EXPLICIT EQUATIONS OF A FAKE PROJECTIVE PLANE

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Abstract. Fake projective planes are smooth complex surfaces of general type with Betti numbers equal to those of the usual projective plane. They come in complex conjugate pairs and have been classified as quotients of the two-dimensional ball by explicitly written arithmetic subgroups. In this paper we find equations of a projective model of a conjugate pair of fake projective planes by studying the geometry of the quotient of such surface by an order seven automorphism.

1. Introduction

A compact complex surface with the same Betti numbers as the usual complex projective plane is called *a fake projective plane* if it is not isomorphic to the complex projective plane. A fake projective plane has ample canonical divisor, so it is a smooth (and geometrically connected proper) surface of general type with geometric genus $p_g = 0$ and self-intersection of canonical class $K^2 = 9$ (this definition extends to arbitrary characteristic). The existence of a fake projective plane was first proved by Mumford [M]. His method was based on the theory of 2-adic uniformization, and led Ishida and Kato [IK] to prove the existence of two more in the 2-adic approach. Recently Allcock and Kato [AK] used a lattice with torsion in the 2-adic method to construct another fake projective plane. The second author [K06] gave a construction of a fake projective plane as a Galois cover of a singular model of Ishida elliptic surface which, as described by Ishida [I], is covered (non-Galois) by Mumford fake projective plane.

Fake projective planes have Chern numbers $c_1^2 = 3c_2 = 9$ and are complex 2-ball quotients by Aubin [Au] and Yau [Y]. Such ball quotients are strongly rigid by Mostow's rigidity theorem [Mos], i.e., determined by fundamental group up to holomorphic or anti-holomorphic isomorphism. Fake projective planes come in complex conjugate pairs by Kharlamov-Kulikov [KK] and have been classified as quotients of the two-dimensional complex ball by explicitly written co-compact torsion-free arithmetic subgroups of $\mathrm{PU}(2,1)$ by Prasad-Yeung [PY] and Cartwright-Steger [CS], [CS2]. The arithmeticity of their fundamental groups was proved by Klingler [Kl]. There are exactly 100 fake projective planes total, corresponding to 50 distinct fundamental

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groups. Cartwright and Steger also computed the automorphism group of each fake projective plane X, which is given by $\mathrm{Aut}(X) \cong N(X)/\pi_1(X)$, where $N(X)$ is the normalizer of $\pi_1(X)$ in its maximal arithmetic subgroup of $\mathrm{PU}(2,1)$. In particular $\mathrm{Aut}(X) \cong \{1\}$, $\mathbb{Z}_3$, $\mathbb{Z}_3^2$ or G_{21} where $\mathbb{Z}_n$ is the cyclic group of order n and G_{21} is the unique non-abelian group of order 21. Among the 50 pairs exactly 33 admit non-trivial automorphisms: 3 pairs have Aut $\cong G_{21}$, 3 pairs have Aut $\cong \mathbb{Z}_3^2$ and 27 pairs have Aut $\cong \mathbb{Z}_3$. It turns out, for example, that Mumford fake plane and Keum fake plane have fundamental groups in the same maximal arithmetic subgroup of $\mathrm{PU}(2,1)$, but the former has Aut $\cong \{1\}$ and the latter Aut $\cong G_{21}$.

On the other hand, in [K08] all possible quotients of fake projective planes were classified, e.g., the $\mathbb{Z}_7$-quotient of a fake projective plane with Aut $\cong G_{21}$ is a singular model of an elliptic surface with two multiple fibres and one I_9-fibre and three I_1-fibres; the three pairs of fake projective planes with Aut $\cong G_{21}$ produce in this way three such elliptic surfaces, up to complex conjugate, with induced $\mathbb{Z}_3$-action: a $(2,3)$-elliptic surface whose $\mathbb{Z}_3$-quotient is a singular model of Ishida elliptic surface, another $(2,3)$-elliptic surface and a $(2,4)$-elliptic surface. See also [K12] and [K17] for further details.

In this paper we find equations of a projective model of a conjugate pair of fake projective planes by studying the geometry of the quotient of such surface by an order seven automorphism. The equations are given explicitly by 84 cubics in $\mathbb{P}^9$ with coefficients in the field $\mathbb{Q}[\sqrt{-7}]$. Their complex conjugate equations define the complex conjugate surface. This pair has the most geometric symmetries among the 50 pairs, in the sense that its automorphism group is G_{21} and its $\mathbb{Z}_7$-quotient is a singular model of a $(2,4)$-elliptic surface, which is not simply connected. The universal double cover of the $(2,4)$-elliptic surface has only one multiple (double) fibre, has the same Hodge numbers as K3 surfaces, but Kodaira dimension 1. This pair is different from those of Mumford and Keum fake planes, and was discussed in [K11].

It is an open problem to determine whether the bicanonical map of a given fake projective plane gives an embedding into $\mathbb{P}^9$. It has been confirmed affirmatively for several pairs of fake projective planes, including the one in this paper, by the vanishing result of [K13], [K17], [CK] and the theorem of Reider [R] (see also [GKMS], [DBDC], where the authors use the term 'Keum's fake projective planes' for all fake projective planes with Aut $\cong G_{21}$). The equations in this paper also provide an explicit proof for the embeddability for the pair.

The paper is organized as follows. We describe our main result in Section 2 by presenting the equations of a subscheme in $\mathbb{CP}^9$ and indicate the computer calculations that allow one to verify that this subscheme is a fake projective plane. In Section 3 we start the explanation of the process that led us to the equations. Specifically, we discuss the geometry of the minimal resolution of the quotient of a certain fake projective plane by $\mathbb{Z}_7$ and its universal double cover X. In Section 4 we describe the breakthrough calculation that

allowed us to identify the image of X under a certain map to $\mathbb{CP}^3$ as a specific singular sextic surface. In Sections 5 and 6 we describe additional features of the surface X and explain how we found the field of rational functions of the fake projective plane. In Section 7 we finally explain how we obtained the 84 cubic equations of Section 2. We make a minor comment in Section 8.

Computer files for our computation were uploaded as "Ancillary files" in the arXiv site [BK], and a README is given in Section 9.

2. Equations

In this section we write down 84 explicit degree three equations in ten variables. We argue that they cut out a fake projective plane Z with an automorphism group of order 21 with $H_1(Z,\mathbb{Z}) = \mathbb{Z}_2^4$. Here $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$.

The 84 equations with complex conjugate coefficients cut out another fake projective plane that is complex conjugate to the former. We identify this pair as the pair of fake projective planes which is $(a = 7, p = 2, \emptyset, D_3 2_7)$ in Cartwright-Steger classification [CS2, registerofgps.txt], or as one of the three pairs in the class $(k = \mathbb{Q}, \ell = \mathbb{Q}(\sqrt{-7}), p = 2, \mathcal{T}_1 = \emptyset)$ [CS], [PY]. This pair does not belong to the class $(a = 7, p = 2, \{7\})$ which contains Mumford fake plane $(a = 7, p = 2, \{7\}, 7_{21})$ and Keum fake plane $(a = 7, p = 2, \{7\}, D_3 2_7)$.

Let $\mathbb{CP}^9$ be a projective space with homogeneous coordinates denoted by $(U_0, U_1, \ldots, U_9)$. Consider the non-abelian group G_{21} of order 21 which is a semi-direct product of $\mathbb{Z}_7$ and $\mathbb{Z}_3$. We define its action on $\mathbb{CP}^9$ by its action on the homogeneous coordinates by

$$\begin{aligned} &g_7(U_0 : U_1 : U_2 : U_3 : U_4 : U_5 : U_6 : U_7 : U_8 : U_9) := \\ &\quad (U_0 : \xi^6 U_1 : \xi^5 U_2 : \xi^3 U_3 : \xi U_4 : \xi^2 U_5 : \xi^4 U_6 : \xi U_7 : \xi^2 U_8 : \xi^4 U_9) \\ &g_3(U_0 : U_1 : U_2 : U_3 : U_4 : U_5 : U_6 : U_7 : U_8 : U_9) := \\ &\quad (U_0 : U_2 : U_3 : U_1 : U_5 : U_6 : U_4 : U_8 : U_9 : U_7) \end{aligned} \tag{2.1}$$

where $\xi = \exp(\frac{2\pi \mathrm{i}}{7})$ is the primitive seventh root of 1.

Theorem 2.1. Eighty four cubic equations of Tables 1 and 2 give equations of a fake projective plane Z in $\mathbb{CP}^9$ embedded by its bicanonical linear system.

Proof. Let Z be the subscheme of $\mathbb{CP}^9$ cut out by these eighty four equations. We use Magma to calculate the Hilbert series of Z to give

$$\dim H^0(Z, \mathcal{O}(k)) = 18k^2 - 9k + 1$$

for all $k \geq 0$.

We then use reduction modulo 263 with $\mathrm{i}\sqrt{7} = 16 \bmod 263$ (which is chosen just because it is a decent size prime with a clear root of -7). We

TABLE 1. Equations of the fake projective plane 1-24

$$
\begin{aligned}
eq_1 = \;& U_1U_2U_3 + (1-\mathrm{i}\sqrt{7})(U_3^2U_4 + U_1^2U_5 + U_2^2U_6) + (10-2\mathrm{i}\sqrt{7})U_4U_5U_6 \\
eq_2 = \;& (-3+\mathrm{i}\sqrt{7})U_0^3 + (7+\mathrm{i}\sqrt{7})(-2U_1U_2U_3 + U_7U_8U_9 - 8U_4U_5U_6) \\
& +8U_0(U_1U_4 + U_2U_5 + U_3U_6) + (6+2\mathrm{i}\sqrt{7})U_0(U_1U_7 + U_2U_8 + U_3U_9) \\
eq_3 = \;& (11-\mathrm{i}\sqrt{7})U_0^3 + 128U_4U_5U_6 - (18+10\mathrm{i}\sqrt{7})U_7U_8U_9 + 64(U_2U_4^2 + U_3U_5^2 + U_1U_6^2) \\
& +(-14-6\mathrm{i}\sqrt{7})U_0(U_1U_7 + U_2U_8 + U_3U_9) + 8(1+\mathrm{i}\sqrt{7})(U_1^2U_8 + U_2^2U_9 + U_3^2U_7 - 2U_1U_2U_3) \\
eq_4 = \;& -(1+\mathrm{i}\sqrt{7})U_0U_3(4U_6 + U_9) + 8(U_1U_2U_3 + U_1U_6U_9 + U_5U_7U_9) + 16(U_5U_6U_7 - U_1^2U_5 - U_3U_5^2) \\
eq_5 = \;& g_3(eq_4) \\
eq_6 = \;& g_3^2(eq_4) \\
eq_7 = \;& (12+4\mathrm{i}\sqrt{7})U_1U_2U_3 + (4+4\mathrm{i}\sqrt{7})(U_3U_5U_8 - U_0U_2U_5 + 4U_4U_5U_6) + (3-\mathrm{i}\sqrt{7})U_0U_1U_7 \\
& +8(U_2U_4U_7 + U_6U_7U_8 - U_1^2U_8 - 2U_4U_6U_8) + (2+2\mathrm{i}\sqrt{7})(U_3U_8^2 - U_0U_2U_8) \\
eq_8 = \;& g_3(eq_7) \\
eq_9 = \;& g_3^2(eq_7) \\
eq_{10} = \;& (2+6\mathrm{i}\sqrt{7})U_1U_2U_3 + 4(-5+\mathrm{i}\sqrt{7})U_5(U_1^2 + 2U_4U_6) - 8U_0(U_2U_5 + U_3U_6) + 8(-1+\mathrm{i}\sqrt{7})U_3U_5^2 \\
& +2(3-\mathrm{i}\sqrt{7})U_0U_1U_7 - 8U_1^2U_8 + (-1-\mathrm{i}\sqrt{7})U_8(U_0U_2 + 4U_4U_9) + 8(1+\mathrm{i}\sqrt{7})U_3U_5U_8 - 32U_4U_6U_8 \\
& +2(1-\mathrm{i}\sqrt{7})(2U_6U_7U_8 + 4U_5U_7U_9 + 4U_5U_6U_7 + U_7U_8U_9) + 2(3+\mathrm{i}\sqrt{7})U_3U_8^2 - 16U_4U_5U_9 + 4U_1U_9^2 \\
eq_{11} = \;& g_3(eq_{10}) \\
eq_{12} = \;& g_3^2(eq_{10}) \\
eq_{13} = \;& -8\mathrm{i}\sqrt{7}U_1^2U_3 + (-7+5\mathrm{i}\sqrt{7})U_0U_2U_3 + 4(-7+\mathrm{i}\sqrt{7})U_0U_6^2 + 4U_0^2U_7 + (8-8\mathrm{i}\sqrt{7})U_1U_4U_7 \\
& +4(-5-\mathrm{i}\sqrt{7})U_2U_5U_7 + (8+8\mathrm{i}\sqrt{7})U_3U_6U_7 + (-1-5\mathrm{i}\sqrt{7})U_1U_7^2 - 8U_2U_7U_8 + (6+6\mathrm{i}\sqrt{7})U_3U_7U_9 \\
eq_{14} = \;& 8U_1^2U_3 + 2(3-\mathrm{i}\sqrt{7})U_0U_1U_5 + 16U_3U_4U_6 - 16U_5^2U_6 + 2(1+\mathrm{i}\sqrt{7})U_2U_5U_7 - 8U_3U_6U_7 \\
& +2(-1-\mathrm{i}\sqrt{7})U_3^2U_8 + 2(-1+\mathrm{i}\sqrt{7})U_0U_6U_9 + (-5-\mathrm{i}\sqrt{7})U_3U_7U_9 \\
eq_{15} = \;& 2(-3-\mathrm{i}\sqrt{7})U_1^2U_3 + 2(3-\mathrm{i}\sqrt{7})U_0U_2U_3 + 4(-1+\mathrm{i}\sqrt{7})U_0U_1U_5 + 4(-1-\mathrm{i}\sqrt{7})U_3^2U_5 \\
& +8U_1U_2U_6 + 4(1+\mathrm{i}\sqrt{7})U_0U_6^2 - 4U_0^2U_7 + (1+\mathrm{i}\sqrt{7})U_1U_7^2 + 2(-1+\mathrm{i}\sqrt{7})U_0U_1U_8 + 4U_3U_7U_9 \\
eq_{16} = \;& (-3+\mathrm{i}\sqrt{7})U_2^3 + (-3+\mathrm{i}\sqrt{7})U_1^2U_3 + 4U_0U_2U_3 + (-2-2\mathrm{i}\sqrt{7})U_0^2U_4 + 8U_1U_4^2 + 8U_0U_1U_5 \\
& +(-5-\mathrm{i}\sqrt{7})U_1U_2U_6 + (4+4\mathrm{i}\sqrt{7})U_3U_4U_6 + 2U_0U_1U_8 + (3-\mathrm{i}\sqrt{7})U_2U_7U_8 + (2+2\mathrm{i}\sqrt{7})U_3U_4U_9 \\
eq_{17} = \;& 4(-1-\mathrm{i}\sqrt{7})U_2^3 + (5+\mathrm{i}\sqrt{7})U_0U_2U_3 + 4(3-\mathrm{i}\sqrt{7})U_3^2U_5 + 16(1-\mathrm{i}\sqrt{7})U_2U_4U_5 \\
& +4(-1-\mathrm{i}\sqrt{7})U_2U_5U_7 - 8U_1U_2U_9 + 4(1+\mathrm{i}\sqrt{7})U_3U_4U_9 - 32U_5^2U_9 - 16U_5U_8U_9 \\
eq_{18} = \;& 8U_1^2U_3 + (-5-\mathrm{i}\sqrt{7})U_0U_2U_3 + 4(1+\mathrm{i}\sqrt{7})U_3^2U_5 + 4(1+\mathrm{i}\sqrt{7})U_1U_2U_6 \\
& +16(-1+\mathrm{i}\sqrt{7})U_5^2U_6 + 8U_2U_5U_7 - 16U_3U_6U_7 + 8(-1+\mathrm{i}\sqrt{7})U_5U_6U_8 - 8U_3U_7U_9 \\
eq_{19} = \;& (-5-\mathrm{i}\sqrt{7})U_0^2U_4 - 8U_2U_5U_7 + (-1-\mathrm{i}\sqrt{7})U_1U_7^2 + 4U_0U_1U_8 - 4U_2U_7U_8 + (-5+\mathrm{i}\sqrt{7})U_1U_2U_9 \\
& +2(1-\mathrm{i}\sqrt{7})U_3U_4U_9 + 2(1-\mathrm{i}\sqrt{7})U_0U_6U_9 + 4U_3U_7U_9 + 2U_8^2U_9 + 2U_0U_9^2 \\
eq_{20} = \;& 4(1+\mathrm{i}\sqrt{7})U_1^2U_3 + 2(1-\mathrm{i}\sqrt{7})U_0U_2U_3 - 8U_0^2U_4 + 4(-3-\mathrm{i}\sqrt{7})U_1U_4^2 - 8\mathrm{i}\sqrt{7}U_0U_1U_5 \\
& +8(1-\mathrm{i}\sqrt{7})U_2U_4U_5 + (5-\mathrm{i}\sqrt{7})U_0U_1U_8 + 2(-5+\mathrm{i}\sqrt{7})U_3^2U_8 + 16U_5U_8U_9 + 8U_8^2U_9 \\
eq_{21} = \;& (1-\mathrm{i}\sqrt{7})U_1^2U_3 - 4U_0U_1U_5 - 8U_3U_4U_6 - 8U_0U_6^2 + 4U_1U_4U_7 + (2-2\mathrm{i}\sqrt{7})U_2U_5U_7 \\
& +2U_1U_7^2 - 2U_0U_1U_8 + (1+\mathrm{i}\sqrt{7})U_3^2U_8 + (1-\mathrm{i}\sqrt{7})U_2U_7U_8 + (-1+\mathrm{i}\sqrt{7})U_3U_7U_9 \\
eq_{22} = \;& -8U_1^2U_3 + 16U_2U_4U_5 - 8U_1U_2U_6 + 4(1+\mathrm{i}\sqrt{7})U_0U_6^2 + (1+\mathrm{i}\sqrt{7})U_0U_1U_8 \\
& +8U_2U_4U_8 - 8U_5U_6U_8 + 4U_1U_2U_9 - 8U_3U_4U_9 + 2(1+\mathrm{i}\sqrt{7})U_0U_6U_9 \\
eq_{23} = \;& (-3+\mathrm{i}\sqrt{7})(U_2^3 + U_1^2U_3) + 4(-1-\mathrm{i}\sqrt{7})U_1U_4^2 + (-1+3\mathrm{i}\sqrt{7})U_1U_2U_6 + 2(-1-\mathrm{i}\sqrt{7})U_1U_4U_7 \\
& +(1+\mathrm{i}\sqrt{7})U_3^2U_8 + 8U_2U_4U_8 + 4(-1+\mathrm{i}\sqrt{7})U_5U_6U_8 + 4U_2U_7U_8 + 4U_1U_2U_9 \\
eq_{24} = \;& 2U_0U_2U_3 + (-1-\mathrm{i}\sqrt{7})U_0^2U_4 + 2(1-\mathrm{i}\sqrt{7})U_0U_1U_5 + 2(1-\mathrm{i}\sqrt{7})U_1U_2U_6 + 2U_0U_1U_8 \\
& -4U_3^2U_8 - 4U_2U_4U_8 + 2(1-\mathrm{i}\sqrt{7})U_5U_6U_8 + 4U_5U_8U_9 + 2U_8^2U_9
\end{aligned}
$$

TABLE 2. Equations of the fake projective plane 25-84

$eq_{25} =$	$(-1+3\mathrm{i}\sqrt{7})U_0^2U_1+(44-4\mathrm{i}\sqrt{7})U_2^2U_3+64U_3U_4U_5+(36-12\mathrm{i}\sqrt{7})U_1U_3U_6+(16+16\mathrm{i}\sqrt{7})U_4^2U_6$ $+(-4-4\mathrm{i}\sqrt{7})U_0U_2U_7-32U_3U_4U_8+(4+4\mathrm{i}\sqrt{7})U_0U_6U_8-16U_3U_7U_8+(8-8\mathrm{i}\sqrt{7})U_1U_3U_9+16U_4U_7U_9$
$eq_{26} =$	$(-1+3\mathrm{i}\sqrt{7})U_0^2U_1+(-4-4\mathrm{i}\sqrt{7})U_2^2U_3+(40-8\mathrm{i}\sqrt{7})U_1U_2U_5+(4-12\mathrm{i}\sqrt{7})U_1U_3U_6+96U_4^2U_6$ $+(-24-8\mathrm{i}\sqrt{7})U_2U_6^2+16U_1^2U_7+(-2+2\mathrm{i}\sqrt{7})U_0U_2U_7+64U_4U_6U_7+(20-4\mathrm{i}\sqrt{7})U_1U_2U_8-8U_0U_6U_8$ $+16U_4U_7U_9$
$eq_{27} =$	$(5+\mathrm{i}\sqrt{7})U_0^2U_1+(-4-4\mathrm{i}\sqrt{7})U_2^2U_3+(16-16\mathrm{i}\sqrt{7})U_3U_4U_5+(-20-4\mathrm{i}\sqrt{7})U_1U_3U_6+32U_4^2U_6$ $+32U_0U_5U_6+8U_0U_6U_8-16U_1U_3U_9+16U_0U_5U_9+8U_0U_8U_9$
$eq_{28} =$	$8U_2^2U_3+(-3+\mathrm{i}\sqrt{7})U_0U_3^2+(-4-4\mathrm{i}\sqrt{7})U_1U_2U_5+(4+4\mathrm{i}\sqrt{7})U_3U_4U_5+32U_5^3+(4+4\mathrm{i}\sqrt{7})U_3U_5U_7$ $+16U_5^2U_8+(3-\mathrm{i}\sqrt{7})U_1U_3U_9+8U_2U_6U_9$
$eq_{29} =$	$(-3+\mathrm{i}\sqrt{7})U_2^2U_3+(5+\mathrm{i}\sqrt{7})U_0U_2U_4+8U_1U_2U_5-8U_2U_6^2+2U_0U_2U_7+(-1-\mathrm{i}\sqrt{7})U_1U_2U_8+8U_5^2U_8$ $+(3-\mathrm{i}\sqrt{7})U_1U_3U_9+(4+4\mathrm{i}\sqrt{7})U_4^2U_9-8U_2U_6U_9+(2+2\mathrm{i}\sqrt{7})U_4U_7U_9-2U_0U_8U_9+(-3+\mathrm{i}\sqrt{7})U_2U_9^2$
$eq_{30} =$	$8U_2^2U_3+(4-4\mathrm{i}\sqrt{7})U_1^2U_4+(-12-4\mathrm{i}\sqrt{7})U_1U_2U_5+(-4-12\mathrm{i}\sqrt{7})U_4^2U_6+(12+4\mathrm{i}\sqrt{7})U_2U_6^2$ $+(2-2\mathrm{i}\sqrt{7})U_1^2U_7-8U_1U_2U_8-16U_3U_4U_8+(1+3\mathrm{i}\sqrt{7})U_0U_6U_8+(-3-\mathrm{i}\sqrt{7})U_3U_7U_8+4U_1U_3U_9$ $+(6+2\mathrm{i}\sqrt{7})U_2U_6U_9$
$eq_{31} =$	$(-4+4\mathrm{i}\sqrt{7})U_1^2U_4-4U_1U_2U_5+(-4+4\mathrm{i}\sqrt{7})U_3U_4U_5+16U_5^3+(-8+8\mathrm{i}\sqrt{7})U_4^2U_6+(2+2\mathrm{i}\sqrt{7})U_0U_5U_6$ $-4U_1^2U_7+(2+2\mathrm{i}\sqrt{7})U_6U_7^2+8U_3U_4U_8-4U_0U_6U_8-4U_5U_8^2+(1+\mathrm{i}\sqrt{7})U_7^2U_9$
$eq_{32} =$	$(-5-\mathrm{i}\sqrt{7})U_0^2U_1+(-6+2\mathrm{i}\sqrt{7})U_0U_3^2+(-24+8\mathrm{i}\sqrt{7})U_3U_4U_5+(20+4\mathrm{i}\sqrt{7})U_1U_3U_6-32U_4^2U_6$ $-32U_0U_5U_6+32U_2U_6^2+(2+2\mathrm{i}\sqrt{7})U_0U_2U_7+(4+4\mathrm{i}\sqrt{7})U_1U_2U_8-8U_0U_6U_8+(10+2\mathrm{i}\sqrt{7})U_1U_3U_9$ $+16U_2U_6U_9$
$eq_{33} =$	$(7-5\mathrm{i}\sqrt{7})U_0^2U_1+(-56-24\mathrm{i}\sqrt{7})U_1^2U_4+32\mathrm{i}\sqrt{7}U_1U_2U_5+(28+4\mathrm{i}\sqrt{7})U_1U_3U_6+(28+28\mathrm{i}\sqrt{7})U_0U_5U_6$ $+(-84-4\mathrm{i}\sqrt{7})U_1^2U_7+(7+7\mathrm{i}\sqrt{7})U_0U_2U_7-56U_3U_5U_7+56U_6U_7^2+24\mathrm{i}\sqrt{7}U_1U_2U_8+56U_0U_6U_8$ $+(14-18\mathrm{i}\sqrt{7})U_1U_3U_9+28U_7^2U_9$
$eq_{34} =$	$(-5-\mathrm{i}\sqrt{7})U_0^2U_1+48U_1U_2U_5+(-16-16\mathrm{i}\sqrt{7})U_3U_4U_5+32U_4^2U_6+(2+10\mathrm{i}\sqrt{7})U_1^2U_7$ $+(-48+16\mathrm{i}\sqrt{7})U_4U_6U_7+(28-4\mathrm{i}\sqrt{7})U_1U_2U_8+(-12-12\mathrm{i}\sqrt{7})U_3U_4U_8+(-16-8\mathrm{i}\sqrt{7})U_0U_6U_8$ $+(-22+2\mathrm{i}\sqrt{7})U_1U_3U_9+(-8-8\mathrm{i}\sqrt{7})U_2U_6U_9+(-8+8\mathrm{i}\sqrt{7})U_4U_7U_9$
$eq_{35} =$	$(10+2\mathrm{i}\sqrt{7})U_2^2U_3+(-11+\mathrm{i}\sqrt{7})U_0U_2U_4-16U_1U_2U_5+(20+4\mathrm{i}\sqrt{7})U_3U_4U_5-16U_2U_6^2$ $+(-1-\mathrm{i}\sqrt{7})U_0U_2U_7+(-2-2\mathrm{i}\sqrt{7})U_1U_2U_8-16U_5^2U_8+(-4-4\mathrm{i}\sqrt{7})U_4^2U_9+(3-\mathrm{i}\sqrt{7})U_2U_9^2$
$eq_{36} =$	$(2+2\mathrm{i}\sqrt{7})U_0U_3^2+(-6+2\mathrm{i}\sqrt{7})U_0U_2U_4+(4-4\mathrm{i}\sqrt{7})U_1U_3U_6+32U_4^2U_6+(-12-4\mathrm{i}\sqrt{7})U_2U_6^2+2U_0U_2U_7$ $+16U_4U_6U_7+(7-\mathrm{i}\sqrt{7})U_1U_2U_8-8U_5^2U_8+4U_1U_3U_9+(4-4\mathrm{i}\sqrt{7})U_0U_5U_9+4U_4U_7U_9-2\mathrm{i}\sqrt{7}U_0U_8U_9$
$eq_k =$	$g_3(eq_{k-24}),\ k=37,\dots,60$
$eq_k =$	$g_3^2(eq_{k-48}),\ k=61,\dots,84$

calculate (by Macaulay2) the projective resolution of $\mathcal{O}_Z$ as

$$0\to\mathcal{O}(-9)^{\oplus 28}\to\mathcal{O}(-8)^{\oplus 189}\to\mathcal{O}(-7)^{\oplus 540}\to\mathcal{O}(-6)^{\oplus 840}\to$$
$$\to\mathcal{O}(-5)^{\oplus 756}\to\mathcal{O}(-4)^{\oplus 378}\to\mathcal{O}(-3)^{\oplus 84}\to\mathcal{O}\to\mathcal{O}_Z\to 0.$$

By semicontinuity, the resolution is of the same shape over $\mathbb{C}$. Since all of the sheaves $\mathcal{O}(-k),\ k=3,\dots,9$ are acyclic, we see that for all $i\geq 0$

$$h^i(Z,\mathcal{O}_Z)=h^i(\mathbb{CP}^9,\mathcal{O}).$$

That is, $h^1(Z,\mathcal{O}_Z) = h^2(Z,\mathcal{O}_Z) = 0$ and $h^0(Z,\mathcal{O}_Z) = 1$, which implies that the scheme Z is connected. Since the Hilbert polynomial has degree 2, its irreducible components have dimension at most 2.

We also verify that Z is smooth. It is a somewhat delicate calculation. In theory, one can take the 7×7 minors of the 84×10 matrix of partial derivatives of the equations and verify that, together with the equations themselves, they generate the ideal which coincides with $\mathbb{C}[U_0,\dots,U_9]$ for large degrees. In practice, such direct calculation is impossible, since the number of minors is too large. Instead, we pick three 7×7 minors of the Jacobian matrix and show that they have no common zeros on Z by a Hilbert polynomial calculation. The minors are chosen so that they do not vanish at the fixed points of the automorphism g_7, namely at the three points

$$(U_0,\dots,U_7,U_8,U_9) \in \{(0,\dots,0,0,1),(0,\dots,0,1,0),(0,\dots,1,0,0)\}.$$

The subsets of equations and variables that define the minors are given in Table 3. This calculation can be performed in Magma software package modulo 263 with $\mathrm{i}\sqrt{7} = 16$. The Hilbert polynomial of the quotient drops from $18k^2 - 9k + 1$ to $504k - 3654$, then to 7056 and finally to 0 as one adds the three minors to the ideal. If the equations generate the ring modulo 263, then they also generate it with exact coefficients. This calculation means that all geometric points of Z have tangent space of dimension at most two, which together with $h^0(\mathcal{O}_Z) = 1$ implies that Z is a smooth surface.

Table 3. Three 7×7 minors used to verify smoothness

$\{8,19,29,43,55,61,79\};\{U_0,U_1,U_2,U_3,U_5,U_6,U_7\}$
$\{7,19,31,37,55,67,77\};\{U_0,U_1,U_2,U_3,U_4,U_5,U_9\}$
$\{9,13,31,43,53,67,79\};\{U_0,U_1,U_2,U_3,U_4,U_8,U_9\}$

Thus we have a smooth surface Z and a very ample divisor class

$$D := \mathcal{O}_Z(1)$$

on it. The Hilbert polynomial together with the Riemann-Roch implies that

$$D^2 = 36, \;\; DK_Z = 18, \;\; \chi(Z,\mathcal{O}_Z) = 1.$$

Note that this shows that Z is not isomorphic to $\mathbb{CP}^2$. We also know that $h^{0,1}(Z) = h^{0,2}(Z) = 0$, so it remains to prove that $h^{1,1}(Z) = 1$.

To figure out this last Hodge number we used Macaulay to calculate $\chi(Z,\mathcal{O}(2K_Z)) = 10$ (again working modulo 263). For this calculation, we used the resolution to compute the canonical bundle K_Z as in Hartshorne's book, as Ext from the canonical bundle of the ambient space to $\mathcal{O}_Z$, then tensored it with self to get $2K_Z$, then calculated the Hilbert polynomial of the corresponding graded module to get $\chi(Z,2K_Z)$. Now by Riemann-Roch

$\chi(Z, 2K_Z) = K_Z^2 + \chi(Z, \mathcal{O}_Z)$ and we know that $\chi(Z, \mathcal{O}_Z) = 1$, thus $K_Z^2 = 9$. Now Noether's formula finishes the proof that Z is a fake projective plane.

We see that $2K$ is numerically equivalent to D. We calculated

$$\mathrm{Hom}(\mathcal{O}(K), \mathcal{O}(D)) = 0$$

by working modulo 263 and semi-continuity. This implies

$$h^0(Z, \mathcal{O}(D-K)) = 0 = h^2(Z, \mathcal{O}(2K-D)).$$

This implies that $h^0(Z, \mathcal{O}(2K-D)) \geq 1$, so $\mathcal{O}(2K) \simeq \mathcal{O}(D)$. So the fake projective plane Z is embedded via a bicanonical embedding. □

Remark 2.2. Consider the closed subscheme C of Z cut out by $U_0 = 0$ and the following 18 quadrics, which fall into 6 orbits under the $\mathbb{Z}_3$-action

$$\langle U_0,\ U_1^2 - U_6U_7 + \frac{1}{8}(-5 - \mathrm{i}\sqrt{7})U_7U_9,\ U_4U_6 - \frac{1}{8}(1 + \mathrm{i}\sqrt{7})U_3U_8,$$
$$U_2U_4 + \frac{1}{8}(1 + \mathrm{i}\sqrt{7})U_8U_9,\ U_1U_4 + U_3U_6 + \frac{1}{8}(5 + \mathrm{i}\sqrt{7})U_3U_9,$$
$$U_1U_2 + U_5U_8,\ U_4^2 + \frac{1}{8}(1 + \mathrm{i}\sqrt{7})U_2U_9 + \frac{1}{8}(5 + \mathrm{i}\sqrt{7})U_4U_7,$$
$$\text{12 images of the six quadrics under } \mathbb{Z}_3\rangle.$$

By calculating its Hilbert polynomial, we see that it is one-dimensional, with the total degree of one-dimensional components equal to 18. This means that C is a (manifestly $\mathbb{Z}_7$-invariant) curve on Z. Moreover, by computing Hilbert polynomials of $\langle U_0\rangle + I^2$ and $\langle U_0\rangle$, we see that the square of this ideal I lies in $\langle U_0\rangle$. Therefore, we see that the bicanonical divisor $2K_Z$ is linearly equivalent to $2C$. By Lemma 2.3 this implies that $Z/\mathbb{Z}_7$ has minimal model which is a $(2,4)$-elliptic surface. It also identifies Z as the pair of fake projective planes which is $(a = 7, p = 2, \emptyset, D_3 2_7)$ in Cartwright-Steger classification [CS2], or as one of the three pairs in the class $(k = \mathbb{Q}, \ell = \mathbb{Q}(\sqrt{-7}), p = 2, \mathcal{T}_1 = \emptyset)$ [CS], [PY].

Lemma 2.3. Let W be a fake projective plane with $\mathrm{Aut}(W) = \mathbb{Z}_7 : \mathbb{Z}_3$. Then the following are equivalent.

(1) W contains an effective $\mathbb{Z}_7$-invariant curve C with $C^2 = 9$.
(2) The action of $\mathbb{Z}_7$ on W fixes a non-trivial element in $H_1(W, \mathbb{Z})$.
(3) $H_1(W, \mathbb{Z}) = \mathbb{Z}_2^4$.
(4) The minimal resolution of $W/\mathbb{Z}_7$ is a $(2,4)$-elliptic surface.

Proof. On a fake projective plane an effective curve C with $C^2 = 9$ is a member of the linear system $|K_W + t|$ for some non-zero $t \in \mathrm{Tor}\,\mathrm{Pic}(W) \cong H_1(W, \mathbb{Z})$. For a subgroup G of $\mathrm{Aut}(W)$ the linear system $|mK_W + t|$ is G-invariant if and only if so is t. For a cyclic subgroup G of $\mathrm{Aut}(W)$ a complete linear system is G-invariant if and only if a member of the system is G-invariant. This proves the equivalence of (1) and (2). These two are equivalent to (3) by [CK, Corollary 3.4], then to (4) by the classification

of [K08] on the possible geometric structures of quotients of fake projective planes. □

Furthermore, if $H_1(W,\mathbb{Z}) = \mathbb{Z}_2^4$, a $\mathbb{Z}_7$-invariant non-trivial 2-torsion is unique [CK, Corollary 3.4], hence is $\mathrm{Aut}(W)$-invariant.

Remark 2.4. (1) It is well known (cf. [F, Example 9.1.3(ii)]) that a local complete intersection subscheme of $\mathbb{P}^N$ is the scheme-theoretic intersection of $N+1$ hypersurfaces. Thus our surface Z can be defined scheme-theoretically by 10 equations, and the 84 equations seem too many. However, it is important for constructing the resolution to cut it out ideal-theoretically. Moreover, Macaulay 2 works smoothly with the saturated ideal generated by the 84 cubics.

(2) A referee kindly informed us that only 15 among the 84 equations were enough to define Z scheme-theoretically, for example,

$$\{\,3,\ 7,\ 11,\ 19,\ 23,\ 31,\ 35,\ 40,\ 43,\ 53,\ 55,\ 67,\ 71,\ 79,\ 83\,\}.$$

3. Explanation begins: the double cover of the resolution of the $\mathbb{Z}_7$-quotient of a fake projective plane.

The equations of the previous section appear quite mysterious, and we will spend the rest of the paper explaining their origin. Our general construction can be roughly summarized in the following commutative diagram of morphisms, with notations that will be used throughout the paper

$\mathbb{P}^2_{\mathrm{fake}}$: a fake projective plane with $\mathrm{Aut} = \mathbb{Z}_7 : \mathbb{Z}_3$ such that the minimal resolution Y of $\mathbb{P}^2_{\mathrm{fake}}/\mathbb{Z}_7$ is a $(2,4)$-elliptic surface;

X: the universal double cover of Y.

$$\begin{tikzcd}
\mathcal{B}^2 \arrow[dr] & & \widehat{\mathbb{P}^2_{\mathrm{fake}}} \arrow[dl] \arrow[dr] & & X \arrow[dl] \arrow[dr] \arrow[rr, "\pi"] & & \mathbb{P}^3 \\
& \mathbb{P}^2_{\mathrm{fake}} \arrow[dr] & & Y \arrow[dl] \arrow[dr] & & \mathbb{P}^1 \arrow[dl] & \\
& & \mathbb{P}^2_{\mathrm{fake}}/\mathbb{Z}_7 & & \mathbb{P}^1 & &
\end{tikzcd} \tag{3.1}$$

In this section we describe the known results of [K08], [K11], [K17], on the quotients of fake projective planes with automorphism group of order 21 by the subgroup of order 7. Specifically, we describe the aspects of the geometry of Y and X in (3.1) that will be later used to find the equation of $\pi(X) \subset \mathbb{P}^3$.

Let $\mathbb{P}^2_{\mathrm{fake}}$ be a fake projective plane with non-commutative automorphism group $G_{21} \cong \mathbb{Z}_7 : \mathbb{Z}_3$. Consider the quotient $\mathbb{P}^2_{\mathrm{fake}}/\mathbb{Z}_7$ of $\mathbb{P}^2_{\mathrm{fake}}$ by the (normal) Sylow 7-subgroup of G_{21}. It is a singular surface of Kodaira dimension one with three quotient singular points of type $\frac{1}{7}(1,3)$ and inherits an order three automorphism which permutes these singular points. The minimal resolution Y of $\mathbb{P}^2_{\mathrm{fake}}/\mathbb{Z}_7$ is an elliptic surface over $\mathbb{CP}^1$ with $h^{2,0}(Y) = h^{1,0}(Y) = 0$,

and two multiple fibers with multiplicities $(2,3)$ or $(2,4)$, as shown in [K08]. The Hodge numbers of Y are given by

$$h^{0,0}(Y) = h^{2,2}(Y) = 1,\ h^{1,1}(Y) = 10,\ h^{p,q}(Y) = 0 \text{ otherwise.}$$

Throughout the rest of the paper we will consider the fake projective planes which lead to elliptic surfaces Y with multiple fibers of multiplicities $(2,4)$. By the classification of [PY] and [CS] there is exactly one such conjugate pair of fake projective planes. (The other two conjugate pairs with $\mathrm{Aut} \cong G_{21}$ lead to $(2,3)$-elliptic surfaces.) Let us denote by $4F_Y$ the multiplicity four fiber and by $2F_{2,Y}$ the multiplicity two fiber. We summarize the results of [K08], [K11] and [K17].

The preimages of $\frac{1}{7}(1,3)$ singular points in Y are three pairwise disjoint chains of spheres

$$S - B - C,\ S' - B' - C',\ S'' - B'' - C''$$

with $S^2 = (S')^2 = (S'')^2 = -3$ and the squares of the rest equal to -2. The canonical class K_Y is numerically equivalent to F_Y, and the elliptic fibration $Y \to \mathbb{P}^1$ is given by the linear system

$$|4F_Y| = |2F_{2,Y}| = |4K_Y|,$$

i.e., a general fiber is linearly equivalent to $4F_Y$. The curves S, S' and S'' are 4-sections of the fibration, i.e.

$$F_Y S = F_Y S' = F_Y S'' = 1.$$

The curves B, C and their translates are part of an I_9-fiber of $Y \to \mathbb{P}^1$ and the order 3 automorphism group acts fiberwise. There are three additional I_1-fibers, some of which may be the multiple fibers.

The structure of the I_9-fiber will be very important in what follows. We denote its nine components by

$$A - B - C - A' - B' - C' - A'' - B'' - C'' - A.$$

The curve S intersects B transversely and does not intersect C, B', C', B'', C''. The I_9-fiber is not a multiple fiber, i.e., is equivalent to $4F_Y$ by [K17, Theorem 2.3], thus we see that S must intersect A, A' and A'' in three points total. These intersection numbers determine the intersection numbers of S' and S'' with A, A', A'' because of the order three automorphism.

It is easy to see that the classes of the curves

$$A, B, C, A', B', C', A'', B'', C'', S, S', S''$$

generate a sublattice of rank ≥ 10 inside the Picard lattice of Y, the Néron-Severi group of Y modulo torsion. (The first 9 curves already generate a rank 9 sublattice.) By Poincaré duality the Picard lattice of Y is unimodular of signature $(1,9)$, thus the sublattice must have rank 10 and discriminant a

square integer, which puts strong restrictions on the intersection numbers. It was observed in [K11, p 1676] that the only possibilities are

$$\begin{array}{l} \text{Case 1: } SA = 1,\ SA' = 0,\ SA'' = 2; \\ \text{Case 2: } SA = 0,\ SA' = 2,\ SA'' = 1. \end{array} \tag{3.2}$$

In the following, we will show that Case 1 cannot occur.

The fundamental group of a $(2,4)$-elliptic surface is of order 2 [D], thus the surface Y has an unramified double cover X which is part of the diagram (3.1). It comes from a double cover $\mathbb{P}^1 \to \mathbb{P}^1$ of the base of the fibration ramified over the images of F_Y and $F_{2,Y}$. The preimage of the canonical divisor class K_Y is the canonical class K_X of X and is numerically equivalent to the preimage F of F_Y. Since on a simply connected surface a numerical equivalence is a linear equivalence, K_X is linearly equivalent to F. We will denote by F_2 the preimage of $F_{2,Y}$. Then the elliptic fibration $X \to \mathbb{P}^1$ is given by the linear system

$$|2F| = |F_2| = |2K_X|,$$

and has only one multiple fiber $2F$ (with multiplicity 2). In particular X has Kodaira dimension 1. Since X is simply connected, $h^{1,0}(X) = 0$. Since $\chi(X, \mathcal{O}_X) = 2\chi(Y, \mathcal{O}_Y) = 2$ we get $h^0(X, K_X) = 1$. This implies that

$$h^{0,0}(X) = h^{2,2}(X) = h^{2,0}(X) = h^{0,2}(X) = 1,\ \ h^{1,1}(X) = 20,$$

$$h^{p,q}(X) = 0 \text{ otherwise},$$

i.e., X has the Hodge numbers of K3 surfaces. Its Jacobian fibration is an elliptic surface over $\mathbb{P}^1$ with a section, with no multiple fibre, and with singular fibers of the same type as those of X (this is true for Jacobian fibration of any genus one fibration, cf. [CD]), thus has trivial canonical class and the sum of Euler numbers of singular fibres 24, hence is a K3 surface.

The preimage under $X \to Y$ of the curve S is $S_1 + S_2$ where S_i are disjoint smooth rational curves with $S_i^2 = -2$. Each of the curves S_i is a 2-section of $X \to \mathbb{CP}^1$. Similarly, we define S_1', S_2', S_1'' and S_2''. Preimage of the I_9-fiber $A - B - \ldots - C'' - A$ is two disjoint I_9-fibers

$$A_1 - B_1 - \ldots - C_1'' - A_1, \quad A_2 - B_2 - \ldots - C_2'' - A_2.$$

We arrange the indexing so that we get six $(-3) - (-2) - (-2)$ chains of $\mathbb{CP}^1$ curves

$$S_i - B_i - C_i,\ S_i' - B_i' - C_i',\ S_i'' - B_i'' - C_i'',\ i \in \{1, 2\}.$$

As before, we would like to determine the possible intersection numbers of the 24 curves

$$S_1, \ldots, S_2'', A_1, \ldots, C_2''$$

with each other. These intersections are uniquely determined by the non-negative integers $S_1A_1, S_1A_1', S_1A_1'', S_2A_1, S_2A_1', S_2A_1''$ which are subject to $S_i(A_1 + A_2) = SA$, $S_i(A_1' + A_2') = SA'$, $S_i(A_1'' + A_2'') = SA''$ from (3.2). The

resulting intersection matrix has to have rank at most 20, because the rank of the Picard group does not exceed $h^{1,1}(X)$.

A simple computer calculation shows that only Case 2 of (3.2) is possible and, moreover, there holds

$$S_1A_1 = S_2A_1 = 0,\ S_1A_1' = S_2A_1' = 1,\ S_1A_1'' = 0,\ S_2A_1'' = 1, \tag{3.3}$$

i.e., S_1 intersects at one point exactly B_1 and A_1' of the first I_9-fiber, and exactly A_2' and A_2'' of the second. (S_2 intersects exactly B_2 and A_2' of the second I_9-fiber, and exactly A_1' and A_1'' of the first.) This gives a rank 19 intersection matrix. This rank is not the maximum possible $h^{1,1}(X) = 20$, thus leaves a possibility that F or F_2 is of type I_2, i.e., F_Y or $F_{2,Y}$ on Y is of type I_1.

The following is crucial in our approach.

Proposition 3.1. Let D be the divisor $3F + S_1 + S_2$ on X which is the pullback of the divisor $3F_Y + S$ from Y. Then $D^2 = 6$, $h^0(D) = 4$ and the linear system $|D|$ is base point free. It gives a birational map $\pi : X \to \mathbb{CP}^3$ whose image is a sextic surface. Moreover,

(1) F is an elliptic curve and maps $2:1$ onto a line;
(2) each I_9-fiber maps to a union of a conic and two lines;
(3) a general fiber maps birationally onto a plane quartic curve with nodes at the points $\pi(S_1)$ and $\pi(S_2)$;
(4) F_2, if irreducible, maps $2:1$ onto a conic.

Proof. We see immediately that

$$D^2 = (3F + S_1 + S_2)^2 = 6FS_1 + 6FS_2 + S_1^2 + S_2^2 = 12 - 3 - 3 = 6.$$

Therefore, $\chi(D) = \frac{1}{2}D(D - F) + \chi(\mathcal{O}_X) = \frac{1}{2}(6 - 2) + 2 = 4$.

Consider the short exact sequence

$$0 \to \mathcal{O}(3F) \to \mathcal{O}(D) \to \mathcal{O}(D)|_{S_1} \oplus \mathcal{O}(D)|_{S_2} \to 0. \tag{3.4}$$

We know that the bicanonical map of X is the elliptic fibration and has $\mathbb{P}^1$ as its image. Thus the canonical ring of X is a polynomial ring with generators of weights 1 and 2, corresponding to F and F_2, so $h^0(3F) = 2$. Together with the Euler characteristics calculation and $h^2(3F) = h^0(-2F) = 0$ this implies that $h^1(3F) = 0$.

Because of $(3F + S_i)S_i = 3 - 3 = 0$, we know that the restrictions of the sheaf $\mathcal{O}(D)$ to either S_i is isomorphic to the structure sheaf. Thus the long exact sequence in cohomology associated to (3.4) implies that $\dim H^0(X, \mathcal{O}(D)) = 2 + 1 + 1 = 4$. The same long exact sequence implies $h^1(D) = h^2(D) = 0$.

Let us now prove that $H^0(X, \mathcal{O}(D))$ is base point free. The long exact sequence associated to (3.4) implies that there are sections which restrict to non-zero constants on S_1 and S_2, and the base locus of $H^0(X, \mathcal{O}(D))$ is contained in that of $H^0(3F)$. We know that this space is generated by

the sections with divisors $3F$ and $F + F_2$. Therefore, the base locus of $H^0(X, \mathcal{O}(D))$ is contained in F. Consider the short exact sequence

$$0 \to \mathcal{O}(2F + S_1 + S_2) \to \mathcal{O}(D) \to \mathcal{O}(D)|_F \to 0.$$

Since $S_i(2F + S_1 + S_2) < 0$, either S_i is a base component of $|2F + S_1 + S_2|$, hence $h^0(2F + S_1 + S_2) = h^0(2F) = 2$. Since $h^2(2F + S_1 + S_2) = 0$, Riemann-Roch implies that $h^1(2F + S_1 + S_2) = 0$ and $h^0(\mathcal{O}(D)|_F) = 2$. If F is irreducible, then it is an elliptic curve and the restriction of $\mathcal{O}(D)$ to F is the full linear system of degree two, hence is base point free, which implies that so is $H^0(X, \mathcal{O}(D))$. Furthermore F maps $2 : 1$ onto a line, which passes through the points $\pi(S_1)$ and $\pi(S_2)$. If F is of type I_2, i.e., $F = R_1 + R_2$ for two (-2)-curves R_i, then the restriction of $\mathcal{O}(D)$ to either R_i is the full linear system of degree one, hence it is base point free and so is $H^0(X, \mathcal{O}(D))$, and R_i maps $1 : 1$ onto a line L_i. Since R_1 and R_2 intersect at two distinct points, we see that $L_1 = L_2$, but then S_1 must intersect both R_1 and R_2, contradicting $S_1F = 1$.

Looking at the intersection number of D with each component of the I_9-fibers, we easily see that each I_9-fiber maps to a union of a conic and two lines. Since the image of a fiber is contained in a hyperplane section of $\pi(X)$, the degree of $\pi(X)$ is at least 4, hence must be 6.

The restriction of D to a general smooth fiber H of $X \to \mathbb{P}^1$ gives the short exact sequence

$$0 \to \mathcal{O}(F + S_1 + S_2) \to \mathcal{O}(D) \to \mathcal{O}(D)|_H \to 0.$$

The corresponding long exact sequence shows that $H^0(X, \mathcal{O}(D))$ restricts to a 3-dimensional linear subspace of the 4-dimensional space of sections of a degree four line bundle on the elliptic curve H. The corresponding $\mathbb{P}^2$ contains the line which is the image of F. The images of the fibers H are degree four curves in $\mathbb{P}^2$ of genus one, unless they are double covers of conics. In the latter case, $\pi(X)$ would have degree < 6, a contradiction.

Assume that F_2 is irreducible. Then it is an elliptic curve and the corresponding long exact sequence shows that $H^0(X, D)$ restricts to a 3-dimensional linear subspace of the 4-dimensional space $H^0(F_2, D|_{F_2})$. Let a, a', possibly $a = a'$, be the intersection points of F_2 and S_1. Then $F_2 \cap S_2 = \{a + t, a' + t\}$ for a fixed 2-torsion point $t \in F_2$, because the deck transformation of X acts freely on F_2 and switches S_1 and S_2. Let $\nu : F_2 \to \mathbb{P}^1$ be the double cover given by the degree two linear system $|a + a'|$ on F_2. We claim that

$$H^0(X, D)|_{F_2} = \nu^* H^0(\mathbb{P}^1, \mathcal{O}(2))$$

as 3-dimensional subspaces of $H^0(F_2, D|_{F_2})$. To prove this, consider the subspace

$$W_1 := H^0(X, 3F + S_1) \times H^0(S_2) \subset H^0(X, D).$$

Since $h^0(3F) = 2$, $h^1(3F) = 0$, and $\mathcal{O}(3F + S_1)$ restricts to the structure sheaf of S_1, we see that $h^0(3F + S_1) = 3$, hence $\dim W_1 = 3$. It is easy to

compute $h^0(F+S_1)=1$, $h^1(F+S_1)=0$, which implies that $H^0(3F+S_1)$ restrict to the full linear system of $H^0(F_2,(3F+S_1)|_{F_2})$. The latter space equals the 2-dimensional space $H^0(F_2,a+a')$ of the degree two line bundle $\mathcal{O}_{F_2}(a+a')$. Thus W_1 restricts to the 2-dimensional space corresponding to the 1-dimensional linear system $|a+a'|+(a+t)+(a'+t)$. Since $(a+t)+(a'+t)\in|a+a'|$, this 1-dimensional linear system belongs to the linear system of $\nu^*H^0(\mathbb{P}^1,\mathcal{O}(2))$. Similarly, $W_2:=H^0(X,3F+S_2)\times H^0(S_1)\subset H^0(X,D)$ restricts to the 2-dimensional space corresponding to the linear system $a+a'+|(a+t)+(a'+t)|$. The two 2-dimensional spaces $W_i|_{F_2}$ in $\nu^*H^0(\mathbb{P}^1,\mathcal{O}(2))$ have 1-dimensional intersection, which corresponds to the unique divisor $a+a'+(a+t)+(a'+t)$. The claim and the last assertion is proved. □

We remark that F_2, if reducible, maps onto a union of two conics.

Remark 3.2. We eventually expected that a fake projective plane can be identified as such once we have its explicit equations, as we did in Section 2. As a consequence, we felt free to pursue the most likely scenarios rather than try to exhaustively exclude all degenerate cases, since the justification of our approach will be in its final result. This liberating philosophy is similar to the physicists' approach to mathematics: anything goes as long as the final answers concur with experiments. In particular, we assume that F_2 is irreducible.

4. Breakthrough: equation of the image of the double cover X

In this section we describe the major breakthrough that allowed us to eventually write down the equations of the fake projective plane. Specifically, we describe the method that allowed us to find the $\mathbb{Z}_2$-invariant sextic in $\mathbb{CP}^3$ which gives a (highly singular) birational model of the double cover X of the resolution of the $\mathbb{Z}_7$-quotient.

The action of the covering involution σ on X leads to an involution on $H^0(X,D)$ which has two-dimensional eigenspaces. We observe that there are two natural, up to scaling, elements y_0 and y_1 of $H^0(X,\mathcal{O}(D))$ which correspond to divisors

$$F+F_2+S_1+S_2,\ 3F+S_1+S_2$$

respectively. We will linearize the action of the covering involution σ so that $\sigma(y_0)=y_0$ and $\sigma(y_1)=-y_1$. We pick other basis elements of the eigenspaces and denote them by y_2 and y_3.

We know that the images of S_1 and S_2 are disjoint points on $(0:0:*:*)$ which are permuted by the involution. We can scale y_2 and y_3 to ensure that these are $(0:0:-1:1)$ and $(0:0:1:1)$ respectively. For generic a the divisor of y_0-ay_1 is $F_1+S_1+S_2+H_a$ where H_a is a fiber of $X\to\mathbb{CP}^1$. Note that S_1 and S_2 intersect H_a in two points each. These points need to map to the the same point in $\mathbb{CP}^3$ which leads to the statement in Proposition 3.1 that the image of H_a is a nodal plane quartic with two nodes.

We also know that F_2 maps $2:1$ onto a conic.

Putting it all together, the geometry of $\pi : X \to \mathbb{CP}^3$ implies the following.

- The involution acts by $y_i \mapsto (-1)^i y_i$. The sextic f is invariant with respect to this involution.
- The sections y_0 and y_1 are zero on S_1 and S_2. These are automatically zero on F.
- The section $y_1 = 0$ corresponds to the divisor $3F + S_1 + S_2$ and the section $y_0 = 0$ corresponds to $F + F_2 + S_1 + S_2$. The image of F is $(0:0:*:*)$. This is a $2:1$ cover, so $f = 0$ has singularities along $(0:0:*:*)$.
- For $a \neq 0$ the restriction of f to $x_0 = ax_1$ is
$$f(ay_1, y_1, y_2, y_3) = y_1^2 g_a(y_1, y_2, y_3)$$
where $g_a = 0$ is a degree four curve which has nodes at $(0:\pm 1:1)$.
- For $a \neq 0$ the quartic $g_a = 0$ is irreducible, except for $a = \pm 1$ that correspond to the images of the I_9 fibers. (We can fix $a = \pm 1$ for the location of I_9 fiber by scaling y_0 and y_1.)
- The restriction to $y_1 = 0$ is given by
$$f(y_0, 0, y_2, y_3) = y_0^6.$$
Indeed, we must have a multiple of F_1 (since S_1 and S_2 map to points). This means that this should be a multiple of y_0 and we can scale it to be y_0^6.
- The restriction of f to $y_0 = 0$ is given by
$$f(0, y_1, y_2, y_3) = y_1^2 h_0^2(y_1, y_2, y_3)$$
where $h_0 = 0$ is a σ-invariant conic that passes through $(0:\pm 1:1)$. The surface $f = 0$ has singularities along $y_0 = h_0 = 0$.

There are additional restrictions on $f = 0$ that come from the geometry of the I_9 fibers. Without loss of generality, let us assume that the fiber at $y_0 = y_1$ corresponds to the image of the cycle of curves
$$A_1 - B_1 - C_1 - A_1' - B_1' - C_1' - A_1'' - B_1'' - C_1'' - A_1$$
and the $y_0 = -y_1$ fiber corresponds to the cycle $A_2 - \ldots - C_2'' - A_2$.

The intersection numbers imply that
$$DA_1' = 2,\ DA_1'' = 1, DB_1 = 1$$
so the degree four genus one curve with two nodes degenerates into two lines $\pi(A_1'')$ and $\pi(B_1)$ and a conic $\pi(A_1')$. The other six rational curves of this I_9 fiber are contracted to singular points. The line $\pi(B_1)$ must pass through $\pi(S_1) = (0:0:-1:1)$ as does the conic $\pi(A_1')$. The line $\pi(A_1'')$ passes through the other node $\pi(S_2) = (0:0:1:1)$. These lines intersect at some point P which we can set to be $P = (1:1:0:0)$ by adding multiples of y_0 and y_1 to y_2 and y_3 respectively. Moreover we see that
$$P = \pi(B_1'') = \pi(C_1'') = \pi(A_1)$$

so the surface $\pi(X)$ has at least an A_3 type singularity at P. In particular, the partial derivatives and the derivative of the Hessian matrix vanish at P. In addition, we have a singular point $\pi(C_1)$ at the intersection of the line $\pi(B_1)$ and the conic $\pi(A_1')$ which is different from $\pi(S_1) = (0:0:-1:1)$. We also have a singular point

$$\pi(B_1') = \pi(C_1')$$

at the intersection of the conic $\pi(A_1'')$ and the line $\pi(A_1'')$ that is different from $\pi(S_2) = (0:0:1:1)$.

We immediately see from the intersection numbers that

$$DS_1' = DS_2' = DS_1'' = DS_2'' = 3.$$

We focus specifically on S_1''. Note that S_1'' intersects both B_1'' and A_1, which means that $\pi(S_1'')$ passes through $(1:1:0:0)$ twice. Thus it should be a planar degree three rational nodal curve. This turned out to be a key observation that allowed us to get enough equations on the coefficients of f to solve for it.

Proposition 4.1. The sextic equation $f(y_0, y_1, y_2, y_3) = 0$ where

$$\begin{gathered} f = 28y_0^6 - (42 - 2\mathrm{i}\sqrt{7})y_0^4y_1^2 - 4\mathrm{i}\sqrt{7}y_0^2y_1^4 + 56y_0^2y_1^2y_2^2 - (14 + 22\mathrm{i}\sqrt{7})y_0^4y_1y_3 \\ -(7 - 13\mathrm{i}\sqrt{7})y_0^2y_1^2y_3^2 - (77 + 17\mathrm{i}\sqrt{7})y_1^4y_3^2 + (21 - 31\mathrm{i}\sqrt{7})(y_0^3y_1y_2y_3 - y_0y_1^3y_2y_3) \\ -(28 - 20\mathrm{i}\sqrt{7})y_1^3y_3(y_1^2 + y_2^2 - y_3^2) + (14 + 2\mathrm{i}\sqrt{7})y_1^2(y_1^4 + 2y_1^2y_2^2 + (y_2^2 - y_3^2)^2) \\ +(42 + 2\mathrm{i}\sqrt{7})(y_0^2y_1^3y_3 + y_0y_1^2y_2(-y_0^2 + y_1^2 + y_2^2 - y_3^2)) \end{gathered}$$

cuts out a surface which has the same expected properties as the image of the double cover X under the map given by $|3F + S_1 + S_2|$.

Remark 4.2. It is clear that complex conjugation provides another surface with the same properties that comes from the complex conjugate fake projective plane.

We remark that the formula of Proposition 4.1 was obtained by writing down a generic invariant sextic that satisfied the properties and then using Mathematica software package to write down equations on the coefficients. The equations are too complicated to be solved symbolically, but numerical solutions give values that "look like" algebraic numbers. This allow us to identify a putative equation, which can then be checked to give the desired properties.

We now describe the images of the 24 curve $S_1, \ldots, S_2'', A_1, \ldots, C_2''$ on $\pi(X)$. The curve S_1'' was found in the process of getting Proposition 4.1. The curve S_2'' is obtained by simply applying the involution σ. It took a bit of effort to find S_1'. The idea is that there should be an order three automorphism that acts fiberwise on $X \to \mathbb{CP}^1$ and sends $S_1 \to S_1' \to S_1''$. This automorphism is a lift of the order 3 automorphism acting on the quotient $\mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7$. Each of the curves S_1, S_1' and S_1'' have two points in the

generic fiber, which give two orbits under addition of an element of order three. Thus, if we parameterize S_1'' as $S_1''(t)$ there should be a point $S_1'(t)$ in the fiber so that

$$S_1'(t) + S_1''(t) = (S_1)_1 + (S_1)_2$$

where $(S_1)_i$ are two preimages of the node $\pi(S_1)$. Since the preimage of the class of the line in $\mathbb{CP}^2$ that contains the fiber is $(S_1)_1+(S_1)_2+(S_2)_1+(S_2)_2$ we see that the fourth intersection point of the line through the node $\pi(S_2) = (0:0:1:1)$ and $S_1''(t)$ with the quartic image of the fiber should give parameterization of S_1'. We write the corresponding equations in Table 4.

Remark 4.3. The construction of S_1' has an additional advantage of providing us with a rational function on Y which has well-understood zeros and poles. Specifically, the section

$$\left(y_0^3-y_0^2y_1-y_0y_1^2+y_1^3+\frac{1}{2}(1+\mathrm{i}\sqrt{7})(y_0-y_1)y_1(y_2-y_3)+\frac{1}{4}(-1+\mathrm{i}\sqrt{7})y_1(y_2-y_3)^2\right)$$

defines a (nonnormal) cubic cone with vertex $(0:0:1:1)$ that contains S_1' and S_1''. Its symmetrization $f_{cones}(y_0,y_1,y_2,y_3)$ given by

$$\left(y_0^3-y_0^2y_1-y_0y_1^2+y_1^3+\frac{1}{2}(1+\mathrm{i}\sqrt{7})(y_0-y_1)y_1(y_2-y_3)+\frac{1}{4}(-1+\mathrm{i}\sqrt{7})y_1(y_2-y_3)^2\right)$$

$$\left(y_0^3+y_0^2y_1-y_0y_1^2-y_1^3-\frac{1}{2}(1+\mathrm{i}\sqrt{7})(y_0+y_1)y_1(y_2+y_3)-\frac{1}{4}(-1+\mathrm{i}\sqrt{7})y_1(y_2+y_3)^2\right)$$

gives a σ-invariant section of $H^0(X,6D)$ which contains $S_1'+S_2'+S_1''+S_2''$. In fact, we were able to show that its degree 36 intersection curve with $\pi(X)$ is fully accounted for by the curves from our list of 24 rational curves as well as F. The σ-invariant rational function

$$\frac{f_{cones}(y_0,y_1,y_2,y_3)}{(y_0^2-y_1^2)^3}$$

on X gives a rational function on Y whose divisor is

$$2A-3A'+A''-B-B'+2B''-2C+2C''-2S+S'+S''.$$

The curves $A_1,\dots,C_1''$ are either contracted to points or map isomorphically to lines or conics in the plane $y_0=y_1$, as indicated in Table 4.

An important part of our calculations will be based on finding a putative normalization of the ring

$$\mathbb{C}[y_0,y_1,y_2,y_3]/\langle f(y_0,y_1,y_2,y_3)\rangle.$$

Proposition 4.4. The rational functions

$$\hat{y}_4 = \frac{y_0^3}{y_1}$$

$$\hat{y}_5 = \frac{(y_1^2+y_2^2+\frac{1}{4}(-1+3\mathrm{i}\sqrt{7})y_1y_3-y_3^2)y_1}{y_0}$$

TABLE 4. Images of curves on X under the map $\pi : X \to \mathbb{CP}^3$ (The equations are either parametric or non-parametric; the curves $\pi(S_2), \ldots, \pi(C_2'')$ can be found by applying $\sigma(y_0 : y_1 : y_2 : y_3) = (y_0 : -y_1 : y_2 : -y_3)$ to $\pi(S_1), \ldots, \pi(C_1'')$.)

Curves	Equations
$\pi(F)$	$y_0 = y_1 = 0$
$\pi(F_2)$	$y_0 = 0,\ y_1^2 + y_2^2 + \frac{1}{4}(-1 + 3\mathrm{i}\sqrt{7})y_1y_3 - y_3^2 = 0$
$\pi(S_1)$	$(0 : 0 : -1 : 1)$
$\pi(S_1')$	$y_0 = \frac{1}{8}(11 - \mathrm{i}\sqrt{7})t + \frac{1}{8}(-3 + \mathrm{i}\sqrt{7})t^3$ $y_1 = t^3$ $y_2 = \frac{1}{8}(11 - \mathrm{i}\sqrt{7}) + \frac{1}{8}(-1 + 3\mathrm{i}\sqrt{7})t - \frac{1}{8}(5 + \mathrm{i}\sqrt{7})t^2 + \frac{1}{8}(3 - \mathrm{i}\sqrt{7})t^3,$ $y_3 = -\frac{1}{16}(9 + 5\mathrm{i}\sqrt{7}) + \frac{1}{16}(11 - \mathrm{i}\sqrt{7})t + \frac{1}{16}(21 + \mathrm{i}\sqrt{7})t^2 - \frac{1}{16}(7 - 5\mathrm{i}\sqrt{7})t^3$
$\pi(S_1'')$	$y_0 = \frac{1}{8}(11 - \mathrm{i}\sqrt{7})t + \frac{1}{8}(-3 + \mathrm{i}\sqrt{7})t^3$ $y_1 = t^3$ $y_2 = \frac{1}{16}(-9 - 5\mathrm{i}\sqrt{7} + (11 - \mathrm{i}\sqrt{7})t)(-1 + t^2)$ $y_3 = \frac{1}{8}(11 - \mathrm{i}\sqrt{7} + (-1 + 3\mathrm{i}\sqrt{7})t)(-1 + t^2)$
$\pi(A_1)$	$(1 : 1 : 0 : 0)$
$\pi(B_1)$	$y_0 = y_1,\ y_2 = -y_3$
$\pi(C_1)$	$(1 : 1 : -\frac{1}{4}(3 + \mathrm{i}\sqrt{7}), \frac{1}{4}(3 + \mathrm{i}\sqrt{7}))$
$\pi(A_1')$	$y_0 = y_1,$ $\frac{1}{2}(11 - \mathrm{i}\sqrt{7})y_1^2 + \frac{1}{4}(11 - \mathrm{i}\sqrt{7})y_1y_2 + y_2^2 + \frac{1}{2}(-1 + 3\mathrm{i}\sqrt{7})y_1y_3 - y_3^2 = 0$
$\pi(B_1')$	$(-1 : -1 : \frac{1}{2}(1 - \mathrm{i}\sqrt{7}) : \frac{1}{2}(1 - \mathrm{i}\sqrt{7}))$
$\pi(C_1')$	$(-1 : -1 : \frac{1}{2}(1 - \mathrm{i}\sqrt{7}) : \frac{1}{2}(1 - \mathrm{i}\sqrt{7}))$
$\pi(A_1'')$	$y_0 = y_1,\ y_2 = y_3$
$\pi(B_1'')$	$(1 : 1 : 0 : 0)$
$\pi(C_1'')$	$(1 : 1 : 0 : 0)$

lie in the normalization of $\mathbb{C}[y_0, y_1, y_2, y_3]/\langle f(y_0, y_1, y_2, y_3)\rangle$ in its field of fractions. These elements are odd with respect to the involution σ and are homogeneous with grading 2.

Proof. It is straightforward to see that $\hat{y}_4$ and $\hat{y}_5$ satisfy monic quadratic equations with coefficients in the ring. The parity and grading are obvious. $\square$

Remark 4.5. We believe that $y_0, \dots, y_3, \hat{y}_4, \hat{y}_5$ generate the normalization of the ring $\mathbb{C}[y_0, y_1, y_2, y_3]/\langle f(y_0, y_1, y_2, y_3)\rangle$ which is isomorphic to

$$\bigoplus_{k\geq 0} H^0(X, \mathcal{O}(kD)).$$

Moreover, we have calculated generators of the ideal of relations between $y_0, \dots, y_3, \hat{y}_4, \hat{y}_5$. Since we do not need this information for our purposes, we will not present it in the paper. However, we do use the fact that $\hat{y}_4$ and $\hat{y}_5$ give odd sections of $H^0(X, \mathcal{O}(2D))$.

5. Order three automorphism.

An important feature of X is an order three automorphism which is a lift of the order 3 automorphism acting on the quotient $\mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7$. In this section we describe how to find an explicit formula for it in terms of the birational automorphism of the sextic surface $\pi(X) \subset \mathbb{CP}^3$.

Proposition 5.1. Let $Y_0 = \frac{y_0}{y_1}$, $Y_2 = \frac{y_2}{y_1}$ and $Y_3 = \frac{y_3}{y_1}$ be the generators of the field extension $\mathrm{Rat}(X) \supset \mathbb{C}$. The automorphism of order three sends (Y_0, Y_2, Y_3) to (Y_0, Y_2', Y_3') given by Table 5. Its inverse sends (Y_0, Y_2, Y_3) to (Y_0, Y_2'', Y_3'') given by Table 6.

Table 5. Automorphism of order 3 : $(Y_0, Y_2, Y_3) \mapsto (Y_0, Y_2', Y_3')$

$$\begin{aligned}
Y_2' = \;& \tfrac{(3+\mathrm{i}\sqrt{7})}{8}\, Y_0^{-1}((-21\mathrm{i} + 31\sqrt{7})Y_2^2 + (-35\mathrm{i} + 9\sqrt{7})Y_3^2))^{-1} \\
& \Big(7(9\mathrm{i} + 5\sqrt{7})Y_0^4 Y_3 + 2Y_0^2(4(21\mathrm{i} + \sqrt{7})Y_2^2 - (7\mathrm{i} + 11\sqrt{7})Y_3) + \\
& Y_3(-49\mathrm{i} - 13\sqrt{7} - (49\mathrm{i} + 13\sqrt{7})Y_2^2 + 8(-7\mathrm{i} + 5\sqrt{7})Y_3 + (49\mathrm{i} + 13\sqrt{7})Y_3^2) \\
& -Y_0((-21\mathrm{i} + 31\sqrt{7})Y_2^3 + Y_2Y_3(112\mathrm{i} + 48\sqrt{7} + 21\mathrm{i}Y_3 - 31\sqrt{7}Y_3))\Big) \\
Y_3' = \;& \tfrac{(-3\mathrm{i}+\sqrt{7})}{8} Y_0^{-1}((-21\mathrm{i} + 31\sqrt{7})Y_2^2 + (-35\mathrm{i} + 9\sqrt{7})Y_3^2))^{-1} \\
& \Big((-21 - 31\mathrm{i}\sqrt{7})Y_2^3 + Y_0Y_2^2(-168 + 8\mathrm{i}\sqrt{7} + 49Y_3 - 13\mathrm{i}\sqrt{7}Y_3) \\
& +Y_0Y_3^2(56 + 40\mathrm{i}\sqrt{7} - 49Y_3 + 13\mathrm{i}\sqrt{7}Y_3) + Y_2(-21 - 31\mathrm{i}\sqrt{7} \\
& +7(13 + 7\mathrm{i}\sqrt{7})Y_0^4 + 8(21 - \mathrm{i}\sqrt{7})Y_3 + (21 + 31\mathrm{i}\sqrt{7})Y_3^2 \\
& +Y_0^2(-70 - 18\mathrm{i}\sqrt{7} + (-56 - 40\mathrm{i}\sqrt{7})Y_3))\Big)
\end{aligned}$$

Remark 5.2. While formulas of Tables 5 and 6 are not particularly inspiring, they are far preferable to some other formulas for the automorphism that we initially found.

TABLE 6. Inverse automorphism of order 3 : $(Y_0, Y_2, Y_3) \mapsto (Y_0, Y_2'', Y_3'')$

$$
\begin{aligned}
Y_2'' = \; & \Big(-20 - 4\mathrm{i}\sqrt{7} - 4\mathrm{i}(-9\mathrm{i} + \sqrt{7})Y_0^5 Y_2 + (34 - 30\mathrm{i}\sqrt{7})Y_3 + (134 + 14\mathrm{i}\sqrt{7})Y_3^2 \\
& -(15 - 43\mathrm{i}\sqrt{7})Y_3^3 - 48Y_3^4 - (1 + 3\mathrm{i}\sqrt{7})Y_3^5 + 4\mathrm{i}Y_0^6(5\mathrm{i} - \sqrt{7} + 2\sqrt{7}Y_3) \\
& +Y_2^4(-20 - 4\mathrm{i}\sqrt{7} + (-1 - 3\mathrm{i}\sqrt{7})Y_3) + 2Y_0^3 Y_2(36 + 4\mathrm{i}\sqrt[7]{+}(3 + 5\mathrm{i}\sqrt{7})Y_2^2 \\
& +(-5 + 15\mathrm{i}\sqrt{7})Y_3 + (-16 + 2\mathrm{i}\sqrt{7})Y_3^2) + Y_2^2(-40 - 8\mathrm{i}\sqrt{7} + 33(1 - \mathrm{i}\sqrt{7})Y_3 \\
& +(68 + 4\mathrm{i}\sqrt{7})Y_3^2 + (2 + 6\mathrm{i}\sqrt{7})Y_3^3) + 2Y_0^2(10 + 2\mathrm{i}\sqrt{7} + 8Y_2^4 + (-26 + 10\mathrm{i}\sqrt{7})Y_3 \\
& +(-29 - 9\mathrm{i}\sqrt{7})Y_3^2 + (8 - 4\mathrm{i}\sqrt{7})Y_3^3 - Y_2^2 Y_3(17 + \mathrm{i}\sqrt{7} + 8Y_3)) + Y_0^4(20 + 4\mathrm{i}\sqrt{7} \\
& +2(9 + \mathrm{i}\sqrt{7})Y_3 + 4(3 - \mathrm{i}\sqrt{7})Y_3^2 + (7 + 5\mathrm{i}\sqrt{7})Y_3^3 + Y_2^2(-48 + 16\mathrm{i}\sqrt{7} - (7 + 5\mathrm{i}\sqrt{7})Y_3)) \\
& +Y_0 Y_2(-36 - 4\mathrm{i}\sqrt{7} + (5 - \mathrm{i}\sqrt{7})Y_2^4 + 10(1 - 3\mathrm{i}\sqrt{7})Y_3 + 52Y_3^2 + (5 - \mathrm{i}\sqrt{7})Y_3^4 \\
& +2\mathrm{i}Y_2^2(13\mathrm{i} - 7\sqrt{7} + (5\mathrm{i} + \sqrt{7})Y_3^2))\Big) / \Big(2Y_0(-3\mathrm{i} - \sqrt{7} + (3\mathrm{i} + \sqrt{7})Y_0^2 - 2\mathrm{i}Y_2^2 \\
& +(-5\mathrm{i} + \sqrt{7})Y_3 + (\mathrm{i} + \sqrt{7})Y_3^2 - Y_2(-5\mathrm{i} + \sqrt{7} + (-\mathrm{i} + \sqrt{7})Y_3)) \\
& (-3\mathrm{i} - \sqrt{7} + (3\mathrm{i} + \sqrt{7})Y_0^2 - 2\mathrm{i}Y_2^2 - (5\mathrm{i} - \sqrt{7})Y_3 + (\mathrm{i} + \sqrt{7})Y_3^2 + Y_2(-5\mathrm{i} + \sqrt{7} \\
& +(-\mathrm{i} + \sqrt{7})Y_3))\Big) \\
Y_3'' = \; & \Big(8\mathrm{i}\sqrt{7}Y_0^6 Y_2 + Y_0^5(-40 - 8\mathrm{i}\sqrt{7} + (26 + 2\mathrm{i}\sqrt{7})Y_3) + 2Y_0^3(40 + 8\mathrm{i}\sqrt{7} \\
& +2(-17 + 11\mathrm{i}\sqrt{7} + (1 - 2\mathrm{i}\sqrt{7})Y_2^2)Y_3 + (-39 - 11\mathrm{i}\sqrt{7})Y_3^2 + (11 - 3\mathrm{i}\sqrt{7})Y_3^3) \\
& +2Y_0^2 Y_2(-4\mathrm{i}\sqrt{7} + (33 - 3\mathrm{i}\sqrt{7})Y_3 + (25 + 9\mathrm{i}\sqrt{7})Y_3^2 + 4\mathrm{i}(\mathrm{i} + \sqrt{7})Y_3^3 \\
& +4Y_2^2(-4 - \mathrm{i}\sqrt{7} + (1 - \mathrm{i}\sqrt{7})Y_3)) + Y_2(8\mathrm{i}\sqrt{7} + (5 - \mathrm{i}\sqrt{7})Y_2^4 + 2\mathrm{i}(27\mathrm{i} + \sqrt{7})Y_3 \\
& +(-23 - 17\mathrm{i}\sqrt{7})Y_3^2 + (8 - 8\mathrm{i}\sqrt{7})Y_3^3 + (5 - \mathrm{i}\sqrt{7})Y_3^4 + Y_2^2(5 + 7\mathrm{i}\sqrt{7} + 8\mathrm{i}(\mathrm{i} + \sqrt{7})Y_3 \\
& +2\mathrm{i}(5\mathrm{i} + \sqrt{7})Y_3^2)) + Y_0^4((7 - 3\mathrm{i}\sqrt{7})Y_2^3 + \mathrm{i}Y_2(-8\sqrt{7} + 4(3\mathrm{i} + \sqrt{7})Y_3 + (7\mathrm{i} + 3\sqrt{7})Y_3^2)) \\
& +Y_0(-40 - 8\mathrm{i}\sqrt{7} + (42 - 46\mathrm{i}\sqrt{7})Y_3 + 2(83 + 7\mathrm{i}\sqrt{7})Y_3^2 + (-14 + 46\mathrm{i}\sqrt{7})Y_3^3 \\
& -48Y_3^4 + (-1 - 3\mathrm{i}\sqrt{7})Y_3^5 + Y_2^4(-4 - 4\mathrm{i}\sqrt{7} + (-1 - 3\mathrm{i}\sqrt{7})Y_3) \\
& +2Y_2^2(-44 + 4\mathrm{i}\sqrt{7} + (-6 - 16\mathrm{i}\sqrt{7})Y_3 + (26 + 2\mathrm{i}\sqrt{7})Y_3^2 + (1 + 3\mathrm{i}\sqrt{7})Y_3^3))\Big) / \\
& \Big(2Y_0(-3\mathrm{i} - \sqrt{7} + (3\mathrm{i} + \sqrt{7})Y_0^2 - 2\mathrm{i}Y_2^2 - (5\mathrm{i} - \sqrt{7})Y_3 + (\mathrm{i} + \sqrt{7})Y_3^2 \\
& -Y_2(-5\mathrm{i} + \sqrt{7} + (-\mathrm{i} + \sqrt{7})Y_3))(-3\mathrm{i} - \sqrt{7} + (3\mathrm{i} + \sqrt{7})Y_0^2 - 2\mathrm{i}Y_2^2 - (5\mathrm{i} - \sqrt{7})Y_3 \\
& +(\mathrm{i} + \sqrt{7})Y_3^2 + Y_2(-5\mathrm{i} + \sqrt{7} + (-\mathrm{i} + \sqrt{7})Y_3))\Big)
\end{aligned}
$$

Proof. It is a straightforward computer calculation to check that the formulas provide automorphisms. However, it takes too long to verify that the cube of it is identity symbolically. It is, however, trivial to do so heuristically by taking a random point on $\pi(X)$ calculated to high precision and iterating the automorphism three times.

To find the automorphism we used the fact that Y_2 and Y_3 are rational functions with poles along $3F + S_1 + S_2$. So their transforms should be rational functions with poles along $3F + S_1' + S_2'$ and $3F + S_1'' + S_2''$. We also know that $Y_2 + Y_3$ is zero on S_1 and $Y_2 - Y_3$ is zero on S_2. This allows us to fix the transforms up to constants, which can then be recovered. □

6. Double cover of the fake projective plane.

In this section we explain how we found the function field of the fake projective plane.

According to [K11, p 1676] we need to attach the seventh root of the rational function which has divisor

$$5S + B + 4C + 6S' + 4B' + 2C' + 3S'' + 2B'' + C''$$

up to multiples of 7. This divisor is divisible by 7 in the Picard group and corresponds to the third possibility for the divisor B in [ibid], where the curves $A_1, A_2, E_1, B_1, B_2, E_2, C_1, C_2, E_3$ correspond to $C'', B'', S'', C', B', S', C, B, S$ in our notation. The first possibility for B was ruled out, because the I_9-fibre has multiplicity $\mu = 1$ by [K17, p 2 and Theorem 2.3], and the second possibility corresponds to Case 1 of (3.2) which was ruled out in Section 3. We found this function by looking at the equation of the cubic cone with vertex $(0 : 0 : 1 : 1)$ that contains S_1'' and S_1'. When divided by y_1^3 it gives a divisor whose zeros and poles occur only at the named divisors. By symmetrizing it via σ and using the automorphism we were able to get the desired function. We denote the seventh root of this function by z; the function z^7 is given in Table 7.

To find the function field of the fake projective plane we simply need to take the invariants with respect to σ that preserves z and Y_3 and negates Y_0 and Y_2.

We also found a lift of the action of the order three automorphism to the field generated by Y_0, Y_2, Y_3, z. Specifically, the action on z is given in Table 8.

7. Embedding of the fake projective plane into $\mathbb{CP}^9$

Let us now describe the method that allowed us to construct the equations of the fake projective plane.

By a Riemann-Roch calculation, the dimension of the bicanonical linear system on $\mathbb{P}^2_{\text{fake}}$ is 10.

The pullback of the ($\mathbb{Q}$-Cartier) canonical divisor via $\mu : Y \to \mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7$ satisfies

$$K_Y = \mu^* K_{\mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7} - \frac{3}{7}(S+S'+S'') - \frac{2}{7}(B+B'+B'') - \frac{1}{7}(C+C'+C''). \tag{7.1}$$

TABLE 7. Formula for z^7

$$
\begin{aligned}
z^7 = \ & \Big((-315\mathrm{i}+47\sqrt{7})^2(-1+Y_0^2)^5(2795\mathrm{i}+287\sqrt{7}-5590\mathrm{i}Y_0-574\sqrt{7}Y_0+11573\mathrm{i}Y_0^2+2689\sqrt{7}Y_0^2 \\
& -17556\mathrm{i}Y_0^3-4804\sqrt{7}Y_0^3+14357\mathrm{i}Y_0^4+5601\sqrt{7}Y_0^4-11158\mathrm{i}Y_0^5-6398\sqrt{7}Y_0^5 \\
& +5579\mathrm{i}Y_0^6+3199\sqrt{7}Y_0^6+5590\mathrm{i}Y_2^2+574\sqrt{7}Y_2^2-5590\mathrm{i}Y_0Y_2^2-574\sqrt{7}Y_0Y_2^2 \\
& +5994\mathrm{i}Y_0^2Y_2^2-510\sqrt{7}Y_0^2Y_2^2+5590\mathrm{i}Y_0^3Y_2^2+574\sqrt{7}Y_0^3Y_2^2+2795\mathrm{i}Y_2^4+287\sqrt{7}Y_2^4 \\
& +1616\mathrm{i}Y_3-4336\sqrt{7}Y_3+5568\mathrm{i}Y_0Y_3+5824\sqrt{7}Y_0Y_3+3232\mathrm{i}Y_0^2Y_3-8672\sqrt{7}Y_0^2Y_3 \\
& -448\mathrm{i}Y_0^3Y_3+11584\sqrt{7}Y_0^3Y_3-9968\mathrm{i}Y_0^4Y_3-4400\sqrt{7}Y_0^4Y_3+11584\mathrm{i}Y_0^2Y_2Y_3 \\
& +64\sqrt{7}Y_0^2Y_2Y_3-17600\mathrm{i}Y_0^3Y_2Y_3+5696\sqrt{7}Y_0^3Y_2Y_3+1616\mathrm{i}Y_2^2Y_3 \\
& -4336\sqrt{7}Y_2^2Y_3+7184\mathrm{i}Y_0Y_2^2Y_3+1488\sqrt{7}Y_0Y_2^2Y_3-17174\mathrm{i}Y_3^2-638\sqrt{7}Y_3^2 \\
& +11606\mathrm{i}Y_0Y_3^2-5186\sqrt{7}Y_0Y_3^2-5994\mathrm{i}Y_0^2Y_3^2+510\sqrt{7}Y_0^2Y_3^2+5994\mathrm{i}Y_0^3Y_3^2 \\
& -510\sqrt{7}Y_0^3Y_3^2-5590\mathrm{i}Y_2^2Y_3^2-574\sqrt{7}Y_2^2Y_3^2-1616\mathrm{i}Y_3^3+4336\sqrt{7}Y_3^3 \\
& -7184\mathrm{i}Y_0Y_3^3-1488\sqrt{7}Y_0Y_3^3+2795\mathrm{i}Y_3^4+287\sqrt{7}Y_3^4)(2795\mathrm{i}+287\sqrt{7}+5590\mathrm{i}Y_0 \\
& +574\sqrt{7}Y_0+11573\mathrm{i}Y_0^2+2689\sqrt{7}Y_0^2+17556\mathrm{i}Y_0^3+4804\sqrt{7}Y_0^3+14357\mathrm{i}Y_0^4 \\
& +5601\sqrt{7}Y_0^4+11158\mathrm{i}Y_0^5+6398\sqrt{7}Y_0^5+5579\mathrm{i}Y_0^6+3199\sqrt{7}Y_0^6+5590\mathrm{i}Y_2^2 \\
& +574\sqrt{7}Y_2^2+5590\mathrm{i}Y_0Y_2^2+574\sqrt{7}Y_0Y_2^2+5994\mathrm{i}Y_0^2Y_2^2-510\sqrt{7}Y_0^2Y_2^2-5590\mathrm{i}Y_0^3Y_2^2 \\
& -574\sqrt{7}Y_0^3Y_2^2+2795\mathrm{i}Y_2^4+287\sqrt{7}Y_2^4+1616\mathrm{i}Y_3-4336\sqrt{7}Y_3-5568\mathrm{i}Y_0Y_3 \\
& -5824\sqrt{7}Y_0Y_3+3232\mathrm{i}Y_0^2Y_3-8672\sqrt{7}Y_0^2Y_3+448\mathrm{i}Y_0^3Y_3-11584\sqrt{7}Y_0^3Y_3 \\
& -9968\mathrm{i}Y_0^4Y_3-4400\sqrt{7}Y_0^4Y_3-11584\mathrm{i}Y_0^2Y_2Y_3-64\sqrt{7}Y_0^2Y_2Y_3-17600\mathrm{i}Y_0^3Y_2Y_3 \\
& +5696\sqrt{7}Y_0^3Y_2Y_3+1616\mathrm{i}Y_2^2Y_3-4336\sqrt{7}Y_2^2Y_3-7184\mathrm{i}Y_0Y_2^2Y_3 \\
& -1488\sqrt{7}Y_0Y_2^2Y_3-17174\mathrm{i}Y_3^2-638\sqrt{7}Y_3^2-11606\mathrm{i}Y_0Y_3^2+5186\sqrt{7}Y_0Y_3^2 \\
& -5994\mathrm{i}Y_0^2Y_3^2+510\sqrt{7}Y_0^2Y_3^2-5994\mathrm{i}Y_0^3Y_3^2+510\sqrt{7}Y_0^3Y_3^2 \\
& -5590\mathrm{i}Y_2^2Y_3^2-574\sqrt{7}Y_2^2Y_3^2-1616\mathrm{i}Y_3^3+4336\sqrt{7}Y_3^3+7184\mathrm{i}Y_0Y_3^3 \\
& +1488\sqrt{7}Y_0Y_3^3+2795\mathrm{i}Y_3^4+287\sqrt{7}Y_3^4)\Big)\Big/\Big(4096Y_0^4(-4\mathrm{i}+4\mathrm{i}Y_0 \\
& +4\mathrm{i}Y_0^2-4\mathrm{i}Y_0^3+2\mathrm{i}Y_2-2\sqrt{7}Y_2-2\mathrm{i}Y_0Y_2+2\sqrt{7}Y_0Y_2+\mathrm{i}Y_2^2+\sqrt{7}Y_2^2-2\mathrm{i}Y_3 \\
& +2\sqrt{7}Y_3+2\mathrm{i}Y_0Y_3-2\sqrt{7}Y_0Y_3-2\mathrm{i}Y_2Y_3-2\sqrt{7}Y_2Y_3+\mathrm{i}Y_3^2+\sqrt{7}Y_3^2)^2(-4\mathrm{i}-4\mathrm{i}Y_0 \\
& +4\mathrm{i}Y_0^2+4\mathrm{i}Y_0^3-2\mathrm{i}Y_2+2\sqrt{7}Y_2-2\mathrm{i}Y_0Y_2+2\sqrt{7}Y_0Y_2+\mathrm{i}Y_2^2+\sqrt{7}Y_2^2-2\mathrm{i}Y_3 \\
& +2\sqrt{7}Y_3-2\mathrm{i}Y_0Y_3+2\sqrt{7}Y_0Y_3+2\mathrm{i}Y_2Y_3+2\sqrt{7}Y_2Y_3+\mathrm{i}Y_3^2+\sqrt{7}Y_3^2)^2(-21\mathrm{i}Y_2^2 \\
& +31\sqrt{7}Y_2^2-35\mathrm{i}Y_3^2+9\sqrt{7}Y_3^2)^2\Big)
\end{aligned}
$$

TABLE 8. Automorphism of order 3 : $(Y_0, Y_2, Y_3, z) \mapsto (Y_0, Y_2', Y_3', z'')$

$$\boxed{\begin{aligned} z'' = \quad & z^2(-1+Y_0^2)^{-3}(1-Y_0-Y_0^2+Y_0^3-\tfrac{1}{2}(1+\mathrm{i}\sqrt{7})(Y_2-Y_3) \\ & +\tfrac{1}{2}(1+\mathrm{i}\sqrt{7})Y_0(Y_2-Y_3)+\tfrac{1}{4}(-1+\mathrm{i}\sqrt{7})(Y_2-Y_3)^2)(-1-Y_0+Y_0^2+Y_0^3 \\ & -\tfrac{1}{2}(1+\mathrm{i}\sqrt{7})(Y_2+Y_3)-\tfrac{1}{2}(1+\mathrm{i}\sqrt{7})Y_0(Y_2+Y_3)+\tfrac{1}{4}(1-\mathrm{i}\sqrt{7})(Y_2+Y_3)^2) \end{aligned}}$$

This shows that the preimage of $\mu(F_Y)$ on $\mathbb{P}^2_{\text{fake}}$ is numerically equivalent to a canonical divisor. (It is actually a section of a canonical line bundle twisted by an invertible torsion line bundle). In particular, to calculate

$$H^0(\mathbb{P}^2_{\text{fake}}, 2K_{\mathbb{P}^2_{\text{fake}}})$$

we can look for rational functions on $\mathbb{P}^2_{\text{fake}}$ which have poles of order at most two on the curve F^{FPP} which is the preimage of $\mu(F_Y)$ and no other poles.

The action of $\mathbb{Z}_7$ splits the space of such functions into seven eigenspaces. Each eigenspace consists of functions of the form $z^i g$ where g is a function from the function field of Y, as i runs over residues modulo 7. The residual $\mathbb{Z}_3$ action allows us to reduce the calculation to that of $i = -1, 0, 1$.

The $i = 0$ case is easy. The only such function up to scaling is 1.

Now let us calculate such functions of the form zg. Consider the Cartesian product diagram below

$$\begin{array}{ccc} \widehat{\mathbb{P}^2_{\text{fake}}} & \to & Y \\ \downarrow & & \downarrow \\ \mathbb{P}^2_{\text{fake}} & \to & \mathbb{P}^2_{\text{fake}}/\mathbb{Z}_7 \end{array}$$

where $\widehat{\mathbb{P}^2_{\text{fake}}}$ is the singular Galois cover of Y ramified at the nine curves $S, \ldots, C''$ given by normalization of Y in the field of fractions of $\mathbb{P}^2_{\text{fake}}$. We can calculate the global sections of an invertible sheaf on $\mathbb{P}^2_{\text{fake}}$ in terms of the pullback of these sections on $\widehat{\mathbb{P}^2_{\text{fake}}}$.

In view of (7.1) we see that the pullback of $2F^{FPP}$ on $\widehat{\mathbb{P}^2_{\text{fake}}}$ is equal to twice its proper preimage $\widehat{F^{FPP}}$ plus

$$\frac{6}{7}(S+S'+S'')+\frac{4}{7}(B+B'+B'')+\frac{2}{7}(C+C'+C''),$$

where $\frac{1}{7}S$ is the reduced preimage of S under $\widehat{\mathbb{P}^2_{\text{fake}}} \to Y$, and similarly for the other eight curves. The divisor of z on $\widehat{\mathbb{P}^2_{\text{fake}}}$ is

$$-A+A'+\frac{5}{7}S-\frac{1}{7}S'-\frac{4}{7}S''+\frac{1}{7}B+\frac{4}{7}B'-\frac{5}{7}B''+\frac{4}{7}C+\frac{2}{7}C'-\frac{6}{7}C''.$$

This means that the divisor of g on $\widehat{\mathbb{P}^2_{\text{fake}}}$ must be greater or equal to

$$-2\widehat{F^{FPP}} - \frac{6}{7}(S+S'+S'') - \frac{4}{7}(B+B'+B'') - \frac{2}{7}(C+C'+C'') - \mathrm{d}iv(z)$$

$$= -2\widehat{F^{FPP}} + A - A' - \frac{11}{7}S - \frac{5}{7}S' - \frac{2}{7}S'' - \frac{5}{7}B - \frac{8}{7}B' + \frac{1}{7}B'' - \frac{6}{7}C - \frac{4}{7}C' + \frac{4}{7}C''.$$

Since g is a rational function on Y, this translates into the condition that the divisor of g on Y is greater or equal than

$$-2F_Y + A - A' - S - B' + B'' + C'',$$

in other words, it can be computed as a global section of the invertible sheaf

$$\mathcal{O}_Y(2F_Y + S - A + A' + B' - B'' - C'')$$

on Y, or equivalently σ-invariant sections of

$$\mathcal{O}_X(2F + S_1 + S_2 - A_1 - A_2 + A_1' + A_2' + B_1' + B_2' - B_1'' - B_2'' - C_1'' - C_2'').$$

Note that the rational function $Y_0^2 - 1$ on Y has pole of order 2 at F_Y and zeros of order 1 at the nine curves $A, \dots, C''$ of the I_9 fiber. As a result, the σ-invariant section $y_0^2 - y_1^2$ of $H^0(X, 2D)$ is $2F + I_9 + 2S_1 + 2S_2$. Since

$$(2F_Y + I_9 + 2S) - (2F_Y - A + A' + B' - B'' - C'' + S) = S + 2A + A'' + B + 2B'' + C + C' + 2C'',$$

we can find σ-invariant sections of

$$\mathcal{O}_X(2F + S_1 + S_2 - A_1 - A_2 + A_1' + A_2' + B_1' + B_2' - B_1'' - B_2'' - C_1'' - C_2'').$$

by looking at σ-invariant sections of $2D$ which vanish on $(S + 2A + A'' + B + 2B'' + C + C' + 2C'')$. By using the calculation of Table 4 it can be seen that such sections are multiples of $y_2^2 - y_3^2$, so the rational function in question is

$$\frac{(y_2^2 - y_3^2)z}{y_0^2 - y_1^2},$$

up to a multiplicative constant.

Similarly, for the $z^{-1}g$, we end up looking at g which are global sections of

$$\mathcal{O}_Y(2F_Y + A - A' - C + B'' + C'' + S' + S'').$$

We can construct these functions as

$$\frac{(y_0^2 - y_1^2)r(y_0, y_1, y_2, y_3)}{f_{cones}(y_0, y_1, y_2, y_3)}$$

where $r(y_0, y_1, y_2, y_3)$ is a σ-invariant section of $H^0(X, 4D)$ and f_{cones} is given in Remark 4.3. The denominator f_{cones} is a σ-invariant element of $H^0(X, 6D)$ which vanishes on $S' + S''$ given by

$$\left(y_0^3 - y_0^2 y_1 - y_0 y_1^2 + y_1^3 + \frac{1}{2}(1+\mathrm{i}\sqrt{7})(y_0 - y_1)y_1(y_2 - y_3) + \frac{1}{4}(-1+\mathrm{i}\sqrt{7})y_1(y_2 - y_3)^2\right)$$

$$\left(y_0^3 + y_0^2 y_1 - y_0 y_1^2 - y_1^3 - \frac{1}{2}(1+\mathrm{i}\sqrt{7})(y_0 + y_1)y_1(y_2 + y_3) - \frac{1}{4}(-1+\mathrm{i}\sqrt{7})y_1(y_2 + y_3)^2\right).$$

We know that the section $(y_0^2 - y_1^2)$ of $H^0(Y, D)$ has divisor $2F_Y + 2S + I_9$ where $I_9 = A + \ldots + C''$ is the sum of the curves in the I_9 fiber. As a result, the section r should be vanishing on

$$\begin{aligned}(4F_Y + 4S + 2I_9) + (2A - 3A' + A'' - B - B' + 2B'' - 2C + 2C'' - 2S + S' + S'')\\ -(2F_Y + A - A' - C + B'' + C'' + S' + S'')\\ = 2F_Y + 2S + 3A' + 3A'' + B + B' + 3B'' + C + 2C' + 3C''.\end{aligned}$$

Importantly, we need to use not just polynomial r but also elements of the normalization, namely products of σ-antiinvaritant degree two polynomials in y_i with $\hat{y}_4$ and $\hat{y}_5$ from Proposition 4.4.

This is a rather delicate calculation that led us to the results in Table 9. Note that these functions are only determined up to linear changes of variables. We have reduced the ambiguity a bit by requiring that the first of these sections vanishes at the fixed points of $\mathbb{Z}_7$ action on $\mathbb{P}^2_{\text{fake}}$ and have chosen constants in a noble but not very successful attempt to make the equations more palatable.

TABLE 9. Rational functions $z^{-1}g$

$\Big(4\mathrm{i}(-1+Y_0)(1+Y_0)(-266\mathrm{i}Y_0 + 34\sqrt{7}Y_0 + 532\mathrm{i}Y_0^3 - 68\sqrt{7}Y_0^3 - 266\mathrm{i}Y_0^5 + 34\sqrt{7}Y_0^5$ $-70\mathrm{i}Y_2 + 46\sqrt{7}Y_2 - 126\mathrm{i}Y_0^2Y_2 - 58\sqrt{7}Y_0^2Y_2 + 196\mathrm{i}Y_0^4Y_2 + 12\sqrt{7}Y_0^4Y_2 - 469\mathrm{i}Y_0Y_2^2$ $+97\sqrt{7}Y_0Y_2^2 - 63\mathrm{i}Y_0^3Y_2^2 - 29\sqrt{7}Y_0^3Y_2^2 - 70\mathrm{i}Y_2^3 + 46\sqrt{7}Y_2^3 + 238\mathrm{i}Y_0Y_3 + 266\sqrt{7}Y_0Y_3$ $-238\mathrm{i}Y_0^3Y_3 - 266\sqrt{7}Y_0^3Y_3 + 259\mathrm{i}Y_2Y_3 + 41\sqrt{7}Y_2Y_3 - 259\mathrm{i}Y_0^2Y_2Y_3 - 41\sqrt{7}Y_0^2Y_2Y_3$ $+56\mathrm{i}Y_0Y_2^2Y_3 + 104\sqrt{7}Y_0Y_2^2Y_3 + 728\mathrm{i}Y_0Y_3^2 - 56\sqrt{7}Y_0Y_3^2 - 196\mathrm{i}Y_0^3Y_3^2 - 12\sqrt{7}Y_0^3Y_3^2$ $+70\mathrm{i}Y_2Y_3^2 - 46\sqrt{7}Y_2Y_3^2 - 56\mathrm{i}Y_0Y_3^3 - 104\sqrt{7}Y_0Y_3^3)\Big)/\Big((-35\mathrm{i} + 23\sqrt{7})Y_0(4 - 4Y_0 - 4Y_0^2$ $+4Y_0^3 - 2Y_2 - 2\mathrm{i}\sqrt{7}Y_2 + (2 + 2\mathrm{i}\sqrt{7})Y_0(Y_2 - Y_3) + 2Y_3 + 2\mathrm{i}\sqrt{7}Y_3 + \mathrm{i}(\mathrm{i} + \sqrt{7})(Y_2 - Y_3)^2)$ $(-4\mathrm{i} - 4\mathrm{i}Y_0 + 4\mathrm{i}Y_0^2 + 4\mathrm{i}Y_0^3 - 2\mathrm{i}Y_0Y_2 + 2\sqrt{7}Y_0Y_2 - 2\mathrm{i}Y_0Y_3 + 2\sqrt{7}Y_0Y_3 + 2(-\mathrm{i} + \sqrt{7})$ $(Y_2 + Y_3) + (\mathrm{i} + \sqrt{7})(Y_2 + Y_3)^2)z\Big)$
$\Big(16\mathrm{i}(-1+Y_0)(1+Y_0)(-133\mathrm{i} + 17\sqrt{7} + 266\mathrm{i}Y_0^2 - 34\sqrt{7}Y_0^2 - 133\mathrm{i}Y_0^4 + 17\sqrt{7}Y_0^4$ $-133\mathrm{i}Y_0Y_2 + 17\sqrt{7}Y_0Y_2 + 133\mathrm{i}Y_0^3Y_2 - 17\sqrt{7}Y_0^3Y_2 - 217\mathrm{i}Y_2^2 + 37\sqrt{7}Y_2^2 - 49\mathrm{i}Y_0^2Y_2^2$ $-3\sqrt{7}Y_0^2Y_2^2 + 119\mathrm{i}Y_3 + 133\sqrt{7}Y_3 - 119\mathrm{i}Y_0^2Y_3 - 133\sqrt{7}Y_0^2Y_3 + 217\mathrm{i}Y_3^2 - 37\sqrt{7}Y_3^2$ $+49\mathrm{i}Y_0^2Y_3^2 + 3\sqrt{7}Y_0^2Y_3^2)\Big)/\Big((-35\mathrm{i} + 23\sqrt{7})(4 - 4Y_0 - 4Y_0^2 + 4Y_0^3 - 2Y_2 - 2\mathrm{i}\sqrt{7}Y_2$ $+(2 + 2\mathrm{i}\sqrt{7})Y_0(Y_2 - Y_3) + 2Y_3 + 2\mathrm{i}\sqrt{7}Y_3 + \mathrm{i}(\mathrm{i} + \sqrt{7})(Y_2 - Y_3)^2)(-4\mathrm{i} - 4\mathrm{i}Y_0 + 4\mathrm{i}Y_0^2$ $+4\mathrm{i}Y_0^3 - 2\mathrm{i}Y_0Y_2 + 2\sqrt{7}Y_0Y_2 - 2\mathrm{i}Y_0Y_3 + 2\sqrt{7}Y_0Y_3 + 2(-\mathrm{i} + \sqrt{7})(Y_2 + Y_3)$ $+(\mathrm{i} + \sqrt{7})(Y_2 + Y_3)^2)z\Big)$

The rational functions we have constructed so far lead to the variables U_0, U_1, U_4, U_7 of Theorem 2.1. The other sections are obtained by applying the order three automorphism. We used Mathematica to tabulate numerically several dozens points on $\mathbb{P}^2_{\text{fake}}$ by first picking random values for Y_2 and Y_3, then solving for (one of the) values of Y_0, then solving for one of the values of z by taking a seventh root of z^7. Then we looked for degree two and three polynomial equations that vanish on these points. Mathematica is able to work with these numerical approximations by keeping accuracy estimates. As a result, it can give solutions of expected dimension to linear system whose coefficients are only known approximately by assuming that all minors within the accuracy bound of zero are in fact zero. After finding approximations of the resulting expressions by algebraic numbers, we arrived at 84 degree three equations of Theorem 2.1.

8. Concluding remarks.

We have also calculated 147 degree seven equations among sections of $4H$ on the unramified double cover of $\mathbb{P}^2_{\text{fake}}$. There were no degree six equations.

It seems difficult to compute explicit equations of the unramified $\mathbb{Z}_2^4$-cover of $\mathbb{P}^2_{\text{fake}}$.

9. Computer files.

Nine computer files for our computation were uploaded as "Ancillary files" in the arXiv site [BK].

The file "Magma84FinalFPPexact" contains the calculation of the Hilbert polynomial of the surface, as well as the check of Remark 2.2. This is done with exact coefficients, i.e. in $\mathbb{Z}[\sqrt{-7}]$.

The file "Magma84FinalFPPmodular" contains the smoothness calculation. Specifically, it calculates three size seven minors of the Jacobian matrix and verifies that they have no common zeros on the surface. The calculation of each minor takes approximately one hour on our hardware.

The file "M284FinalFPP" is Macaulay2 file. It computes the projective resolution over a finite field, calculates the canonical bundle using this resolution, calculates bicanonical bundle and its Euler characteristic, and, finally, calculates $H^0(Z, \mathcal{O}(D-K)) = 0$. The last calculation is time consuming, it runs between one and two hours on our hardware.

The choices of Magma vs. Macaulay are somewhat idiosyncratic. We are not experts in either language and used what was accessible. It is conceivable that many of the calculations can be performed in either platform. Some linear algebra calculations appeared faster in Magma, which also allowed us to work with exact coefficients. On the other hand, some schemes and sheaf cohomology calculations were more natural in Macaulay.

References

[BK] L. Borisov, J. Keum, *Explicit equations of a fake projective plane*, arXiv:1802.06333v3 [math.AG].

[AK] D. Allcock, F. Kato, *A fake projective plane via* 2*-adic uniformization with torsion*, Tohoku Math. J. 69 (2017) 221-237.

[Au] T. Aubin, *Équations du type Monge-Ampère sur les variétés kähleriennes compactes*, C. R. Acad. Sci. Paris Ser. A-B **283** (1976), no. 3, Aiii, A119–A121.

[BHPV] W. Barth, K. Hulek, Ch. Peters, A. Van de Ven, *Compact complex surfaces. Second edition.* Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics , 4. Springer-Verlag, Berlin, 2004. xii+436 pp.

[CS] D. Cartwright, T. Steger, *Enumeration of the 50 fake projective planes*, C. R. Acad. Sci. Paris, Ser. I **348** (2010) 11-13.

[CS2] D. Cartwright, T. Steger, http://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes

[CK] F. Catanese, J. Keum, *The Bicanonical map of fake projective planes with an automorphism.* arXix:1801.05291, International Mathematics Research Notices, published online.

[CD] F. Cossec, I. Dolgachev, *Enriques surfaces I*, Birkhäuser 1989.

[DBDC] G. Di Brino, L. Di Cerbo, *Exceptional collections and the bicanonical map of Keum's fake projective planes*, Communications in Contemporary Mathematics **20** no. 1 (2018), 1650066 (13 pages), World Scientific Publishing Company, DOI: 10.1142/S0219199716500668.

[D] I. Dolgachev, *Algebraic surfaces with* $q = p_g = 0$, C.I.M.E. Algebraic surfaces, pp 97-215, Liguori Editori, Napoli 1981.

[F] W. Fulton, *Intersection Theory*, Springer-Verlag 1984.

[GKMS] S. Galkin, L. Katzarkov, A. Mellit, E. Shinder, *Derived categories of Keum's fake projective planes* Adv. Math. **278** (2015) 238-253.

[I] M-N Ishida, *An elliptic surface covered by Mumford's fake projective plane*, Tohoku Math.J. (2) **40** no. 3 (1988) 367-396.

[IK] M-N Ishida, F Kato, *The strong rigidity theorem for non-Archimedean uniformization*, Tohoku Math. J. (2) **50** (1998) 537–555.

[K06] J. Keum, *A fake projective plane with an order 7 automorphism*, Topology **45** (2006) 919-927.

[K08] J. Keum, *Quotients of fake projective planes*, Geom. Topol. **12** (2008) 2497-2515.

[K11] J. Keum, *A fake projective plane constructed from an elliptic surface with multiplicities* $(2, 4)$, Sci. China Math. **54** (2011) 1665-1678.

[K12] J. Keum, *Toward a geometric construction of fake projective planes*, Rend. Lincei Mat. Appl. **23** (2012) 137-155.

[K13] J. Keum, *Every fake projective plane with an order 7 automorphism has* $H^0(2L) = 0$ *for any ample generator* L, manuscript circulated on July 8 2013.

[K17] J. Keum, *Vanishing theorem on fake projective planes with enough automorphisms*, Trans. Amer. Math. Soc. **369** (2017) 7067-7083.

[KK] V. S. Kharlamov, V. M. Kulikov, *On real structures on rigid surfaces*, Izv. Russ. Akad. Nauk. Ser. Mat. **66**, no. 1, (2002) 133-152; Izv. Math. **66**, no. 1, (2002) 133-150.

[Kl] B. Klingler, *Sur la rigidité de certains groupes fondamentaux, l'arithméticité des réseaux hyperboliques complexes, et les "faux plans projectifs"*, Invent. Math. **153** (2003) 105-143.

[Mos] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Annals Math. Studies **78**, Princeton Univ. Press, Princeton, N.J.; Univ. Tokyo Press, Tokyo 1973.

[M] D. Mumford, *An algebraic surface with K ample,* $K^2 = 9$*,* $p_g = q = 0$, Amer. J. Math. **101** (1979) 233-244.

[PY] G. Prasad, S.-K. Yeung, *Fake projective planes*, Invent. Math. **168** (2007) 321-370; Addendum, **182** (2010) 213-227.

[R] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces.* Ann. of Math. (2) **127** (1988), no. 2, 309–316

[Y] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Natl. Acad. Sci. USA **74** (1977) 1798–1799.

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