APPLICATIONS OF HOMOLOGICAL MIRROR SYMMETRY TO HYPERGEOMETRIC SYSTEMS: DUALITY CONJECTURES

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Abstract. Homological mirror symmetry for crepant resolutions of Gorenstein toric singularities leads to a pair of conjectures on certain hypergeometric systems of PDEs. We explain these conjectures and verify them in some cases.

1. Introduction

Mirror symmetry is the string theory statement that two N=(2,2) superconformal field theories are isomorphic, up to a so-called mirror involution of the theory. Classically, this is a statement about a type IIB theory on one compact Calabi-Yau variety X and a type IIA theory on a mirror compact Calabi-Yau variety Y at the level of closed strings. Homological mirror symmetry of Kontsevich is an open-string version of mirror symmetry. It asserts that the derived category of coherent sheaves $D^b(X)$ on X (which is the category of boundary conditions on type IIB open strings propagating in X) is equivalent to the derived Fukaya category $DFuk(Y)$ of Y, which is a remarkable statement that relates two a priori unrelated constructs. Since $D^b(X)$ and $DFuk(Y)$ depend non-trivially on the complex parameters of X and on the Käbler parameters of Y respectively, mirror symmetry implies a (at least local) bijection between these sets of parameters.

The less well-known but equally fascinating aspect of homological mirror symmetry concerns the dependence of the boundary conditions of IIB open strings on X on the Käbler parameters of X. Such dependence is locally trivial, but the global structure of this dependence leads to a number of conjectural consequences. One of them is that birational Calabi-Yau varieties are derived equivalent, which is a prototypical case of Kawamata's D-equivalence conjecture [Ka]. One also

The first author was partially supported by the NSF grant DMS-1201466. The second author was supported by the grants NSF DMS 0854977 FRG, NSF DMS 0600800, NSF DMS 0652633 FRG, NSF DMS 0854977, NSF DMS 0901330, FWF P 24572 N25, FWF P20778 and by an ERC Grant.

expects to have a locally trivial family of triangulated categories over the base given by the complex moduli of the mirror Calabi-Yau. After passing to Grothendieck groups this should simply be the variation of Hodge structures on the cohomology of the mirror variety. Our paper deals with this aspect of homological mirror symmetry at the level of Grothendieck groups in the important case of so-called toric Calabi-Yau manifolds.

Toric Calabi-Yau manifolds are noncompact toric varieties with trivial canonical bundle, such as crepant resolutions of toric Gorenstein singularities. The lack of compactness of these Calabi-Yau varieties causes some subtlety in the formulation of homological mirror symmetry. One also generally needs to use smooth Deligne-Mumford stacks rather than varieties since higher-dimensional toric singularities often do not admit crepant scheme resolutions. In this paper, we greatly clarify the situation and formulate a set of conjectures that relate different projective resolutions of a Gorenstein toric singularity. We verify the conjectures in a non-trivial example by a brute force calculation and produce other evidence in favor of the conjectures.

To formulate the results of the paper we need to introduce the combinatorial data that underlie crepant resolutions of toric singularities. Let C be a rational polyhedral cone in a lattice N which defines an affine toric variety $X = \operatorname{Spec}\mathbb{C}[C^\vee \cap N^\vee]$. This variety has Gorenstein singularities if and only if there exists a linear function $\deg : N \to \mathbb{Z}$ which equals 1 on the generators of all one-dimensional rays of C. Let $\{v_1, \ldots, v_n\}$ be a subset of the set of lattice points of degree 1 in C which includes all generators of the rays of C. To any projective simplicial subdivision Σ of C such that the generators of the one-dimensional cones are among these v_i, one can associate a smooth toric Deligne-Mumford stack $\mathbb{P}_\Sigma$ with a crepant birational morphism

$$\pi : \mathbb{P}_\Sigma \to X.$$

To such Σ, we associate the Grothendieck group $K_0(\mathbb{P}_\Sigma)$ of $D(\mathbb{P}_\Sigma)$ as well as the Grothendieck group $K_0^c(\mathbb{P}_\Sigma)$ of the subcategory of complexes whose cohomology sheaves are supported on the compact set $\pi^{-1}(0)$. We prove that Euler characteristics provides a perfect pairing between these spaces, see Corollary 4.6.

Crucially, we define isomorphisms between $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ and the solutions to so-called better behaved GKZ hypergeometric systems $bbGKZ(C, 0)$ and $bbGKZ(C^\circ, 0)$ (see Definition 6.1) respectively given by certain Gamma series Γ and Γ°. We claim that these Gamma series are remarkably compatible with the natural maps between the K_0 and

K_0^c spaces, for different resolutions of singularities of X. Specifically, for two adjacent triangulations Σ_1 and Σ_2, there are natural pullback-pushforward maps pp between the Grothendieck groups which we expect to be compatible with the analytic continuation of the solutions of $bbGKZ$.

Conjecture I (=7.1). The following diagrams of isomorphisms are commutative

$$\begin{array}{ccc} K_0(\mathbb{P}_{\Sigma_2})^\vee & \xrightarrow{pp^\vee} & K_0(\mathbb{P}_{\Sigma_1})^\vee \\ \Gamma\downarrow & & \downarrow\Gamma \\ bbGKZ(C,0) & \xrightarrow{a.c.} & bbGKZ(C,0) \end{array}$$

$$\begin{array}{ccc} K_0^c(\mathbb{P}_{\Sigma_2})^\vee & \xrightarrow{pp^\vee} & K_0^c(\mathbb{P}_{\Sigma_1})^\vee \\ \Gamma^\circ\downarrow & & \downarrow\Gamma^\circ \\ bbGKZ(C^\circ,0) & \xrightarrow{a.c.} & bbGKZ(C^\circ,0) \end{array}$$

where the top rows are the duals of the maps induced by the pullback-pushforward derived functors, and the bottom rows are analytic continuations along a certain path in the domain of parameters considered in [BH2].

The pairing between $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ should itself be given by some conjectural pairing between the solutions of $bbGKZ(C,0)$ and $bbGKZ(C^\circ,0)$.

Conjecture II (=7.3). There exists a collection $p_{c,d}(x_1,\ldots,x_n)$ of polynomials indexed by $c\in C$, $d\in C^\circ$, such that the following hold.

- All but a finite number of $p_{c,d}$ are zero.
- For any pair of solutions (Φ_c) and (Ψ_d) of $bbGKZ(C,0)$ and $bbGKZ(C^\circ,0)$ respectively the sum

$$(\sharp)\qquad \sum_{c,d} p_{c,d}\Phi_c\Psi_d$$

  is constant as a function of $(x_1,\ldots,x_n)$.
- The pairing provided by $(\sharp)$ is non-degenerate.
- For any projective simplicial subdivision Σ, the pairing provided by $(\sharp)$ is the inverse of the Euler characteristics pairing between $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ under the Γ and Γ° series.

The paper is organized as follows. In Sections 2 and 3, we discuss a combinatorial approach to cohomology with compact support (using piecewise polynomial functions) and Grothendieck groups in the case of semi–projective toric varieties. The technical details describing the crucial perfect pairing between the regular Grothendieck group and Grothendieck group with compact support are given in Section 4. The derived category origin of the latter Grothendieck group is explained in Section 5. Finally, after describing the Gamma series solutions to the two classes of associated better behaved GKZ systems in Section 6, we present the conjectures in Section 7. In the the last two sections of the paper, we exhibit evidence in certain classes of examples lending strong support for the conjectures.

2. Stanley-Reisner calculation for cohomology with compact support

In this preliminary section we describe a Stanley-Reisner type presentation of cohomology with compact support of semi-projective toric varieties. While this presentation is undoubtedly known to some experts in the field, we were unable to find a suitable reference.

Let C be a finite rational polyhedral cone[1] in the lattice N. We assume that $\dim C = \mathrm{rk}(N)$ but we do not assume $C \cap (-C) = \{0\}$, and in fact we allow C to be all of N. Let v_i be a set of n elements of C and let a simplicial fan Σ correspond to a simplicial complex on the set of indices i. We assume that the support of Σ is all of C. We denote the simplicial complex (i.e. a set of subsets of the set of indices) by Σ as well, as it will be clear from the context what we consider. Some v_i may not form a cone in Σ.

Definition 2.1. Denote by A the ring of Σ-piecewise polynomial functions on C (with values in $\mathbb{C}$). These are collections of polynomial functions on cones $\sigma \in \Sigma$ which are compatible with restrictions. Denote by A^c the ideal of A of functions f with the property that $f|_{\partial C} = 0$.

Remark 2.2. When C fills the whole space, we have $\partial C = \emptyset$ and $A^c = A$. In particular, when $\mathrm{rk}N = 0$, we have $A^c = A = \mathbb{C}$.

The goal of the rest of the section is to describe natural presentations of A and A^c. The former is well-known, while the latter does not seem to appear in the literature.

For each i such that v_i generates a ray of the fan, consider a piecewise linear function D_i on C which takes the value 1 on v_i and 0 on the other generators of rays of Σ. The piecewise linear function D_i is called the Courant function associated to v_i, see [Bill]. If v_i is not a ray

[1] We will be sloppy in our notation in that C will sometimes denote the set of lattice points of the cone and sometimes the set of real points of it. We hope that this does not lead to confusion.

generator of Σ, set $D_i = 0$. The following is the standard Stanley-Reisner presentation of A, see [Bill, 2.3, 3.6], [Br2, 1.3].

Proposition 2.3. *The ring A is naturally isomorphic to the quotient of $\mathbb{C}[D_1, \ldots, D_n]$ by the ideal generated by the monomials*

$$\prod_{i \in I} D_i$$

for all $I \notin \Sigma$.

We now give an analogous statement for A^c.

Proposition 2.4. *The module A^c over $\mathbb{C}[D_1, \ldots, D_n]$ has the following natural presentation using generators F_I, for $I \in \Sigma$ with $\sigma_I^\circ \subseteq C^\circ$. It is the quotient of the free module*

$$\bigoplus_{I \in \Sigma, \sigma_I^\circ \subseteq C^\circ} \mathbb{C}[D_1, \ldots, D_n] F_I$$

by the submodule generated by relations

$$D_i F_I - F_{I \cup \{i\}}, \text{ for } i \notin I, I \cup \{i\} \in \Sigma \tag{2.1}$$

and

$$D_i F_I, \text{ for } i \notin I, I \cup \{i\} \notin \Sigma. \tag{2.2}$$

Proof. Note that there is a natural morphism of A–modules from

$$\bigoplus_{I \in \Sigma, \sigma_I^\circ \subseteq C^\circ} \mathbb{C}[D_1, \ldots, D_n] F_I / \langle \text{relations } (2.1)(2.2) \rangle \tag{2.3}$$

to A^c which sends $F_I \mapsto \prod_{i \in I} D_i$.

Further observe that the rings $\mathbb{C}[D_1, \ldots, D_n]$, A and the module A^c are naturally $\mathbb{Z}^n$ graded where n is the number of indices i. The underlying reason for such grading is the $(\mathbb{R}_{>0})^n$ action on C that scales the rays of the fan and scales the rest of C linearly with it. This has an effect of separately scaling D_i.

The support of a monomial $\prod_i D_i^{k_i}$ is the set of indices i such that $k_i > 0$. The non-zero monomials that occur in A are exactly the ones whose support lies in Σ. Clearly, A^c is a $\mathbb{Z}^n$ graded ideal of A. A non-zero monomial $\prod_i D_i^{k_i}$ lies in it iff its support lies in Σ but is not a boundary simplex of C. Thus every monomial m in A^c is divisible by the image of $F_{\mathrm{supp}(m)}$. This shows that the map (2.3) is surjective, i.e. the images of F_I generate A^c.

Any monomial ideal has a resolution whose first step is given by the lcm of generators, the Taylor's resolution of [Eis, 17.11]. Thus, it suffices to show that for any I, J with $\sigma_I^\circ, \sigma_J^\circ \subseteq C^\circ$ the corresponding relation

$$(\prod_{i\in I\setminus J} D_i)F_J - (\prod_{i\in J\setminus I} D_i)F_I$$

lies in the module generated by (2.1) and (2.2). If $I\cup J\notin\Sigma$, then it is easy to see that both terms lie in the module generated by (2.2). Otherwise, all of the intermediate sets $J\subseteq I_1\subseteq I\cup J$ have the property $\sigma_{I_1}^\circ\subseteq C^\circ$, and the first term is equal to $F_{I\cup J}$ modulo (2.1). The same is true for the second term, which finishes the proof. □

Remark 2.5. We will now observe that A^c is a dualizing module for A. Recall that A is Cohen-Macaulay, a fact that goes back to the works of Reisner, Stanley and Hochster (but see [Br2, 1.3] for an approach closer to the spirit of this paper). Let $\partial\Sigma$ denote the subfan as well as the simplicial subcomplex of Σ consisting of sub-cones/simplices supported on the boundary ∂C. Topologically, the support of the simplex $\partial\Sigma$ is a sphere, so a result of Munkres [Mun, Theorem 2.1] shows that it is Gorenstein over $\mathbb{C}$. According to a theorem of Hochster [BrHe, Theorem 5.7.2], this means that the dualizing module of the Stanley–Reisner ring A is isomorphic as a $\mathbb{Z}^n$-graded A–module to the graded ideal of A generated by $\prod_{i\in I} D_i$ with $I\in\Sigma\setminus\partial\Sigma$. But these are exactly the simplices $I\in\Sigma$ such that $\sigma_I^\circ\subseteq C^\circ$, and the proof of the previous proposition provides the claim.

Consider the grading on A and A^c with $\deg D_i = 1$ and $\deg F_I = |I|$. Global linear functions on N form a dimension $\mathrm{rk}N$ subspace in the degree one component of A. They are given in terms of the generators by

$$\sum_i (m\cdot v_i)D_i,\ m\in N^\vee$$

and form a regular sequence for any basis of $N^\vee$. Denote by Z the ideal they generate. Then A/ZA is a graded Artinian ring, and A^c/ZA^c is its dualizing module. Thus, A^c/ZA^c has one-dimensional socle and the pairing induced by multiplication and evaluation at the socle is nondegenerate. We can explicitly calculate it as follows.

Proposition 2.6. *For each $I\in\Sigma$ with $|I| = \mathrm{rk}N$ we define Vol_I to be the index of the sublattice in N generated by $v_i, i\in I$. There exists a unique linear function $\int : A^c/ZA^c\to\mathbb{C}$ that takes values $\frac{1}{\mathrm{Vol}_I}$ on F_I for all $I\in\Sigma$ with $|I| = \mathrm{rk}N$. This function is nonzero on the socle element of A^c/ZA^c and induces the pairing.*

Proof. The function $\int$ is defined as follows, see [Br1]. For a continuous piecewise polynomial function $f = (f_J)$ consider a rational function on $N_{\mathbb{C}}$ given by

$$\sum_{|J|=\mathrm{rk}N} \frac{1}{|\mathrm{Vol}(J)|} f_J \prod_{i\in J} \frac{1}{u_{i,J}} \tag{2.4}$$

where $u_{i,J}$ is a linear function on N with value 1 at v_i and 0 on other $v_j \in J$ (in other words, the restriction of D_i to the cone J). The rational function (2.4) is in fact polynomial. Indeed, the singularities of individual terms occur along the interior walls of Σ. However, for every such wall the two adjacent terms cancel each other. Thus the resulting function has singularities at (complex) codimension two only and is therefore nonsingular. Then

$$\int f = \left(\sum_{|J|=\mathrm{rk}N} \frac{1}{|\mathrm{Vol}(J)|} f_J \prod_{i\in J} \frac{1}{u_{i,J}} \right)(0)$$

provides a map $\int : A^c \to \mathbb{C}$.

We will now verify the properties of $\int$. The above argument shows that for a global polynomial function g we have

$$\int fg = g(0) \int f.$$

In particular, $\int$ passes through to the map $A^c/ZA^c \to \mathbb{C}$. We can also easily see that $\int$ vanishes on all but the top degree component of A^c/ZA^c. Indeed, for $\lambda \in \mathbb{C}^*$ consider the scaling $\lambda : N_{\mathbb{C}} \to N_{\mathbb{C}}$. We have

$$\int f \circ \lambda = \lambda^{\mathrm{rk}N} \int f$$

because of the scaling of $u_{i,j}$ in (2.4). Thus $\int f = 0$ for f in all but the top eigenspace of $\mathbb{C}^*$ action. It remains to observe that for $|I| = \mathrm{rk}N$, $\int F_I = \frac{1}{|\mathrm{Vol}(I)|}$ because the function in (2.4) is constant. $\square$

3. K-theory

In this section we consider more sophisticated combinatorial objects associated to C, N and Σ. The first one is the Grothendieck group of the corresponding toric DM stack $\mathbb{P}_\Sigma$. The second one is new and will be later shown to be some kind of Grothendieck group with compact support. At the end of the section we will introduce an example, to which we will be returning frequently throughout the rest of the paper.

We start with what is by now a standard definition of the toric stack associated to C, N, v_i and Σ, see [BCS].

Definition 3.1. Consider the open subset U of $\mathbb{C}^n$ given by

$$U := \{(z_1, \ldots, z_n),\ \text{such that } \{i, z_i = 0\} \in \Sigma\}.$$

Define the group G as the subgroup of $(\mathbb{C}^*)^n$ given by

$$G = \{(\lambda_1, \ldots, \lambda_n),\ \text{such that } \prod_{i=1}^{n} \lambda_i^{\langle m, v_i \rangle} = 1 \text{ for all } m \in N^\vee\}.$$

Define $\mathbb{P}_\Sigma$ to be the *stack* quotient of U by G.

Remark 3.2. The notation $\mathbb{P}_\Sigma$ is often used to denote the GIT quotient U/G which is typically a singular toric variety. However, the reader can be assured that we always work on the smooth toric DM stack.

The Grothendieck group of $\mathbb{P}_\Sigma$ (equivalently, the Grothendieck group of G-equivariant coherent sheaves on U) has been calculated in [BH1].

Proposition 3.3. *Let C, v_i and Σ be as before. Consider the smooth toric DM stack $\mathbb{P}_\Sigma$ defined by Σ and v_i. Then the K-theory $K_0(\mathbb{P}_\Sigma)$ (with complex coefficients) is the quotient of the ring $\mathbb{C}[R_i^{\pm 1}]$ by the relations*

$$\prod_{i=1}^{n} R_i^{m \cdot v_i} - 1,\ m \in N^\vee \ \textit{ and } \ \prod_{i \in I} (1 - R_i),\ I \notin \Sigma. \tag{3.1}$$

We will now describe the structure of $K_0(\mathbb{P}_\Sigma)$ in more detail in combinatorial terms. The ring $K_0(\mathbb{P}_\Sigma)$ is semilocal, with the local summands given by so-called twisted sectors, see the definition below.

Definition 3.4. For every cone $\sigma \in \Sigma$ we denote by $\mathrm{Box}(\Sigma)$ the set of points of $\gamma \in N$ that can be written as $\gamma = \sum_{i \in \sigma} \gamma_i v_i$ with $0 \le \gamma_i < 1$. We denote by $\mathrm{Box}(\Sigma)$ the union of $\mathrm{Box}(\sigma)$ for all $\sigma \in \Sigma$.

The elements of $\mathrm{Box}(\Sigma)$ are in one-to-one correspondence with components of inertia stack of $\mathbb{P}_\Sigma$. In the following definition we will introduce the cohomology of these components.

Definition 3.5. To each $\gamma \in \mathrm{Box}(\Sigma)$ we associate a toric stack called *the twisted sector* defined as the closed toric substack associated to the minimum cone in Σ that contains γ. We denote the corresponding rings (ideals) of piecewise polynomial functions (vanishing on the boundary) by A_γ (A_γ^c). We denote the corresponding Artinian rings and modules by H_γ and H_γ^c respectively.

Remark 3.6. More specifically, for $\gamma \in \mathrm{Box}(\Sigma)$ we define $\sigma(\gamma)$ to be the minimum cone of Σ that contains γ. We define $N_\gamma = N/\mathrm{Span}(\sigma(\gamma))$ to be quotient lattice and Σ_γ be the quotient fan [2]. It is the image of the star of $\sigma(\gamma)$ and consists of $I - \sigma(\gamma)$ for all I in Σ with $I \supseteq \sigma(\gamma)$. The map from N to N_γ is denoted by $\bar{}$. We also use $\bar{D}_i$ and $\bar{F}_I$ for the corresponding elements of $A_\gamma, H_\gamma, A^c_\gamma, H^c_\gamma$.

The key to understanding $K_0(\mathbb{P}_\Sigma)$ is provided by the following result.

Proposition 3.7. *[BH] There is a natural algebra isomorphism*

$$ch : K_0(\mathbb{P}_\Sigma) \xrightarrow{\sim} \bigoplus_{\gamma \in \mathrm{Box}(\Sigma)} H_\gamma.$$

Let us recall the construction of this isomorphism ch, since it will be useful later. One can show that the algebra homomorphisms from $K_0(\mathbb{P}_\Sigma)$ to $\mathbb{C}$ correspond to $\gamma = \sum_{i\in\sigma(\gamma)} \gamma_i v_i$ in $\mathrm{Box}(\Sigma)$ by $R_i \mapsto \mathrm{e}^{2\pi \mathrm{i} \gamma_i}$. The cohomology $H_\gamma = A_\gamma/Z_\gamma A_\gamma$ of the corresponding twisted sector is generated by $\bar{D}_i$ for $i \in \mathrm{Star}(\sigma(\gamma)) - \sigma(\gamma)$ with the relations

$$\prod_{i\in J} \bar{D}_i = 0$$

for J not in a cone in $\mathrm{Star}(\sigma(\gamma))$, and

$$\sum_{i\in \mathrm{Star}(\sigma(\gamma))-\sigma(\gamma)} (m \cdot v_i)\bar{D}_i = 0$$

for $m \in \mathrm{Ann}(v_i, i \in \sigma(\gamma))$. The projection $\mathrm{ch}_\gamma : K_0(\mathbb{P}_\Sigma) \to A_\gamma/Z_\gamma A_\gamma$ is given by

$$\begin{aligned}
&\mathrm{ch}_\gamma(R_i) = 1, && i \notin \mathrm{Star}(\sigma(\gamma))\\
&\mathrm{ch}_\gamma(R_i) = \mathrm{e}^{\bar{D}_i}, && i \in \mathrm{Star}(\sigma(\gamma)) - \sigma(\gamma)\\
&\mathrm{ch}_\gamma(R_i) = \mathrm{e}^{2\pi \mathrm{i}\gamma_i} \textstyle\prod_{j\notin\sigma(\gamma)} \mathrm{ch}_\gamma(R_j)^{(m_i\cdot v_j)}, && i \in \sigma(\gamma)
\end{aligned} \tag{3.2}$$

where m_i is any $\mathbb{Q}$-valued linear function on N which takes values -1 on v_i and 0 on all other $v_j, j \in \sigma(\gamma)$. Here, in the last line, the rational powers of unipotent elements are well-defined.

Remark 3.8. The first line in (3.2) can be thought of as a particular case of the second line, under $\bar{D}_i = 0$ for i not in the induced fan.

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[2]More accurately, one may consider the quotient by the span of v_i in $\sigma(\gamma)$. The resulting lattice has torsion and leads to toric DM stacks which are not schemes at generic point, see [BCS]. This leads to various factors $\frac{1}{|\mathrm{Box}(\sigma(\gamma))|}$ in the formulas of the paper.

We now define a module $K_0^c(\mathbb{P}_\Sigma)$ over $K_0(\mathbb{P}_\Sigma)$ which will turn out to be isomorphic to the sum of cohomologies with compact support for all of the twisted sectors.

Definition 3.9. Consider the module $K_0^c(\mathbb{P}_\Sigma)$ over the ring $K_0(\mathbb{P}_\Sigma)$ generated by G_I, $I \in \Sigma, \sigma_I^\circ \subseteq C^\circ$, with relations for all i, I with $i \notin I$

$$(1 - R_i^{-1})G_I = G_{I\cup\{i\}}, \text{ if } I \cup \{i\} \in \Sigma, \tag{3.3}$$

$$(1 - R_i^{-1})G_I = 0, \text{ if } I \cup \{i\} \notin \Sigma. \tag{3.4}$$

Remark 3.10. We will later see that $K_0^c(\mathbb{P}_\Sigma)$ is isomorphic to the Grothendieck group of a certain category of "compactly supported" sheaves on $\mathbb{P}_\Sigma$, which will justify the notation.

Proposition 3.11. *There is a natural isomorphism*

$$ch^c : K_0^c(\mathbb{P}_\Sigma) \tilde{\to} \bigoplus_{\gamma \in \mathrm{Box}(\Sigma)} H_\gamma^c$$

compatible with the ring isomorphism $ch : K_0(\mathbb{P}_\Sigma) \tilde{\to} \bigoplus_{\gamma\in\mathrm{Box}(\Sigma)} H_\gamma$.

Proof. We define $ch^c = \bigoplus_\gamma ch_\gamma^c$ where $ch_\gamma^c : K_0^c(\mathbb{P}_\Sigma) \to H_\gamma^c$ is given by

$$ch_v^c(\prod_{i=1}^n R_i^{l_i} G_I) = 0$$

for $I \not\subseteq \mathrm{Star}(\sigma(v))$ and

$$\begin{aligned} ch_\gamma^c(\textstyle\prod_{i=1}^n R_i^{l_i} G_I) = \ & \textstyle\prod_{i=1}^n ch_\gamma(R_i)^{l_i} \prod_{i\in I, i\notin\sigma(\gamma)} \left(\frac{1-e^{-\bar{D}_i}}{\bar{D}_i}\right) \\ & \cdot \textstyle\prod_{i\in I\cap\sigma(\gamma)} (1 - ch_\gamma(R_i)^{-1}) \bar{F}_{\bar{I}} \end{aligned} \tag{3.5}$$

for $I \subseteq \mathrm{Star}(\sigma(\gamma))$. The cone $\bar{I}$ in the induced fan is defined by the set of indices in I, but not in $\sigma(\gamma)$. The $\bar{F}_{\bar{I}}$ indicates the generator of H_v^c that corresponds to $\bar{I}$ in the induced fan Σ_γ.

We need to prove that the above ch^c satisfies the claim of the proposition. First of all, we need to prove that ch_γ^c is well-defined for any $\gamma \in \mathrm{Box}(\Sigma)$. Because $ch_\gamma : K_0(\mathbb{P}_\Sigma) \to H_\gamma$ is a ring homomorphism, we only need to check that the above prescription agrees on two sides of (3.3) and (3.4). If $I \not\subseteq \mathrm{Star}(\sigma(\gamma))$, then both sides map to 0. If $I \subseteq \mathrm{Star}(\sigma(\gamma))$, but $I \cup \{i\} \not\subseteq \mathrm{Star}(\sigma(\gamma))$, then the right hand side is always sent to 0 by the above prescription. In this case $i \notin \mathrm{Star}(\sigma(\gamma))$, thus $ch_\gamma(R_i)^{-1} = 1$, so the left hand side is trivially sent to 0. We are left to consider the case $I \cup \{i\} \subseteq \mathrm{Star}(\sigma(\gamma))$. If $I \cup \{i\} \notin \Sigma$, then $i \notin \sigma(\gamma)$. The left hand side of (3.4) goes to zero because $(1 - ch_\gamma(R_i)^{-1}) = (1 - \mathrm{e}^{-\bar{D}_i}) \sim \bar{D}_i$ and $\bar{D}_i\bar{F}_{\bar{I}} = 0$ in H_γ^c. If $I \cup \{i\} \in \Sigma$

(and in $\mathrm{Star}(\sigma(\gamma))$), then there are two possibilities. If $i \notin \sigma(v)$, the statement follows from $\bar{D}_i \bar{F}_{\bar{I}} = \bar{F}_{\overline{I \cup \{i\}}}$. If $i \in \sigma(\gamma)$, then $\bar{I} = \overline{I \cup \{i\}}$ and the statement is immediate.

To show that the map ch^c is an isomorphism, observe that every finitely generated module over an Artinian ring is a direct sum of its localizations at maximal ideals. In our case, the maximum ideals correspond to $\gamma \in \mathrm{Box}(\Sigma)$, so it suffices to show that the induced map

$$ch^c_\gamma : (K^c_0(\mathbb{P}_\Sigma))_\gamma \to H^c_\gamma$$

is an isomorphism. The localization with respect to this maximal ideal is characterized by the nilpotency of $R_i - \mathrm{e}^{2\pi \mathrm{i} \gamma_i}$ where $\gamma = \sum_{i \in \sigma(\gamma)} \gamma_i v_i$, and $\gamma_i = 0$ for $i \notin \sigma(\gamma)$. The localization of $K^c_0(\mathbb{P}_\Sigma)$ is generated as a module over the localization of $K_0(\mathbb{P}_\Sigma)$ (that we know to be isomorphic to H_v) by G_I with the relations (3.3) and (3.4). Observe that if $\lambda_i \neq 0$, then $(1 - R_i^{-1})$ is invertible, thus (3.4) implies that $G_I = 0$ for $I \not\ni i$. Consequently, we may restrict our attention to $I \subseteq \mathrm{Star}(\sigma(\gamma))$. Note that we are only using I such that $\sigma^\circ_I \subseteq C^\circ$, and the images modulo the span of $\sigma(\gamma)$ have this property. Moreover, $(1 - R_i^{-1})$ are invertible for $i \in \sigma(\gamma)$, which allows to pass from I to $I - \sigma(\gamma)$. It remains to observe that for $I \subseteq \mathrm{Star}(\sigma(\gamma)) - \sigma(\gamma)$ the relations (3.3) and (3.4) become the defining relations on H^c_γ under ch_γ and $G_I \to \prod_{i \in I} \left(\frac{1 - e^{-\bar{D}_i}}{\bar{D}_i} \right) \bar{F}_{\bar{I}}$. □

Remark 3.12. The choice of ch^c is also compatible with the natural maps $H^c_\gamma \to H_\gamma$ and $K^c_0(\mathbb{P}_\Sigma) \to K_0(\mathbb{P}_\Sigma)$. We will not use this fact in the paper, but it is an important motivation behind the definition of ch^c.

As promised, we now introduce the key example that will be featured prominently throughout the paper.

Example 3.13. Let $C \subset \mathbb{Z}^2$ be the cone generated by $v_1 = (0, 1)$, $v_2 = (1, 1)$ and $v_3 = (3, 1)$. Let Σ be the simplicial complex with the maximum sets $\{1, 2\}$ and $\{2, 3\}$. Let us first construct $\mathbb{P}_\Sigma$. It is the stack quotient $[U/G]$ where

$$U = \{(z_1, z_2, z_3),\ (z_1, z_3) \neq (0, 0)\},\ G = \{(\lambda^2, \lambda^{-3}, \lambda),\ \lambda \in \mathbb{C}^*\}.$$

Let us now consider the Grothendieck groups of $\mathbb{P}_\Sigma$. The group $K_0(\mathbb{P}_\Sigma)$ is written as

$$\begin{aligned} &\mathbb{C}[R_1^{\pm 1}, R_2^{\pm 1}, R_3^{\pm 1}]/\langle (1 - R_1)(1 - R_3), R_1 R_2 R_3 - 1, R_2 R_3^3 - 1 \rangle \\ &= \mathbb{C}[R_3^{\pm 1}]/\langle (1 - R_3)^2 (1 + R_3) \rangle = \mathbb{C} \oplus \mathbb{C} R_3 \oplus \mathbb{C} R_3^2, \end{aligned}$$

and

$$K_0^c(\mathbb{P}_\Sigma) = K_0(\mathbb{P}_\Sigma)G_2 = \mathbb{C}G_2 \oplus \mathbb{C}R_3G_2 \oplus \mathbb{C}R_3^2G_2.$$

There are two twisted sectors, which correspond to $\gamma = (0,0)$ and $\gamma = (2,1) = \frac{1}{2}v_2 + \frac{1}{2}v_3$. The $(0,0)$ sector is sometimes called the untwisted sector. [3] We have

$$H_{(0,0)} = \mathbb{C}[D_1, D_2, D_3]/\langle D_1, D_3, D_2 + 3D_3, D_1 + D_2 + D_3\rangle = \mathbb{C} \oplus \mathbb{C}D_3$$
$$H_{(2,1)} = \mathbb{C}[\emptyset] = \mathbb{C}.$$

The cohomology with compact support $H^c_{(0,0)}$ is generated by F_2 and we have

$$H^c_{(0,0)} = H_{(0,0)}F_2 = \mathbb{C}F_2 \oplus \mathbb{C}D_3F_2,\ H^c_{(2,1)} = \mathbb{C}\bar{F}_\emptyset.$$

The integration maps $\int$ are given by

$$\int_{(0,0)} F_2 = 0,\ \int_{(0,0)} D_3F_2 = \frac{1}{2},\ \int_{(2,1)} \bar{F}_\emptyset = 1.$$

The maps ch and ch^c are given by

(3.6)

$$\begin{aligned}
&ch(1) = 1 \oplus 1,\ ch(R_3) = (1 + D_3) \oplus (-1),\ ch(R_3^2) = (1 + 2D_3) \oplus 1,\\
&ch^c(G_2) = (1 + \tfrac{3}{2}D_3)F_2 \oplus 2\bar{F}_\emptyset,\ ch^c(R_3G_2) = (1 + \tfrac{5}{2}D_3)F_2 \oplus (-2)\bar{F}_\emptyset,\\
&ch^c(R_3^2G_2) = (1 + \tfrac{7}{2}D_3)F_2 \oplus 2\bar{F}_\emptyset.
\end{aligned}$$

4. Euler characteristics

In this section we define the pairing between $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$. It is based on the linear map

$$\chi : K_0^c(\mathbb{P}_\Sigma) \to \mathbb{C}$$

which will later be reinterpreted as the Euler characteristic of some sheaves on $\mathbb{P}_\Sigma$. We then calculate it for the Example 3.13.

For $\alpha \in \mathbb{Z}^n$ and G_I we define $\chi(\prod_i R_i^{\alpha_i} G_I)$ as follows. Let $M = N^\vee$ be the dual lattice. Consider the following element in the field of rational functions $\mathbb{C}(M_\mathbb{Q})$

(4.1)

$$\sum_{\substack{J\supseteq I,\\ |J|=\mathrm{rk}N}} \frac{1}{|\mathrm{Box}(J)|} \sum_{\substack{\gamma\in\mathrm{Box}(J)\\ \gamma=\sum_{j\in J}\gamma_j v_j}} \prod_{i\in J} q^{-u_{i,J}\alpha_i} \mathrm{e}^{-2\pi \mathrm{i}\gamma_i\alpha_i} \prod_{i\in J-I} \frac{1}{1 - q^{u_{i,J}}\mathrm{e}^{2\pi\mathrm{i}\gamma_i}}$$

where $u_{i,J}$ form a dual basis to $v_i, i \in J$. The sum is a polynomial in $\mathbb{C}[M_\mathbb{Q}]$, rather than a rational function in $\mathbb{C}[M_\mathbb{Q}]$. We then evaluate it

[3] By a quirk of terminology, we consider the untwisted sector to be one element of the larger set of twisted sectors.

at $q = 0$ to get the the value of χ. We will later see that this is an integer.

Our first goal is prove the key properties of χ, starting from the fact that it is well-defined.

Proposition 4.1. *The above definition of* $\chi(\prod_i R_i^{\alpha_i} G_I)$ *makes sense.*

Proof. We need to show that the resulting function is a Laurent polynomial with fractional exponents in $\mathbb{C}[M_{\mathbb{Q}}]$. It can be easily seen to be the Euler characteristics of the line bundle $\bigotimes_i \mathcal{L}_i^{\alpha_i}$ on the smooth toric DM stack associated to I. More precisely, it calculates this Euler characteristics as the Euler characteristics of the pushforward of this bundle to $\mathbb{P}_\Sigma$. Recall that $\mathbb{P}_\Sigma$ is the stack quotient $[U/G]$ for the following U and G. The scheme U is an open subscheme of $\mathbb{C}^n$ whose closed points $(z_1, \ldots, z_n)$ have the property that there exists a subset in Σ that contains all of the indices for which $z_i = 0$. The group G is the subgroup of $(\mathbb{C}^*)^n$ that acts diagonally on $\mathbb{C}^n$ described by $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\prod_{i=1}^n \lambda_i^{m \cdot v_i} = 1$ for all $m \in M$. The aforementioned pushforward corresponds to the G-equivariant module on $\mathbb{C}^n$ which is isomorphic to

$$F = \mathbb{C}[z_1, \ldots, z_n]/\langle z_i, i \in I\rangle$$

with the linearization $\lambda^* \prod_j z_j^{r_j} = \prod_j \lambda_j^{r_j - \alpha_j} z_j^{r_j}$.

We consider the Čech cover of U by U_J, $J \in \Sigma$, defined as the subsets with $z_j \neq 0, j \notin J$. We calculate the equivariant cohomology of F by the complex of invariant sections. If $J \supsetneq I$, then the observe that the sections are zero. We apply the delta function trick of [BL] to show that only the contributions of maximum-dimensional J appear. Finally, to calculate the graded dimensions of the contribution of such U_J we need to account for the sublattice, which is where the summation over $\mathrm{Box}(J)$ comes in.

The resulting Euler characteristics is finite, because of projectivity of the corresponding twisted sector, which is assured by the assumption on I. $\square$

Remark 4.2. An alternative proof that χ is well-defined follows from the argument of Proposition 4.5.

Proposition 4.3. *The linear function* χ *descends to a map* $K_0^c(\mathbb{P}_\Sigma) \to \mathbb{C}$.

Proof. We first observe that for any $m \in M$

$$\chi(\prod_i R_i^{l_i} G_I) = \chi(\prod_i R_i^{l_i+m\cdot v_i} G_I)$$

because the expression (4.1) is changed by q^m. The analogous statement for (3.4) can be observed directly from (4.1) or from the Koszul complexes and the geometric description of χ. $\square$

We now define a pairing between $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$.

Definition 4.4. We define the Euler characteristics pairing

$$\chi : K \times K^c \to \mathbb{Z}$$

by

$$\chi(\prod_i R_i^{\beta_i}, \prod_i R_i^{\alpha_i} G_I) = \chi(\prod_i R_i^{\alpha_i - \beta_i} G_I).$$

We now would like to compare the pairing between $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ with the pairings between H_γ and H_γ^c for the twisted sectors. The key to this is to view χ as a linear function on $\bigoplus_{\gamma\in \mathrm{Box}(\Sigma)} H_\gamma^c$, which is essentially a combinatorial statement of a version of the Hirzebruch-Riemann-Roch theorem.

Recall that we have defined maps $\int_\gamma : H_\gamma^c \to \mathbb{C}$.

Proposition 4.5. *For an arbitrary $v \in K_0^c(\mathbb{P}_\Sigma)$ we have*

$$\chi(v) = \sum_\gamma \frac{1}{|\mathrm{Box}(\sigma(\gamma))|} \int_\gamma \prod_{i\in\sigma(\gamma)} \frac{1}{(1-ch(R_i^{-1}))}$$

$$\prod_{\substack{i\in \mathrm{Star}(\sigma(\gamma))\\ i\notin\sigma(\gamma)}} \Big(\frac{D_{\bar i}}{1-ch(R_i^{-1})}\Big) ch_\gamma^c(v).$$

Proof. Consider $v = (\prod_i R_i^{\alpha_i}) G_I$. Then we have from (4.1) that $\chi(v)$ is given by the evaluation at $\mathbf{u} = 0$ of the expression

$$\sum_{\substack{J\supseteq I,\\ |J|=\mathrm{rk}N}} \frac{1}{|\mathrm{Box}(J)|} \sum_{\substack{\gamma\in\mathrm{Box}(J)\\ \gamma=\sum_{j\in J}\gamma_j v_j}} \prod_{i\in J} \mathrm{e}^{\alpha_i u_{i,J} + 2\pi \mathrm{i}\gamma_i\alpha_i} \prod_{i\in J-I} \frac{1}{1-\mathrm{e}^{-u_{i,J}-2\pi\mathrm{i}\gamma_i}}$$

where $u_{i,J}$ is a linear function on $N_\mathbb{C}$ with values 1 at i and 0 at other elements of J. Note that we have changed the sign in $\mathbf{u}$, which will not matter since we are evaluating at 0. We also switched from γ to a dual twisted sector with $\mathrm{e}^{2\pi\mathrm{i}\gamma_i^\vee} = \mathrm{e}^{-2\pi\mathrm{i}\gamma_i}$. Let us rewrite this as a summation over the twisted sectors, i.e. over the elements of $\mathrm{Box}(\Sigma)$.

$$\chi(v) = \sum_{\gamma=\sum_{i\in\sigma(\gamma)}\gamma_i v_i \in \mathrm{Box}(\Sigma)} \chi_\gamma(v),$$

where $\chi_\gamma(v)$ is the evaluation at $\mathbf{u}=0$ of

$$\sum_{J\supseteq\sigma(\gamma)\cup I, |J|=\mathrm{rk}N} \frac{1}{|\mathrm{Box}(J)|}\prod_{i\in J} \mathrm{e}^{\alpha_i u_{i,J}+2\pi\mathrm{i}\gamma_i\alpha_i} \prod_{i\in J-I}\frac{1}{1-\mathrm{e}^{-u_{i,J}-2\pi\mathrm{i}\gamma_i}}.$$

Note that $\chi_\gamma(v)$ is well-defined. Indeed, in the neighborhood of $\mathbf{u}=0$ the terms of the summation have poles of order one along divisors $u_{i,J}=0$ for some $J\supseteq I\cup\sigma(\gamma)$ with $i\notin I\cup\sigma(\gamma)$. Each such occurrence corresponds to a codimension one cone in σ given by $J-\{i\}$. This cone contains I and thus an interior point of C. Thus, each cone appears in two terms, for adjacent J_1 and J_2 with $J_1-\{i_1\}=J_2-\{i_2\}$. The corresponding contributions are the same at a generic point of $u_{i,J}=0$, except for $\frac{1}{|\mathrm{Box}(J_1)|}\frac{1}{1-\mathrm{e}^{-u_{i_1,J_1}}}$ and $\frac{1}{|\mathrm{Box}(J_2)|}\frac{1}{1-\mathrm{e}^{-u_{i_2,J_2}}}$, which give opposite residues on $u_{i,J}=0$ in view of

$$|\mathrm{Box}(J_1)|u_{i_1,J_1} = -|\mathrm{Box}(J_2)|u_{i_2,J_2}.$$

Thus, the function used to define $\chi_\gamma(v)$ has no poles in codimension one and is holomorphic in the neighborhood of $\mathbf{u}=0$.

As before, we see that χ_γ is well-defined on $K_0^c(\mathbb{P}_\Sigma)$. Observe also that χ_γ factors through $ch_\gamma : K_0^c(\mathbb{P}_\Sigma)\to H_\gamma^c$. To prove this, observe that for any $w=f(\log R)G_I$ we have

$$\chi_\gamma(w) = \Big(\sum_{\substack{J\supseteq\sigma(\gamma)\cup I,\\ |J|=\mathrm{rk}N}} \frac{f(2\pi\mathrm{i}\gamma_i+\delta(i\in J)u_{i,J})}{|\mathrm{Box}(J)|}\prod_{i\in J-I}\frac{1}{1-\mathrm{e}^{-u_{i,J}-2\pi\mathrm{i}\gamma_i}}\Big)_{\mathbf{u}=0},$$

so one only needs to know the power series expansion of f near $(2\pi\mathrm{i}\gamma)$, i.e. the information about the γ local component of $K_0^c(\mathbb{P}_\Sigma)$.

Consider the fan Σ_γ in the quotient lattice $N_\gamma = N/(N\cap\mathbb{Q}(\sigma(\gamma)))$ which comes from $J\supseteq\sigma(\gamma)$. Denote the corresponding cones and simplicial sets by $\bar{J}$. In the γ component of $K_0^c(\mathbb{P}_\Sigma)$ we have $R_i=1$ for $i\notin\mathrm{Star}(\sigma(\gamma))$. We can also use the polynomial relations on R_i to solve for R_i in $\sigma(\gamma)$. We can thus consider only $v=(\prod_{i\in\mathrm{Star}(\sigma(\gamma))-\sigma(\gamma)} R_i^{\alpha_i})G_I$. In fact, we can use the invertibility of $(1-R_i^{-1})$ on the γ component of $K_0^c(\mathbb{P}_\Sigma)$ for $i\in\sigma(\gamma)$ to only consider I with $I\cap\sigma(\gamma)=\emptyset$. For each such $i\in\mathrm{Star}(\sigma(\gamma))-\sigma(\gamma)$ we will denote by $\bar{i}$ the corresponding index in the quotient fan and define $\alpha_{\bar{i}} = \alpha_i$. We have

$$|\mathrm{Box}(J)| = |\mathrm{Vol}(\bar{J})|\,|\mathrm{Box}(\sigma(\gamma))|,$$

so using (2.4) and (3.5) we get

$$\chi_\gamma(v) = \int_\gamma f$$

where f is a piecewise linear function on Σ_γ whose component $f_{\bar{J}}$ is given by

$$\frac{1}{|\mathrm{Box}(\sigma(\gamma))|}\prod_{i\in\sigma(\gamma)}\frac{1}{(1-ch_\gamma(R_i^{-1}))}\prod_{\bar{i}\in\bar{J}}\frac{u_{\bar{i},\bar{J}}}{(1-\mathrm{e}^{-u_{\bar{i},\bar{J}}})}\,ch_\gamma^c(\prod_i R_i^{\alpha_i}(G_I))_{\bar{J}}.$$

The statement of proposition then follows. □

Corollary 4.6. *The pairing between $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ given by Definition* 4.4 *is non-degenerate.*

Proof. Observe that

$$\chi(w,v) = \chi(w^\vee v)$$

where $w \to w^\vee$ is the duality automorphism on $K_0(\mathbb{P}_\Sigma)$ which sends $R_i \to R_i^{-1}$. The duality does not affect the nondegeneracy of the pairing. The rest follows from the fact that $(w,v)\mapsto\int_\gamma wv$ is a duality between H_γ and H_γ^c, and the fact that the corrections

$$\frac{1}{|\mathrm{Box}(\sigma(\gamma))|}\int_\gamma\prod_{i\in\sigma(\gamma)}\frac{1}{(1-ch(R_i^{-1}))}\prod_{\substack{i\in\mathrm{Star}(\sigma(\gamma))\\ i\notin\sigma(\gamma)}}\Big(\frac{D_{\bar{i}}}{1-ch(R_i^{-1})}\Big)$$

of Proposition 4.5 are invertible in H_γ. □

Remark 4.7. There are natural integer structures on $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ given by looking at the integer linear combinations of monomials. It is clear from the proof of Proposition 4.1 that the pairing on such monomials is integral. We do not know whether the pairing in question is unimodular.

Example 4.8. We now calculate in detail the case of $C\subset\mathbb{Z}^2$ with $v_1=(0,1)$, $v_2=(1,1)$ and $v_3=(3,1)$ considered in Example 3.13.

We will consider the natural bases of $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ given by $1, R_3, R_3^2$ and $G_2, R_3G_2, R_3^2G_2$. Our goal is to calculate the pairing between these basis elements. The key step is the following calculation of $\chi(R^kG_2)$. Let us denote $q^{(a,b)}=s^at^b$. The Euler characteristic in question is the value at $s=t=1$ of

$$\frac{1}{1-s^{-1}t}+\frac{1}{2}\frac{1}{1-s^{\frac12}t^{-\frac12}}s^{-\frac{k}{2}}t^{\frac{k}{2}}+\frac{1}{2}\frac{1}{1+s^{\frac12}t^{-\frac12}}(-1)^ks^{-\frac{k}{2}}t^{\frac{k}{2}}.$$

We can set $t = 1$ to get

$$\frac{1}{1-s}\Big(- s + \frac{1}{2}s^{-\frac{k}{2}}(1+s^{\frac{1}{2}}) + \frac{1}{2}(-1)^k s^{-\frac{k}{2}}(1-s^{\frac{1}{2}})\Big)$$

$$= \frac{1}{1-s}\Big(- s + \frac{1}{2}(1+(-1)^k)s^{-\frac{k}{2}} + \frac{1}{2}(1-(-1)^k)s^{-\frac{k}{2}+\frac{1}{2}}\Big).$$

This means that

$$\chi(R_3^k G_2) = \begin{cases} \frac{k}{2}+1, & k = 0 \mod 2 \\ \frac{k+1}{2}, & k = 1 \mod 2 \end{cases}$$

and the pairings are given by the following table.

(4.2)

	1	R_3	R_3^2
G_2	1	0	0
R_3G_2	1	1	0
$R_3^2G_2$	2	1	1

Remark 4.9. A motivated reader may want to check that Proposition 4.5 holds in this example. An unmotivated reader can rest assured that we checked it.

5. Derived categories

The space $K_0(\mathbb{P}_\Sigma)$ is the (complexified) Grothendieck group of the triangulated category $D(\mathbb{P}_\Sigma)$ of bounded complexes of coherent sheaves on the stack $\mathbb{P}_\Sigma$. In this section we will describe the triangulated category $D^c(\mathbb{P}_\Sigma)$ whose Grothendieck group is naturally isomorphic to $K_0^c(\mathbb{P}_\Sigma)$.

Note that the coarse moduli space of $\mathbb{P}_\Sigma$ is not compact. However, it has a closed, proper over $\mathrm{Spec}(\mathbb{C})$, subscheme $\pi^{-1}(0)$ which is the union of the torus orbits for cones $\sigma \in \Sigma$ such that the interior of σ lies in the interior of C. Geometrically, this is the zero fiber of the map π from $\mathbb{P}_\Sigma$ to the affine toric variety $\mathrm{Spec}\,\mathbb{C}[C^\vee \cap N^\vee]$ defined by C, which motivates the notation. The fiber $\pi^{-1}(0)$ need not be irreducible or even equidimensional. The following definition is natural.

Definition 5.1. Denote by $D^c(\mathbb{P}_\Sigma)$ the full subcategory of $D(\mathbb{P}_\Sigma)$ which consists of objects whose cohomology sheaves are supported on $\pi^{-1}(0)$.

Theorem 5.2. *The complexified Grothendieck group of $D^c(\mathbb{P}_\Sigma)$ is isomorphic to $K_0^c(\mathbb{P}_\Sigma)$.*

Proof. We will denote the Grothendieck group of $D^c(\mathbb{P}_\Sigma)$ by $G_0(D^c)$. This is a temporary notation, which will be later replaced by $K_0^c(\mathbb{P}_\Sigma)$ in view of this theorem.

There is a map $\mu : K_0^c(\mathbb{P}_\Sigma) \to G_0(D^c)$ which sends $\prod_i R_i^{k_i} G_I$ to the image of the pushforward of the appropriate line bundle on the closed substack of $\mathbb{P}_\Sigma$ which corresponds to I. The relations get sent to zero in view of Koszul complexes. We now want to show that this map μ is an isomorphism.

First, we tackle the surjectivity of μ. Every complex is equal in the Grothendieck group to the sum of its cohomology sheaves. Images of sheaves supported on $\pi^{-1}(0)$ in the Grothendieck group are generated by sheaves supported on irreducible components of $\pi^{-1}(0)$. Moreover, one can filter by powers of the corresponding ideal, to reduce to push-forwards from the said irreducible components. Finally, sheaves on toric stacks are resolved by line bundles, see [BH].

Injectivity requires a bit more work. We can use the pairing between $K_0^c(\mathbb{P}_\Sigma)$ and $K_0(\mathbb{P}_\Sigma)$ to do this. The key to this argument is the following lemma.

Lemma 5.3. *There is a natural pairing between $D(\mathbb{P}, \Sigma)$ and $D^c(\mathbb{P}_\Sigma)$ given by*

$$\langle F, G\rangle = \sum_i (-1)^k \dim_{\mathbb{C}} \mathrm{Hom}(F, G[i]).$$

This descends to the Grothendieck groups and coincides with the pairing defined in Corollary 4.6 on the image of $K_0^c(\mathbb{P}_\Sigma)$ defined above.

Proof. There is a spectral sequence that calculates $\mathrm{Hom}(F, G[i])$ starting from Hom-s between F and the shifts of the direct sum of cohomology sheaves of G. There is then a local-to-global spectral sequence that calculates these spaces in terms of cohomology of the local Ext sheaves, which are finite-dimensional because $\pi^{-1}(0)$ is proper. Thus, the pairing is well-defined, and the passage to Grothendieck groups is then trivial.

To verify that this pairing reduces to the pairing of Corollary 4.6, it is enough to consider the pairing between $\prod_i R_i^{\beta_i}$ and $\prod_i R_i^{\alpha_i} G_I$. It then follows from the proof of Proposition 4.1. □

We now complete the proof of Theorem 5.2 by proving the injectivity of μ. If an element of $K_0^c(\mathbb{P}_\Sigma)$ is mapped to zero, then its pairing with any element of $K_0(\mathbb{P}_\Sigma)$ is zero. However, the pairing between $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ is perfect by Corollary 4.6, which finishes the proof. □

6. Gamma series with values in $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$

In this section we describe the Gamma series solution to an appropriate version of GKZ hypergeometric system. Then we calculate these solutions explicitly for the Example 3.13 in terms of Gamma function. We also calculate the leading terms of some of these series, which will be useful in Section 8.

Our combinatorial setup is as before. However, we now require that v_i lie in a hyperplane in N. We also assume that the fan Σ is projective, in the sense that it corresponds to a chamber of the secondary fan. We recall the definitions of the better behaved hypergeometric system of equations from [BH1].

Definition 6.1. Consider the system of partial differential equations on the infinite collection of functions $\Phi_c(x_1,\ldots,x_n)$ of n variables. The functions are indexed by lattice elements $c\in C$. The differential equations are the following. For all $m\in M$, $c\in C$, $i\in\{1,\ldots,n\}$

$$\partial_i\Phi_c=\Phi_{c+v_i},\qquad \sum_{i=1}^n\langle m,v_i\rangle x_i\partial_i\Phi_c+\langle m,c\rangle\Phi_c=0. \tag{6.1}$$

We call this system $bbGKZ(C,\beta=0)$ or $bbGKZ(C,0)$ for short. It was observed in [BH1] that this system reduces to a system of holonomic PDEs of a finite collection of functions. Similarly, we define $bbGKZ(C^\circ,0)$ by considering $c\in C^\circ$ only.

Remark 6.2. There is a restriction natural map from the space of solutions of $bbGKZ(C,0)$ to that of $bbGKZ(C^\circ,0)$. However, this map is never an isomorphism, because it sends the trivial solution $\{\Phi_c=\delta_c^0\}$ to zero. The dimension of the image has been shown to be related to Erkhardt and Stanley polynomials of C, see [Bo2].

A modification of the original Gamma series of [GKZ, S, BH2] yields the following solution of $bbGKZ(C,0)$ with values in K_0^c. We assume that we are given a projective [4] simplicial subdivision Σ of C. We also fix a choice of a branch of $\log(x_i)$.

Definition 6.3. Consider for each c in C and each twisted sector $\gamma=\sum_i\gamma_iv_i$ the set $L_{c,\gamma}$ of $(l_i)\in\mathbb{Q}^n$ such that $\sum_i l_iv_i=-c$ and $l_i-\gamma_i\in\mathbb{Z}$. Define a collection of functions of $(x_1,\ldots,x_n)$ indexed by $c\in C$ with

[4] Projectivity is necessary to assure uniform convergence in an open set of parameters.

values in $\oplus_\gamma H_\gamma$

$$(\Gamma(x_1,\ldots,x_n))_c = \bigoplus_\gamma \sum_{(l_i)\in L_{c,\gamma}} \prod_{i=1}^n \frac{x_i^{l_i+\frac{D_i}{2\pi \mathrm{i}}}}{\Gamma(1+l_i+\frac{D_i}{2\pi \mathrm{i}})}$$

where x^a is defined by $\mathrm{e}^{a\log x}$ after picking a branch of $\log x$. Here $D_i = \log ch_\gamma(R_i \mathrm{e}^{-2\pi \mathrm{i}\gamma_i})$, in particular it is nilpotent. Thus $x_i^{\frac{D_i}{2\pi \mathrm{i}}}$ is well defined once the branch of $\log x$ is fixed.

Remark 6.4. We are using the natural isomorphism $ch : K_0(\mathbb{P}_\Sigma) \tilde{\to} \oplus_\gamma H_\gamma$. It allows us to view the above Gamma series as taking values in $K_0(\mathbb{P}_\Sigma)$.

Proposition 6.5. *The Gamma series of Definition* 6.3 *defines a solution of bbGKZ$(C,0)$ with values in $K_0(\mathbb{P}_\Sigma)$. By composing with linear functions on $K_0(\mathbb{P}_\Sigma)$ one gets all solutions of bbGKZ$(C,0)$.*

Proof. Convergence is routine and is proved along the lines of [BH2] where the case of $c=0$ is treated in detail. We leave this to the reader. The fact that this is a solution follows from $L_{c+v_i,\gamma} = L_{c,\gamma} - v_i$ and the defining property of the Gamma function. The second statement is a special case of [H, Theorem 3]. □

We now define an analogous Gamma series with values in $K_c^0(\mathbb{P}_\Sigma)$. Again, we fix the branches of $\log x_i$.

Definition 6.6. For each $c \in C^\circ$, each twisted sector γ and each element of $L_{c,\gamma}$ consider the set σ of i with $l_i \in \mathbb{Z}_{<0}$. If $\sigma \cup \sigma(\gamma)$ is not in a cone of Σ, then we set the notation F_σ to zero. We define

$$(\Gamma^\circ(x_1,\ldots,x_n))_c = \bigoplus_\gamma \sum_{(l_i)\in L_{c,\gamma}} \prod_{i=1}^n \frac{x_i^{l_i+\frac{D_i}{2\pi \mathrm{i}}}}{\Gamma(1+l_i+\frac{D_i}{2\pi \mathrm{i}})} (\prod_{i\in\sigma} D_i^{-1}) F_\sigma.$$

with $D_i = \log ch_\gamma(R_i \mathrm{e}^{-2\pi \mathrm{i}\gamma_i})$.

Proposition 6.7. *The collection of series $(\Gamma^\circ)_c$ for $c \in C^\circ$ defines a solution of bbGKZ$(C^\circ,0)$ with values in $K_0^c(\mathbb{P}_\Sigma)$. If one composes this with linearly independent linear functions on $K_0^c(\mathbb{P}_\Sigma)$, one gets linearly independent solutions.*

Proof. We have $\sum_i l_i v_i = -c$, so σ consisting of i with $l_i \in \mathbb{Z}_{<0}$ can not be a boundary simplex of Σ_γ. Indeed, since $c \in C^\circ$, the simplex σ is non–empty. If C_γ denotes the support of the fan Σ_γ and $h : C_\gamma \to \mathbb{R}_{\geq 0}$ is supporting function vanishing on σ, then

$$0 > h(-c) = \sum_i l_i h(v_i) = \sum_{i\notin\sigma(\gamma)} l_i h(v_i) = \sum_{i\notin\sigma\cup\sigma(\gamma)} l_i h(v_i) \geq 0.$$

Thus $\Gamma^\circ(x)_c$ is well-defined as an element of $\oplus_\gamma H^c_\gamma$, which is naturally isomorphic to $K_0^c(\mathbb{P}_\Sigma)$ by Prop. 3.11.

Convergence is again straightforward. It is also clear that this gives a solution of $bbGKZ(C^\circ, 0)$. For the linear independence statement, it is clearly enough to prove the statement for a fixed twisted sector. The statement is then a direct consequence of [H, Lemma 1 ii) and Prop. 4] applied to the case $\beta = 0$. For, it is enough to note that, for fixed λ and c, a term in the above summation defining the Gamma series Γ° is non-zero if and only if the corresponding term in the Gamma series summation for $\beta = 0$ in loc. cit. is non–zero. In fact, the terms coincide up to a rescaling of the classes D_i by the $2\pi i$ factor. The proof of [H, Prop. 4] applies then without any changes. □

Corollary 6.8. *If $bbGKZ(C, 0)$ and $bbGKZ(C^\circ, 0)$ denote the spaces of solutions to the corresponding systems, then the Gamma series functions $\Gamma : K_0(\mathbb{P}_\Sigma)^\vee \to bbGKZ(C, 0)$ and $\Gamma^\circ : K_0^c(\mathbb{P}_\Sigma)^\vee \to bbGKZ(C^\circ, 0)$ are isomorphisms of linear spaces.*

Proof. The first part simply restates the result of Proposition 6.5. For the second part, note that for each γ, H^c_γ is the linear dual to H_γ, by the results of Section 2. The isomorphisms of Propositions 3.7 and 3.11 imply that the complex vector spaces $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ have the same dimension, and the result follows from the previous proposition. □

Example 6.9. We will now calculate the Gamma series in the case of $C \subset \mathbb{Z}^2$ with $v_1 = (0, 1)$, $v_2 = (1, 1)$ and $v_3 = (3, 1)$ that we considered in Example 3.13. We first calculate $\Gamma(x_1, x_2, x_3)$ with values in $\oplus_\gamma H_\gamma = K_0(\mathbb{P}_\Sigma)$. The solution is uniquely determined by $(\Gamma(x_1, x_2, x_3))_{(0,0)}$ and $(\Gamma(x_1, x_2, x_3))_{(2,1)}$, because all other components of the solution are partial derivatives of one or both of these two functions.

We have $\gamma_1 = (0, 0)$ and $\gamma_2 = (2, 1)$. The sets $L_{c,\gamma}$ are summarized in the following table in terms of (l_1, l_2, l_3). We also include in the table the case of $c = (1, 1)$ which is used in the later calculation of the Gamma series solution of $bbGKZ(C^\circ, 0)$.

c, γ	$L_{c,\gamma}$
$(0,0),(0,0)$	$\mathbb{Z}(2,-3,1)$
$(0,0),(2,1)$	$(-1,\frac{3}{2},-\frac{1}{2})+\mathbb{Z}(2,-3,1)$
$(2,1),(0,0)$	$(-1,1,-1)+\mathbb{Z}(2,-3,1)$
$(2,1),(2,1)$	$(0,-\frac{1}{2},-\frac{1}{2})+\mathbb{Z}(2,-3,1)$
$(1,1),(0,0)$	$(0,-1,0)+\mathbb{Z}(2,3,-1)$
$(1,1),(2,1)$	$(-1,\frac{1}{2},-\frac{1}{2})+\mathbb{Z}(2,3,-1)$

We write the corresponding summations. To shorten the notation, we use

$$x = x_1^2 x_2^{-3} x_3.$$

For the twisted sector $\gamma = (2,1)$ the D_i are 0, therefore

$$\begin{aligned}
\Gamma(x_1,x_2,x_3)_{(0,0)} &= \sum_k \frac{x^k x_1^{\frac{D_1}{2\pi i}} x_2^{\frac{D_2}{2\pi i}} x_3^{\frac{D_3}{2\pi i}}}{\Gamma(1+2k+\frac{D_1}{2\pi i})\Gamma(1-3k+\frac{D_2}{2\pi i})\Gamma(1+k+\frac{D_3}{2\pi i})} \\
&\oplus (x_1^{-1} x_2^{\frac{3}{2}} x_3^{-\frac{1}{2}}) \sum_k \frac{x^k}{\Gamma(2k)\Gamma(\frac{5}{2}-3k)\Gamma(\frac{1}{2}+k)} \\
&= \sum_{k\geq 0} \frac{x^k(1+\frac{D_3}{2\pi i}\log x)}{\Gamma(1+2k+\frac{2D_3}{2\pi i})\Gamma(1-3k-\frac{3D_3}{2\pi i})\Gamma(1+k+\frac{D_3}{2\pi i})} \\
&\oplus (x_1^{-1} x_2^{\frac{3}{2}} x_3^{-\frac{1}{2}}) \sum_{k\geq 1} \frac{x^k}{(2k-1)!\Gamma(\frac{5}{2}-3k)\Gamma(\frac{1}{2}+k)} \\
&= 1 + \frac{D_3}{2\pi i}\log x + \frac{3D_3}{2\pi i} \sum_{k\geq 1} \frac{x^k(3k-1)!(-1)^k}{(2k)!k!} \\
&\oplus (x_1^{-1} x_2^{\frac{3}{2}} x_3^{-\frac{1}{2}}) \sum_{k\geq 1} \frac{x^k}{(2k-1)!\Gamma(\frac{5}{2}-3k)\Gamma(\frac{1}{2}+k)} \\
&= 1 + \frac{D_3}{2\pi i}(\log x - 3x + \frac{15}{2}x^2 + \ldots) \\
&\oplus (x_1^{-1} x_2^{\frac{3}{2}} x_3^{-\frac{1}{2}}) \frac{1}{\pi}(-x + \frac{35}{24}x^2 - \frac{3003}{640}x^3 + \ldots).
\end{aligned}$$

A similar calculation which we omit shows that

$$\begin{aligned}
\Gamma(x_1,x_2,x_3)_{(2,1)} &= \frac{D_3}{2\pi i}(x_1^{-1}x_2x_3^{-1})(3x - 12x^2 + 63x^3 + \ldots) \\
&\oplus (x_2^{-\frac{1}{2}} x_3^{-\frac{1}{2}}) \frac{1}{\pi}(1 - \frac{15}{8}x + \frac{1155}{128}x^2 + \ldots).
\end{aligned}$$

We will now write down the Gamma series solution Γ° of $bbGKZ(C^\circ, 0)$ with values in $K_0^c(\mathbb{P}_\Sigma)$. It is determined uniquely by $\Gamma^\circ(x_1,x_2,x_3)_{(1,1)}$ and $\Gamma^\circ(x_1,x_2,x_3)_{(2,1)}$. This calculation is more delicate, as we need to use the values of the derivative of the Gamma function. We use the standard results about the special values of the logarithmic derivative ψ of the Gamma function to find the following.

$$\begin{aligned}
\Gamma^\circ_{(1,1)} &= x_2^{-1} \sum_k \frac{x^k(1+\frac{D_3}{2\pi i}\log x)}{\Gamma(1+2k+\frac{D_1}{2\pi i})\Gamma(-3k+\frac{D_2}{2\pi i})\Gamma(1+k+\frac{D_3}{2\pi i})} D_2^{-1} F_2 \\
&\oplus x_1^{-1} x_2^{\frac{1}{2}} x_3^{-\frac{1}{2}} \sum_k \frac{x^k}{\Gamma(2k)\Gamma(\frac{3}{2}-3k)\Gamma(\frac{1}{2}+k)} \bar{F}_\emptyset \\
&= x_2^{-1} \sum_{k\geq 0} \frac{x^k(1+\frac{D_3}{2\pi i}\log x)\Gamma(3k+1+3\frac{D_3}{2\pi i})(-1)^k}{\Gamma(1+2k+\frac{2D_3}{2\pi i})\Gamma(1+k+\frac{D_3}{2\pi i})} \frac{1}{2\pi i} F_2 \\
&\oplus x_1^{-1} x_2^{\frac{1}{2}} x_3^{-\frac{1}{2}} \sum_{k\geq 1} \frac{x^k}{\Gamma(2k)\Gamma(\frac{3}{2}-3k)\Gamma(\frac{1}{2}+k)} \bar{F}_\emptyset \\
&= x_2^{-1} \sum_{k\geq 0} \frac{x^k(-1)^k(3k)!}{(2k)!k!} \frac{1}{2\pi i} F_2 + x_2^{-1} \sum_{k\geq 0} \frac{x^k(-1)^k(3k)!}{(2k)!k!} \\
&\cdot (\log x + 3\psi(3k+1) - 2\psi(2k+1) - \psi(k+1)) \frac{D_3}{2\pi i} F_2 \\
&\oplus x_1^{-1} x_2^{\frac{1}{2}} x_3^{-\frac{1}{2}} \sum_{k\geq 1} \frac{x^k}{\Gamma(2k)\Gamma(\frac{3}{2}-3k)\Gamma(\frac{1}{2}+k)} \bar{F}_\emptyset \\
&= x_2^{-1}(1 - 3x + 15x^2 + \ldots) \frac{1}{2\pi i} F_2 \\
&+ x_2^{-1}(\log x(1 - 3x + 15x^2 + \ldots) - \frac{9}{2}x + \frac{101}{4}x^2 + \ldots) \frac{D_3}{2\pi i} \frac{1}{2\pi i} F_2 \\
&\oplus x_1^{-1} x_2^{\frac{1}{2}} x_3^{-\frac{1}{2}} (\frac{3}{2}x - \frac{105}{16}x^2 + \frac{9009}{256}x^3 + \ldots) \frac{1}{\pi} \bar{F}_\emptyset
\end{aligned}$$

Similarly,

$$\Gamma^\circ_{(2,1)} = x_1^{-1}x_2x_3^{-1}\sum_{k\geq 1}\frac{x^k(1+\frac{D_3}{2\pi\mathrm{i}}\log x)}{\Gamma(2k+\frac{D_1}{2\pi\mathrm{i}})\Gamma(2-3k+\frac{D_2}{2\pi\mathrm{i}})\Gamma(k+\frac{D_3}{2\pi\mathrm{i}})}D_2^{-1}F_2$$
$$\oplus x_2^{-\frac12}x_3^{-\frac12}\sum_{k\geq 0}\frac{x^k}{\Gamma(1+2k)\Gamma(\frac12-3k)\Gamma(\frac12+k)}\bar F_\emptyset$$
$$= x_1^{-1}x_2x_3^{-1}\sum_{k\geq 1}\frac{x^k(1+\frac{D_3}{2\pi\mathrm{i}}\log x)\Gamma(3k-1+\frac{3D_3}{2\pi\mathrm{i}})(-1)^k}{\Gamma(2k+\frac{D_1}{2\pi\mathrm{i}})\Gamma(k+\frac{D_3}{2\pi\mathrm{i}})}\frac{1}{2\pi\mathrm{i}}F_2$$
$$\oplus x_2^{-\frac12}x_3^{-\frac12}\sum_{k\geq 0}\frac{x^k}{\Gamma(1+2k)\Gamma(\frac12-3k)\Gamma(\frac12+k)}\bar F_\emptyset$$
$$= x_1^{-1}x_2x_3^{-1}\sum_{k\geq 1}\frac{x^k\Gamma(3k-1)(-1)^k}{\Gamma(2k)\Gamma(k)}\frac{1}{2\pi\mathrm{i}}F_2$$
$$+x_1^{-1}x_2x_3^{-1}\sum_{k\geq 1}\frac{x^k\Gamma(3k-1)(-1)^k}{\Gamma(2k)\Gamma(k)}$$
$$\cdot(\log x+3\psi(3k-1)-2\psi(2k)-\psi(k))\frac{D_3}{2\pi\mathrm{i}}\frac{1}{2\pi\mathrm{i}}F_2$$
$$\oplus\frac{1}{\pi}x_2^{-\frac12}x_3^{-\frac12}(1-\tfrac{15}{8}x+\tfrac{1155}{128}x^2+\ldots)\bar F_\emptyset$$
$$= x_1^{-1}x_2x_3^{-1}(-x+4x^2-21x^3+\ldots)\frac{1}{2\pi\mathrm{i}}F_2$$
$$+x_1^{-1}x_2x_3^{-1}(\log x(-x+4x^2-21x^3+\ldots)-x+\tfrac{19}{3}x^2-\ldots)\frac{D_3}{2\pi\mathrm{i}}\frac{1}{2\pi\mathrm{i}}F_2$$
$$\oplus x_2^{-\frac12}x_3^{-\frac12}(1-\tfrac{15}{8}x+\tfrac{1155}{128}x^2+\ldots)\frac{1}{\pi}\bar F_\emptyset.$$

Remark 6.10. The solutions to $bbGKZ(C,0)$ can be written in terms of elementary functions. We will spare the reader the lengthy calculation that we performed using various symbolic manipulation software packages. Our method was to rewrite the PDE as third order ordinary differential equations with respect to x and then solve them symbolically using standard software. The solutions are then matched to the Gamma series using the asymptotic expansions at $x=0$. The final answers are that $\Gamma_{(0,0)}$ can be written as the function

$$1+\frac{D_3}{2\pi\mathrm{i}}\Big(3\log\Big((\sqrt{x}+\sqrt{4/27+x})^{\frac13}-(-\sqrt{x}+\sqrt{4/27+x})^{\frac13}\Big)$$
$$-\frac12\log(4x)\Big)\oplus(-\tfrac{2}{\pi})\arctan\Big(\frac{-2^{\frac23}+3(\sqrt{x}+\sqrt{4/27+x})^{\frac23}}{\sqrt{3}(2^{\frac23}+3(\sqrt{x}+\sqrt{4/27+x})^{\frac23})}\Big),$$

and $\Gamma_{(2,1)}$ as

$$\frac{D_3}{2\pi\mathrm{i}}x_1^{-1}x_2x_3^{-1}\frac{9\cdot 2^{\frac43}x}{2^{4/3}+(-\sqrt{27x}+\sqrt{4+27x})^{\frac43}+(\sqrt{27x}+\sqrt{4+27x})^{\frac43}}$$
$$\oplus\frac{x_2^{-\frac12}x_3^{-\frac12}}{2^{\frac23}\pi}\Big(\frac{(-\sqrt{27x}+\sqrt{4+27x})^{\frac23}+(\sqrt{27x}+\sqrt{4+27x})^{\frac23})}{\sqrt{4+27x}}\Big).$$

We were unable to find an analogous description of Γ°. It appears that the coefficient of D_3F_2 is not an elementary function.

7. The conjectures

We are now ready to formulate several conjectures regarding the Gamma series constructions of this paper. In the subsequent sections we will explain the evidence that supports them.

We expect the Gamma series solutions of Definitions 6.3 and 6.6 to be compatible with pullback-pushforward and analytic continuation. Specifically, let Σ_1 and Σ_2 be two adjacent triangulations, so that the stacks $\mathbb{P}_{\Sigma_1}$ and $\mathbb{P}_{\Sigma_2}$ differ by a flop that's a composition of weighted blowup and weighted blowdown. There is a natural pullback-pushforward functor from the derived category of coherent sheaves on $\mathbb{P}_{\Sigma_1}$ to that of $\mathbb{P}_{\Sigma_2}$. This functor preserves the property of support of cohomology supported on a compact toric substack. Indeed, the maps are compatible with the restriction to the preimage of the complement of the origin. As a consequence of [Ka1, Theorem 4.2], the pullback-pushforward functors $D(\mathbb{P}_{\Sigma_1}) \to D(\mathbb{P}_{\Sigma_2})$ and $D^c(\mathbb{P}_{\Sigma_1}) \to D^c(\mathbb{P}_{\Sigma_2})$ are equivalences. They induce natural group isomorphisms $pp : K_0(\mathbb{P}_{\Sigma_1}) \to K_0(\mathbb{P}_{\Sigma_2})$ and $pp : K_0^c(\mathbb{P}_{\Sigma_1}) \to K_0^c(\mathbb{P}_{\Sigma_2})$.

Conjecture 7.1. The following diagrams of isomorphisms are commutative

$$\begin{array}{ccc} K_0(\mathbb{P}_{\Sigma_2})^\vee & \xrightarrow{pp^\vee} & K_0(\mathbb{P}_{\Sigma_1})^\vee \\ \Gamma\downarrow & & \downarrow\Gamma \\ bbGKZ(C,0) & \xrightarrow{a.c.} & bbGKZ(C,0) \end{array}$$

$$\begin{array}{ccc} K_0^c(\mathbb{P}_{\Sigma_2})^\vee & \xrightarrow{pp^\vee} & K_0^c(\mathbb{P}_{\Sigma_1})^\vee \\ \Gamma^\circ\downarrow & & \downarrow\Gamma^\circ \\ bbGKZ(C^\circ,0) & \xrightarrow{a.c.} & bbGKZ(C^\circ,0) \end{array}$$

where the top rows are the duals of the maps induced by the pullback-pushforward derived functors, and the bottom rows are analytic continuations along a certain path in the domain of parameters considered in [BH2].

Remark 7.2. The motivation behind this conjecture comes from Homological Mirror Symmetry. A particular case of this phenomenon was first observed in the second author's thesis. It has been extended to the case of stacks in [BH2]. However, in that paper we were using the original GKZ system, rather than a better-behaved one, so the vertical maps were not isomorphisms in general. The advantage of the better-behaved version is that all of the maps in the above conjecture are isomorphisms. It appears plausible that methods similar to those of [BH2] will suffice in our case. However, the details are far from settled at this time.

It seems plausible from the approach of [MMW, W] that the systems of PDEs $bbGKZ(C,0)$ and $bbGKZ(C^\circ,0)$ are dual to each other. We formulate here a more specific conjectural mechanism of such duality.

Conjecture 7.3. There exists a collection $p_{c,d}(x_1,\ldots,x_n)$ of polynomials indexed by $c\in C$, $d\in C^\circ$, such that the following hold.

- All but a finite number of $p_{c,d}$ are zero.
- For any pair of solutions (Φ_c) and (Ψ_d) of $bbGKZ(C,0)$ and $bbGKZ(C^\circ,0)$ respectively the sum
$$\sum_{c,d} p_{c,d}\Phi_c\Psi_d \tag{7.1}$$
is constant as a function of $(x_1,\ldots,x_n)$.
- The pairing provided by (7.1) is non-degenerate.
- For any projective simplicial subdivision Σ, the pairing provided by (7.1) is the inverse of the Euler characteristics pairing between $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ under the Γ and Γ° series.

Remark 7.4. The polynomials $p_{c,d}$ are not unique in view of the relations between the components of Φ and Ψ. However, one may hope for a somewhat special choice of p.

Remark 7.5. Conjectures 7.1 and 7.3 can likely be categorified to produce two families of triangulated categories over the parameter space of x such that in the neighborhoods of toric degeneracy points the Gamma series provide some kind of character map to the corresponding K-theory. However, such categories have not yet been constructed.

8. Example: The pairing

Consider the spaces of solutions $(\Phi)_c$ and $(\Psi)_d$ for $c\in C$, $d\in C^\circ$ of $bbGKZ(C,0)$ and $bbGKZ(C^\circ,0)$ respectively for the cone spanned by $v_1=(0,1)$, $v_2=(1,1)$, $v_3=(3,1)$. Recall that these are collections of functions of x_1,x_2,x_3.

Proposition 8.1. *For each Φ and Ψ define*
$$\langle\Phi,\Psi\rangle = -2x_2x_3\Phi_{(2,1)}\Psi_{(2,1)} + \sum_{\substack{c\in C,d\in C^\circ\\ c+d=v_1+v_2}} x_1x_2\Phi_c\Psi_d(-1)^{\deg c}$$
$$+\sum_{\substack{c\in C,d\in C^\circ\\ c+d=v_1+v_3}} 9x_1x_3\Phi_c\Psi_d(-1)^{\deg c} + \sum_{\substack{c\in C,d\in C^\circ\\ c+d=v_2+v_3}} 4x_2x_3\Phi_c\Psi_d(-1)^{\deg c}$$
where $\deg(a,b)=b$. *Then this function of (x_1,x_2,x_3) is constant. Moreover, the resulting pairing of the solution spaces is perfect.*

Proof. The linear relations on Φ_c and Ψ_d imply that $\langle\Phi,\Psi\rangle$ depends on $x_1^2x_2^{-3}x_3$ only. So to prove that it is constant we only need to verify

$\frac{\partial}{\partial x_2}\langle\Phi,\Psi\rangle = 0$. This is a consequence of the relations on Φ and Ψ as follows. We have

$$\begin{aligned}\langle\Phi,\Psi\rangle = \; & x_1x_2(\Phi_{(0,0)}\Psi_{(1,2)} - \Phi_{(0,1)}\Psi_{(1,1)})\\ + \; & 9x_1x_3(\Phi_{(0,0)}\Psi_{(3,2)} - \Phi_{(1,1)}\Psi_{(2,1)} - \Phi_{(2,1)}\Psi_{(1,1)})\\ + \; & x_2x_3(4\Phi_{(0,0)}\Psi_{(4,2)} - 6\Phi_{(2,1)}\Psi_{(2,1)} - 4\Phi_{(3,1)}\Psi_{(1,1)}),\end{aligned}$$

so

$$\begin{aligned}\frac{\partial}{\partial x_2} \; & \langle\Phi,\Psi\rangle = x_1(\Phi_{(0,0)}\Psi_{(1,2)} - \Phi_{(0,1)}\Psi_{(1,1)})\\ + \; & x_3(4\Phi_{(0,0)}\Psi_{(4,2)} - 6\Phi_{(2,1)}\Psi_{(2,1)} - 4\Phi_{(3,1)}\Psi_{(1,1)})\\ + \; & x_1x_2(\Phi_{(1,1)}\Psi_{(1,2)} + \Phi_{(0,0)}\Psi_{(2,3)} - \Phi_{(1,2)}\Psi_{(1,1)} - \Phi_{(0,1)}\Psi_{(2,2)})\\ + \; & 9x_1x_3(\Phi_{(0,0)}\Psi_{(4,3)} - \Phi_{(2,2)}\Psi_{(2,1)} - \Phi_{(3,2)}\Psi_{(1,1)} - \Phi_{(2,1)}\Psi_{(2,2)})\\ + \; & x_2x_3(4\Phi_{(1,1)}\Psi_{(4,2)} + 4\Phi_{(0,0)}\Psi_{(5,3)} - 6\Phi_{(3,2)}\Psi_{(2,1)}\\ - \; & 6\Phi_{(2,1)}\Psi_{(3,2)} - 4\Phi_{(4,2)}\Psi_{(1,1)} - 4\Phi_{(3,1)}\Psi_{(2,2)}).\end{aligned}$$

We use equations

$$x_2\Phi_{(a+1,b+1)} + 3x_3\Phi_{(a+3,b+1)} + a\Phi_{(a,b)} = 0$$

and similarly for Ψ to get rid of x_1x_2 terms above. We use equations

$$3x_1\Phi_{(a,b+1)} + 2x_2\Phi_{(a+1,b+1)} + (3b-a)\Phi_{(a,b)} = 0$$

and similarly for Ψ to get rid of x_2x_3 terms. After cancellations, we get

$$\frac{\partial}{\partial x_2}\langle\Phi,\Psi\rangle = 3x_1x_3(\Phi_{(3,1)}\Psi_{(1,2)} - \Phi_{(0,1)}\Psi_{(4,2)})$$

which equals zero in view of relations

$$x_1\Phi_{(0,1)} - 2x_3\Phi_{(3,1)} = x_1\Psi_{(1,2)} - 2x_3\Psi_{(4,2)} = 0.$$

We have thus proved that $\langle\Phi,\Psi\rangle$ is constant.

To verify that the pairing is perfect, we will analyze its results on the Gamma series solutions from Section 6. It is enough to calculate the leading terms of the expressions. We use $e_{(0,0)}$ and $e_{(2,1)}$ as notation to designate the identity parts in different twisted sectors for the Γ series. This calculation shows

$$\langle\Gamma,\Gamma^\circ\rangle = -\frac{3}{2\pi^2}(e_{(0,0)}\otimes D_3F_2 + 4e_{(2,1)}\otimes\bar{F}_\emptyset - D_3e_{(0,0)}\otimes F_2) \tag{8.1}$$

in the tensor product $(\oplus_\gamma H_\gamma)\otimes(\oplus_\gamma H^c_\gamma)$. □

We now want to verify Conjecture 7.3 in our case. Recall that $ch = ch_{(0,0)}\oplus ch_{(2,1)}$ (resp. $ch^c = ch^c_{(0,0)}\oplus ch^c_{(2,1)}$) identifies $K_0(\mathbb{P}_\Sigma)$ (resp.

$K_0^c(\mathbb{P}_\Sigma)$) with $H_{(0,0)} \oplus H_{(2,1)}$ (resp. $H^c_{(0,0)} \oplus H^c_{(2,1)}$). They have been calculated in (3.6). We use their inverses to rewrite (8.1) as

$$\begin{aligned}
\langle \Gamma, \Gamma^\circ \rangle = \; & -\tfrac{3}{2\pi^2}(\tfrac{1}{4}(3+2R_3-R_3^2)\otimes \tfrac{1}{2}(R_3^2G_2-G_2) \\
& +4\tfrac{1}{4}(1-2R_3+R_3^2)\otimes \tfrac{1}{8}(G_2-2R_3G_2+R_3^2G_2) \\
& -\tfrac{1}{2}(R_3^2-1)\otimes \tfrac{1}{2}(3G_2+R_3G_2-2R_3^2G_2)) \\
= \; & -\tfrac{3}{16\pi^2}((3+2R_3-R_3^2)\otimes (R_3^2G_2-G_2) \\
& +(1-2R_3+R_3^2)\otimes (G_2-2R_3G_2+R_3^2G_2) \\
& -2(R_3^2-1)\otimes (3G_2+R_3G_2-2R_3^2G_2)) \\
= \; & -\tfrac{3}{4\pi^2}(1\otimes G_2 - R_3\otimes G_2 - R_3^2\otimes R_3G_2 + R_3\otimes R_3G_2 \\
& -R_3^2\otimes G_2 + R_3^2\otimes R_3^2G_2)
\end{aligned}$$

It remains to observe that this is precisely $-\frac{3}{4\pi^2}$ times the inverse of the pairing on $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$ calculated in (4.2). Thus we have found a pairing that satisfies Conjecture 7.3.

While we have verified the conjecture for Σ that uses v_2, we also need to check that *the same pairing* works for the only other possible fan, namely the one that does not use v_2. In this case, there are three twisted sectors, indexed by $(0,0)$, $(1,1)$ and $(2,1)$. Each of the corresponding spaces H_γ and H^c_γ is one-dimensional. We will briefly go through the key steps of the calculation while leaving the details to the reader. We recycle the notation from the other examples, but they are now applied to a different situation. We use the notation $w = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}}$.

$$\begin{aligned}
& K_0(\mathbb{P}_\Sigma) = \mathbb{C}[R_3]/\langle 1-R_3^3\rangle,\ K_0^c(\mathbb{P}_\Sigma) = \mathbb{C}[R_3]G_{13}/\langle (1-R_3^3)G_{13}\rangle \\
& \chi(R_3^kG_{13}) = \delta_k^{0 \bmod 3} \\
& ch(a_1+a_2R_3+a_3R_3^2) = (a_1+a_2+a_3)e_{(0,0)} \\
& \quad \oplus (a_1+wa_2+w^2a_3)e_{(1,1)} \oplus (a_1+w^2a_2+wa_3)e_{(2,1)} \\
& ch^c(G_{13}) = F_{13} \oplus 3\bar{F}_{\emptyset,(1,1)} \oplus 3\bar{F}_{\emptyset,(2,1)}.
\end{aligned}$$

The Gamma series are expanded in terms of the powers of x^{-1}.

$$\begin{aligned}
\Gamma_{(0,0)} &= e_{(0,0)} \oplus x_1^{-\frac{4}{3}} x_2^2 x_3^{-\frac{2}{3}} (-\tfrac{\sqrt{3}}{12\pi} + \tfrac{7\sqrt{3}}{2430\pi} x^{-1} + \ldots) e_{(1,1)} \\
&\quad \oplus x_1^{-\frac{2}{3}} x_2 x_3^{-\frac{1}{3}} (\tfrac{\sqrt{3}}{2\pi} - \tfrac{5\sqrt{3}}{648\pi} x^{-1} + \ldots) e_{(2,1)} \\
\Gamma_{(2,1)} &= 0 e_{(0,0)} \oplus x_1^{-\frac{1}{3}} x_3^{-\frac{2}{3}} (\tfrac{\sqrt{3}}{2\pi} - \tfrac{2\sqrt{3}}{81\pi} x^{-1} + \ldots) e_{(1,1)} \\
&\quad \oplus x_1^{-\frac{5}{3}} x_2^2 x_3^{-\frac{4}{3}} (\tfrac{\sqrt{3}}{18\pi} - \tfrac{4\sqrt{3}}{729\pi} x^{-1} + \ldots) e_{(2,1)} \\
\Gamma^\circ_{(1,1)} &= x_1^{-2} x_2^2 x_3^{-1} (\tfrac{1}{8\pi^2} - \tfrac{1}{80\pi^2} x^{-1} + \ldots) F_{13} \\
&\quad \oplus x_1^{-\frac{4}{3}} x_2 x_3^{-\frac{2}{3}} (-\tfrac{\sqrt{3}}{6\pi} + \tfrac{7\sqrt{3}}{486\pi} x^{-1} + \ldots)^{-1} \bar{F}_{\emptyset,(1,1)} \\
&\quad \oplus x_1^{-\frac{2}{3}} x_3^{-\frac{1}{3}} (\tfrac{\sqrt{3}}{2\pi} - \tfrac{5\sqrt{3}}{162\pi} x^{-1} + \ldots) \bar{F}_{\emptyset,(2,1)} \\
\Gamma^\circ_{(2,1)} &= x_1^{-1} x_2 x_3^{-1} (-\tfrac{1}{4\pi^2} + \tfrac{1}{48\pi^2} x^{-1} + \ldots) F_{13} \\
&\quad \oplus x_1^{-\frac{1}{3}} x_3^{-\frac{2}{3}} (\tfrac{\sqrt{3}}{2\pi} - \tfrac{2\sqrt{3}}{81\pi} x^{-1} + \ldots)^{-1} \bar{F}_{\emptyset,(1,1)} \\
&\quad \oplus x_1^{-\frac{5}{3}} x_2^2 x_3^{-\frac{4}{3}} (\tfrac{\sqrt{3}}{18\pi} - \tfrac{4\sqrt{3}}{729\pi} x^{-1} + \ldots) \bar{F}_{\emptyset,(2,1)}.
\end{aligned}$$

The pairing gives

$$\langle \Gamma, \Gamma^\circ \rangle = -\frac{9}{4\pi^2}(3e_{(2,1)} \otimes \bar{F}_{\emptyset,(1,1)} + e_{(0,0)} \otimes F_{13} + 3e_{(1,1)} \bar{F}_{\emptyset,(2,1)})$$

$$= -\frac{3}{4\pi^2}(1 \otimes G_{13} + R_3 \otimes R_3 G_{13} + R_3^2 \otimes R_3^2 G_{13})$$

as predicted.

Proposition 8.2. *Conjectures* 7.1 *and* 7.3 *hold in the Example* $v_1 = (0,1)$, $v_2 = (1,1)$, $v_3 = (3,1)$.

Proof. We have verified Conjecture 7.3 above. Observe that the solutions of $bbGKZ(C,0)$ are uniquely determined by $(0,0)$ term. As a result, the system is equivalent to the usual GKZ system. The analytic continuation statement is already proved in [BH2], thus it holds for our Γ series. Because both Euler characteristics and the pairing of Conjecture 7.3 are unchanged under the pullback-pushforward and analytic continuation respectively, the statement for Γ° follows. ☐

9. Pairing with 1

In this section we provide important evidence in favor of Conjecture 7.3 which gives hints at a possible explicit description of the pairing. Specifically, we can be reasonably certain of the part of the pairing in Conjecture 7.3 that involves Φ_0 only. In terms of K-theory this corresponds to a well-defined linear function on $K_0(\mathbb{P}_\Sigma)$, namely the rank at the generic point as we see in the following proposition.

Proposition 9.1. *Consider the linear function*

$$\mathrm{rk} : K_0(\mathbb{P}_\Sigma) \to \mathbb{C}$$

defined by $\mathrm{rk}(\prod_i R_i^{l_i}) = 1$ *for any collection* $(l_1, \ldots, l_n) \in \mathbb{Z}^n$. *Then* $\mathrm{rk}(\Gamma(x_1, \ldots, x_n))_c = \delta_c^0$.

Proof. First, we observe that rk is well-defined. Indeed, it clearly vanishes on all relations (3.1). Second, under the isomorphism of $K_0(\mathbb{P}_\Sigma) = \oplus H_\gamma$, the corresponding linear functions on the $\gamma \neq 0$ sectors are zero, because at least one $R_i - 1$ is invertible there. Similarly, for the H_0 sector the function rk reads off the degree zero component only.

Since the function rk vanishes on H_γ, $\gamma \neq 0$, we only need to see what happens to the terms

$$\mathrm{rk}\Big(\prod_i \frac{x_i^{l_i + \frac{D_i}{2\pi \mathrm{i}}}}{\Gamma(1 + l_i + \frac{D_i}{2\pi \mathrm{i}})}\Big)$$

with $(l_i) \in L_{c,0}$. Since rk vanishes on all positive degree monomials in D_i, we need $l_i \geq 0$. Since $\sum_i l_i v_i = -c$, the only nonzero terms occur for $l_i = 0$ for all i and $c = 0$. In that case, we have $\mathrm{rk}(1) = 1$. ☐

If we consider the pairing of Conjecture 7.3 applied to $\Phi_c = \delta_c^0$ and arbitrary Ψ, then this pairing only involves the $p_{0,d}$. In what follows we show that there is a natural choice of such $p_{0,d}$ so that $\sum_{d \in C^\circ} p_{0,d} \Psi_d$ is constant and is compatible with Gamma series and the pairing on K-theories.

Recall, see for example [BM, Def. 2.3], that the logarithmic Hessian of $\sum_i x_i [v_i]$ is an element of the semigroup ring $\mathbb{C}[C]$ given by

$$\mathrm{Hessian}(x_1, \ldots, x_n) = \sum_{|I| = \mathrm{rk} N} \mathrm{Vol}_I^2 (\prod_{i \in I} x_i) [\sum_{i \in I} v_i]$$

where the sum is taken over subsets of $\{1, \ldots, n\}$ and where Vol_I denotes the wedge of $v_i, i \in I$, in the increasing order, as a multiple of the generator of $\Lambda^{\mathrm{rk} N} N$. We can use the coefficients of Hessian to build a constant linear function on the solutions of $bbGKZ(C^\circ, 0)$ as follows.

Proposition 9.2. *For any solution* Ψ *of* $bbGKZ(C^\circ, 0)$ *the function of* $x = (x_1, \ldots, x_n)$

$$\sum_{d \in C^\circ} \mathrm{Coeff}_d(\mathrm{Hessian}(x)) \Psi_d(x) \tag{9.1}$$

is constant.

Proof. The expression of (9.1) is given by

$$\sum_{|I|=\mathrm{rk}N} \Psi_{\sum_{i\in I} v_i}(\prod_{i\in I} x_i)\mathrm{Vol}_I^2.$$

To prove the proposition, we will show that the partial derivative of the above function with respect to x_1 vanishes. This derivative is given by

$$S = \sum_{|I|=\mathrm{rk}N} \Psi_{v_1+\sum_{i\in I} v_i}(\prod_{i\in I} x_i)\mathrm{Vol}_I^2 + \sum_{|I|=\mathrm{rk}N, I\ni 1} \Psi_{\sum_{i\in I} v_i}(\prod_{1\neq i\in I} x_i)\mathrm{Vol}_I^2.$$

For each $I \ni 1$ of size $\mathrm{rk}N$ consider the relations (6.1) for μ given by composing $\wedge(\Lambda_{1\neq i\in I})$ with $\Lambda^{\mathrm{rk}N}N \to \mathbb{Z}$ and $c = \sum_{i\in I} v_i$. We will also multiply each relation by $(\prod_{1\neq i\in I} x_i)\mathrm{Vol}_I$. Notice that this allows us to consider c which are not in the interior of C, since in that case we multiply the fictitious quantities by zero. When we add these for all I as above we get

$$\sum_{|I|=\mathrm{rk}N, I\ni 1} \sum_{j\notin I\backslash\{1\}} \Psi_{\sum_{i\in I} v_i + v_j}(\prod_{i\in\{j\}\cup(I\backslash\{1\})} x_i)\mathrm{Vol}_I \mathrm{Vol}_{\{j\}\cup(I\backslash\{1\})}\sigma(I,1,j)$$
$$+ \sum_{|I|=\mathrm{rk}N, I\ni 1} \Psi_{\sum_{i\in I} v_i}(\prod_{1\neq i\in I} x_i)\mathrm{Vol}_I^2 = 0,$$

where $\sigma(I,1,j)$ is ± 1 depending of whether the location of j in $\{j\}\cup(I\backslash\{1\})$ is odd or even. Thus, we have

$$S = \sum_{|I|=\mathrm{rk}N} \Psi_{v_1+\sum_{i\in I} v_i}(\prod_{i\in I} x_i)\mathrm{Vol}_I^2 - \sum_{|I|=\mathrm{rk}N, I\ni 1} \sum_{j\notin I\backslash\{1\}} \Psi_{\sum_{i\in I} v_i + v_j}$$
$$(\prod_{i\in\{j\}\cup(I\backslash\{1\})} x_i)\mathrm{Vol}_I \mathrm{Vol}_{\{j\}\cup(I\backslash\{1\})}\sigma(I,1,j)$$
$$= \sum_{|I|=\mathrm{rk}N, I\not\ni 1} \Psi_{v_1+\sum_{i\in I} v_i}(\prod_{i\in I} x_i)\mathrm{Vol}_I^2 - \sum_{|I|=\mathrm{rk}N, I\ni 1} \sum_{j\notin I} \Psi_{\sum_{i\in I} v_i + v_j}$$
$$(\prod_{i\in\{j\}\cup(I\backslash\{1\})} x_i)\mathrm{Vol}_I \mathrm{Vol}_{\{j\}\cup(I\backslash\{1\})}\sigma(I,1,j)$$
$$= \sum_{|J|=\mathrm{rk}N, J\not\ni 1} \Psi_{v_1+\sum_{i\in J} v_i}(\prod_{i\in J} x_i)\mathrm{Vol}_J \sum_{k\in J\cup\{1\}} \mathrm{Vol}_{\{1\}\cup(J\backslash\{k\})}\sigma(J,k,1).$$

It thus remains to show that for each such J

$$\mathrm{Vol}_J \sum_{k\in J\cup\{1\}} \mathrm{Vol}_{\{1\}\cup(J\backslash\{k\})}\sigma(J,k,1) = 0. \tag{9.2}$$

If $\mathrm{Vol}_J \neq 0$, then $v_i, i \in J$ span $N_{\mathbb{R}}$. As a result, v_1 and $v_i, i \in J$ satisfy a linear dependence relation

$$\sum_{k \in J \cup \{1\}} \alpha_j v_j = 0.$$

with $\alpha_1 \neq 0$. Since all the elements v_i lie in a hyperplane, it follows that $\sum_{k \in J \cup \{1\}} \alpha_k = 0$. It remains to observe that

$$\mathrm{Vol}_{\{1\} \cup (J \setminus \{k\})} \sigma(J, k, 1) = -\mathrm{Vol}_J \alpha_k \alpha_1^{-1}.$$

□

In view of Proposition 9.1, if one has a pairing that satisfies the condition of Conjecture 7.3, then plugging in $\Phi_c = \delta_c^0$

$$\langle \delta_c^0, \Gamma^\circ \rangle$$

must produce the element of $K_0^c(\mathbb{P}_\Sigma)$ which is the contraction of the dual to the Euler characteristics pairing with the rank function. Consider the element $ch^c[\mathcal{O}_p]$ of $H_0^c \subseteq \oplus_\gamma H_\gamma^c$ which is given by $\mathrm{Vol}_I F_I$ for any cone I of Σ of maximum dimension. We should view $[\mathcal{O}_p]$ as a class of a generic point on $\mathbb{P}_\Sigma$ (even thought the definition of the abelian category only used complexes with cohomology supported on the $\pi^{-1}(0)$). Euler characteristics with this $[\mathcal{O}_p]$ is precisely the rank function on $K_0(\mathbb{P}_\Sigma)$. As a consequence, the following proposition is consistent with Conjecture 7.3.

Proposition 9.3. *For any choice of projective subdivision* Σ, *we have*

$$\sum_{d \in C^\circ} \mathrm{Coeff}_d(\mathrm{Hessian}(x)) \Gamma_d^\circ(x_1 \ldots, x_n) = \frac{\mathrm{Vol}(\mathrm{conv}(v_1, \ldots, v_n))}{(2\pi \mathrm{i})^{\mathrm{rk} N}} [\mathcal{O}_p].$$

where Vol *is the normalized volume.*

Proof. By Proposition 9.2, the left hand side of the equation is a constant. Therefore, it is enough to prove the statement asymptotically as $(x_1, \ldots, x_n)$ approaches the degeneracy point that corresponds to the triangulation Σ. Specifically, let $\mu_i \in \mathbb{R}$ be chosen compatible with Σ in the following sense. We extend these values to a Σ piecewise linear function μ on $\mathbb{C}$ by using $\mu(v_i) = \mu_i$ for $\{i\} \in \Sigma$. For every positive linear combination $\sum_{j \in J} \alpha_j v_j$ there holds

$$\sum_{j \in J} \alpha_j \mu_j \geq \mu(\sum_{j \in J} \alpha_j v_j) \tag{9.3}$$

with equality if and only if $J \in \Sigma$. The fact that such collection of μ_i exists is precisely the projectivity condition on Σ.

Consider $x_i = \mathrm{e}^{t\mu_i}$ and let $t \to +\infty$. We will study the asymptotic behavior of Γ°. Uniform convergence allows one to consider limits of individual terms.

The terms of $\mathrm{Vol}_I^2(\prod_{i\in I} x_i)\Gamma^\circ(x_1,\ldots,x_n)_{\sum_{i\in I} v_i}$ for $(l_j) \in L_{\sum_{i\in I} v_i,\gamma}$ are

$$\mathrm{Vol}_I^2(\prod_{i\in I} x_i)\Big(\prod_{j=1}^n \frac{x_j^{l_j+\frac{D_j}{2\pi\mathrm{i}}}}{\Gamma(1+l_j+\frac{D_j}{2\pi\mathrm{i}})}\Big)(\prod_{j,l_j\in\mathbb{Z}_{<0}} D_j^{-1})F_{\{j,l_j\in\mathbb{Z}_{<0}\}}.$$

As $t\to\infty$ this is, up to an invertible element,

$$\mathrm{e}^{t(\sum_{i=1}^n l_i\mu_i+\sum_{i\in I}\mu(v_i))}F_{\{j,l_j\in\mathbb{Z}_{<0}\}}.$$

In the limit $t\to+\infty$ only the terms with $\sum_{i=1}^n l_i\mu_i+\sum_{i\in I}\mu_i \geq 0$ contribute. In addition, to get a nonzero term, we must have $\{j,l_j\in\mathbb{Z}_{<0}\}$ be a cone in Σ that contains $\sigma(\gamma)$.

Consider the linear relation

$$\sum_{i=1}^n l_i v_i + \sum_{i\in I} v_i = \sum_{i=1}^n \hat{l}_i v_i = 0$$

on v_i. The negative coefficients of this linear combination are a subset of $\{j,l_j\in\mathbb{Z}_{<0}\}\cup\sigma(\gamma)$. Thus we have

$$\sum_{\hat{l}_i\geq 0}\hat{l}_i\mu_i \geq \sum_{\hat{l}_i<0}(-\hat{l}_i)\mu_i = \mu(-\sum_{\hat{l}_i<0}\hat{l}_i v_i) = \mu(\sum_{\hat{l}_i\geq 0}\hat{l}_i v_i).$$

The convexity of μ then implies that inequality is in fact equality and there exists a cone of Σ which contains all v_i for which $\hat{l}_i\geq 0$. Thus we have that both for i with $l_i\geq 0$ and for i with $\hat{l}_i<0$ there are cones of Σ that contain them. The two sides of

$$\sum_{i,\hat{l}_i\geq 0}\hat{l}_i v_i = \sum_{i,\hat{l}_i\geq 0}(-\hat{l}_i)v_i$$

are contained in, necessarily disjoint cones of Σ. The only way this can happen is when all $\hat{l}_i = 0$. This means that we are working in the non twisted sector, $\gamma = 0$. It also means that $l_i = -1$ for $i\in I$ and $l_i = 0$ for $i\notin I$. Note that F_I is annihilated by all D_j because $|I| = \mathrm{rk}N$. Thus the limit of the corresponding term of the Gamma series is

$$\mathrm{Vol}_I^2\prod_{i\in I}(\frac{1}{\Gamma(\frac{D_i}{2\pi\mathrm{i}})}D_i^{-1})F_I = \frac{1}{(2\pi\mathrm{i})^{\mathrm{rk}N}}\mathrm{Vol}_I ch^c([O_p]).$$

When the above terms are added over all I that form a basis of a maximum dimensional cone of Σ, the claim follows. $\square$

Remark 9.4. Proposition 9.3 explains the coefficient $-\frac{3}{4\pi^2}$ that we saw in Section 8. We originally thought that a multiple of

$$\langle\Phi,\Psi\rangle=\sum_{c,d}\Phi_c\Psi_d(-1)^{\deg c}\mathrm{Coeff}_{c+d}(H(x))$$

would always provide the pairing of Conjecture 7.3, but the example showed this to not be the case. However, Proposition 8.1 gives hope that some modification of the above guess that takes into account twisted sectors may work in general.

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