

ON THE K -THEORY OF SMOOTH TORIC DM STACKS

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ABSTRACT. We explicitly calculate the Grothendieck K -theory ring of a smooth toric Deligne-Mumford stack and define an analog of the Chern character. In addition, we calculate K -theory push-forwards and pullbacks for weighted blowups of reduced smooth toric DM stacks.

1. INTRODUCTION

In this paper we calculate the Grothendieck K -theory rings of coherent sheaves on smooth toric Deligne-Mumford stacks. These stacks, which generalize the notion of a smooth toric variety, have been defined in [BCS].

We are mostly interested in the reduced case, which is characterized by the condition that there is an open substack which is a subscheme. However, since our technique is applicable to non-reduced stacks as well, we extend the result to the general case in a separate section. We find that the Grothendieck group of a smooth toric Deligne-Mumford stack \mathbb{P}_Σ is generated by classes of invertible sheaves, and we find the generators of the ideal of relations satisfied by these sheaves. In the reduced case $K_0(\mathbb{P}_\Sigma)$ is generated by the classes R_i of the invertible sheaves \mathcal{L}_i which correspond to the one-dimensional cones of the fan Σ . These sheaves generalize the sheaves $\mathcal{O}(D_i)$ for codimension one strata D_i in a smooth toric variety. Moreover, we find the generators of the ideal of relations among $R_i^{\pm 1}$.

Theorem 4.10. Let B be the quotient of the Laurent polynomial ring $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ by the ideal generated by the relations

- $\prod_{i=1}^n x_i^{f(v_i)} = 1$ for any linear function $f : N \rightarrow \mathbb{Z}$,
- $\prod_{i \in I} (1 - x_i) = 0$ for any set $I \subseteq [1, \dots, n]$ such that $v_i, i \in I$ are not contained in any cone of Σ .

Then the map $\rho : B \rightarrow K_0(\mathbb{P}_\Sigma)$ which sends x_i to R_i is an isomorphism.

We also show that the K -theory with complex coefficients is a semilocal Artinian \mathbb{C} -algebra and explicitly describe its local components.

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We produce a *vector space* isomorphism between the K -theory with complex coefficients and a combinatorially defined ring dubbed SR-cohomology, which we call *combinatorial Chern character*. It generalizes the usual Chern character in the case of projective toric varieties and is expected to coincide with the equivariant Chern character of [AR] (see also the very recent preprint [JKK]) in the projective DM stack case.

The paper is structured as follows. In Section 2 we recall the definition and basic properties of smooth toric DM stacks. We restrict our attention to the reduced case, which makes the Gale duality construction of [BCS] significantly easier to describe. In Section 3 we introduce the SR-cohomology in the reduced case and describe its decomposition into sectors. In Section 4 we calculate the K -theory of the reduced smooth toric DM stack. The key idea is to use the homogenization trick of [E] to resolve any coherent sheaf on \mathbb{P}_Σ by direct sums of invertible sheaves. Section 5 defines the combinatorial Chern character, and Section 6 extends the results of Sections 4 and 5 to the nonreduced case. Sections 7, 8 and 9 describe the K -theory pullbacks and pushforwards for special classes of morphisms between reduced smooth toric DM stacks. These results will be used in a subsequent paper [BH] on homological mirror symmetry and the GKZ hypergeometric system of partial differential equations.

There is some overlap between the results of this paper and those of the recent preprint [Ba], where the equivariant K -theory of smooth toric varieties is studied with the help of the Merkurjev's spectral sequence.

2. REVIEW OF REDUCED SMOOTH TORIC DM STACKS

In this section we will briefly review the definitions of toric Deligne-Mumford stacks, as developed in [BCS]. We are specifically interested in the reduced case, which simplifies the construction significantly.

Let N be a free abelian group, and let Σ be a simplicial fan in N . See [F] for the definition of a fan. A *stacky fan* Σ is defined as the data $(\Sigma, \{v_i\})$ where $\{v_i\}$ is a collection of lattice points, one for each one-dimensional cone $C_i \in \Sigma$. We require $v_i \in C_i \cap (N - \{\mathbf{0}\})$, but v_i may or may not be the minimum lattice point on C_i . The paper [BCS] makes an additional technical assumption

- (1) *elements v_i span a finite index subgroup of N .*

Remark 2.1. It is likely that this assumption (1) is just an artifact of the technique of [BCS]. In general, one can always look at the preimage N_f in N of the torsion part of $N/\text{Span}(v_i, i = 1, \dots, n)$. The quotient N/N_f splits off (noncanonically) as a free direct summand. It

is reasonable to expect that the DM stack that corresponds to Σ is the product of $(\mathbb{C}^*)^{\text{rk}(N/N_f)}$ and the toric DM stack of [BCS] constructed for Σ in N_f instead of N . However, the functoriality of this construction is a bit unclear, in view of the fact that the splitting is non-canonical.

The collection $\{v_i\}, i = 1, \dots, n$ gives a map

$$\phi : \mathbb{Z}^n \rightarrow N$$

with finite cokernel. We dualize to get an injective map

$$\phi^\vee : N^\vee \rightarrow (\mathbb{Z}^n)^\vee.$$

The Gale dual of ϕ (see [BCS]) is simply the map $(\mathbb{Z}^n)^\vee \rightarrow \text{Coker}(\phi^\vee)$. Denote by G the algebraic group $\text{Hom}(\text{Coker}(\phi)^\vee, \mathbb{C}^*)$. The exact sequence

$$(2) \quad 0 \rightarrow N^\vee \rightarrow (\mathbb{Z}^n)^\vee \rightarrow \text{Coker}(\phi^\vee) \rightarrow 0$$

leads to the exact sequence

$$0 \rightarrow G \rightarrow (\mathbb{C}^*)^n \rightarrow \text{Hom}(N^\vee, \mathbb{C}^*).$$

Hence, G can be thought of as a subgroup of $(\mathbb{C}^*)^n$. Its \mathbb{C} -points can be thought of as collections of n nonzero complex numbers $\lambda_i, i = 1, \dots, n$ which satisfy the condition

$$(3) \quad \prod_{i=1}^n \lambda_i^{f(v_i)} = 1$$

for all linear functions $f : N \rightarrow \mathbb{Z}$.

Consider the subset Z of \mathbb{C}^n that consists of all the points $\mathbf{z} = (z_1, \dots, z_n)$ such that the set of v_i for the zero coordinates of \mathbf{z} is contained in a cone of Σ . Then the toric DM stack \mathbb{P}_Σ that corresponds to the stacky fan $\Sigma = (\Sigma, \{v_i\})$ is defined as the stack quotient $[Z/G]$ where Z and G are endowed with the natural reduced scheme structures. The action of (λ_i) on (z_i) is given by $(\lambda_i z_i)$. It has been shown in [BCS] that \mathbb{P}_Σ is a Deligne-Mumford stack whose moduli space is the simplicial toric variety \mathbb{P}_Σ .

To a cone in C one can associate a closed substack of \mathbb{P}_Σ by looking at the quotient of N by the sublattice spanned by $v_i \in C$. The new fan is defined as the image of the link of C in Σ , and the new v_j are the images of the old v_j from the link. Unfortunately, this procedure may introduce torsion into N , so the resulting substack is not reduced. Moreover, in order for the construction of [BCS] to work, the images of v_j from the link of C need to generate a finite index subgroup in

$N/\text{Span}(v_i \in C)$. The easiest way to assure this is by imposing a stronger technical condition

(4) *every cone of Σ is contained in a cone of dimension $\text{rk}N$,*

see [BCS]. This also assures that \mathbb{P}_Σ is covered by open substacks of the form $\mathbb{C}^{\text{rk}N}/G_i$ where G_i are finite abelian groups. These open substacks can be indexed by the cones in Σ of maximum dimension. On the other hand, one can definitely talk about closed toric subvarieties of the coarse moduli space \mathbb{P}_Σ of \mathbb{P}_Σ without the additional assumption (4). Throughout this paper we will only use the assumption (1).

3. ORBIFOLD COHOMOLOGY AND SR-COHOMOLOGY OF REDUCED SMOOTH TORIC DM STACKS

In this section we introduce some combinatorial invariants of toric DM stacks which we call SR-cohomology rings. They coincide with the orbifold cohomology rings in the projective case but are generally different, even for smooth toric varieties.

A natural combinatorial invariant of a fan is the partial semigroup ring $\mathbb{Z}[N, \Sigma]$ which is defined as a free abelian group with the basis $\{[w], w \in N \cap \Sigma\}$ indexed by the lattice elements inside the support of the fan and multiplication

$$[w_1] * [w_2] = \begin{cases} [w_1 + w_2], & \text{if there exists } C \in \Sigma, C \ni w_1, w_2 \\ 0, & \text{otherwise.} \end{cases}$$

The rings $\mathbb{Q}[N, \Sigma]$ and $\mathbb{C}[N, \Sigma]$ are defined analogously. For a stacky fan Σ , these rings are given additional structure of graded rings. The grading can take nonnegative rational values. It is defined by $\deg([w]) = \sum_i \alpha_i$ where $w = \sum_i \alpha_i v_i$ is the unique way of writing w as a rational linear combination of v_i that lie in the minimum cone of Σ that contains it.

To any stacky fan Σ one can associate a *SR-cohomology ring*, to be denoted by $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$.

Definition 3.1. Pick a basis $\{f_1, \dots, f_{\text{rk}N}\}$ of $\text{Hom}(N, \mathbb{Z})$. Each f_i gives rise to a degree one element $t_i := \sum_{j=1}^n f_i(v_j)[v_j]$ in $\mathbb{C}[N, \Sigma]$. Then

$$H_{SR}(\mathbb{P}_\Sigma, \mathbb{C}) := \mathbb{C}[N, \Sigma] / \langle t_1, \dots, t_{\text{rk}N} \rangle.$$

The rings $H_{SR}(\mathbb{P}_\Sigma, \mathbb{Q})$ and $H_{SR}(\mathbb{P}_\Sigma, \mathbb{Z})$ are defined analogously.

It is clear that the above defined SR-cohomology rings are independent of the choice of the basis $\{f_i\}$. If \mathbb{P}_Σ is a smooth projective toric *variety*, then its combinatorial cohomology rings are isomorphic to its usual cohomology rings. Indeed, this is precisely the Stanley-Reisner

presentation of the cohomology ring of a smooth projective toric variety. This is the motivation behind our notation H_{SR} . We call our rings SR-cohomology rings, as opposed to Stanley-Reisner cohomology rings, to avoid confusion with the term Stanley-Reisner ring that is already used in the literature.

Remark 3.2. We abuse notation slightly and still use $H_{SR}(\mathbb{P}_\Sigma, *)$ even if Σ does not satisfy (1). Hopefully, one will eventually define toric stacks without this restriction.

The main technical result of [BCS] connects the SR-cohomology of a smooth toric DM stack with its orbifold cohomology, see [CR] and [AGV]. However, one needs an important additional assumption that the coarse moduli space \mathbb{P}_Σ is projective.

Theorem 3.3. ([BCS]) *For any \mathbb{P}_Σ such that \mathbb{P}_Σ is projective, there holds*

$$H_{orb}(\mathbb{P}_\Sigma, \mathbb{Q}) \cong H_{SR}(\mathbb{P}_\Sigma, \mathbb{Q}).$$

Remark 3.4. In general, the SR-cohomology and orbifold or usual cohomology rings are different. The simplest example is given by the fan Σ which is the union of the first and the third quadrants in \mathbb{Z}^2 . The elements v_i are $(\pm 1, 0)$ and $(0, \pm 1)$. The corresponding variety is $\mathbb{P}^1 \times \mathbb{P}^1 - \{(0, \infty), (\infty, 0)\}$. The SR-cohomology is three-dimensional, with dimension 2 graded component of degree one. It is unclear what geometrically defined cohomology theory on $\mathbb{P}^1 \times \mathbb{P}^1 - \{(0, \infty), (\infty, 0)\}$ can produce such a ring.

Despite Remark 3.4, the SR-cohomology ring $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$ is very well behaved.

Proposition 3.5. *The SR-cohomology ring $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$ of any (reduced) stacky fan Σ is a finite dimensional complex vector space. It is a local Artinian graded \mathbb{C} -algebra with the maximum ideal given by the span of the elements of positive degree.*

Proof. Since $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$ is nonnegatively graded and its degree zero part is isomorphic to \mathbb{C} , it is enough to show that it is an Artinian ring. Hence, let us study \mathbb{C} -algebra homomorphisms $\phi : H_{SR}(\mathbb{P}_\Sigma, \mathbb{C}) \rightarrow \mathbb{C}$.

In view of relations in $\mathbb{C}[N, \Sigma]$, the values of $\phi([w])$ are nonzero only for w in some cone C of Σ . Then the relations t_i show that these values must be zero for $w \in C$ that are integer linear combinations of $v_j \in C$. Finally, for any other nonzero $w \in C$, some positive multiple of it is an integer linear combination of $v_j \in C$. This leads to $\phi([w])^l = 0$, so $\phi([w]) = 0$ for all nonzero n . This shows that the only maximum ideal in $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$ is the span of the elements of positive degree. \square

In the case of projective \mathbb{P}_Σ , there is a natural decomposition of the orbifold and SR-cohomology into direct sum of sectors. Each sector corresponds to the usual cohomology of some closed toric subvariety of \mathbb{P}_Σ . We observe that this decomposition still occurs for the SR-cohomology, even without the projectivity assumption. Moreover, we will not even assume (1).

Definition 3.6. The *untwisted sector* of SR-cohomology is defined as the subring of $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$ generated by the images of $[v_i], i = 1, \dots, n$.

Remark 3.7. It is easy to see that the choice of v_i in the corresponding one-dimensional cone does not change the untwisted sector much, if one works with rational or complex coefficients. Indeed, one can simply rescale the corresponding variables. Consequently, we can talk about the SR-cohomology of the toric variety \mathbb{P}_Σ , in analogy with the projective case. We can define it as the untwisted sector of the SR-cohomology of the stack \mathbb{P}_Σ obtained by picking the v_i to be the smallest lattice points on the one-dimensional cones of Σ . We denote these rings by $H_{SR}(\mathbb{P}_\Sigma, \mathbb{Q}$ or $\mathbb{C})$.

For a cone $C \in \Sigma$, let $\text{Box}(C)$ be the set of elements v of N which are linear combinations with rational coefficients in the range $[0, 1)$ of elements $v_i \in C$. Let $\text{Box}(\Sigma)$ be the union of $\text{Box}(C)$ for all $C \in \Sigma$.

Proposition 3.8. *As a module over the untwisted sector, the ring $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$ is a direct sum of the modules H_v generated by the images in $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$ of the elements $[v] \in \mathbb{C}[N, \Sigma]$ for all $v \in \text{Box}(\Sigma)$.*

Proof. The statement follows from the analogous result for $\mathbb{C}[N, \Sigma]$, where it is obvious. \square

In the projective case, each of the modules H_v is isomorphic to the cohomology of the toric subvariety of \mathbb{P}_Σ that corresponds to the minimum cone of Σ that contains v . Remarkably, this is true in general if one works in SR-cohomology with complex or rational coefficients.

Proposition 3.9. *The module H_v is isomorphic to the SR-cohomology with complex coefficients of the (closed) toric subvariety of \mathbb{P}_Σ that corresponds to the minimum cone of Σ that contains v .*

Proof. The proof is analogous to that of [BCS, Proposition 5.2], but we briefly sketch it here for the benefit of the reader. Let N_1 denote the quotient of N by the subgroup generated by $v_i \in C$, and let $N' = N_1/\text{torsion}$ be its torsion-free part. Let Σ' in N' be the image of the link of C in Σ and let v'_i be the images of v_i from the link. The SR-cohomology of the closed toric subvariety that corresponds to C is

isomorphic to the untwisted sector in the SR-cohomology $H_{SR}(\mathbb{P}_{\Sigma'}, \mathbb{C})$ of Σ' .

Observe that H_v is a quotient of the submodule M_v of $\mathbb{C}[N, \Sigma]$ which is supported on the star of C . Moreover, it is generated by the monomial of the form $[n + v]$ where n is a lattice point in the star of C in Σ which is an *integer* linear combination of the v_j in the minimum cone that contains it. This identifies H_v with the quotient of the polynomial ring in the variables D_1, \dots, D_k that correspond to one-dimensional faces of C and cones in its link by the ideal with generators

- $\prod_{i \in I} D_i$, if no cone $C' \supseteq C$ contains all $v_i, i \in I$,
- $\sum_{i, v_i \in \text{star}(C)} f(v_i) D_i$ for any linear map $f : N \rightarrow \mathbb{Z}$.

On the other hand, the untwisted sector of $H_{SR}(\mathbb{P}_{\Sigma'}, \mathbb{C})$ is isomorphic to the quotient of the polynomial ring in the variables $D'_1, \dots, D'_{k'}$ which correspond to one-dimensional faces of cones in the link of C (hence not in C), by the ideal generated by

- $\prod_{i \in I} D'_i$, if no cone $C' \supseteq C$ contains all $v_i, i \in I$,
- $\sum_{i, v_i \in \text{link}(C)} f'(\rho(v_i)) D'_i$ for any linear map $f' : N' \rightarrow \mathbb{Z}$.

Here $\rho : N \rightarrow N'$ is the projection.

To connect these two spaces, observe that linear functions on N' lift to linear functions on N . Pick a complementary sublattice in the lattice of linear functions on N and pick its basis $f_1, \dots, f_{\dim C}$. These f_i provide relations on D_i that allow one to express $D_i, v_i \in C$, in terms of $D_i, v_i \in \text{link}(C)$, in the relations for H_v . The remaining relations are precisely those of the untwisted sector of $H_{SR}(\mathbb{P}_{\Sigma'}, \mathbb{C})$. Moreover, this isomorphism is independent of the choice of the complementary lattice or its basis. \square

Remark 3.10. Propositions 3.8 and 3.9 still hold for SR-cohomology with rational coefficients. The proofs are unchanged.

Remark 3.11. Suppose that the fan Σ is projective, or that it is a subdivision of a cone. Under these assumptions, if one picks a basis $\{f_j\}$ of N^\vee , then the corresponding elements $\sum_{i=1}^n f(v_i)[v_i]$ form a regular sequence in $\mathbb{Q}[N]^\Sigma$. As a result, the graded dimension of $H_{SR}(\Sigma, \mathbb{C})$, defined as $\sum_{d \in \mathbb{Q}} t^d \dim_{\mathbb{C}} H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})_{\deg=d}$, equals

$$\text{gr.dim} H_{SR}(\mathbb{P}_\Sigma, \mathbb{C}) = (1 - t)^{\text{rk} N} \sum_{\mathbf{n} \in N \cup \Sigma} t^{\deg(\mathbf{n})}.$$

The proof of this regularity is sketched in [BCS] for the projective case and in [BM] for the cone case. In general, the graded dimension of $H_{SR}(\Sigma, \mathbb{C})$ is the sum over sectors of the graded dimensions of the

SR-cohomology of the sector, but it lacks such a nice combinatorial formula.

4. K -THEORY OF REDUCED TORIC DM STACKS

The goal of this section is to prove a combinatorial description of the K -theory of reduced toric DM stack which is analogous to the Stanley-Reisner presentation of the cohomology of a smooth toric variety. Analogous statements for smooth toric varieties are contained in [Ba]. The resulting ring is then compared to the SR-cohomology of the stack. We use the notations from the previous sections. We do not make any assumptions on the stacky fan apart from (1).

Definition 4.1. Let \mathcal{X} be a smooth Deligne-Mumford stack. Define the (Grothendieck) K -theory group $K_0(\mathcal{X})$ to be the quotient of the free abelian group generated by coherent sheaves \mathcal{F} on \mathcal{X} by the relations $[\mathcal{F}_1] - [\mathcal{F}_2] + [\mathcal{F}_3]$ for all exact sequences $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$.

Remark 4.2. The K -theory of \mathcal{X} admits a product structure by

$$[\mathcal{F}_1] * [\mathcal{F}_2] = \sum_{i=0}^{\dim \mathcal{X}} (-1)^i [\mathcal{T}or^i(\mathcal{F}_1, \mathcal{F}_2)].$$

The image of the structure sheaf \mathcal{O} plays the role of the identity.

We recall that the category of coherent sheaves on $[Z/G]$ is equivalent to that of G -linearized coherent sheaves on Z , see [V, Example 7.21]. We will always implicitly use this equivalence.

Definition 4.3. For each i from 1 to n we define a G -linearized invertible sheaf \mathcal{L}_i on Z as follows. As a sheaf, it will be isomorphic to the structure sheaf \mathcal{O}_Z . For $g = (\lambda_1, \dots, \lambda_n) \in G$ the isomorphism $\mathcal{O}_Z \rightarrow g^*\mathcal{O}_Z = \mathcal{O}_Z$ sends 1 to λ_i . Here we have used the canonical isomorphism $g^*\mathcal{O}_Z = \mathcal{O}_Z$. We define R_i to be the image of \mathcal{L}_i in the K -theory of \mathbb{P}_Σ .

Remark 4.4. The sheaf \mathcal{L}_i has a G -invariant global section given by the restriction to Z of the i -th coordinate function z_i on \mathbb{C}^n . Indeed, $g^*z_i = \lambda_i z_i$, so z_i is compatible with the isomorphisms of the above definition. In the case of a smooth toric variety, \mathcal{L}_i corresponds to $\mathcal{O}(D_i)$ where D_i is the divisor of the i -th dimension one cone of Σ .

Proposition 4.5. *The elements R_i of $K_0(\mathbb{P}_\Sigma)$ satisfy the following relations.*

- $\prod_{i=1}^n R_i^{f(v_i)} = 1$ for any linear function $f : N \rightarrow \mathbb{Z}$,

- $\prod_{i \in I} (1 - R_i) = 0$ for any set $I \subseteq [1, \dots, n]$ such that the $v_i, i \in I$ are not contained in any cone of Σ .

Proof. The relations (3) and the definition of \mathcal{L}_i show that $\otimes_i \mathcal{L}_i^{f(v_i)}$ are in fact trivially linearized on Z for every linear function $f : N \rightarrow \mathbb{Z}$. Passing to K -theory, we get the first set of relations on R_i .

To get the second set of relations, consider the Koszul complex of the set of elements $z_i, i \in I$ in the ring $A = \mathbb{C}[z_1, \dots, z_n]$. It gives an exact sequence

$$\begin{aligned} \dots &\rightarrow \bigoplus_{J \subseteq I, |J|=k} A \rightarrow \bigoplus_{J \subseteq I, |J|=k-1} A \rightarrow \dots \\ \dots &\rightarrow \bigoplus_{j \in I} A \rightarrow A \rightarrow A / \langle z_i, i \in I \rangle \rightarrow 0 \end{aligned}$$

of A -modules. We can pass to the associated sheaves on \mathbb{C}^n and then restrict them to Z to get

$$0 \rightarrow \dots \rightarrow \bigoplus_{J \subseteq I, |J|=k} \mathcal{O}_Z \rightarrow \bigoplus_{J \subseteq I, |J|=k-1} \mathcal{O}_{\mathbb{C}^n} \rightarrow \dots \rightarrow 0$$

where we have used the condition on I to see that $A / \langle z_i, i \in I \rangle$ is supported outside of Z . The maps between the copies of \mathcal{O}_Z for J_1 and J_2 are zero unless $J_2 \subset J_1$ and are $\pm z_{J_1 - J_2}$ otherwise. We can make this into a complex of G -linearized sheaves

$$0 \dots \rightarrow \bigoplus_{J \subseteq I, |J|=k} \otimes_{j \in J} \mathcal{L}_j^{-1} \rightarrow \bigoplus_{J \subseteq I, |J|=k-1} \otimes_{j \in J} \mathcal{L}_j^{-1} \rightarrow \dots \rightarrow 0$$

by using the G -equivariant maps $z_j : \mathcal{L}_j^{-1} \rightarrow \mathcal{O}_Z$ which are twists by \mathcal{L}_j^{-1} of the maps of Remark 4.4.

The alternating sum of any long exact sequence of sheaves is zero in the K -theory, which implies the second set of relations, after multiplying by the invertible element $\prod_{i \in I} R_i$. \square

Theorem 4.6. *The elements R_i generate the K_0 -theory of the reduced toric DM stack \mathbb{P}_Σ .*

Proof. Consider a G -linearized coherent sheaf \mathcal{F} on the open subset Z of \mathbb{C}^n . Let us denote the embedding of Z into \mathbb{C}^n by i . We observe that $i^* i_* \mathcal{F} \rightarrow \mathcal{F}$ is an isomorphism, simply because i is an open embedding. Since \mathbb{C}^n is affine, $i_* \mathcal{F}$ is the sheaf associated to $\mathbb{C}[z_1, \dots, z_n]$ -module $M = H^0(Z, \mathcal{F})$.

Lemma 4.7. *In the notations above, M is a finitely generated module.*

Proof of Lemma 4.7. We know by [H, Exercise II.5.15] that \mathcal{F} is a restriction to Z of some coherent sheaf on \mathbb{C}^n . This coherent sheaf can be resolved by free sheaves on \mathbb{C}^n . When we restrict to Z we see that \mathcal{F} is resolved by direct sums of \mathcal{O}_Z . Hence, it is enough to see that all the cohomology groups of \mathcal{O}_Z are finitely generated modules over $\mathbb{C}[z_1, \dots, z_n]$. By using the $(\mathbb{C}^*)^n$ -action, we see that $H^0(Z, \mathcal{O}_Z)$ is

generated by some monomials $\prod_j z_j^{r_j}$, if one thinks of it as the subspace of the quotient field of $\mathbb{C}[z_1, \dots, z_n]$. We easily see that $\mathbb{C}^n - Z$ is a union of coordinate subspaces of codimension at least two, which implies that r_i have to be nonnegative. Then we see that $H^0(Z, \mathcal{O}_Z) = \mathbb{C}[z_1, \dots, z_n]$. To calculate the higher cohomology $H^*(Z, \mathcal{O}_Z)$ groups, we can cover Z by open affine subsets U_C of the form

$$U_C = \{(z_1, \dots, z_n), z_i \neq 0 \text{ for } v_i \notin C\}$$

where C runs over the set of nonzero cones of Σ . The sections of \mathcal{O}_Z on U_C are the linear span of monomials $\prod_j z_j^{r_j}$ such that the set of i with $r_i < 0$ is contained in the set of i with $v_i \notin C$. Various intersections $U_{C_1 C_2 \dots C_k}$ of U_C give the spans of monomials with the condition that for each j with $r_j < 0$ the lattice element v_j is not contained in $\cap_l C_l$. We use the cover by sets of type U_C to calculate the cohomology of \mathcal{O}_Z as Čech cohomology. For a given (r_1, \dots, r_n) , the cohomology of \mathcal{O}_Z that corresponds to this \mathbb{Z}^n -grading is given by the reduced homology of the following simplicial complex. Its set of vertices is the set Σ_+ of all cones of positive dimension in Σ , and its maximum simplices are the complements in Σ_+ of the one-element subsets $\{C\}$ that correspond to cones that do not contain any v_i with $r_i < 0$. So we have a simplicial complex which is a union of a collection of complements of one-dimensional subsets. If not all the elements of Σ_+ are in this collection, then the resulting complex is a cone, since adding such an element has no effect on whether a subset of Σ_+ belongs to the complex. Hence, it has trivial reduced homology. If all elements of Σ_+ are a part of the collection, then there can be no v_i with $r_i < 0$. The resulting complex is a sphere, and we have a one-dimensional top reduced homology that corresponds to the monomial in $H^0(Z, \mathcal{O}_Z)$. This shows that the higher cohomology $H^{>0}(Z, \mathcal{O}_Z)$ groups vanish, which finishes the proof of the lemma. \square

Proof of Theorem 4.6 continues. The G -linearization on \mathcal{F} gives rise to a G -action on M , which is compatible with the G -action on $A = \mathbb{C}[z_1, \dots, z_n]$, in the sense that $g(rm) = g(r)g(m)$ for all $g \in G$, $r \in A$ and $m \in M$. Moreover, we claim that M is generated by a finite set of *eigenelements* m_j with $g(m_j) = \chi_j(g)m_j$ for some character $\chi_j : G \rightarrow \mathbb{C}^*$. In view of the structure of G , all of its algebraic finite-dimensional actions are diagonalizable. As a consequence, a module M is generated by a finite set of eigenelements if and only if

- (5) *for any $m \in M$ the linear span of $gm, g \in G$ is finite-dimensional, and the action of G on it is algebraic.*

It is clear that this finiteness condition (5) for a G -equivariant Noetherian A -module implies (5) for all equivariant submodules and quotients.

Recall (see [MF]) that the G -linearization of \mathcal{F} on Z is given by an isomorphism $\phi : \mu^* \mathcal{F} \rightarrow \pi_2^* \mathcal{F}$ on $G \times Z$ where μ is the multiplication and π_2 is the second projection. Denote by A_G the ring of regular functions on G . We will use the fact that $H^0(Z, \mathcal{O}_Z) = H^0(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = A$ from the proof of Lemma 4.7. The isomorphism ϕ induces a map

$$\phi_{global} : M \otimes_A (A_G \otimes_{\mathbb{C}} A) \rightarrow M \otimes_{\mathbb{C}} A_G$$

where the tensor multiplication on the left is via the map $A \rightarrow A_G \times A$ induced by the action of G on Z . Indeed, the left hand side maps to the global sections of $\mu^*(\mathcal{F})$, whereas the right hand side consists of the global sections of $\pi_2^* \mathcal{F}$. This map ϕ_{global} encodes the action of G on M by mapping $m \otimes (f \otimes a) \mapsto \sum_k m_k \otimes f_k$ such that for any $g \in G$ there holds $ag(m) f(g) = \sum_k f_k(g) m_k$. By taking $a = f = 1$ we see that there is a finite sum $\sum_k m_k \otimes f_k$ such that $g(m) = \sum_k f_k(g) m_k$ for all $g \in G$. In particular, the span of $gm, g \in G$ is finite-dimensional. By picking a basis of it, and applying the above, we see that the action of G is algebraic, which shows (5).

Consequently, there is a presentation

$$F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

where F_i is a direct sum of rank one $\mathbb{C}[z_1, \dots, z_n]$ -modules generated by eigenelements of G . Indeed, the kernel of $F_0 \rightarrow M$ also satisfies (5). Hence, it is generated by a finite number of eigenelements, which allows one to construct F_1 . We will now use the homogenization trick of [E, Corollary 19.8]. Namely, consider the ring $\mathbb{C}[z_0, \dots, z_n]$ and extend the G -action on it by $gz_0 = z_0$ for all $g \in G$. The map $F_1 \rightarrow F_0$ is given by a matrix S of polynomials in z_1, \dots, z_n . Define the matrix \tilde{S} of homogeneous polynomials of some large fixed degree d by multiplying each monomial in S by an appropriate power of z_0 . We will then have a presentation of some module \tilde{M} over $\mathbb{C}[z_0, \dots, z_n]$

$$\tilde{F}_1 \rightarrow \tilde{F}_0 \rightarrow \tilde{M} \rightarrow 0.$$

Here \tilde{F}_i are direct sums of rank one G -equivariant $\mathbb{C}[z_0, \dots, z_n]$ -modules generated by homogeneous eigenelements. As in [E, Corollary 19.8], we observe that

$$\mathbb{C}[z_1, \dots, z_n] = \mathbb{C}[z_0, \dots, z_n]/(1 - z_0)$$

and $\tilde{M}/(1 - z_0)\tilde{M} = M$. Given a G -equivariant homogeneous module \tilde{F} over $\mathbb{C}[z_0, \dots, z_n]$, we can look at the finite-dimensional vector space

$$\tilde{F}/\langle z_0, \dots, z_n \rangle \tilde{F}.$$

This vector space inherits the G -action. It also inherits the the grading, which is preserved by the G -action. As a result, this vector space has a basis of homogeneous eigenelements. Each such eigenelement can be lifted to an element r_j of \tilde{F} . By looking at the (finite-dimensional) G -span of r_j in \tilde{F} , we can modify r_j to be itself a homogeneous eigenelement. The elements r_j generate \tilde{F} by the graded Nakayama lemma.

We apply this procedure to the kernel K of $\tilde{F}_1 \rightarrow \tilde{F}_0$, and continue on to build a resolution of \tilde{M} . Since the end of this resolution is the minimal graded resolution of the homogenous module K , it terminates. We then get a free resolution

$$0 \rightarrow \tilde{F}_l \rightarrow \dots \rightarrow \tilde{F}_0 \rightarrow \tilde{M} \rightarrow 0$$

of \tilde{M} such that each \tilde{F}_i is freely generated by homogeneous eigenelements r_{ij} and the maps are G -equivariant and are compatible with the grading. We mod out this resolution by $1 - z_0$ to get a resolution of M . The exactness follows from $\mathrm{Tor}_{\mathbb{C}[z_0, \dots, z_n]}^{>0}(M, \mathbb{C}[z_1, \dots, z_n]) = 0$, as in [E, Corollary 19.8]. We thus get a G -equivariant resolution of M by direct sums of free modules of rank one generated by eigenelements. We claim that each such module can be identified with the global sections of a tensor product of powers of line bundles \mathcal{L}_i . Indeed, one simply needs to show that every character of G is a restriction of a character of $(\mathbb{C}^*)^n$, which follows from the exact sequence (2).

We pass to the corresponding exact sequence of G -linearized sheaves on \mathbb{C}^n and then restrict it to Z to see that \mathcal{F} is a linear combination of products of R_i . \square

Corollary 4.8. *The bounded derived category of the category of coherent sheaves on \mathbb{P}_Σ is generated by the invertible sheaves $\otimes_i \mathcal{L}_i^{r_i}$.*

Proof. The argument of Theorem 4.6 shows that every sheaf on \mathbb{P}_Σ admits a free resolution by direct sums of invertible sheaves of the above type. \square

Remark 4.9. We refer the reader to [K2] for much stronger results concerning these derived categories.

Theorem 4.10. *Let B be the quotient of the Laurent polynomial ring $\mathbb{Z}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$ by the ideal generated by the relations*

- $\prod_{i=1}^n x_i^{f(v_i)} = 1$ for any linear function $f : N \rightarrow \mathbb{Z}$,
- $\prod_{i \in I} (1 - x_i) = 0$ for any set $I \subseteq [1, \dots, n]$ such that $v_i, i \in I$ are not contained in any cone of Σ .

Then the map $\rho : B \rightarrow K_0(\mathbb{P}_\Sigma)$ which sends x_i to R_i is an isomorphism.

Proof. Theorem 4.6 shows that ρ is surjective. It is therefore sufficient to show its injectivity. We will define a map $\rho_1 : K_0(\mathbb{P}_\Sigma) \rightarrow B$ and prove that $\rho_1 \circ \rho = \mathbf{id}$.

For any G -linearized sheaf \mathcal{F} on Z , consider the G -equivariant module $M = H^0(Z, \mathcal{F})$ over $A = \mathbb{C}[z_1, \dots, z_n]$. Consider the A -module \mathbb{C} with trivial G -action. For each i from 0 to n the finite-dimensional vector space $\mathrm{Tor}_A^i(M, \mathbb{C})$ is acted upon by G . It is a direct sum over the characters of $\chi : G \rightarrow \mathbb{C}^*$ of the eigenspaces $V_{i,\chi}$. The group $\mathrm{Hom}(G, \mathbb{C}^*)$ is a quotient of \mathbb{Z}^n by N^\vee . Because of the first set of relations on x_i , every character χ gives a well-defined monomial $x_\chi \in B$. We then define

$$\rho_1 : \mathcal{F} \mapsto \sum_{\chi} \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}}(V_{i,\chi}) x_\chi.$$

We would like to show that ρ_1 passes down to a well-defined map on K -theory, i.e. it is additive on short exact sequences. If

$$(6) \quad 0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow 0$$

is an exact sequence of G -linearized sheaves on Z with G -equivariant morphisms, then the sequence of A -modules

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3$$

is only exact on the left. We complete it to a long exact sequence

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow M_4 \rightarrow 0.$$

This long exact sequence of modules splits into short exact sequences, which, in turn, give long exact sequences of $\mathrm{Tor}^i(*, \mathbb{C})$. This shows that

$$\rho_1(M_1) + \rho_1(M_3) - \rho_1(M_2) = \sum_{\chi} \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}}(\mathrm{Tor}^i(M_4, \mathbb{C})_{\chi}) R_{\chi}.$$

Because of (6), M_4 is associated to the G -equivariant sheaf \mathcal{F}_4 which is supported on $\mathbb{C}^n - Z$. We will show that

$$(7) \quad \sum_{\chi} \sum_{i=0}^n (-1)^i \dim_{\mathbb{C}}(\mathrm{Tor}^i(M_4, \mathbb{C})_{\chi}) x_{\chi} = 0$$

in B by Noetherian induction on M_4 . Since the above element is additive on short exact sequences, it is enough to find a G -equivariant submodule M of M_4 which satisfies (7). Pick an associated prime p of M_4 and consider an element $m \in M_4$ such that $\mathrm{Ann}(m) = p$. In general, we can not expect m to be an eigenelement, nor can we expect p to be G -invariant. However, let us look at the vector space V which is the linear span of $gm, g \in G$. Observe that for each $g \in G$, the

annihilator gp of gm is also an associated prime of M_4 . Since G may not be connected, it could conceivably permute the associated primes. Consider the ideal $I \subset A$ given by

$$I = \bigcap_{g \in G} gp.$$

It is clear that I is G -invariant and that it annihilates V . Hence the submodule M of M_4 generated by V is an A/I -module.

Since M_4 is supported on $\mathbb{C}^n - Z$, each of its associated primes contains a prime ideal of A that corresponds to an irreducible component of $\mathbb{C}^n - Z$. More specifically, this is an ideal J which is generated by z_i with indices i , such that v_i do not lie in a cone C of Σ . Since $p \supseteq J$, we get $I \supseteq J$, and M is also an A/J -module. We now repeat the argument of Theorem 4.6 to resolve M by direct sums of rank one free G -equivariant A/J -modules generated by eigenelements. In view of the long exact sequences of Tor, it suffices to show that (7) holds true for such rank one A/J -module. But this is precisely a relation from the second set of relations on x_i , times a monomial in x_i to account for possible character of the action of G on the generator.

It remains to show that $\rho_1 \circ \rho$ is the identity. Since B is additively generated by monomials $\prod_i x_i^{r_i}$, it is enough to check this for a monomial. Observe that the global sections of $\otimes_i \mathcal{L}_i^{r_i}$ form a free module A_χ over A of rank one. Hence $\text{Tor}^{>0}(A_\chi, \mathbb{C})$ are zero and $\text{Tor}^0(A_\chi, \mathbb{C}) = \mathbb{C}$, with character the χ that corresponds to $\prod_i R_i^{r_i}$. This finishes the proof. \square

5. COMBINATORIAL CHERN CHARACTER

In this section we study the K -theory of the reduced toric DM stack \mathbb{P}_Σ in more detail. More specifically, we show that $K_0(\mathbb{P}_\Sigma, \mathbb{C}) := K_0(\mathbb{P}_\Sigma) \otimes_{\mathbb{Z}} \mathbb{C}$ is isomorphic as a vector space to $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$.

By Theorem 4.10, we have that $K_0(\mathbb{P}_\Sigma, \mathbb{C}) \cong B_{\mathbb{C}} := B \otimes_{\mathbb{Z}} \mathbb{C}$. Its maximum ideals correspond to points $(y_1, \dots, y_n) \in \mathbb{C}^n$ that satisfy $\prod_{i=1}^n y_i^{f(v_i)} = 1$ and $\prod_{i \in I} (1 - y_i) = 0$, for f and I in the definition of B .

Lemma 5.1. *The ring $B_{\mathbb{C}}$ is Artinian. Its maximum ideals are in one-to-one correspondence with the elements of $\text{Box}(\Sigma)$ as follows. A point $v = \sum_{v_i \in C} \alpha_i v_i$ corresponds to the n -tuple $(y_1, \dots, y_n) \in \mathbb{C}^n$ with $y_i = e^{2\pi i \alpha_i}$ for $v_i \in C$ and $y_i = 1$ otherwise.*

Proof. We need to solve for $(y_1, \dots, y_n) \in \mathbb{C}^n$ that satisfy $\prod_{i=1}^n y_i^{f(v_i)} = 1$ and $\prod_{i \in I} (1 - y_i) = 0$ as in the definition of B . Because of the second

set of equations, y_i are equal to 1 for all v_i outside some cone $C \in \Sigma$. We can assume that C is generated precisely by v_i for indices i with $y_i \neq 1$. To simplify notations, let us assume that these v_i are v_1, \dots, v_k for some $k \leq \text{rk}N$.

The first set of relations now reads

$$(8) \quad \prod_{i=1}^k y_i^{f(v_i)} = 1$$

for any linear function $f : N \rightarrow \mathbb{Z}$. Let N_1 be the intersection of N and the rational span of v_1, \dots, v_k . It is enough to look at $f : N_1 \rightarrow \mathbb{Z}$. By looking at some f_i which is zero for $j \in [1, \dots, k] - \{i\}$, we conclude that y_i is a root of 1. We introduce $\alpha_i \in [0, 1)$, such that $y_i = e^{2\pi i \alpha_i}$. Then the relations (8) amount to $\sum_i f(v_i) \alpha_i \in \mathbb{Z}$ for all $f \in N_1^\vee$. This is true if and only if $v = \sum_{i=1}^k \alpha_i v_i \in N_1$. Hence, the solutions to (8) are in one-to-one correspondence with elements of $\text{Box}(C)$. The condition $y_i \neq 1$ for $v_i \in C$ assures that v does not lie in $\text{Box}(C_1)$ for any proper face C_1 of C .

Since we are looking at all possible cones C here, the description of maximum ideals follows. Finally, the ring $B_{\mathbb{C}}$ is Artinian, since it is Noetherian of Krull dimension zero. \square

Since $B_{\mathbb{C}}$ is Artinian, it is a direct sum of Artinian local rings obtained by localizing at maximum ideals, which we denote by $(B_{\mathbb{C}})_v$. We have

$$B_{\mathbb{C}} = \bigoplus_{v \in \text{Box}(\Sigma)} (B_{\mathbb{C}})_v$$

The next lemma describes the structure of $(B_{\mathbb{C}})_v$.

Lemma 5.2. *Let C be the minimum cone of Σ that contains v . Then the ring $(B_{\mathbb{C}})_v$ is isomorphic as a \mathbb{C} -algebra to the SR-cohomology with complex coefficients of the (closed) subvariety of \mathbb{P}_{Σ} that corresponds to C .*

Proof. To simplify notations, we assume that $v = \sum_{i=1}^k \alpha_i v_i$ with $\alpha_i \in (0, 1)$. We will also index the rest of v_i in such a way that v_{k+1}, \dots, v_l are contained in some cone $C_1 \supset C$, and v_{l+1}, \dots, v_n are not.

We can localize first and then apply our relations. In fact, since $B_{\mathbb{C}}$ is Artinian, we may assume to be working in the quotient of the power series ring in $x_i - y_i$ by a sufficiently high power of the maximum ideal. This makes $x_i - 1$ nilpotent in $(B_{\mathbb{C}})_v$ for $i > k$ and it makes $x_i - e^{2\pi i \alpha_i}$ nilpotent for $1 \leq i \leq k$. We define $z_i = \log(x_i) := \sum_{m>0} \frac{1}{m} (x_i - 1)^m$ for $i > k$ and $z_i = \log(x_i e^{-2\pi i \alpha_i}) := \sum_{m>0} \frac{1}{m} (x_i e^{-2\pi i \alpha_i} - 1)^m$ for $i = 1, \dots, k$. The elements z_i are also nilpotent and we can assume to be

working in the quotient B_1 of $\mathbb{C}[[z_1, \dots, z_n]]$ by a sufficiently high power of the maximum ideal.

We further observe that $z_j = 0$ in $(B_{\mathbb{C}})_v$ for $j > l$. Indeed, we have

$$(x_j - 1) \prod_{i=1}^k (x_i - 1) = 0$$

which translates into

$$(e^{z_j} - 1) \prod_{i=1}^k (e^{2\pi i \alpha_i} e^{z_i} - 1) = 0.$$

Since $\alpha_i \in (0, 1)$, we see that $(e^{2\pi i \alpha_i} e^{z_i} - 1)$ is invertible in B_1 , so $e^{z_j} - 1 = 0$. This gives $z_j(1 + \frac{1}{2}z_j + \dots) = 0$, which leads to $z_j = 0$. As a result, we may just ignore z_j for $j > l$ in our calculations and work in the quotient B_2 of $\mathbb{C}[[z_1, \dots, z_l]]$ by a sufficiently high power of the maximum ideal.

The relations $\prod_{i \in I} (x_i - 1) = 0$ are now only nontrivial for $I \subseteq [1, \dots, l]$. Then every such set can be enlarged by adding all of $[1, \dots, i]$. Consider the quotient fan Σ_C in N/N_1 which is made from the images of the cones that contain C . Since $(x_i - 1)$ is invertible for $i \leq k$, the relations $\prod_{i \in I} (x_i - 1) = 0$ become relations of the form $\prod_{i \in I_1} (e^{z_i} - 1)$ for $I_1 \subseteq [k+1, \dots, l]$ such that $z_i, i \in I_1$ are not contained in any cone of Σ_C . As before, this is equivalent to $\prod_{i \in I_1} z_i = 0$ for these $I_1 \subseteq [k+1, \dots, l]$. This completely describes the second set of relations on z_i .

We now need to rewrite the first set of relations in terms of z_i . Let $f : N \rightarrow \mathbb{Z}$ be a linear function. Then we have

$$\prod_{i=1}^k (e^{2\pi i \alpha_i})^{f(v_i)} \prod_{i=1}^{k+l} e^{z_i f(v_i)} - 1 = 0.$$

Since $f(v) \in \mathbb{Z}$, this becomes simply

$$e^{\sum_{i=1}^{k+l} z_i f(v_i)} - 1 = 0,$$

which is further equivalent to

$$\sum_{i=1}^{k+l} z_i f(v_i) = 0.$$

We can now ignore the integrality condition on f and simply look at all linear functions $f : N \rightarrow \mathbb{Z}$. Consider the subspace V of linear functions on N that satisfy $f(v_i) = 0$ for all $i \leq k$, and let V_1 be a complement of it. We can use elements of V_1 to express z_1, \dots, z_k as linear combinations of z_{k+1}, \dots, z_l . This will allow us to write $(B_{\mathbb{C}})_v$

as the quotient of the ring $B_3 = \mathbb{C}[z_{k+1}, \dots, z_l]$ by a high power of a maximum ideal, and by the relations

- $\prod_{i \in I_1} z_i$, for I_1 such that $z_i, i \in I_1$ do not lie in a cone of Σ_C ,
- $\sum_{i=1}^{k+l} z_i f(v_i)$ for any linear function $f : N/N_1 \rightarrow \mathbb{Q}$.

This is immediately recognized as the SR-cohomology ring of the toric subvariety of \mathbb{P}_Σ that corresponds to the cone C . \square

Theorem 5.3. *There is a natural vector space isomorphism between $K_0(\mathbb{P}_\Sigma, \mathbb{C})$ and $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$.*

Proof. Follows directly from Theorem 4.10, Lemmas 5.1 and 5.2 and Propositions 3.8 and 3.9. \square

Remark 5.4. We call the map of Theorem 5.3 *combinatorial Chern character*. While it is an isomorphism of vector spaces, we stress that this map is not a ring homomorphism, since $B_{\mathbb{C}}$ is semilocal and $H_{SR}(\mathbb{P}_\Sigma, \mathbb{C})$ is local. One motivation behind our construction is that it generalizes the Chern character for projective toric varieties (which, however, is a ring isomorphism).

Remark 5.5. If \mathbb{P}_Σ is projective, one can use the isomorphism of SR- and orbifold cohomology to construct a map $K_0(\Sigma, \mathbb{C}) \rightarrow H_{orb}(\mathbb{P}_\Sigma, \mathbb{C})$. We suspect that this map is the Chern character map of [AR], which also motivated our terminology. But since our technique is quite different, we found it hard to make the connection explicit.

Remark 5.6. It is an interesting question as to under what conditions on the fan the K -theory is torsion-free. We do not know the answer to this, even in the case of smooth toric *varieties*.

Remark 5.7. The additive isomorphism between $K_0(\mathbb{P}_\Sigma, \mathbb{C})$ and the SR-cohomology $H_{SR}(N, \Sigma, \mathbb{C})$ indicates that the K -theory of a reduced toric DM stack possesses an alternative product structure that makes it into a local ring. One wonders what the geometric meaning of this structure may be. A related open question is whether there is a structure like this for the K -theory of any DM stack, not necessarily a toric one.

6. THE K -THEORY OF NONREDUCED TORIC DM STACKS

In this section we extend the calculation of K -theory of toric stacks to the nonreduced case. Since this is not the main focus of this paper, we only sketch the changes necessary to extend the results.

The principal feature of the nonreduced case is that N is now just a finitely generated abelian group, i.e. it is allowed to have torsion. A

fan in N is a pullback of a fan in $N/\text{torsion}$. A stacky fan is defined by a choice of a nonzero element v_i in each of the one-dimensional cones of Σ . We still have a map

$$\mathbb{Z}^n \rightarrow N$$

with finite coindex. This allows one to define a Gale dual

$$(\mathbb{Z}^n)^\vee \rightarrow N'$$

which is the analog of $(\mathbb{Z}^n)^\vee \rightarrow \text{Coker}\phi$ of Section 2, but we no longer have the surjectivity. We refer the reader to [BCS] for the details. Consequently, the group $G = \text{Hom}(N', \mathbb{C})$ is no longer a subgroup of $(\mathbb{C}^*)^n$, but rather maps to it with a finite kernel.

Otherwise, the definition of the open subset $Z \subseteq \mathbb{C}^n$ is the same as in the reduced case. This allows one to define line bundles \mathcal{L}_i on \mathbb{P}_Σ . Unfortunately, they will no longer generate the K -theory. The problem is that in the proof of Theorem 4.6 we have used that every character of G lifts to a character of $(\mathbb{C}^*)^n$. This is no longer the case. However, we can still look at the ring B which is the quotient of the character ring of G by the relations

$$\prod_{i \in I} (x_i - 1) = 0$$

for all $I \subseteq [1, \dots, n]$, such that $v_i, i \in I$ are not contained in any cone of Σ . Here x_i correspond to \mathcal{L}_i as in nonreduced case. With this modification, the proofs of both Theorems 4.6 and 4.10 are extended to the nonreduced case without any major changes to show that $K_0(\mathbb{P}_\Sigma) \cong B$.

The description of the maximum ideals of $B_{\mathbb{C}}$ from Lemma 5.1 still holds in the nonreduced case, but the proof is a bit more complicated, especially is one does not assume the condition (4). Specifically, these ideals correspond to elements of G whose action on \mathbb{C}^n has eigenvalues one outside of the set of indices i for v_i in some cone $C \in \Sigma$. So we have a morphism $\psi : N' \rightarrow \mathbb{C}^*$ such that the composition $(\mathbb{Z}^n)^\vee \rightarrow N' \rightarrow \mathbb{C}^*$ takes value one on the basis elements that correspond to v_i outside of C . The value of ψ on any element of N' is a root of one. Indeed, $\psi \in G$ fixes a point in Z , and it has been shown in [BCS] that all isotropy subgroups of G -action are finite. Consequently, we can think of ψ as a map $N' \rightarrow \mathbb{Q}/\mathbb{Z}$. Conversely, every such ψ gives rise to a group element with the above eigenvalue properties, and hence to a local subring of $B_{\mathbb{C}}$.

We split $(\mathbb{Z}^n)^\vee$ into $(\mathbb{Z}^{\dim C})^\vee \oplus (\mathbb{Z}^{n-\dim C})^\vee$ according to whether the corresponding basis elements lie in C . Consider the short exact

sequence of complexes

$$\begin{array}{ccccccc} 0 \rightarrow & (\mathbb{Z}^{n-\dim C})^\vee & \rightarrow & (\mathbb{Z}^{\dim C})^\vee \oplus (\mathbb{Z}^{n-\dim C})^\vee & \rightarrow & (\mathbb{Z}^{\dim C})^\vee & \rightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \rightarrow & N' & \rightarrow & N' & \rightarrow & 0 & \rightarrow 0 \end{array}$$

where we assume that the vertical lines are extended by zeroes in both directions. Gale duality with torsion (see [BCS]) implies that N can be canonically identified with the H^1 of the derived $\text{Hom}(*, \mathbb{Z})$ of the middle complex. The above exact sequence shows that $N/\text{Span}(v_i \in C)$ is H^1 of the derived $\text{Hom}((\mathbb{Z}^{n-\dim C})^\vee \rightarrow N', \mathbb{Z})$.

We observe that $\psi : N' \rightarrow \mathbb{Q}/\mathbb{Z}$ constructed earlier can be identified with elements of H^0 of derived $\text{Hom}((\mathbb{Z}^{n-\dim C})^\vee \rightarrow N', \mathbb{Q}/\mathbb{Z})$. The short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ yields an exact sequence

$$\begin{aligned} H^0(\text{Hom}((\mathbb{Z}^{n-\dim C})^\vee \rightarrow N', \mathbb{Z})) &\rightarrow H^1(\text{Hom}((\mathbb{Z}^{n-\dim C})^\vee \rightarrow N', \mathbb{Z})) \rightarrow \\ &\rightarrow H^1(\text{Hom}((\mathbb{Z}^{n-\dim C})^\vee \rightarrow N', \mathbb{Q})). \end{aligned}$$

Since the last term of the above sequence is torsion free, the group $H^0(\text{Hom}((\mathbb{Z}^{n-\dim C})^\vee \rightarrow N', \mathbb{Z}))$ is precisely the torsion subgroup of $N/\text{Span}(v_i \in C)$. Finally, elements of this torsion subgroup are in one-to-one correspondence with elements of $\text{Box}(C)$. We leave to the reader to check that these identifications are compatible with embeddings $\text{Box}(C_1) \subseteq \text{Box}(C_2)$ for $C_1 \subseteq C_2$ and that they coincide with the construction of Section 5 in the reduced case.

The SR-cohomology of \mathbb{P}_Σ is again given as the quotient of $\mathbb{C}[N, \Sigma]$ by the linear relations $\sum_i f(v_i)[v_i]$. Note that it is no longer a local ring, because of the graded zero part is the group ring of the torsion subgroup of N . SR-cohomology again splits into the untwisted and twisted sectors according to elements of $\text{Box}(\Sigma)$, and Propositions 3.8 and 3.9 still hold.

The combinatorial Chern character map of Theorem 5.3 still exists in the nonreduced case. It sends the local subring of the Artinian ring B_C which corresponds to the point $v \in \text{Box}(\Sigma)$ to the SR-cohomology of the corresponding twisted sector. More specifically, let v be this point and let C be the minimum cone of Σ that contains it. The map from G to G' is a finite unramified cover, so the local ring summand for B_C is isomorphic to the local ring of a semi-local ring B'_C which is obtained by cutting the character ring of G' by the same set of relations. Then the calculation of Lemma 5.2 shows that $(B_C)_v \cong (B'_C)_v$ is isomorphic to the SR-cohomology of the twisted sector of the reduction of \mathbb{P}_Σ that corresponds to the reduction of v modulo torsion. However, this reduction does not change the coarse moduli space of the twisted sector,

which finishes the construction of the isomorphism. The details are left to the reader.

7. BIRATIONAL MORPHISMS OF REDUCED TORIC DM STACKS

The statements of this section are implicit in [K1], but we adjust the exposition for our notations.

Let N be a lattice and let $\Sigma = (\Sigma, \{v_i\})$ be a stacky fan in N . We will assume that every cone of Σ is contained in a cone of Σ of dimension $\text{rk}N$. As before, let n be the number of v_i and let $Z \subset \mathbb{C}^n$ consist of the points $\mathbf{z} = (z_1, \dots, z_n)$ such that the set of v_i for the zero coordinates of \mathbf{z} is contained in a cone of Σ . Let G be the subgroup of $(\mathbb{C}^*)^n$ described by (3). The toric DM stack \mathbb{P}_Σ is defined as the stack quotient $[Z/G]$ where Z and G are endowed with the natural reduced scheme structures and the action of (λ_i) on (z_i) is given by $(\lambda_i z_i)$.

Let $\Sigma' = (\Sigma', \{v'_j\})$ be another stacky fan in the same lattice N . Let us further assume that every cone of Σ' is contained in a cone of Σ . In particular, for every j there is a unique way of writing v'_j in the form

$$v'_j = \sum_i \alpha_{i,j} v_i, \quad \alpha_{i,j} \in \mathbb{Q}$$

under the assumption that $\alpha_{i,j}$ are zero unless v_i lies in the minimum cone of Σ that contains v'_j . Moreover, let us assume that all $\alpha_{i,j}$ above are *integer*.

Definition 7.1. Under the above assumptions, there is a morphism

$$\mathbb{P}_{\Sigma'} \rightarrow \mathbb{P}_\Sigma$$

defined as follows. Define a homomorphism $(\mathbb{C}^*)^{n'} \rightarrow (\mathbb{C}^*)^n$ which sends (λ'_j) to $(\prod_j (\lambda'_j)^{\alpha_{i,j}})$. It is easy to see that this homomorphism sends points of G' to points of G . We also consider the map from $\mathbb{C}^{n'}$ to \mathbb{C}^n which sends (z'_j) to $(\prod_j (z'_j)^{\alpha_{i,j}})$. This map sends points in Z' to points in Z . Indeed, if the C' is a cone of Σ' that contains all v'_j for which $z'_j = 0$, then $\prod_j (z'_j)^{\alpha_{i,j}}$ is nonzero unless v_i lies in the minimum cone of Σ that contains C' . The map $Z' \rightarrow Z$ is compatible with the map of groups $G' \rightarrow G$ and their actions. This gives a map of quotient stacks, since one has the map between the corresponding groupoids.

Remark 7.2. We call the toric morphisms of Definition 7.1 *birational*, because they induce isomorphisms on the big strata $(\mathbb{C}^*)^{\text{rk}N}$ in both stacks.

8. K -THEORY PULLBACKS FOR BIRATIONAL MORPHISMS

Let $\mu : \mathbb{P}_{\Sigma', \{v'_j\}} \rightarrow \mathbb{P}_{\Sigma, \{v_i\}}$ be a toric birational morphism of toric DM stacks. Let R_i be defined as the elements of $K_0(\mathbb{P}_{\Sigma, \{v_i\}}, \mathbb{Q})$ that correspond to the invertible sheaves that correspond to v_i , and similarly for R'_j . There is a pullback map

$$\mu^* : K_0(\mathbb{P}_{\Sigma, \{v_i\}}, \mathbb{Q}) \rightarrow K_0(\mathbb{P}_{\Sigma', \{v'_j\}}, \mathbb{Q})$$

defined by pulling back the coherent sheaves. As we saw earlier in Theorem 4.10, the elements $\prod_i R_i^{r_i}$ span $K_0(\mathbb{P}_{\Sigma, \{v_i\}}, \mathbb{Q})$.

Proposition 8.1. *The pullback map μ^* is given by*

$$\mu^* \prod_i R_i^{r_i} = \prod_j (R'_j)^{\sum_i \alpha_{i,j} r_i}$$

Proof. The category of coherent sheaves on $\mathbb{P}_{\Sigma, \{v_i\}}$ is equivalent to that of G -linearized coherent sheaves on Z , see [V, Example 7.21]. Under this equivalence, $\prod_i R_i^{r_i}$ corresponds to the trivial sheaf on Z , linearized by $\mathcal{O}_Z \rightarrow g^* \mathcal{O}_Z = \mathcal{O}_Z$ which send 1 to $\prod_i (\lambda_i)^{r_i}$. These isomorphisms pull back to the isomorphisms of $\mathcal{O}_{Z'}$ that, for given $(\lambda'_1, \dots, \lambda'_{n'})$, send 1 to $\prod_{i,j} ((\lambda'_j)^{\alpha_{i,j}})^{r_i}$ on Z' . Since the power of λ'_j is $\sum_i \alpha_{i,j} r_i$, the result follows. \square

Remark 8.2. It is an amusing, though unnecessary, combinatorial exercise to see that the relations among $\prod_i R_i^{r_i}$ get mapped in the ideal of the relations among $\prod_j (R'_j)^{r'_j}$.

9. WEIGHTED BLOWUPS AND PUSHFORWARD FORMULAS

Let $(\Sigma, \{v_i\})$ be a stacky fan in lattice N . Let C be a cone in Σ of dimension $d > 1$ and let $\{v_1, \dots, v_d\}$ be the set of $v_i \in C$. Let h_1, \dots, h_d be some positive integers. Consider

$$v'_0 := \sum_{i=1}^d h_i v_i.$$

The cone C decomposes into a union of d cones of dimension d which are given by positive linear combinations of v'_0 and all but one of v_i , $1 \leq i \leq d$.

We define a stacky fan $(\Sigma', \{v'_j\})$ as follows. We add an extra ray which corresponds to v'_0 and keep the rest of v_i unchanged. Thus $n' = n + 1$ and $v'_i = v_i$ for $i > 0$. Every cone \hat{C} of Σ that does not contain C is still a cone of Σ' . Every cone $\hat{C} \supseteq C$ is subdivided into d cones C'_1, \dots, C'_d according to the above decomposition of C .

Our definitions assure that there is a birational morphism

$$\mu : \mathbb{P}_{\Sigma', \{v'_j\}} \rightarrow \mathbb{P}_{\Sigma, \{v_i\}}$$

which we call the *weighted blowdown* morphism. We will also denote by π the corresponding morphism $Z' \rightarrow Z$. Our goal is to calculate the K -theory pushforward of μ . It is best described in terms of generating functions.

Theorem 9.1. *Let $R = R'_0$ be the K -theory class of the invertible sheaf that corresponds to the extra ray v'_0 . There holds*

$$\mu_* \left(\frac{1}{1 - R^{-1}t} \right) = \frac{1}{1 - t} - \frac{t}{1 - t} \prod_i \frac{1 - R_i^{-1}}{1 - R_i^{-1}t^{h_i}}$$

which should be interpreted as an identity of formal power series in t with values in K -theory.

Proof. We will again identify the abelian categories of sheaves of $\mathbb{P}_{\Sigma', \{v'_j\}}$ and $\mathbb{P}_{\Sigma, \{v_i\}}$ with the categories of G' - and G -linearized sheaves on Z' and Z respectively. The K -theory pushforward of a sheaf F' is defined as the K -theory image of the alternating sum of the higher direct images of F' under μ . In order to describe the higher direct images of μ , we need to describe the direct image functor for μ in terms of $\pi : Z' \rightarrow Z$.

We first observe that the kernel H of $G' \rightarrow G$ is isomorphic to \mathbb{C}^* . Indeed, the map comes from $(\mathbb{C}^*)^{n+1} \rightarrow (\mathbb{C}^*)^n$ defined by

$$(\lambda'_0, \lambda'_1, \dots, \lambda'_n) \mapsto (\lambda'_1(\lambda'_0)^{h_1}, \dots, \lambda'_d(\lambda'_0)^{h_d}, \lambda'_{d+1}, \dots, \lambda'_n).$$

The kernel is given by $\lambda'_0 = \lambda$, $\lambda'_i = \lambda^{-h_i}$ for $1 \leq i \leq d$, $\lambda'_{>d} = 1$, which clearly lies in G' . Moreover, the map $G' \rightarrow G$ is split surjective. Indeed, given $(\lambda_i) \in G$, the collection $(1, \lambda_1, \dots, \lambda_n)$ will lie in G' , which produces the splitting. We will only need surjectivity for our arguments.

For every G' -linearized coherent sheaf \mathcal{F}' on Z' , its pushforward $\mathcal{F} = \pi_* \mathcal{F}'$ on Z inherits the G' -linearization under the action of G' on Z induced from $G' \rightarrow G$. Its subsheaf \mathcal{F}^H of H -invariants is therefore given the structure of a G -linearized sheaf on Z . The K -theory pushforward of a sheaf on $\mathbb{P}_{\Sigma', \{v'_j\}}$ that corresponds to \mathcal{F}' is given by \mathcal{F}^H . Indeed, consider the following commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{\pi} & Z \\ q' \downarrow & & \downarrow q \\ [Z'/G'] & \xrightarrow{\mu} & [Z/G] \end{array}$$

The sheaf on $[Z'/G']$ that corresponds to \mathcal{F}' is the subsheaf of G' -invariants of $q'_*\mathcal{F}'$. Consequently, its direct image in $[Z/G]$ is the subsheaf of G' -invariants of $q_*\pi_*\mathcal{F}' = q_*\mathcal{F}$. The subgroup H of G' acts trivially on Z , so the G' -invariants of $q_*\mathcal{F}$ are the G -invariants of $q_*\mathcal{F}^H$. This corresponds to the G -equivariant sheaf \mathcal{F}^H on Z .

We have thus described the direct image functor in terms of the composition of the direct image functor π_* from $\text{Coh}_{G'}(Z')$ to $\text{Coh}_{G'}(Z)$ and the functor of H -invariants from $\text{Coh}_{G'}(Z)$ to $\text{Coh}_G(Z)$. The latter is exact, therefore, the higher direct images of μ are given by $(R^k\pi_*(\mathcal{F}'))^H$.

In order to prove the theorem, we need to calculate the higher direct images of the G' -linearized invertible sheaf \mathcal{L} on Z' with the linearization that sends 1 to $(\lambda'_0)^{-l}$ for some integer $l \geq 0$. We claim that

$$(R^{>0}\pi_*\mathcal{L})^H = 0.$$

It is sufficient to check the statement at a fiber. Let $\mathbf{z} = (z_1, \dots, z_n)$ be a point in Z . The fiber of π consists of points $(z'_0, z'_1, \dots, z'_n) \in Z'$ such that

$$\mathbf{z} = (z'_1(z'_0)^{h_1}, \dots, z'_d(z'_0)^{h_d}, z'_{d+1}, \dots, z'_n).$$

Hence if one or more of z_1, \dots, z_d are nonzero, then the fiber is isomorphic to \mathbb{C}^* , which is an orbit of H . The structure sheaf of \mathbb{C}^* has no higher cohomology, so $R^{>0}\pi_*\mathcal{L}$ will have zero fibers at such \mathbf{z} , and $R^{>0}\pi_*\mathcal{L}$ are supported over $z_1 = \dots = z_d = 0$. For these \mathbf{z} , the fiber is described by $z'_0 = 0$, $z'_i = z_i$ for $i > d$, and $(z'_1, \dots, z'_d) \neq (0, \dots, 0)$. The fiber is isomorphic to $\mathbb{C}^d - \mathbf{0}$, with the group H acting by multiplications of the i -th coordinate by λ^{-h_i} . The cohomology of the structure sheaf \mathcal{O} on $\mathbb{C}^d - \mathbf{0}$ occurs at H^0 and H^{d-1} only, as can be easily calculated by the Čech complex for the covering by the open sets $(z'_i \neq 0)$. Moreover, there is a natural isomorphism

$$H^{d-1}(\mathbb{C}^d - \mathbf{0}, \mathcal{O}) \cong \left(\prod_i (z'_i)^{-1} \right) \mathbb{C}[(z'_1)^{-1}, \dots, (z'_d)^{-1}].$$

This means that the action of $\lambda \in H$ multiplies monomial generators of $H^{d-1}(\mathbb{C}^d - \mathbf{0}, \mathcal{O})$ by $\lambda^{-\sum_i h_i s_i}$ for some $s_i < 0$. Consequently, it multiplies generators of $H^{d-1}(\mathbb{C}^d - \mathbf{0}, \mathcal{L})$ by $\lambda^{l - \sum_i h_i s_i}$, which shows that the space of H -invariants is zero. Of course, the above is basically a calculation of cohomology of a line bundle on a weighted projective space.

As a result, to calculate the K -theory pushforward of R^{-l} for $l \geq 0$ it is enough to calculate the direct image of the corresponding invertible sheaf. In the notations of the preceding paragraph, \mathcal{L} is naturally embedded into \mathcal{O} as an ideal sheaf of $(z'_0)^l$. As a result, $(\pi_*\mathcal{L})^H$ is embedded into $(\pi_*\mathcal{O})^H$. We first show that the latter is isomorphic

to \mathcal{O}_Z , which is just the statement that $\mu_*\mathcal{O} = \mathcal{O}$ for a birational morphism.

The isomorphism will be glued from the isomorphisms on $(\mathbb{C}^*)^n$ -invariant affine subsets $U_\sigma \subseteq Z$ which correspond to the maximum-dimensional cones σ of Σ . The subset U_σ is defined by the condition that $z_i \neq 0$ for all $v_i \notin \sigma$ and is isomorphic to $(\mathbb{C}^*)^{n-\text{rk}N} \times \mathbb{C}^{\text{rk}N}$. If $\sigma \not\supseteq C$, then the preimage $\pi^{-1}U_\sigma$ in Z' is given by the conditions $z'_0 \neq 0$ and $z'_i \neq 0$ for $i > 0$ and $v_i \notin \sigma$. Then $\pi^{-1}U_\sigma$ is isomorphic to $(\mathbb{C}^*)^{n+1-\text{rk}N} \times \mathbb{C}^{\text{rk}N}$. The sections of $\pi_*\mathcal{O}_{Z'}$ on U_σ are spanned by the monomials $(z'_0)^{s_0} \prod_i (z'_i)^{s_i}$ with $s_i \geq 0$ for all $v_i \in \sigma$. The H -invariant sections in addition satisfy $s_0 = \sum_{i=1}^d h_i s_i$. The isomorphism from \mathcal{O}_{U_σ} to $(\pi_*\mathcal{O}_{Z'})^H$ is constructed by sending $\prod_{i=1}^n z_i^{s_i}$ to $(z'_0)^{\sum h_i s_i} \prod_{i=1}^n (z'_i)^{s_i}$. In the case when $\sigma \supseteq C$, the preimage $\pi^{-1}U_\sigma$ is given by the conditions $z'_i \neq 0$ for $v_i \notin \sigma$, and $(z'_1, \dots, z'_d) \neq \mathbf{0}$. Consequently, it is isomorphic to $\mathbb{C} \times (\mathbb{C}^d - \mathbf{0}) \times (\mathbb{C}^*)^{n-\text{rk}N} \times \mathbb{C}^{\text{rk}N-d}$. Because $d \geq 2$, the sections of \mathcal{O} on this space ignore the deletion of $\mathbf{0}$. The sections of $\pi_*\mathcal{O}_{Z'}$ are over U_σ are therefore spanned by monomials $(z'_0)^{s_0} \prod_i (z'_i)^{s_i}$ with $s_0 \geq 0$ and $s_i \geq 0$ for $v_i \in \sigma$. The H -invariance condition simply expresses s_0 in terms of other s_i but imposes no further restrictions on s_1, \dots, s_n . Consequently, we again have an isomorphism between \mathcal{O}_Z and $(\pi_*\mathcal{O}_{Z'})^H$. It is clear that these isomorphisms are compatible on the intersections and hence glue together to show $(\pi_*\mathcal{O}_{Z'})^H \cong \mathcal{O}_Z$.

Obviously, the sheaf \mathcal{O}_Z is a restriction of $\mathcal{O}_{\mathbb{C}^n}$ via the open embedding $Z \subseteq \mathbb{C}^n$. We claim that $(\pi_*\mathcal{L})^H$ is also a restriction of an ideal sheaf \mathcal{I} from \mathbb{C}^n . Namely, consider the ideal sheaf \mathcal{I} on \mathbb{C}^n which corresponds to the submodule over $\mathbb{C}[z_1, \dots, z_n]$ which is the span of monomials $\prod_{i=1}^n z_i^{s_i}$ with

$$\sum_{i=1}^d s_i h_i \geq l.$$

To show that it restricts to $(\pi_*\mathcal{L})^H$ on Z , it is again enough to calculate the sections over the open subsets U_σ . For $\sigma \not\supseteq C$, we have $\mathcal{O}_Z|_{U_\sigma} \cong \mathcal{I}|_{U_\sigma}$. For $\sigma \supseteq C$, the condition on $(z'_0)^{s_0} \prod_i (z'_i)^{s_i}$ to lie in the ideal of $(z'_0)^l$ translates into $s_0 \geq l$. The invariants of that come from the monomials $\prod_i z_i^{s_i}$ with $\sum_{i=1}^d s_i h_i \geq l$, as claimed.

To calculate the pushforward of \mathcal{L} we now simply need to calculate the free graded resolution of the ideal I of the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$ which is the span of the monomials with the condition $\prod_i z_i^{s_i}$ with $\sum_{i=1}^d s_i h_i \geq l$. Clearly, the variables z_{d+1}, \dots, z_n can be ignored. If

$$0 \rightarrow F^d \rightarrow F^{d-1} \rightarrow \dots \rightarrow F^0 \rightarrow I \rightarrow 0$$

is such a free resolution, then the alternating sum of the \mathbb{Z}^d -graded dimensions of F^k is the \mathbb{Z}^d -graded dimension of I , and the same is true for their generating functions. The generating function of a copy A of $\mathbb{C}[z_1, \dots, z_n]$ with the grading shifted so that the multidegree of 1 is (r_1, \dots, r_d) is

$$\sum_{\mathbf{deg} \in \mathbb{Z}^d} \dim_{\mathbb{C}}(A_{\mathbf{deg}}) \mathbf{t}^{\mathbf{deg}} = \prod_{i=1}^d \frac{t_i^{r_i}}{1 - t_i}.$$

On the other hand, such a module A gives rise to the invertible sheaf on Z that gives $\prod_{i=1}^d R_i^{-r_i}$ in K -theory of $\mathbb{P}_{\Sigma, \{v_i\}}$. So the K -theory pushforward of R^{-l} is given by

$$\mu_* R^{-l} = \prod_{i=1}^d (1 - R_i^{-1}) \sum_{(s_1, \dots, s_d) \in \mathbb{Z}_{\geq 0}^d, h_1 s_1 + \dots + h_d s_d \geq l} R_1^{-s_1} \cdot \dots \cdot R_d^{-s_d}$$

where the sum should be interpreted as a formal power series in R_i^{-1} which actually gives a polynomial after being multiplied by $\prod (1 - R_i^{-1})$.

It remains to observe that if we look at the above in terms of the generating functions for all $l \geq 0$, then

$$\begin{aligned} \sum_{l \geq 0} t^l \mu_* R^{-l} &= \prod_{i=1}^d (1 - R_i^{-1}) \sum_{l \geq 0} t^l \sum_{(s_1, \dots, s_d) \in \mathbb{Z}_{\geq 0}^d, \sum_{i=1}^d h_i s_i \geq l} \prod_{i=1}^d R_i^{-s_i} \\ &= \prod_{i=1}^d (1 - R_i^{-1}) \sum_{(s_1, \dots, s_d) \in \mathbb{Z}_{\geq 0}^d} \prod_{i=1}^d R_i^{-s_i} \sum_{0 \leq l \leq \sum_{i=1}^d h_i s_i} t^l \\ &= \prod_{i=1}^d (1 - R_i^{-1}) \sum_{(s_1, \dots, s_d) \in \mathbb{Z}_{\geq 0}^d} \frac{1 - t^{1 + \sum_{i=1}^d h_i s_i}}{1 - t} \prod_{i=1}^d R_i^{-s_i} \\ &= \frac{1}{1 - t} - \frac{t}{1 - t} \prod_{i=1}^d (1 - R_i^{-1}) \sum_{(s_1, \dots, s_d) \in \mathbb{Z}_{\geq 0}^d} t^{\sum_{i=1}^d h_i s_i} \prod_{i=1}^d R_i^{-s_i} \\ &= \frac{1}{1 - t} - \frac{t}{1 - t} \prod_{i=1}^d \frac{1 - R_i^{-1}}{1 - R_i^{-1} t^{h_i}}. \end{aligned}$$

We remark that the above calculations should be interpreted as calculations in formal power series in t and R_i^{-1} with only finitely many terms at any given degree, so convergence is never an issue. This gives the desired formula for $\mu_* \frac{1}{1 - R^{-1} t}$. \square

Remark 9.2. The above theorem allows one to calculate the pushforward of any element of K -theory in view of the description of the pullback in Proposition 8.1 and the pull-push formula. Indeed, every element of $K_0(\mathbb{P}_{\Sigma', \{v'_i\}})$ can be written as a polynomial in R and R^{-1} with coefficients in $\mu^* K_0(\mathbb{P}_{\Sigma, \{v_i\}})$. Since R is quasi-unipotent, it can be expressed in terms of negative powers of R .

So far we have considered the weighted blowups of a stratum of codimension $d > 1$. While blowups in codimension one are isomorphisms in the smooth variety case, this is no longer true for weighted blowups of stacks. Namely, let $(\Sigma, \{v_i\})$ be a stacky fan in N . Let C be a dimension one cone of Σ and let v_1 be the chosen lattice point on C . For a positive integer k consider the stacky fan $(\Sigma', \{v'_i\})$ defined by $\Sigma' = \Sigma$, $v'_1 = kv_1$, $v'_i = v_i$, $i > 1$. There is a birational morphism

$$\mu : \mathbb{P}_{\Sigma', \{v'_i\}} \rightarrow \mathbb{P}_{\Sigma, \{v_i\}}$$

which we again call weighted blowdown morphism. To complete the discussion, we calculate the corresponding pushforward in K -theory. Since the pullback is given by $\mu^* R_1 = R_1^k$, $\mu^* R_i = R_i$ for $i > 1$, it is enough to calculate the pushforward of R_1^m for $m = 1, \dots, k$.

Proposition 9.3. *For $1 \leq m \leq k$ we have $\mu_*(R_1')^{-m} = R_1^{-1}$.*

Proof. The map $G' \rightarrow G$ is given by

$$(\lambda'_1, \lambda'_2, \dots, \lambda'_n) \rightarrow (\lambda'_k, \lambda'_2, \dots, \lambda'_n)$$

and the map $\pi : Z' \rightarrow Z$ is given by

$$\pi : (z_1, z_2, \dots, z_n) \mapsto (z_1^k, z_2, \dots, z_n).$$

We denote by H the kernel of $G' \rightarrow G$. Since π is finite, its higher direct images vanish, and we only need to calculate $(\pi_* \mathcal{L})^H$ for the sheaf \mathcal{L} which is isomorphic to $\mathcal{O}_{Z'}$ with linearization given by multiplication by $(\lambda'_1)^{-m}$. We can think of \mathcal{L} as an ideal sheaf of $\mathcal{O}_{Z'}$ generated by $(z'_1)^m$.

Similar to the proof of Theorem 4.10, $(\pi_* \mathcal{L})^H$ is an ideal sheaf in \mathcal{O}_Z . It is induced from a sheaf on $\mathbb{C}^n \supset Z$. A monomial $\prod_i z_i^{s_i}$ lies in the corresponding ideal of $\mathbb{C}[z_1, \dots, z_n]$ if and only if $s_1 k \geq m$. Since $1 \leq m \leq k$, this is equivalent to $s_1 \geq 1$. This ideal sheaf is then identified with \mathcal{L}_1^{-1} . \square

We rewrite the result of Proposition 9.3 to resemble that of Theorem 9.1. We denote $R = R_1'$.

Corollary 9.4. *For the blowdown of codimension one,*

$$\mu_*\left(\frac{1}{1-R^{-1}t}\right) = \frac{1}{1-t} - \frac{t}{(1-t)} \frac{(1-R_1^{-1})}{(1-R_1^{-1}t^k)}.$$

Proof.

$$\begin{aligned} \mu_*\left(\frac{1}{1-R^{-1}t}\right) &= 1 + \sum_{l>0} \mu_* R^{-l} t^l = 1 + \sum_{l\geq 0} \sum_{m=1}^k \mu_* R^{-lk-m} t^{lk+m} \\ &= 1 + \sum_{l\geq 0} \sum_{m=1}^k \mu_*(R^{-m} \mu^* R_1^{-l}) t^{lk+m} = 1 + \sum_{l\geq 0} \sum_{m=1}^k R_1^{-l-1} t^{lk+m} \\ &= 1 + \frac{R_1^{-1}(t-t^{k+1})}{(1-R_1^{-1}t^k)(1-t)} = \frac{1}{1-t} - \frac{t}{(1-t)} \frac{(1-R_1^{-1})}{(1-R_1^{-1}t^k)}. \end{aligned}$$

□

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