Classical algebraic geometry. A modern view, by Igor V. Dolgachev. Cambridge University Press, Cambridge, 2012. xii+639 pp. ISBN: 978-1-107-01765-8, hardcover. [1]

Modern algebraic geometry is based heavily on the language of schemes, developed by Grothendieck and others in the middle of the twentieth century. However, as a byproduct of the scheme revolution, many beautiful older results were largely forgotten. Dolgachev attempts to mitigate this educational gap by translating advanced classical materials into the language familiar to modern algebraic geometers. The book is enormous in scope; in this review we aim to highlight some of its more accessible elements.

1. What is classical algebraic geometry?

To give the reader unfamiliar with classical algebraic geometry a flavor of the subject matter, we will briefly review a few remarkable geometric constructions which were developed in the nineteenth century.

Complex projective spaces. The favorite playground of algebraic geometry in general and classical algebraic geometry in particular are the complex projective spaces. The projective space $\mathbb{CP}^n$ parametrizes lines in an $(n+1)$-dimensional vector space over $\mathbb{C}$. In coordinates, points of $\mathbb{CP}^n$ are encoded by nonzero $(n+1)$-tuples of complex numbers $(x_0 \colon x_1 \colon \dots \colon x_n)$ up to homotheties

$$(x_0 \colon x_1 \colon \dots \colon x_n) \sim (\lambda x_0 \colon \lambda x_1 \colon \dots \colon \lambda x_n),\ \lambda \in \mathbb{C}\backslash\{0\}. \tag{1.1}$$

Complex projective space $\mathbb{CP}^n$ contains the affine space $\mathbb{C}^n$ as a subset of points with $x_0 \neq 0$ with coordinates $(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$ which are well-defined on the set of equivalence classes of (1.1). However, $\mathbb{CP}^n$ is preferred to $\mathbb{C}^n$ because it is compact in the usual topology. In fact, projective spaces are the simplest complex algebraic varieties which are compact when viewed as complex manifolds. This explains why the bulk of algebraic geometry concerns subvarieties in $\mathbb{CP}^n$.

Subvarieties of $\mathbb{CP}^n$ are given by systems of homogeneous equations in coordinates $(x_0, \dots, x_n)$. For example, Fermat cubic and quartic curves in $\mathbb{CP}^2$ are given by

$$x_0^3 + x_1^3 + x_2^3 = 0 \tag{1.2}$$

and

$$x_0^4 + x_1^4 + x_2^4 = 0 \tag{1.3}$$

respectively. The geometry of the solution sets of these and other equations of small degree was studied extensively in the second half of the nineteenth and the first half of the twentieth centuries, and is now commonly referred to as classical algebraic geometry. Below we list a few of its most prominent results.

Triple tangents of cubics in $\mathbb{CP}^2$. Let $F(x_0, x_1, x_2)$ be a degree three homogeneous polynomial in three variables, so that the corresponding curve [2] $E \subset \mathbb{CP}^2$

$$E = \{(x_0 \colon x_1 \colon x_2) \in \mathbb{CP}^2, \text{ such that } F(x_0, x_1, x_2) = 0\}$$

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[2] The real dimension of E is two, and as a compact Riemann surface E has the topology of the torus $S^1 \times S^1$. Algebraic geometers call E a curve to indicate its dimension over $\mathbb{C}$.

is a smooth complex manifold. Then there are exactly 9 lines in $\mathbb{CP}^2$ which are triple tangent to E, that is the restriction of the polynomial F to the line has a triple root. In the particular case of the Fermat cubic (1.2), these tritangent lines can be found by direct calculation to be

$$x_0 + \xi x_1 = 0,\ x_1 + \xi x_2 = 0,\ x_2 + \xi x_0 = 0$$

for each of the three solutions of $\xi^3 = 1$. In general, this statement is far from obvious. The nine triple tangency points correspond to points of order three once E is given a Lie group structure with one of these points as the identity; this statement is routinely covered in introductory algebraic geometry courses.

Bitangent lines of quartic curves. A (smooth) Riemann surface C given by a degree four equation $G(x_0, x_1, x_2) = 0$ in $\mathbb{CP}^2$ is topologically equivalent to a sphere with three handles. It was observed by Plücker in 1839 [11] that there are exactly 28 lines in $\mathbb{CP}^2$ which have two tangency points with C (or a four-tangency point). These are called bitangent lines of C. In the particular case of Fermat quartic (1.3) a direct calculation shows that twelve of these are four-tangent lines [3]

$$x + \xi y = 0,\ y + \xi z = 0,\ z + \xi x = 0$$

for $\xi^4 = -1$ and sixteen more bitangent lines are

$$x + \xi_1 y + \xi_2 z = 0$$

where ξ_1 and ξ_2 are all possible choices of fourth roots of 1.

In general, these 28 bitangent lines correspond to the so-called odd theta characteristics of C. If one considers a double cover of $\mathbb{CP}^2$ ramified at C, then each of these lines l splits up into two components, because the restriction of G to l is the square of a degree two polynomial. These components then give all 56 holomorphic spheres of self-intersection (-1) on a del Pezzo surface of degree 2. This is already a fairly non-trivial statement, which only occasionally makes it into the graduate curriculum.

Lines on a cubic surface. Let S be a smooth cubic surface, that is a smooth two-dimensional complex manifold given by a degree three equation in the coordinates x_i on $\mathbb{CP}^3$. Continuing with our Fermat-type examples one can look at

$$S = \{(x_0 \colon x_1 \colon x_2 \colon x_3) \in \mathbb{CP}^3,\ \text{such that } x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0\}. \tag{1.4}$$

It has been known at least since the work of Cayley and Salmon [5] that S contains exactly 27 lines. For the specific surface (1.4) these lines are given in the parametric form as images of $\mathbb{CP}^1$ with coordinates $(s \colon t)$ by

$$(s \colon \xi_1 s \colon t \colon \xi_2 t),\ (s \colon t \colon \xi_1 s \colon \xi_2 t),\ (s \colon t \colon \xi_1 t \colon \xi_2 s),$$

where ξ_1 and ξ_2 are nine independent choices of third roots of 1. Also of note are 45 planes which contain triples of these lines. We will describe some less well known results about the cubic surfaces in the next section.

[3] These are also called inflection bitangent lines.

2. Further examples.

The aforementioned basic facts of classical algebraic geometry are well known to most if not all modern day algebraic geometers. However, there is a wealth of other results about low-dimensional subvarieties of small degree in projective spaces which are known only to a small sliver of specialists. We will list several such results in this section.

Polarity construction. The cornerstone of many arguments of Dolgachev's book is the following polarity construction. If a degree d hypersurface X in $\mathbb{CP}^n$ is given by $\{F(x_0,\ldots,x_n)=0\}$ and $a=(a_0\colon\ \ldots : a_n)$ is a point in $\mathbb{CP}^n$, then *the first polar* $P_a(X)$ of X with respect to a is given by $G=0$ where

$$G(x_0,\ldots,x_n) := \frac{d}{dt}(F(x_0+ta_0,\ldots,x_n+ta_n))\Big|_{t=0} = \sum_i a_i \frac{\partial F}{\partial x_i}.$$

It is clear that G is a homogeneous polynomial of degree $d-1$, which is well defined up to a multiplicative constant, so $P_a(X)$ is well-defined. In the case when X is a smooth conic in $\mathbb{CP}^2$ and $a\not\in X$, the geometric meaning of $P_a(X)$ is the line connecting two points in X where tangent lines pass through a.

This construction can be extended to multiple points by iteration. In particular if one repeats it $d-1$ times for $a\in X$, then $P_{a^{d-1}}(X)$ is the tangent hyperplane to X at a if X is nonsingular at a. If X is singular at a, then $P_{a^{d-1}}(X)$ is the whole $\mathbb{CP}^n$. More generally, when one views points in $\mathbb{CP}^n$ as hyperplanes in the dual projective space, this construction associates to a degree d hypersurface in $\mathbb{CP}^n$ and a degree k hypersurface in the dual projective space a degree $d-k$ hypersurface in $\mathbb{CP}^n$ (unless the corresponding derivative is zero, in which case the polar is the whole $\mathbb{CP}^n$). For $k=d$ this is simply the pairing between d-th symmetric powers of dual vector spaces. We will use this pairing in the discussion below of plane quartics represented by sums of five fourth powers.

Quartics as sums of fourth powers and Clebsch quartics. The space of quartic polynomials in three variables is isomorphic to $\mathbb{C}^{15}$. It contains a three-dimensional (non-linear) subvariety of fourth powers of linear forms. Thus, a naive count of dimensions suggests that a generic degree four polynomial in three variables can be written as a sum of five fourth powers of linear polynomials. However, this is not true, as was first observed by Lüroth in 1868 [9]. We will sketch two simple arguments below.

Suppose $F=l_1^4+\cdots+l_5^4$. The tangent space to F in the space of all quartics that can be represented by sums of five fourth powers[4] is the space of terms of degree 1 in ϵ in

$$(l_1+\epsilon\hat{l}_1)^4+\cdots+(l_5+\epsilon\hat{l}_5)^4 \tag{2.1}$$

for all possible choices of $\hat{l}_i$. Consider the 5 points p_i on the dual $\mathbb{CP}^2$ that correspond to l_i. There is a conic that passes through them, and we consider the square H of its equation. If one takes the polar pairing of H and (2.1), the fact that H vanishes twice at p_i implies that the pairing vanishes to first order in ϵ. As

[4] We are being sloppy here. More precisely, if the dimension count were to work, then a generic F would have a finite number of decompositions into five fourth powers, and the linear term in (2.1) gives the tangent space to the corresponding branch.

the result, the tangent space to the space of five fourth powers of linear forms is orthogonal to H and is not the whole space of quartics.

Another argument can be given as follows. If $F = \sum_{i=1}^{5} l_i^4$, then the linear span of six second order partial derivatives of F with respect to x_0, x_1, x_2 lies in the five-dimensional span of l_i^2. However, for a generic quartic F the span of second order partial derivatives is six-dimensional. Therefore a necessary condition for F to be represented as a sum of five fourth powers is the vanishing of the determinant of the coefficient matrix of the second order partial derivatives of F. This determinant is a degree six polynomial in the coefficients of F, known as the catalecticant. Such quartics are known as Clebsch quartics.

It is very natural to ask what is the variety of presentations of a given Clebsch quartic into the sum of fourth powers, up to permutation and scaling of the coefficients. It turns out that for a general Clebsch quartic this variety is a rational curve.

More on cubic surfaces. Consider a smooth cubic surface S in $\mathbb{CP}^3$ given by

$$F(x_0, x_1, x_2, x_3) = 0$$

for a degree 3 homogeneous polynomial F. It turns out that $F = 0$ can always be rewritten in the form

$$l_1 l_2 l_3 + m_1 m_2 m_3 = 0$$

where l_i and m_i are linear combinations of $x_0, \ldots, x_3$. This is known as the Cayley-Salmon equation. In fact, there are 120 ways of writing S in this form, up to rescaling and permuting the linear factors.

Note that $l_i = m_j = 0$ gives 9 out of 27 lines on S. In addition, $l_i = 0$ (or $m_i = 0$) are the so-called tritangent planes of S, which are characterized by the property that they intersect S at three lines. In the case of the Fermat cubic (1.4), one can rewrite its equation as

$$(x_0 + x_1)(x_0 + \xi x_1)(x_0 + \xi^2 x_1) + (x_2 + x_3)(x_2 + \xi x_3)(x_2 + \xi^2 x_3) = 0$$

where $\xi = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$.

On another note, let us further assume that the coefficients of F are chosen generically (for example, they are algebraically independent over $\mathbb{Q}$, although a much weaker condition would suffice). Then a result going back to Sylvester [12] states that F can be written as a sum of cubes of five linear polynomials

$$F = t_1^3 + t_2^3 + t_3^3 + t_4^3 + t_5^3.$$

Moreover, such presentation is unique, up to permutation of the t_i and scaling by third roots of 1. Equivalently, a general cubic surface is isomorphic to a surface in $\mathbb{CP}^4$ with coordinates $(z_0 \colon z_1 \colon z_2 \colon z_3 \colon z_4)$ given by the equations

$$\sum_{i=0}^{4} a_i z_i^3 = \sum_{i=0}^{4} z_i = 0,$$

where a_i are determined uniquely up to permutation and common scaling.

Dolgachev also describes in great detail various degenerations of cubic surfaces. There are 21 different classes of cubic surfaces with isolated singularities. For example, the maximum number of singular points is 4, which is the case for the so-called Cayley surface $x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0$. Dolgachev devotes a whole chapter in the book to the study of cubic surfaces, and the above statements constitute just a tiny portion of the book's material.

3. Comments.

Dolgachev's book is meant to attract attention to the vast knowledge base in algebraic geometry that preceded the by now standard axiomatic approach of Grothendieck and others. As the scheme-theoretic approach to algebraic geometry began to dominate, the old material became neglected, with relatively few experts in classical algebraic geometry still practicing the art.

It is worth pointing out that even though modern topics, such as the minimal model program, mirror symmetry, geometry of moduli spaces of curves, and various flavors of non-commutative algebraic geometry, are the face of the field, there is still considerable amount of work being done on classical objects. Often, new techniques are used to attack old problems, as was the case with the classification of Fano varieties of dimension three due to Iskovskikh and Mori-Mukai, see [8, 10]. While that result is very much in the spirit of classical algebraic geometry, minimal model program techniques were essential to the argument. In a more recent example, the study of moduli spaces of cubic surfaces has been reinvigorated by the work of Allcock, Carlsson, Toledo and others, see [1, 2]. Cremona groups, first considered in 1863 in [6], still remain an active research topic, see for example [4]. So it is fair to say that classical algebraic geometry was never completely forgotten. It has been, however, neglected. Dolgachev's book goes a long way in bringing many beautiful old results back into the limelight.

Dolgachev's book is aimed at algebraic geometers of all levels; however it is not meant to serve as an introduction to algebraic geometry.[5] Familiarity with Hartshorne's book [7] is preferred, as concepts such as projective spaces, invertible sheaves, sheaf cohomology, etc. are used freely. Dolgachev does not attempt to reconstruct the original proofs; rather, the focus of the book is on bringing the old results out of obscure library corners and making them accessible to the current generation of algebraic geometers. This modernization is crucial because it is very difficult to read the original texts of the 19th and early 20th century due to natural terminology drift and differing underlying common knowledge assumptions. Some of the results are given as exercises; there are also extensive historical notes at the end of each chapter.

Dolgachev's book is a labor of love, with the author producing a remarkable in-depth review of his favorite subject. This one-of-a-kind book is likely to be widely used as a reference whenever classical results are used or quoted.

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