(x-y)(-u-v,u) \\
y^2(-v,u+v) - (-y)^2(v,v-u) = {} & (y+x)^2(u-v,u) \\
& -(x-y)^2(-u-v,u)
\end{aligned}$$

which allows us to write $\lambda(PD(\phi))$ as

$$\begin{array}{c}
\frac{1}{24}\sum_{u,v}(\phi(x^2(u-v,u) - x^2(-u-v,u))\lambda(x^2(u,v)) \\
-2\phi(-x(y+x)(u-v,u) - x(x-y)(-u-v,u))\lambda(xy(u,v)) \\
+\phi((y+x)^2(u-v,u) - (x-y)^2(-u-v,u))\lambda(y^2(u,v))).
\end{array}$$

It remains to switch the indexing in $\sum_{u,v}$ so that $\phi(...)$ becomes $\phi(...(u,v))$ and simplify to get $-\phi(PD(\lambda))$. □

Proposition 3.3. *The map PD passes through the spaces of cusp symbols, namely there is a commutative diagram*

$$\begin{array}{ccc}
M_4(l)^* & \stackrel{PD}{\to} & M_4(l) \\
\downarrow & & \uparrow \\
S_4(l)^* & \to & S_4(l)
\end{array}$$

with the side maps coming from the natural inclusions $S_4(l) \to M_4(l)$.

Proof. First, let $\psi_{(a,b)} : M_4(l) \to \mathbb{C}$ be the evaluation at the cusp $\pm(a,b)$ with $a \in \mathbb{Z}/l\mathbb{Z}$ and $b \in \mathbb{Z}/(a,l)\mathbb{Z}$ as in (2.2). Let us show that $\psi_{(a,b)}(PD(\phi)) = 0$ for any ϕ. Using (2.2), we get

$$24\psi_{(a,b)}(PD(\phi)) = 2\phi\Big(\sum_{v=b \bmod (a,l)} ((y-x)^2(-v,a+v) - (y+x)^2(v,v-a))$$

$$- \sum_{u=-b \bmod (a,l)} (y^2(-a,u+a) - y^2(a,a-u))\Big)$$

By switching from u to $u \mp a$ in the two terms of the last sum, we see that it vanishes. For the first sum, we switch from v to $v-a$ in the first term. Then we use $(x-y)^2(c,d) = -(x+y)^2(-d,c)$ to reduce it (up to a constant) to the value of ϕ on $\sum_{v=b \bmod (a,l)}(y-x)^2(a-v,v)$. We now apply the relations on modular symbols to rewrite this as

$$-\sum_{v=b \bmod (a,l)} (x^2(v,-a) + y^2(-a,a-v))$$

$$= - \sum_{v=b \bmod (a,l)} (x^2(v,-a) + y^2(a,v)) = 0.$$

This shows that the image of PD sits inside $S_4(l)$.

By Proposition 3.2, we now see that $\phi(PD(\psi_{a,b})) = 0$ for any ϕ. This shows that PD passes through $S_4(l)^*$ which finishes the proof. □

The key result of this section hinges on a formula for the Petersson inner product of cusp forms in terms of period integrals. Recall that the Petersson inner product of two holomorphic cusp forms of weight four with respect to $\Gamma_1(l)$ is defined as

$$(f,g)_{\mathrm{Petersson}} = \iint_{\mathcal{H}/\Gamma_1(l)} f(\tau)\overline{g(\tau)}\Im(\tau)^2 d\tau d\bar{\tau}.$$

Period integrals define a pairing between the space $\mathcal{S}_4(l)$ of cusp forms of weight four and $M_4(l)$, which we denote by $\langle\ ,\ \rangle$. This pairing is a crucial feature of the theory of modular symbols, and we refer the reader to [M1] for its definitions and properties.

Theorem 3.4. *For any two holomorphic weight four forms f and g there holds*

(1) $$(f,g)_{\mathrm{Petersson}} = -\tfrac{1}{24}\sum_{c,d\in\mathbb{Z}/l\mathbb{Z},\ gcd(c,d,l)=1} \Big(\overline{(\langle g,(y-x)^2(-d,c+d)\rangle - \langle g,(y+x)^2(d,d-c)\rangle)}\langle f,x^2(c,d)\rangle \\ -2\overline{(\langle g,y(y-x)(-d,c+d)\rangle - \langle g,-y(y+x)(d,d-c)\rangle)}\langle f,xy(c,d)\rangle \\ +\overline{(\langle g,y^2(-d,c+d)\rangle - \langle g,(-y)^2(d,d-c)\rangle)}\langle f,y^2(c,d)\rangle\Big)$$

(2) $$0 = -\tfrac{1}{24}\sum_{c,d\in\mathbb{Z}/l\mathbb{Z},\ gcd(c,d,l)=1} \Big((\langle g,(y-x)^2(-d,c+d)\rangle - \langle g,(y+x)^2(d,d-c)\rangle)\langle f,x^2(c,d)\rangle \\ -2(\langle g,y(y-x)(-d,c+d)\rangle - \langle g,-y(y+x)(d,d-c)\rangle)\langle f,xy(c,d)\rangle \\ +(\langle g,y^2(-d,c+d)\rangle - \langle g,(-y)^2(d,d-c)\rangle)\langle f,y^2(c,d)\rangle\Big).$$

Proof. Consider the cosets $\Gamma_1(l)\lambda$ for $\lambda \in Sl_2(\mathbb{Z})$. Then the fundamental domain $\mathcal{H}/\Gamma_1(l)$ can be chosen as $\bigcup_{\Gamma_1(l)\lambda}\lambda(D_0)$ for any fundamental domain D_0 of $\Gamma_1(l)$. Moreover, we can use a union of three different such D_0 to write the Petersson pairing as

$$(f,g)_{\mathrm{Petersson}} = \frac{1}{3}\sum_{\Gamma_1(l)\lambda}\iint_{\lambda(D)} f(\tau)\overline{g(\tau)}\Im(\tau)^2 d\tau d\bar{\tau}$$

where D is the geodesic triangle in $\mathcal{H}$ with vertices $\mathrm{i}\infty$, -1 and 0. The boundary of D consists of the vertical lines $\Re(\tau) = 0$ and $\Re(\tau) = -1$, as well as the upper half of the circle of radius $\frac{1}{2}$ centered at $-\frac{1}{2}$.

We use $\Im(\lambda(\tau))^2 = \Im(\tau)^2|c\tau+d|^{-4}$ where $\lambda(\tau) = \frac{a\tau+b}{c\tau+d}$ to rewrite each term of the above sum as

$$\iint_D f(\lambda(\tau))\overline{g(\lambda(\tau))}\Im(\tau)^2|c\tau+d|^{-8} d\tau d\bar{\tau}.$$

For each such λ we introduce for $i = 0, 1, 2$

$$G_{i,\lambda}(\tau) = \overline{\int_{-1}^{\tau} g(\lambda(s))(cs+d)^{-4} s^i ds}$$

and $f_{i,\lambda}(\tau)) = f(\lambda(\tau))(c\tau+d)^{-4}\tau^i$. Then we write $\Im(\tau)^2 = -\frac{1}{4}(\tau-\bar{\tau})^2$ and use Stokes's Theorem to derive

$$(f,g)_{\text{Petersson}} = -\frac{1}{12}\sum_{\Gamma_1(l)\lambda}\int_{\partial D}\Big(G_{0,\lambda}(\tau)f_{2,\lambda}(\tau) - 2G_{1,\lambda}(\tau)f_{1,\lambda}(\tau) + G_{2,\lambda}(\tau)f_{0,\lambda}(\tau)\Big)d\tau.$$

The boundary of D splits into three geodesics, and our first claim is that the terms of the integration for the $\int_{\mathrm{i}\infty}^{-1}$ and $\int_{-1}^{0}$ of ∂D cancel each other. Consider the map $\sigma(\tau) := -\frac{1}{\tau+2}$. Element $\sigma \in Sl_2(\mathbb{Z})$ acts on the set of cosets $\Gamma_1(l)\lambda$ by right multiplication. We observe that

$$\int_{-1}^{0}(G_{0,\lambda\sigma}(\tau)f_{2,\lambda\sigma}(\tau) - 2G_{1,\lambda\sigma}(\tau)f_{1,\lambda\sigma}(\tau) + G_{2,\lambda\sigma}(\tau)f_{0,\lambda\sigma}(\tau))d\tau$$
$$= \int_{-1}^{0}(G_{0,\lambda}(\sigma(\tau))f_{2,\lambda}(\sigma(\tau)) - 2G_{1,\lambda}(\sigma(\tau))f_{1,\lambda}(\sigma(\tau)) + G_{2,\lambda}(\sigma(\tau))f_{0,\lambda}(\sigma(\tau)))d\sigma(\tau)$$
$$= \int_{-1}^{\mathrm{i}\infty}(G_{0,\lambda}(\tau)f_{2,\lambda}(\tau) - 2G_{1,\lambda}(\tau)f_{1,\lambda}(\tau) + G_{2,\lambda}(\tau)f_{0,\lambda}(\tau))d\tau.$$

The first equality is verified by a lengthy but straightforward calculation, which is left to the reader, since we will perform a similar calculation below. It is crucial that we chose (-1) as the lower limit of integration in the definition of $G_{i,\lambda}$ and that σ preserves (-1).

So now we are left with

$$(f,g)_{\text{Petersson}} = -\frac{1}{12}\sum_{\Gamma_1(l)\lambda}\int_{0}^{\mathrm{i}\infty}\Big(G_{0,\lambda}(\tau)f_{2,\lambda}(\tau) - 2G_{1,\lambda}(\tau)f_{1,\lambda}(\tau) + G_{2,\lambda}(\tau)f_{0,\lambda}(\tau)\Big)d\tau.$$

We will do a similar trick, this time with $\nu(\tau) = -\frac{1}{\tau}$ instead of $\sigma(\tau)$. It has an effect of switching the direction of integration, but since it does not preserve (-1), the functions $G_{i,\lambda}$ acquire extra additive terms. More specifically, one has

$$G_{i,\lambda\nu}(\tau) = \overline{\int_{-1}^{\tau} g(\lambda(-\frac{1}{s}))(ds-c)^{-4}s^i ds} = \overline{\int_{1}^{\nu(\tau)} g(\lambda(s))(cs+d)^{-4}s^{2-i}(-1)^i ds}$$
$$= (-1)^i G_{2-i,\lambda}(\nu(\tau)) + \overline{\int_{1}^{-1} g(\lambda(s))(cs+d)^{-4}s^{2-i}(-1)^i ds}$$
$$f_{i,\lambda\nu}(\tau)d\tau = (-1)^i f_{2-i,\lambda}(\nu(\tau))d\nu(\tau).$$

We rewrite $(f,g)_{\mathrm{Petersson}}$ as

$$-\frac{1}{24}\Big(\sum_{\Gamma_1(l)\lambda}\int_0^{\mathrm{i}\infty}\Big(G_{0,\lambda}(\tau)f_{2,\lambda}(\tau)-2G_{1,\lambda}(\tau)f_{1,\lambda}(\tau)+G_{2,\lambda}(\tau)f_{0,\lambda}(\tau)\Big)d\tau$$

$$+\sum_{\Gamma_1(l)\lambda}\int_0^{\mathrm{i}\infty}\Big(G_{0,\lambda\nu}(\tau)f_{2,\lambda\nu}(\tau)-2G_{1,\lambda\nu}(\tau)f_{1,\lambda\nu}(\tau)+G_{2,\lambda\nu}(\tau)f_{0,\lambda\nu}(\tau)\Big)d\tau\Big)$$

which together with transformation formulas for G and f implies, after cancelling $\pm\int_0^{\mathrm{i}\infty}$

$$(f,g)_{\mathrm{Petersson}}=-\frac{1}{24}\sum_{\Gamma_1(l)\lambda}\Big(\overline{\int_1^{-1}g(\lambda(s))(cs+d)^{-4}s^2ds}\int_0^{\mathrm{i}\infty}f_{0,\lambda}(\tau)d\tau$$

$$-2\overline{\int_1^{-1}g(\lambda(s))(cs+d)^{-4}sds}\int_0^{\mathrm{i}\infty}f_{1,\lambda}(\tau)d\tau$$

$$+\overline{\int_1^{-1}g(\lambda(s))(cs+d)^{-4}ds}\int_0^{\mathrm{i}\infty}f_{2,\lambda}(\tau)d\tau\Big).$$

It remains to write $\int_1^{-1}$ in terms of pairings with modular symbols by writing the arc from 1 to (-1) in terms of the unimodular arcs from 1 to $\mathrm{i}\infty$ and from $\mathrm{i}\infty$ to (-1). Namely, for a homogeneous degree two polynomial $P(x,y)$ one has

$$\int_1^{-1}g(\lambda(s))(cs+d)^{-4}P(s,1)ds$$

$$=\int_1^0 g(\lambda(s))(cs+d)^{-4}P(s,1)ds-\int_{-1}^0 g(\lambda(s))(cs+d)^{-4}P(s,1)ds$$

$$=\int_0^{\mathrm{i}\infty}g(\lambda(\frac{1}{1-s}))(-ds+(c+d))^{-4}P(1,1-s)ds$$

$$-\int_0^{\mathrm{i}\infty}g(\lambda(-\frac{1}{1+s}))(ds+(d-c))^{-4}P(-1,1+s)dt$$

$$=\langle g,P(y,y-x)(-d,c+d)\rangle-\langle g,P(-y,y+x)(d,d-c)\rangle.$$

Finally, one observes that cosets $\Gamma_1(l)\lambda$ are in one-to-one correspondence with pairs (c,d) with $gcd(c,d,l)=1$, and the first claim of the theorem follows.

The second claim of the theorem is proved by the same technique. This time we define $G_{i,\lambda}(\tau)$ as $\int_{-1}^{\tau}g(\lambda(s))(cs+d)^{-4}s^ids$. Consequently, it is holomorphic, and the Stokes's Theorem gives 0 instead of the Petersson product. The rest of the calculations are unchanged. □

Remark 3.5. Similar formulas for Petersson product are already present in the literature. They seem to go back to at least as far as [H] and [KZ]. We learned the argument (in weight two case) from [M2]. In addition to extending it to weight four, we streamlined it just slightly by looking at the union of three fundamental domains for $Sl_2(\mathbb{Z})$, rather than one. This allowed us to avoid integration between elliptic points.

Corollary 3.6. *The pairing on $S_4(l)$ induced by PD is nondegenerate.*

Proof. By [Sh], the integration pairing is a perfect pairing between $S_4(l)$ and the direct sum $V_{hol}\oplus\bar{V}_{hol}$ of the spaces of holomorphic and anti-holomorphic cusp forms. Every element $\phi\in S_4(l)^*$ can therefore be written as $\langle f,\cdot\rangle + \langle\bar{g},\cdot\rangle$. Suppose $PD(\phi)=0$. Denote by $^-$ the anti-isomorphism of $M_4(l)$ that sends $\alpha x^iy^{2-i}(u,v)$ to $\bar{\alpha}x^iy^{2-i}(u,v)$. Then Theorem 3.4 shows that

$$0=\phi(\overline{PD(\phi)})=-\langle f,f\rangle_{\text{Petersson}}-\langle g,g\rangle_{\text{Petersson}}\leq 0$$

with equality only if $f=g=0$. □

Remark 3.7. The arguments of our proof of Theorem 3.4 extend naturally to arbitrary integer weights $k\geq 2$ and arbitrary subgroups Γ of finite index in $Sl_2(\mathbb{Z})$. We expect Corollary 3.6 to extend to arbitrary weight and to arbitrary group Γ, after an appropriate definition of PD. One would need to interpret the arguments of Propositions 3.2 and 3.3 to extend them to this more general setting. For instance, we expect that the bilinear form on $S_k(\Gamma)$ induced by PD is $(-1)^{k+1}$-symmetric. We believe that maps PD can be interpreted as an intersection pairings in the middle cohomology of the Kuga varieties, although we do not need this for the purposes of this paper. Nevertheless, this is why we refer to PD as the Poincaré duality map.

The map PD behaves well with respect to the involution i. We denote by $M_4(l)^*_\pm$ the eigenspaces of the dual involution i^* on $M_4(l)^*$.

Proposition 3.8. $PD(M_4(l)^*_\pm)\subseteq M_4(l)_\mp$.

Proof. If $\phi\in M_4(l)^*_\pm$, then for any modular symbol $P(x,y)(u.v)$ there holds $\phi(P(x,y)(u,v))=\phi(P(x,y)(u,v)_\pm)$. Consequently,

$$\begin{aligned}
PD(\phi)=\frac{1}{24}\sum_{u,v\in\mathbb{Z}/l\mathbb{Z}}&(\phi((y-x)^2(-v,u+v)_\pm-(y+x)^2(v,v-u)_\pm)x^2(u,v)\\
&-2\phi(y(y-x)(-v,u+v)_\pm-(-y)(y+x)(v,v-u)_\pm)xy(u,v)\\
&+\phi(y^2(-v,u+v)_\pm-(-y)^2(v,v-u)_\pm)y^2(u,v))\\
=\frac{1}{24}\sum_{u,v\in\mathbb{Z}/l\mathbb{Z}}&(\phi((y-x)^2(-v,u+v)_\pm\mp(y-x)^2(-v,v-u)_\pm)x^2(u,v)\\
&-2\phi(y(y-x)(-v,u+v)_\pm\mp(-y)(y-x)(-v,v-u)_\pm)xy(u,v)\\
&+\phi(y^2(-v,u+v)_\pm\mp(-y)^2(-v,v-u)_\pm)y^2(u,v))\\
=\frac{1}{24}\sum_{u,v\in\mathbb{Z}/l\mathbb{Z}}&(\phi((y-x)^2(-v,u+v)_\pm)(x^2(u,v)\mp x^2(-u,v))\\
&-2\phi(y(y-x)(-v,u+v)_\pm)(xy(u,v)\pm xy(-u,v))\\
&+\phi(y^2(-v,u+v)_\pm)(y^2(u,v)\mp y^2(-u,v)))
\end{aligned}$$

$$= \frac{1}{12} \sum_{u,v \in \mathbb{Z}/l\mathbb{Z}} (\phi((y-x)^2(-v,u+v)_\pm) x^2(u,v)_\mp$$
$$-2\phi(y(y-x)(-v,u+v)_\pm) xy(u,v)_\mp + \phi(y^2(-v,u+v)_\pm) y^2(u,v)_\mp).$$

□

Remark 3.9. In what follows, we will abuse the notations somewhat to denote the induced map $M_4(l)_-^* \to M_4(l)_+$ by PD as well. By Proposition 3.3, this map comes from a map $S_4(l)_-^* \to S_4(l)_+$.

Corollary 3.10. *The induced map $PD : S_4(l)_-^* \to S_4(l)_+$ is an isomorphism.*

Proof. Combine Corollary 3.6 and Proposition 3.8. □

4. The Wronskian map

In this section we define the map from the modular symbols of weight four to cusp forms of weight four. First, we need to define the Eisenstein series $s_a(q)$, $t_a(q)$ and $r_a(q)$ for $a \in \mathbb{Z}/l\mathbb{Z}$. Our notation for s_a differs from that of [BG1] by a Fricke involution. We recall that quasimodular forms of weight two are linear combinations of the usual modular forms of weight two and the (slightly non-modular) Eisenstein series $E_2(q)$ of weight 2. In weight one, quasimodular forms are modular.

Proposition 4.1. *For each $a \bmod l$ there exist $\Gamma_1(l)$-quasimodular forms $s_a(q)$, $t_a(q)$ and $r_a(q)$ of weights 1, 2 and 2 respectively given by*

$$s_a(q) = (\frac{1}{2} - \{\frac{a}{l}\}) + \sum_{n>0} q^n \sum_{d|n} (\delta_d^{a \bmod l} - \delta_d^{-a \bmod l}),$$
$$\text{if } a \neq 0 \bmod l, s_0(q) = 0,$$
$$t_a(q) = constant + \sum_n q^n \sum_{d|n} \frac{n}{k} (\delta_d^{a \bmod l} + \delta_d^{-a \bmod l})$$
$$r_a(q) = constant + \sum_n q^n \sum_{d|n} d (\delta_d^{a \bmod l} + \delta_d^{-a \bmod l}),$$

where the exact values of the constants depend on a and l and are determined uniquely by the quasimodularity.

Proof. These series are obtained as linear combinations of the weight one and two Eisenstein series considered in [BG1]. Details are left to the reader. □

Definition 4.2. We define the map $\mu : S_4(l) \to \mathcal{S}_4(l)$ by the formula

$$x^2(u,v) \mapsto -2t_u r_v - \frac{1}{l} q \frac{\partial}{\partial q} r_v - \delta_v^{0 \bmod l} q \frac{\partial}{\partial q} t_u, \; xy(u,v) \mapsto \frac{1}{2\pi \mathrm{i}} W(s_u, s_v),$$
$$y^2(u,v) \mapsto 2r_u t_v + \frac{1}{l} q \frac{\partial}{\partial q} r_u + \delta_u^{0 \bmod l} q \frac{\partial}{\partial q} t_v.$$

Note that the ring of quasimodular forms is closed under $q\frac{\partial}{\partial q}$. We will show in Theorem 4.3 below that μ is well-defined, i.e. it is compatible with the relations on modular symbols.

Theorem 4.3. *The map μ of definition 4.2 is well-defined.*

Proof. We need to check that μ maps the relations (2.1) to zero. We have

$$\mu(x^2(u,v)+y^2(v,-u))=0$$

and

$$\mu(xy(u,v)-xy(v,-u))=\frac{1}{2\pi\mathrm{i}}(W(s_u,s_v)-W(s_v,s_{-u}))=0$$

by the symmetry properties $s_{-a}=-s_a$, $r_a=r_{-a}$ and $t_a=t_{-a}$.

The difficult part is to show that μ maps

$$R=xy(v,-u-v)-xy(-u-v,u)+y^2(-u-v,u)+x^2(u,v)-xy(u,v)$$

to zero. For each positive integer n let us denote by $I(n)$ the set of fourtuples $(m_1,k_1,m_2,k_2)\in\mathbb{Z}^4_{>0}$ that satisfy $m_1k_1+m_2k_2=n$. Let us denote by $\sim$ the equality of power series in q up to linear combinations of quasimodular forms of weights 0, 1, 2, and the derivatives of $s_a(q)$ with respect to τ. This allows us to avoid looking at the specific values of the constant terms in Proposition 4.1.

We have

$$\mu(R)\sim -\frac{1}{l}q\frac{\partial}{\partial q}r_v-\delta_v^{0\,\mathrm{mod}l}q\frac{\partial}{\partial q}t_u+\delta_{u+v}^{0\,\mathrm{mod}l}q\frac{\partial}{\partial q}t_u+\frac{1}{l}q\frac{\partial}{\partial q}r_{u+v}$$
$$+\sum_{n>0}q^n\sum_{I_n}\Big((m_1k_1-m_2k_2)(\delta_{k_1}^{v\,\mathrm{mod}l}-\delta_{k_1}^{-v\,\mathrm{mod}l})(\delta_{k_2}^{-u-v\,\mathrm{mod}l}-\delta_{k_2}^{u+v\,\mathrm{mod}l})$$
$$-(m_1k_1-m_2k_2)(\delta_{k_1}^{-u-v\,\mathrm{mod}l}-\delta_{k_1}^{u+v\,\mathrm{mod}l})(\delta_{k_2}^{u\,\mathrm{mod}l}-\delta_{k_2}^{-u\,\mathrm{mod}l})$$
$$+2k_1m_2(\delta_{k_1}^{-u-v\,\mathrm{mod}l}+\delta_{k_1}^{u+v\,\mathrm{mod}l})(\delta_{k_2}^{u\,\mathrm{mod}l}+\delta_{k_2}^{-u\,\mathrm{mod}l})$$
$$-2m_1k_2(\delta_{k_1}^{u\,\mathrm{mod}l}+\delta_{k_1}^{-u\,\mathrm{mod}l})(\delta_{k_2}^{v\,\mathrm{mod}l}+\delta_{k_2}^{-v\,\mathrm{mod}l})$$
$$-(m_1k_1-m_2k_2)(\delta_{k_1}^{u\,\mathrm{mod}l}-\delta_{k_1}^{-u\,\mathrm{mod}l})(\delta_{k_2}^{v\,\mathrm{mod}l}-\delta_{k_2}^{-v\,\mathrm{mod}l})\Big).$$

We introduce the notation $A_{k_1,k_2}=\delta_{k_1}^{u\,\mathrm{mod}l}\delta k_2{}^{v\,\mathrm{mod}l}+\delta_{k_1}^{-u\,\mathrm{mod}l}\delta_{k_2}^{-v\,\mathrm{mod}l}$ to rewrite the above as

$$\mu(R)\sim -\frac{1}{l}q\frac{\partial}{\partial q}r_v-\delta_v^{0\,\mathrm{mod}l}q\frac{\partial}{\partial q}t_u+\delta_{u+v}^{0\,\mathrm{mod}l}q\frac{\partial}{\partial q}t_u+\frac{1}{l}q\frac{\partial}{\partial q}r_{u+v}$$
$$+\sum_{n>0}q^n\sum_{I(n)}\Big((m_2k_2-m_1k_1-2m_1k_2)A_{k_1,k_2}$$
$$+(m_2k_2-m_1k_1+2k_1m_2)A_{k_2,-k_1-k_2}+(m_1k_1-m_2k_2)A_{-k_1-k_2,k_1}$$
$$-(m_2k_2-m_1k_1+2m_1k_2)A_{-k_1,k_2}-(m_2k_2-m_1k_1-2k_1m_2)A_{k_2,k_1-k_2}$$
$$-(m_1k_1-m_2k_2)A_{k_1-k_2,-k_1}\Big)$$

We recall (see [BG2]) that $I(n)$ is a disjoint union of the runs of Euclid algorithm. The algorithm is given by the partially defined map $up : I(n) \to I(n)$

$$up : (m_1,k_1,m_2,k_2) \mapsto \begin{cases} (m_2, k_1+k_2, m_1-m_2, k_1), & m_1 > m_2 \\ (m_2-m_1, k_2, m_1, k_1+k_2), & m_1 < m_2. \end{cases}$$

Repeated applications of this map go from the subset of $I(n)$ with $k_1 = k_2$ to the subset of $I(n)$ with $m_1 = m_2$, where up is not defined. As in [BG2], we will show that for each run of the algorithm the above sum is telescoping. Namely, the "plus" terms with A_{k_1,k_2}, $A_{k_2,-k_1-k_2}$, $A_{-k_1-k_2,k_1}$ for (m_1,k_1,m_2,k_2) cancel the "minus" terms with A_{-k_1,k_2}, A_{k_2,k_1-k_2}, $A_{k_1-k_2,-k_1}$ for $up(m_1,k_1,m_2,k_2)$. There are two cases to check, depending on whether $m_1 > m_2$ or $m_1 < m_2$. In the case of $m_1 > m_2$ the "minus" terms for $up(m_1,k_1,m_2,k_2) = (m_2, k_1+k_2, m_1-m_2, k_1)$ are

$$\begin{aligned}
&-((m_1-m_2)k_1 - m_2(k_1+k_2) + 2m_2k_1)A_{-k_1-k_2,k_1} \\
&-((m_1-m_2)k_1 - m_2(k_1+k_2) - 2(k_1+k_2)(m_1-m_2))A_{k_1,k_2} \\
&-(m_2k_1 + m_2k_2) - (m_1-m_2)k_1)A_{k_2,-k_1-k_2} \\
&= -(m_1k_1 - m_2k_2)A_{-k_1-k_2,k_1} - (-k_1m_1 + k_2m_2 - 2k_2m_1)A_{k_1,k_2} \\
&-(m_2k_2 - m_1k_1 + 2m_2k_1)A_{k_2,-k_1-k_2}
\end{aligned}$$

which is seen to equal the "plus" terms for (m_1,k_1,m_2,k_2). The case of $m_1 < m_2$ is similar and left to the reader. One needs to use there the symmetry $A_{k_1,k_2} = A_{-k_1,-k_2}$. Consequently, the only terms that will not be cancelled are the "plus" terms for the subset of $I(n)$ with $m_1 = m_2$ and the "minus" terms for the subset of $I(n)$ with $k_1 = k_2$. This gives

$$\begin{aligned}
\mu(R) \sim & -\frac{1}{l}q\frac{\partial}{\partial q}r_v - \delta_v^{0\,\mathrm{mod} l}q\frac{\partial}{\partial q}t_u + \delta_{u+v}^{0\,\mathrm{mod} l}q\frac{\partial}{\partial q}t_u + \frac{1}{l}q\frac{\partial}{\partial q}r_{u+v} \\
& + \sum_{n>0} q^n \Big(\sum_{\substack{m_1,k_1,k_2>0 \\ m_1(k_1+k_2)=n}} (-nA_{k_1,k_2} + nA_{k_2,-k_1-k_2} + m_1(k_1-k_2)A_{-k_1-k_2,k_1}) \\
& - \sum_{\substack{m_1,m_2,k_1>0 \\ (m_1+m_2)k_1=n}} (nA_{-k_1,k_1} - nA_{k_1,0} + (m_1-m_2)k_1A_{0,-k_1}) \Big) \\
= & -\frac{1}{l}q\frac{\partial}{\partial q}r_v - \delta_v^{0\,\mathrm{mod} l}q\frac{\partial}{\partial q}t_u + \delta_{u+v}^{0\,\mathrm{mod} l}q\frac{\partial}{\partial q}t_u + \frac{1}{l}q\frac{\partial}{\partial q}r_{u+v} \\
& + \sum_{n>0} nq^n \sum_{d|n} \Big(\sum_{0<k\le d} (-A_{k,d-k} + A_{d-k,-d}) - A_{0,-d} \\
& + \sum_{0<k\le d} (\frac{2k}{d} - 1)A_{-d,k} + \frac{n}{d}(A_{d,0} - A_{-d,d}) \Big).
\end{aligned}$$

We observe that

$$\sum_{0<k\le d} \delta_k^{u\,\mathrm{mod} l} = \frac{d}{l} - \{\frac{d-u}{l}\} + \{-\frac{u}{l}\}$$

and

$$\sum_{0<k\leq d}(\frac{2k}{d}-1)\delta_k^{u\,\mathrm{mod}l}=\frac{l}{d}\Big(\{\frac{d-u}{l}\}^2-\{\frac{d-u}{l}\}-\{-\frac{u}{l}\}^2+\{-\frac{u}{l}\}\Big)$$

$$+\Big(1-\{\frac{d-u}{l}\}-\{-\frac{u}{l}\}\Big).$$

Consequently,

$$\mu(R)\sim-\frac{1}{l}q\frac{\partial}{\partial q}r_v-\delta_v^{0\,\mathrm{mod}l}q\frac{\partial}{\partial q}t_u+\delta_{u+v}^{0\,\mathrm{mod}l}q\frac{\partial}{\partial q}t_u+\frac{1}{l}q\frac{\partial}{\partial q}r_{u+v}$$

$$+\sum_{n>0}nq^n\sum_{d|n}\Big(-\delta_u^{0\,\mathrm{mod}l}(\delta_d^{v\,\mathrm{mod}l}+\delta_d^{-v\,\mathrm{mod}l})$$

$$-\delta_d^{u+v\,\mathrm{mod}l}(\frac{d}{l}-\{\frac{v}{l}\}+\{-\frac{u}{l}\})-\delta_d^{-u-v\,\mathrm{mod}l}(\frac{d}{l}-\{-\frac{v}{l}\}+\{\frac{u}{l}\})$$

$$+\delta_d^{v\,\mathrm{mod}l}(\frac{d}{l}-\{-\frac{u}{l}\}+\{-\frac{u+v}{l}\})+\delta_d^{-v\,\mathrm{mod}l}(\frac{d}{l}-\{\frac{u}{l}\}+\{\frac{u+v}{l}\})$$

$$+\frac{n}{d}\delta_v^{0\,\mathrm{mod}l}(\delta_d^{u\,\mathrm{mod}l}+\delta_d^{-u\,\mathrm{mod}l})-\frac{n}{d}\delta_{u+v}^{0\,\mathrm{mod}l}(\delta_d^{v\,\mathrm{mod}l}+\delta_d^{-v\,\mathrm{mod}l})+\delta_d^{-u\,\mathrm{mod}l}\cdot$$

$$\cdot\Big(\frac{l}{d}(\{\frac{-u-v}{l}\}^2-\{\frac{-u-v}{l}\}-\{-\frac{v}{l}\}^2+\{-\frac{v}{l}\})+(1-\{\frac{-u-v}{l}\}-\{-\frac{v}{l}\})\Big)$$

$$+\delta_d^{u\,\mathrm{mod}l}\Big(\frac{l}{d}(\{\frac{u+v}{l}\}^2-\{\frac{u+v}{l}\}-\{\frac{v}{l}\}^2+\{\frac{v}{l}\})+(1-\{\frac{u+v}{l}\}-\{\frac{v}{l}\})\Big)\Big).$$

We use $\{t\}+\{-t\}=1-\delta_t^{0\,\mathrm{mod}1}$ and, after tedious but straightforward calculations, get

$$\mu(R)\sim 0.$$

Since $\mu(R)$ is quasimodular of weight four and $\sim$ is equality modulo forms of weight less than four, we get $\mu(R)=0$. □

Remark 4.4. There is a natural projection map from the space of quasimodular forms of weight four to the space of modular forms of weight four, which sends all forms divisible by E_2 to zero. So one can compose μ with this projection and have a map μ_1 to the space of modular forms of weight four.

Proposition 4.5. *Map μ sends $M_4(l)_-$ to zero.*

Proof. The statement immediately follows from the symmetry properties of r, s and t. □

Proposition 4.6. *The image of $S_4(l)_+$ under μ is a subspace of $\mathcal{S}_4(l)$ which is the linear span of $W(s_a,s_b)$ with $gcd(a,b,l)=1$.*

Proof. By Proposition 2.2, $\mu(S_4(l))$ is spanned by $\mu(xy(a,b))=W(s_a,s_b)$ for all $gcd(a,b,l)=1$. By Proposition 4.5, $\mu(S_4(l))_+=\mu(S_4(l))$. □

5. The composition map

In this section we calculate the composition of the duality map $PD : S_4(l)^*_- \to S_4(l)_+$ of Section 3 and the Wronskian map μ of Section 4. Our arguments are purely elementary. The result of this calculation will be used in the next section.

We need to introduce some additional notation. For any $\phi \in S_4(l)^*_-$ we will set $\phi(P(x,y)(u,v)_-) = 0$ if $gcd(u,v,l) > 1$. We will also use the notation $\sim$, namely, $f \sim g$ would mean that $f - g$ is a linear combination of quasimodular forms of weight at most two and the quasimodular forms of weight three that are derivatives of $s_u(\tau)$. Finally, for every $n > 0$ we introduce the set $H(n)$ of fourtuples of integers (a,b,c,d) that satisfy $ad - bc = n, a > b \geq 0, d > c \geq 0$.

Proposition 5.1. *For any $\phi \in S_4(l)^*_-$ there holds*

$$\mu \circ PD(\phi) \sim \sum_{n>0} q^n \sum_{H(n)} \phi((ax+by)(cx+dy)(c,d)_-).$$

Proof. We will use the notations A_{k_1,k_2} and $I(n)$ from Section 4. We use the last identity in the proof of Proposition 3.8 to get

$$12\mu \circ PD(\phi) = \mu\Big(\sum_{u,v\in\mathbb{Z}/l\mathbb{Z}} (\phi((y-x)^2(-v,u+v)_-)x^2(u,v)_+$$
$$-2\phi(y(y-x)(-v,u+v)_-)xy(u,v)_+ + \phi(y^2(-v,u+v)_-)y^2(u,v)_+)\Big)$$
$$\sim 2\sum_{n>0} q^n \phi\Big(\sum_{I(n)} \Big((-2m_1k_2(y-x)^2 - 2(m_1k_1 - m_2k_2)y(y-x)$$
$$+2m_2k_1y^2)(-k_2,k_1+k_2)_- + (-2m_1k_2(y-x)^2 + 2(m_1k_1 - m_2k_2)y(y-x)$$
$$+2m_2k_1y^2)(-k_2,-k_1+k_2)_-\Big)\Big)$$
$$-\frac{1}{l}\sum_{u,v\in\mathbb{Z}/l\mathbb{Z}} \phi((y-x)^2(-v,u+v)_- - y^2(-u,v+u)_-)q\frac{\partial r_v}{\partial q}$$
$$-\sum_{u\in\mathbb{Z}/l\mathbb{Z}} \phi((y-x)^2(0,u)_- - y^2(-u,u)_-)q\frac{\partial t_u}{\partial q}.$$

Let us simplify the last two lines of the above equation. We use $y^2(-u,v+u)_- = -x^2(-v,-u)_- - (x-y)^2(v+u,-v)_-$, $y^2(-u,u)_- = -x^2(0,-u)_- - (x-y)^2(u,0)_-$, $x^2(v,u)_- + x^2(v,-u)_- = y^2(v,u)_- + y^2(v,-u)_- = 0$ and $x^2(0,u)_- = y^2(0,u)_- = 0$ and $xy(0,u)_- = xy(u,0)_-$ to rewrite them as

$$\frac{4}{l}\sum_{u,v\in\mathbb{Z}/l\mathbb{Z}} \phi(xy(v,u)_-)q\frac{\partial r_v}{\partial q} + 4\sum_u \phi(xy(u,0)_-)q\frac{\partial t_u}{\partial q}.$$

To handle the sum over $I(n)$, for each n we observe that $I(n)$ can be embedded into the disjoint union of two copies of $H(n)$ in two different ways as follows. The subset of $I(n)$ with $m_1 \geq m_2$ can be identified with the subset of $H(n)$ with $c>0$ via $(m_1,k_1,m_2,k_2)=(a,d-c,a-b,c)$. The subset of $I(n)$ with $m_1<m_2$ can be identified with the subset of $H(n)$ with $bc>0$ via $(m_1,k_1,m_2,k_2)=(a-b,c,a,d-c)$. This describes the first embedding of $I(n)$ into the disjoint union of two copies of $H(n)$. The second embedding is obtained by comparing k_i. Namely, the subset of $I(n)$ with $k_1>k_2$ can be identified with the subset of $H(n)$ with $bc>0$ via $(m_1,k_1,m_2,k_2)=(a-b,d,b,d-c)$, and the subset of $I(n)$ with $k_1\le k_2$ can be identified with the subset of $H(n)$ with $b>0$ via $(m_1,k_1,m_2,k_2)=(b,d-c,a-b,d)$. We will use these embeddings in order to rewrite the above sum over $I(n)$ in terms of $H(n)$ as follows. For the terms with $(-k_2,k_1+k_2)_-$ we will use the first embedding, and for the terms with $(-k_2,-k_1+k_2)_-$ we will use the second one. After some straightforward simplifications, we get

$$
\begin{aligned}
12\mu\circ PD(\phi)\sim 4\sum_{n>0}q^n\phi\Big(\sum_{H(n),bc>0}\Big(&(-acx^2+(ad+bc)xy-bdy^2)(-c,d)_-\\
&+((-ad-bc+ac+bd)x^2+(ad+bc-2bd)xy+bdy^2)(c-d,d)_-\\
&+((-ad-bc+ac+bd)x^2+(ad+bc-2ac)xy+acy^2)(c-d,-c)_-\\
&+(-bdx^2+(ad+bc)xy-acy^2)(-d,c)_-\Big)\\
&+\sum_{H(n),b=0,c>0}(-acx^2+(ad+bc)xy-bdy^2)(-c,d)_-\\
&+\sum_{H(n),b>0,c=0}((-bd)x^2+(ad+bc)xy+(-ac)y^2)(-d,c)_-\\
&+\sum_{d|n}\Big(\frac{2n^2}{d}xy(0,d)_-+\frac{2nd}{l}\sum_{u\in\mathbb{Z}/l\mathbb{Z}}xy(d,u)_-\Big)\Big)\\
=4\sum_{n>0}q^n\phi\Big(\sum_{H(n),bc>0}\Big(&(2acx^2+2(ad+bc)xy+2bdy^2)(c,d)_-\\
&+((-ad-bc+ac+bd)x^2+(ad+bc-2bd)xy+bdy^2)(c-d,d)_-\\
&+((-ad-bc+ac+bd)x^2+(ad+bc-2ac)xy+acy^2))(c-d,-c)_-\Big)\\
&+\sum_{H(n),bc=0}(acx^2+(ad+bc)xy+bdy^2)(c,d)_- -\sum_{d|n}nxy(0,d)_-\\
&+\sum_{d|n}\Big(\frac{2n^2}{d}xy(0,d)_-+\frac{2nd}{l}\sum_{u\in\mathbb{Z}/l\mathbb{Z}}xy(d,u)_-\Big)\Big).
\end{aligned}
$$

We used various symmetries of $(u,v)_-$ to derive the last identity. A fortunate observation allows one to simplify the second and third lines of the last formula. Indeed, the relations on modular symbols imply

$$
\begin{aligned}
&((-ad-bc+ac+bd)x^2+(ad+bc-2bd)xy+bdy^2)(c-d,d)_-\\
&+(acx^2+(ad+bc-2ac)xy+(-ad-bc+ac+bd)y^2)(-c,c-d)_-\\
&=(acx^2+(ad+bc)xy+bdy^2)(c,d)_-.
\end{aligned}
$$

Then one gets

$$12\mu\circ PD(\phi)\sim 4\sum_{n>0}q^n\phi\Big(\sum_{H(n)}(3acx^2+3(ad+bc)xy+3bdy^2)(c,d)_-$$

$$-2\sum_{H(n),bc=0}(acx^2+(ad+bc)xy+bdy^2)(c,d)_- - \sum_{d|n}nxy(0,d)_-$$

$$+\sum_{d|n}\Big(\frac{2n^2}{d}xy(0,d)_- + \frac{2nd}{l}\sum_{u\in\mathbb{Z}/l\mathbb{Z}}xy(d,u)_-\Big)\Big)$$

$$=4\sum_{n>0}q^n\phi\Big(\sum_{H(n)}(3acx^2+3(ad+bc)xy+3bdy^2)(c,d)_-$$

$$-2\sum_{d|n,d>c>0}\frac{n}{d}(cx^2+dxy)(c,d)_- - \sum_{d|n}nxy(0,d)_-$$

$$+\sum_{d|n}\frac{2nd}{l}\sum_{u\in\mathbb{Z}/l\mathbb{Z}}xy(u,d)_-\Big).$$

Using calculations similar to that of Section 4, we can rewrite the last two lines in terms of fractional parts as

$$S=\phi\Big(4\sum_{n>0}nq^n\sum_{d|n}\sum_{u\in\mathbb{Z}/l\mathbb{Z}}\Big((\frac{l}{d}(\{\frac{d-u}{l}\}-\{\frac{d-u}{l}\}^2)-\frac{l}{d}(\{-\frac{u}{l}\}-\{-\frac{u}{l}\}^2)$$

$$+\frac{l-d}{l}-2\{\frac{u-d}{l}\})x^2+(-2\{\frac{u-d}{l}\}+1-\{-\frac{u}{l}\}+\{\frac{u}{l}\})xy\Big)(u,d)_-\Big).$$

We observe that for any t there holds $\{t\}-\{t\}^2=\{-t\}-\{-t\}^2$, and then use symmetries of $P(x,y)(\pm u,d)_-$ to see that

$$S\sim\phi\Big(4\sum_{n>0}nq^n\sum_{d|n}\sum_{u\in\mathbb{Z}/l\mathbb{Z}}\Big(-2\{\frac{u-d}{l}\}x^2+(-2\{\frac{u-d}{l}\}+1)xy\Big)(u,d)_-\Big)$$

$$=\phi\Big(4\sum_{n>0}nq^n\sum_{d|n}\sum_{u\in\mathbb{Z}/l\mathbb{Z}}\Big((\{\frac{-u-d}{l}\}-\{\frac{u-d}{l}\})x^2+$$

$$(-\{\frac{u-d}{l}\}-\{\frac{-u-d}{l}\}+1)xy\Big)(u,d)_-\Big)$$

$$\sim\phi\Big(4\sum_{n>0}nq^n\sum_{d|n}\sum_{u\in\mathbb{Z}/l\mathbb{Z}}\Big((\{1-\delta_d^{-u\,\mathrm{mod}l})x^2+\delta_d^{u\,\mathrm{mod}l}xy\Big)(u,d)_-\Big)$$

$$=\phi\Big(4\sum_{n>0}nq^n\sum_{d|n}(x^2+xy)(d,d)_-)\Big)\sim 0.$$

This finishes the proof. □

6. Relation to Hecke eigenforms of rank zero

In this section we prove our main result that relates Wronskians of weight one Eisenstein series and the Hecke eigenforms of weight four with nonzero central value of L-function.

Let T_n denote the Hecke operators for $\Gamma_1(l)$ and let $L(f,s)$ denote the Hecke L-function. We will normalize it so the central value is $L(f,2) = \int_0^{i\infty} f(\tau)\tau\, d\tau$. We say that a weight four Hecke eigenform f has analytic rank zero if $L(f,2)\neq 0$.

Definition 6.1. Let $f\in\mathcal{S}_4(l)$ be a weight four cusp form for $\Gamma_1(l)$. Define $\rho(f)=\sum_{n>0} L(T_nf,2)q^n$.

The following statements are analogous to the weight two calculation of [BG1].

Proposition 6.2. *Definition 6.1 gives a linear map $\rho:\mathcal{S}_4(l)\to\mathcal{S}_4(l)$, which commutes with $\Gamma_0(l)/\Gamma_1(l)$-action. The image of ρ contains all newforms f with $L(f,2)\neq 0$, and is contained in the span of all Atkin-Lehner lifts of all Hecke eigenforms f of analytic rank zero.*

Proof. The arguments of [BG1, Propositions 4.3 and 4.5] apply to weight four case without any serious changes. □

Similar to [BG1], the key idea of this paper is to relate the map ρ to the map μ of Section 4.

Proposition 6.3. *The map ρ is the composition of the maps*

$$\mathcal{S}_4(l)\stackrel{Int}{\to}(S_4(l)_-)^*\stackrel{PD}{\to}S_4(l)_+\stackrel{\mu}{\to}\mathcal{S}_4(l)$$

where Int is induced by the integration pairing of $\mathcal{S}_4(l)$ and $S_4(l)_-$, the PD is the Poincaré duality map of Section 3, and μ is the Wronskian map of Section 4.

Proof. We denote by $\langle,\rangle$ the integration pairing between $\mathcal{S}_4(l)$ and $S_4(l)_-$. For a given $f\in\mathcal{S}_4(l)$ we calculate

$$\rho(f)=\sum_{n>0}L(T_nf,2)q^n=\sum_{n>0}\langle T_nf,xy(0,1)_-\rangle q^n.$$

By [M1, Theorem 2 and Proposition 10],

$$\langle T_nf,xy(0,1)_-\rangle=\langle f,T_nxy(0,1)_-\rangle=\langle f,\sum_{H(n)}(ax+by)(cx+dy)(c,d)_-\rangle$$

which leads to

$$\rho(f)=\sum_{n>0}q^n\langle f,\sum_{H(n)}(ax+by)(cx+dy)(c,d)_-\rangle$$

Proposition 5.1 now shows $\mu\circ PD\circ Int(f)\sim\rho(f)$ and since both sides are quasimodular forms of weight four, the claim follows. □

Corollary 6.4. *The image of* ρ *equals the linear span of* $W(s_a, s_b)$ *for* $gcd(a, b, l) = 1$.

Proof. Recall that Int and PD are isomorphisms, by [M1] and Corollary 3.10 respectively. Then Proposition 4.6 finishes the proof. □

We are now ready to formulate our main result.

Theorem 6.5. *For arbitrary* $l > 1$ *the span of Hecke eigenforms of weight four and analytic rank zero is equal to the span of the Wronskians* $W(s_a, s_b)$ *for all* $a, b \in \mathbb{Z}/l\mathbb{Z}$.

Proof. In one direction, consider $f = W(s_a, s_b)$. If $gcd(a, b, l) = d$, then Corollary 6.4 applied to $\frac{l}{d}$ shows that f is in $\rho(\mathcal{S}_4(\frac{l}{d}))$. Indeed, f is, up to a nonzero multiple, the d-lift of $W(s_{\frac{a}{d},\frac{l}{d}}, s_{\frac{b}{d},\frac{l}{d}})$ where the second subscript in s is used to indicate the level. By Proposition 6.2, $W(s_{\frac{a}{d},\frac{l}{d}}, s_{\frac{b}{d},\frac{l}{d}})$ lies in the linear span of eigenforms of analytic rank zero, hence f does as well.

To prove the opposite inclusion, it is enough to show that for any $d|l$ and any newform $g(\tau) \in \mathcal{S}_4(\frac{l}{d})$ of analytic rank zero, its lift $g(k\tau) \in \mathcal{S}_4(l)$ lies in the span of $W(s_a, s_b)$ for any $k|d$. By Proposition 6.2, $g \in \rho(\mathcal{S}_4(\frac{l}{d}))$. Then by Corollary 6.4, g is a linear combination of Wronskians of Eisenstein series $s_{i,\frac{l}{d}}$ of level $\frac{l}{d}$. Then $g(k\tau)$ is a linear combination of Wronskians of s-series of level $\frac{kl}{d}$, since $s_{i,\frac{l}{d}}(k\tau) = s_{ki,\frac{kl}{d}}(\tau)$. Finally, s-series of level $\frac{kl}{d}$ are sums of s_a of level l, which shows that $g(k\tau)$ lies in the span of the Wronskians $W(s_a, s_b)$, as claimed. □

Corollary 6.6. *The span of Hecke eigenforms of weight four and analytic rank zero for the group* $\Gamma_0(l)$ *coincides with the span of*

$$\sum_{j \in (\mathbb{Z}/l\mathbb{Z})^*} W(s_{aj}, s_{bj})$$

for all $a, b \in \mathbb{Z}/l\mathbb{Z}$.

Proof. Use the formulas for the action of $\Gamma_0(l)$ on s_a from [BG2]. □

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