TWISTOR GEOMETRY AND WARPED PRODUCT ORTHOGONAL COMPLEX STRUCTURES

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Abstract. The twistor space of the sphere S^{2n} is an isotropic Grassmannian that fibers over S^{2n}. An orthogonal complex structure on a subdomain of S^{2n} (a complex structure compatible with the round metric) determines a section of this fibration with holomorphic image. In this paper, we use this correspondence to prove that any finite energy orthogonal complex structure on $\mathbb{R}^6 \subset S^6$ must be of a special warped product form, and we also prove that any orthogonal complex structure on $\mathbb{R}^{2n}$ that is asymptotically constant must itself be constant. We will also give examples defined on $\mathbb{R}^{2n}$ which have infinite energy, and examples of non-standard orthogonal complex structures on flat tori in complex dimension three and greater.

Contents

1. Introduction

An orthogonal complex structure on a Riemannian manifold (M, g) is a complex structure which is integrable and is compatible with the metric g. In a previous paper of the second and third authors, the case of domains in $\mathbb{R}^4$ with the Euclidean metric was considered, and various Liouville-type theorems were proved [SV09]. In particular, it was shown that if J is an orthogonal complex structure on $\mathbb{R}^4 \setminus K$, where K is a closed set with $\mathcal{H}^1(K) = 0$ (vanishing 1-dimensional Hausdorff measure), then J is conformally equivalent to a constant OCS. This generalized an earlier result of Wood [Woo92] and equivalent arguments of LeBrun–Poon [LP93].

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In this paper, we consider the case of higher-dimensional Euclidean spaces, and then focus on the case of $\mathbb{R}^6$. The 4-dimensional Liouville theorem does not directly generalize to higher dimensions. Indeed, the twistor construction over $\mathbb{R}^4$ itself gives rise to an orthogonal complex structure on $\mathbb{R}^6$ that is not conformally constant, see Remark 5.5 below. Global examples of OCSes on $\mathbb{R}^{2n}$ for $n \geqslant 3$ which are not conformally constant were described explicitly by Baird–Wood [BW95] in the context of harmonic morphisms (see [Sal85, BW03, GW97] for other relevant links with harmonic maps and morphisms).

We next discuss a method for writing down many examples of global OCSes on $\mathbb{R}^6$. Endow $\mathbb{R}^6 = \mathbb{C}^3$ with complex coordinates (z^1, z^2, z^3), and consider an orthogonal almost complex structure of the form

$$J = J_1(z^3) \oplus J_0, \tag{1.1}$$

where $J_1(z^3)$ is an OCS on the space $\mathbb{R}^4$ for which z^3 is constant, and J_0 is the standard OCS on $\mathbb{R}^2$ with coordinate $z^3 = x^3 + iy^3$. Any OCS on $\mathbb{R}^4$ is necessarily constant. Moreover, the OCSes on $\mathbb{R}^4$ consistent with a fixed orientation are parametrized by $\mathrm{SO}(4)/\mathrm{U}(2)$, the complex projective line $\mathbb{P}^1$, so J_1 can be regarded as a map $f : \mathbb{C} \to \mathbb{P}^1$. If f is holomorphic, then (1.1) is integrable and so defines a global OCS on $\mathbb{R}^6$. It is an example of a *warped product orthogonal complex structure*, as defined in Section 5. While a warped product OCS on $\mathbb{R}^6$ is determined by a single meromorphic function, these structures become much more complicated in higher dimensions; this will be discussed further in Section 4.

The differential of any conformal map $\psi : \mathbb{R}^6 \setminus \{p\} \to \mathbb{R}^6$ lies in $\mathrm{CO}(6) = \mathbb{R}_+ \times \mathrm{O}(6)$, thus the conformal group $\mathrm{O}_+(1,7)$ (time-oriented Lorentz transformations) acts on the space of OCSes on subdomains of $\mathbb{R}^6$ by conjugation by the differential ψ_*. The round metric on $S^6 \setminus \{\infty\} = \mathbb{R}^6$ is conformally equivalent to the Euclidean metric g_E. Thus if J is an OCS defined on $\mathbb{R}^6$ away from finitely many points, we can equivalently view J as an OCS on S^6 away from finitely many points.

Definition 1.1. Let J be an orthogonal complex structure defined on $\Omega = S^6 \setminus K$ where K is a finite set of points. Then J is said to have *finite energy* if

$$\int_{S^6 \setminus K} \|\nabla J\|^6 dV < \infty, \tag{1.2}$$

where the covariant derivative, norm and volume form are taken with respect to the round metric on S^6.

The main result in this paper shows that the above warped product construction gives all of the finite energy OCSes on $\mathbb{R}^6$, up to conformal equivalence:

Theorem 1.2. *Let J be an orthogonal complex structure of class C^1 on $S^6 \setminus K$ with finite energy, where K is a finite set of points.*

(i) *J is conformally equivalent to a warped product structure globally defined on $\mathbb{R}^6 = S^6 \setminus \{\infty\}$ with the correct orientation, and determined by a rational function $f : \mathbb{C} \to \mathbb{P}^1$.*

(ii) *J is conformally equivalent to the standard orthogonal complex structure on $\mathbb{R}^6 = \mathbb{C}^3$ if and only if f is constant.*
(iii) *$(\mathbb{R}^6, J)$ is biholomorphic to $\mathbb{C}^3$, and $(\mathbb{R}^6, g_E, J)$ is cosymplectic (the Kähler form is co-closed), but is not locally conformal to a Kähler metric, in particular, it is not Kähler unless f is constant.*

Taking $f : \mathbb{C} \to \mathbb{P}^1$ to be a non-algebraic meromorphic function in the warped product construction, we find examples of infinite energy OCSes globally defined on $\mathbb{R}^6$. Taking products of these with the standard OCS on $\mathbb{R}^{2k}$, one obtains examples in all higher dimensions which violate any reasonable finite energy assumption.

If in particular we choose a doubly-periodic meromorphic function on $\mathbb{C}$, we find the following examples of non-standard complex flat tori.

Theorem 1.3. *Let (T^2, J_2) be an elliptic curve with a compatible flat metric g_2, and let (T^4, g_4) be a flat 4-torus. There is an infinite-dimensional space of complex structures on the 6-torus $(T^6, g_6) = (T^4 \times T^2, g_4 \oplus g_2)$ which are orthogonal relative to the product metric. These structures are of warped product form, determined by a meromorphic function $f : (T^2, J_2) \to \mathbb{P}^1$. Lifting to $\mathbb{R}^6$, they have infinite energy.*

We note that, just as in Theorem 1.2 (iii), these tori are cosymplectic but not Kähler for non-constant f, see Proposition 5.3 below. A similar construction yields non-standard examples on tori in all higher even dimensions, see Subsection 5.1. However, such examples do not exist in real dimension four. Locally conformally flat compact Hermitian surfaces have been classified by Pontecorvo [Pon92]. In the flat case, the OCS must lift to a constant OCS on $\mathbb{R}^4$, thus the space of OCSes on a flat 4-torus is finite dimensional.

A complete classification of finite energy OCSes as in Theorem 1.2 in higher dimensions is much more difficult. We do not attempt such a general classification in this paper, but our work does lead to some natural questions that may point the way towards a possible solution of the problem. In any case, we prove a Liouville theorem under a stronger assumption:

Definition 1.4. Let J be an orthogonal complex structure defined on $\mathbb{R}^{2n}$. We say that J is *asymptotically constant* if

$$\|J(\mathbf{x}) - J'\| \to 0 \text{ as } \mathbf{x} \to \infty,$$

for some constant orthogonal complex structure J'. Here, the norm refers equally to the Euclidean or round metric, as any conformal factor cancels out.

Assuming this condition, we have the following Liouville theorem in all even dimensions.

Theorem 1.5. *Let J be an orthogonal complex structure of class C^1 on $\mathbb{R}^{2n}$ which is asymptotically constant. Then $\pm J$ is isometrically equivalent to the standard constant orthogonal complex structure on $\mathbb{R}^{2n}$.*

Theorem 1.5 will be proved in Section 6. We next give a brief outline of the proof of Theorem 1.2. We use the twistor fibration $\mathbb{P}^3 \to Q^6 \to S^6$, which was studied

in particular detail by Slupinski [Slu96]. The complex 6-quadric Q^6 fibers over S^6, with fiber the complex projective space $\mathbb{P}^3$ that can be identified with $SO(6)/U(3)$, and local sections are orthogonal almost complex structures compatible with a fixed orientation. Such a section is integrable precisely when the graph is a holomorphic subvariety. For simplicity, assume that J is an OCS on $\mathbb{R}^6 = S^6 \setminus \{\infty\}$, so that its graph $J(\mathbb{R}^6)$ lies in $Q^6 \setminus \mathbb{P}^3_\infty$, where $\mathbb{P}^3_\infty = \pi^{-1}(\infty)$ is the fiber over the point at infinity.

Consider the closure $X = \overline{J(\mathbb{R}^6)} \subset Q^6$. The finite energy assumption (1.2) implies that the graph of J has finite area. This in turn implies that its closure is a 3-dimensional analytic subvariety, by a theorem of Bishop [Bis64]. Moreover, by Chow's Theorem, it is algebraic [Cho49]. Now, any 3-dimensional subvariety $X \subset Q^6$ has a bidegree (q, p); see Section 2.3. Since our X arises from an OCS, it hits generic twistor fibers in one point, and this implies that the bidegree of X in Q^6 is $(1, p)$, and the degree of X in $\mathbb{P}^7$ is $p+1$. Theorem 1.2 will then be seen as a consequence of the following result.

Theorem 1.6. *Let X be a threefold of type $(1, p)$ in Q^6. Then X yields an orthogonal complex structure maximally defined on $S^6 \setminus E$, where E is a closed set with real dimension at least 2 unless X corresponds to a warped product structure globally defined on $\mathbb{R}^6$.*

This will be proved in Section 8, using the classification of threefolds of order one in the 6-quadric obtained in [BV08], some of the results of which are tailored to applications in the present paper.

In closing the Introduction, we remark that the above theorem can be applied to give a partial classification of locally conformally flat Hermitian threefolds. There are also applications of our theorems to the theory of harmonic maps from Euclidean spaces. These aspects will be discussed in a forthcoming work.

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2. Background

2.1. **Complex structures, isotropic Grassmannians and spinors.** There is a bijective correspondence between the following objects:

(i) points of the coset space $Z_n^+ = \mathrm{SO}(2n)/\mathrm{U}(n)$,

(ii) constant or linear OCSes on $\mathbb{R}^{2n}$ consistent with a fixed orientation,

(iii) skew-symmetric orthogonal matrices with Pfaffian equal to 1,

(iv) maximal isotropic subspaces in $\mathbb{C}^{2n}$ inducing a fixed orientation.

To formalize the correspondence between (i) and (ii), first note that the isotropy subgroup $\mathrm{U}(n)$ of Z_n^+ may be regarded as the stabilizer of a fixed OCS $\mathbb{J}$ on $\mathbb{R}^{2n}$.

We may express $\mathbb{J}$ as a skew-symmetric orthogonal matrix of size $2n$. By reducing this matrix to standard block-diagonal form (as in (3.16) below), we see that $\det \mathbb{J} = 1$ irrespective of the induced orientation on $\mathbb{R}^{2n}$. The latter is instead encoded in the *Pfaffian* of $\mathbb{J}$. Recall that the determinant of any skew-symmetric matrix M of size $2n$ can be written

$$\det M = (\operatorname{Pf} M)^2, \tag{2.1}$$

where $\operatorname{Pf} M$ is a polynomial of degree n in its entries. Standard expressions for the Pfaffian show that $\operatorname{Pf} \mathbb{J} = 1$ if and only if $\mathbb{J}$ induces a positive orientation on $\mathbb{R}^{2n}$.

Any other OCS will now equal $J = A\mathbb{J}A^{-1}$ for some $A \in O(2n)$, and $\operatorname{Pf} J = (\det A)(\operatorname{Pf} \mathbb{J})$, so to preserve orientation we must take $\det A = 1$. We can then map J to the coset $A\mathrm{U}(n) \in Z_n^+$. For the description in (iv), we associate to J its $+i$-eigenspace $T^{1,0}$, a totally isotropic subspace of $\mathbb{C}^{2n}$. We shall take the remaining mechanics of this correspondence for granted, and explain instead how spinors can be used to add a fifth class of objects to the list above.

Consider the complex representation

$$\Delta = \Delta_+ \oplus \Delta_-$$

of $\mathrm{Spin}(2n)$, where each irreducible summand $\Delta_\pm$ has dimension 2^{n-1}. Given a non-zero spinor ϕ in Δ_+,

$$V_\phi = \{v \in \mathbb{C}^{2n} : v \cdot \phi = 0\} \tag{2.2}$$

is an isotropic subspace, where $\cdot$ denotes Clifford multiplication. This follows because because if $v, w \in V_\phi$ then

$$0 = v \cdot (w \cdot \phi) - w \cdot (v \cdot \phi) = -2 \langle v, w \rangle \phi. \tag{2.3}$$

Using the underlying scalar product, we can identify $\mathbb{C}^{2n}$ with its dual and regard Clifford multiplication as an injection

$$m : \Delta_+ \to \mathbb{C}^{2n} \otimes \Delta_-.$$

Choose a basis (δ_ℓ) of Δ_-, and set

$$m(\phi) = \sum_{\ell=1}^{2^{n-1}} \alpha_\ell \otimes \delta_\ell. \tag{2.4}$$

Then the α_ℓ span the annihilator $(V_\phi)^\circ$ of V_ϕ; the bigger the latter, the smaller its annihilator. In the extreme case, V_ϕ is *maximal* isotropic if and only if $(V_\phi)^\circ$ is *isotropic* (of the same dimension). A nice way to chararacterize when this occurs is through representation theory.

It is well-known that an element of the Clifford algebra $\mathrm{Cl}(\mathbb{C}^{2n})$ is itself an endomorphism of Δ. Furthermore, Cartan established equivariant decompositions

$$\begin{aligned} \Delta_+ \otimes \Delta_+ &= \Lambda^n_+ \ \oplus \ \Lambda^{n-2} \oplus \cdots \\ \Delta_+ \otimes \Delta_- &= \Lambda^{n-1} \oplus \Lambda^{n-3} \oplus \cdots, \end{aligned} \tag{2.5}$$

where $\Lambda^k = \bigwedge^k(\mathbb{C}^{2n})$ denotes exterior power of the basic representation $\Lambda^1 = \mathbb{C}^{2n}$ of $\mathrm{SO}(2n) = \mathrm{Spin}(2n)/\mathbb{Z}_2$, and Λ^n_+ is the $+1$ eigenspace of the Hodge map $*\colon \Lambda^n \to \Lambda^n$.

Theorem 2.1 (Cartan [Car81])**.** *The isotropic subspace V_ϕ is maximal if and only if the only non-zero component of $\phi \otimes \phi$ is in Λ^n_+. This component is decomposable (that is, a simple n-form), and generates the subspace corresponding to ϕ.*

If the condition of the theorem is satisfied, then ϕ is called a *pure* spinor. Sometimes we shall use the symbol ϕ to indicate the projective class of a pure spinor, and we let J_ϕ denote the OCS (or skew-symmetric orthogonal matrix with positive Pfaffian) characterized by

$$T^{1,0} = V_\phi, \quad \Lambda^{0,1} = (V_\phi)^\circ.$$

There are no purity conditions for $n = 2, 3$ because in both cases Λ^n_+ equals the symmetric part of the tensor product; hence the coset spaces in (ii) are complex projective spaces:

$$Z_2^+ = \mathbb{P}^1, \qquad Z_3^+ = \mathbb{P}^3.$$

However, in dimension 8, there is one quadratic relation given by projection to the summand Λ^0, and

$$Z_4^+ \subset \mathbb{P}(\Delta_+) \tag{2.6}$$

is a non-degenerate quadric $Q^6 \subset \mathbb{P}^7$.

2.2. **Twistor fibrations.** We shall first discuss the twistor space $\mathscr{Z} = \mathscr{Z}(\mathbb{R}^{2n})$ of Euclidean space $\mathbb{R}^{2n}$. As a smooth manifold, it is the product

$$\mathscr{Z} = Z_n^+ \times \mathbb{R}^{2n}. \tag{2.7}$$

The "twistor" complex structure $\mathscr{J}$ on $\mathscr{Z}$ is defined as follows. The tangent space to $\mathscr{Z}$ at a point $p = (J, \mathbf{x})$ splits as

$$T_p\mathscr{Z} = V_p \oplus H_p, \tag{2.8}$$

where the vertical space V_p is tangent to the Z_n^+ factor at J, and the horizontal space H_p is tangent to the $\mathbb{R}^{2n}$ factor at $\mathbf{x}$. As the notation suggests, $J \in Z_n^+$ is itself an almost complex structure on the vector space $\mathbb{R}^{2n} \cong H_p$.

The tangent space to Z_n^+ at J can be identified with those skew-symmetric endomorphisms of $\mathbb{R}^{2n}$ that *anti-commute* with J. It follows that, if $H_p^{1,0}$ denotes the $+i$-eigenspace for J, there is a canonical identification

$$V_p \otimes_{\mathbb{R}} \mathbb{C} = \textstyle\bigwedge^2(H_p^{1,0}) \oplus \bigwedge^2(H_p^{0,1}). \tag{2.9}$$

This not only determines the standard complex structure of Z_n^+ but also allows us to fix its sign in the context of the twistor fibration $\mathscr{Z} \to \mathbb{R}^{2n}$. We define $V_p^{1,0} = \bigwedge^2(H_p^{1,0})$, and decree the $+i$-eigenspace of $\mathscr{J}$ to be

$$V_p^{1,0} \oplus H_p^{1,0} \subset (T_p\mathscr{Z}) \otimes_{\mathbb{R}} \mathbb{C}. \tag{2.10}$$

It is known that, with this careful choice, $\mathscr{J}$ is integrable [AHS78, Bes87, dBN98, OR85, Sal85].

An analogous construction can be used to define the twistor space of any Riemannian manifold. The fiber over each point is again Z_n^+. The splitting (2.8) is accomplished by means of the Levi-Civita connection, and this enables one to define a tautological almost complex structure $\mathscr{J}$ in the same way. In particular, the twistor space of the even-dimensional round sphere is the total space $\mathscr{Z}(S^{2n})$ of the fibration

$$Z_n^+ \to \mathscr{Z}(S^{2n}) \to S^{2n}, \tag{2.11}$$

endowed with the structure $\mathscr{J}$. The orthonormal frame bundle of S^{2n} is the Lie group $\mathrm{SO}(2n+1)$, so (2.11) is the fibration

$$\mathrm{SO}(2n)/\mathrm{U}(n) \to \mathrm{SO}(2n+1)/\mathrm{U}(n) \to S^{2n}.$$

This was used in the study of minimal surfaces in S^{2n} [Cal67, Bar75]. On the other hand, it is known that $\mathrm{SO}(2n+1)/\mathrm{U}(n) \cong \mathrm{SO}(2n+2)/\mathrm{U}(n+1)$ (see [Bat90, Sal96]), and so there is a fibration

$$Z_n^+ \to Z_{n+1}^+ \xrightarrow{\pi} S^{2n}. \tag{2.12}$$

We shall give another description of this in Section 4.

Over the 4-sphere, one recovers the "Penrose fibration"

$$\mathbb{P}^1 \to \mathbb{P}^3 \to S^4. \tag{2.13}$$

If we identify S^4 with the quaternionic projective line $\mathbb{HP}^1$, this is merely a Hopf-type map. In dimension 6, we have

$$\mathbb{P}^3 \to Q^6 \to S^6, \tag{2.14}$$

as stated in the Introduction, although we now have the more precise description $Q^6 \subset \mathbb{P}(\Delta_+)$. Pending an explicit formula for the projection to $\mathbb{R}^6 \subset S^6$ in Section 7, we shall study the geometry underlying (2.14) in the next subsection.

2.3. **Linear spaces on 6-quadrics.** In this subsection, we abbreviate to Λ the standard complex representation $\Lambda^1 = \mathbb{C}^8$ of $\mathrm{SO}(8) = \mathrm{Spin}(8)/\mathbb{Z}_2$. Just as $Q^6 \subset \mathbb{P}(\Delta_+)$ parametrizes maximal positively-oriented isotropic subspaces of Λ, so the 6-quadric in $\mathbb{P}(\Delta_-)$ parametrizes maximal negatively-oriented isotropic suspaces in Λ. The triality principal asserts that the representations Δ_+, Δ_-, Λ are equivalent by a cyclic permutation induced by an outer automorphism of $\mathrm{Spin}(8)$, and we deduce that the 6-quadrics in $\mathbb{P}(\Delta_-)$, $\mathbb{P}(\Lambda)$ parametrize different families of maximal isotropic subspaces of Δ_+ or, equivalently, linear $\mathbb{P}^3$-s in the twistor space (2.6).

In this way, we see the classical fact that the set of $\mathbb{P}^3$-s in the twistor space has two components, each of which can itself be identified with a 6-quadric. In the twistor context, this theory was described by Slupinski [Slu96]. The family parametrized by the quadric in $\mathbb{P}(\Lambda)$ contains the twistor fibers (themselves parametrized by the real submanifold $S^6 \subset \mathbb{P}(\Lambda)$) but consists of what generally we shall call "vertical" $\mathbb{P}^3$-s. A vertical $\mathbb{P}^3$ is either a fiber or a twistor space of a 4-sphere conformally embedded in S^6 (via (2.13) or a negatively-oriented version) [Slu96]. On the other hand, the

family parametrized by the quadric in $\mathbb{P}(\Delta_-)$ consists of "horizontal" $\mathbb{P}^3$-s. If P is a horizontal $\mathbb{P}^3$ then there exists a unique $p \in S^6$ such that $P \cap \pi^{-1}(p)$ is a $\mathbb{P}^2$ and $P \cap \pi^{-1}(q)$ a single point for $q \neq p$. Moreover, P is determined by p and $P \cap \pi^{-1}(p)$, so the family of horizontal $\mathbb{P}^3$-s is a "dual" twistor space projecting to S^6 with fiber $(\mathbb{P}^3)^*$ (we have after all just swapped Δ_+ and Δ_-).

We continue to denote the twistor space of S^6 by Q^6, leaving implicit its embedding in $\mathbb{P}(\Delta_+)$. The homology group

$$H_6(Q^6, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \tag{2.15}$$

is generated by a horizontal $\mathbb{P}^3$ (we choose this to represent the first factor), and a vertical $\mathbb{P}^3$ (the second factor). This implies that any 3-dimensional subvariety $X \subset Q^6$ has a well-defined *bidegree* (q, p). Any vertical or horizontal $\mathbb{P}^3$ has zero self-intersection in Q^6, and $\mathbb{P}^3$-s from opposite families have intersection 1.

To illustrate this from Cartan's viewpoint, we use the isomorphisms

$$\Delta_- \otimes \Delta_- = \Lambda^4_- \oplus \Lambda^2 \oplus \Lambda^0, \tag{2.16}$$

$$\Delta_- \otimes \Delta_+ = \Lambda^3 \oplus \Lambda^1. \tag{2.17}$$

They are instances of (2.5) though here (having applied triality) $\Lambda^k = \bigwedge^k \Delta_+$. Two horizontal $\mathbb{P}^3$-s are determined by pure spinors ϕ, $\psi \in \Delta_-$. The non-zero component of $\phi \otimes \psi$ in the smallest summand of (2.16) will always be a simple form that spans the intersection of the corresponding 4-dimensional subspaces in Δ_+. In the generic case, the component $\langle \phi, \psi \rangle \in \Lambda^0$ will be non-zero, and the two $\mathbb{P}^3$-s will have empty intersection. If $\langle \phi, \psi \rangle = 0$ then the component in Λ^2 (formally $\phi \wedge \psi$) will be a simple wedge product of 1-forms, indicating that the two $\mathbb{P}^3$-s intersect in a $\mathbb{P}^1$. Similarly, two $\mathbb{P}^3$-s from different families will intersect generically in a point or (if the Λ^1 component of $\phi \otimes \psi$ in (2.17) vanishes) a $\mathbb{P}^2$. A similar intersection criterion holds in higher dimensions, but we will not require it. We refer the reader to [Car81, Che97] for further details.

The following lemma will be used in Section 8. It describes the geometry of the twistor projection restricted to a plane $\mathbb{P}^2$ in Q^6.

Lemma 2.2. *Every linear $\mathbb{P}^2 \subset Q^6$ is either contained entirely in a fiber of the twistor projection, or intersects exactly one twistor fiber in a $\mathbb{P}^1$ and all other fibers in a point or the empty set.*

Proof. Given any $\mathbb{P}^2$, call it P, there is exactly one horizontal and one vertical $\mathbb{P}^3$ containing P [BV08, Proposition 3.2]. Consider the horizontal $\mathbb{P}^3$ containing P. As mentioned above, a horizontal $\mathbb{P}^3$ in Q^6 hits exactly one fiber in a $\mathbb{P}^2 = P_0$, and hits every other fiber in a point. The planes P and P_0 are then two $\mathbb{P}^2$-s in a $\mathbb{P}^3$; they are either equal or intersect in a $\mathbb{P}^1$. □

Remark 2.3. If we look instead at the vertical $\mathbb{P}^3$ containing P, it is either a twistor fiber or (from above) can identified with the twistor space of an $S^4 \subset S^6$. Any $\mathbb{P}^2$ in this twistor bundle hits exactly one fiber in a $\mathbb{P}^1$ and hits every other fiber over this S^4 in exactly one point [SV09, Proposition 3.3].

3. Coordinates on the twistor fiber

We shall return to consider the twistor space $\mathscr{Z}(\mathbb{R}^{2n})$ in Section 4. But we first describe an atlas of coordinates covering the space $Z_n^+ = \mathrm{SO}(2n)/\mathrm{U}(n)$ that constitutes the twistor fiber over $\mathbb{R}^{2n}$ or S^{2n}. We assume that $n \geqslant 2$.

We will define quantities

$$\xi_{i_1\ldots i_p} \quad \text{for } p = 0, 2, \ldots \text{ even, up to } n \text{ or (if } n \text{ is odd) } n-1,$$

$$\eta_{i_1\ldots i_q} \quad \text{for } q = 1, 3, \ldots \text{ odd, up to } n-1 \text{ or (respectively) } n,$$

both skew-symmetric in all indices running from 1 to n. The ξ-s will be holomorphic coordinates on Z_n^+. At each point of Z_n^+ the η-s will be elements of

$$\mathbb{C}^{2n} = \mathbb{R}^{2n} \otimes \mathbb{C} = \Lambda^{1,0} \oplus \Lambda^{0,1}, \tag{3.1}$$

decomposed relative to the standard complex structure $\mathbb{J}$ on $\mathbb{R}^{2n}$ for which $\Lambda^{1,0}$ is spanned by $dz^1, \ldots, dz^n$.

Having fixed $\mathbb{J}$, there are isomorphisms

$$\Delta_+ \;\cong\; \Lambda^{0,0} \oplus \Lambda^{2,0} \oplus \Lambda^{4,0} \oplus \cdots \tag{3.2}$$

$$\Delta_- \;\cong\; \Lambda^{1,0} \oplus \Lambda^{3,0} \oplus \Lambda^{5,0} \oplus \cdots \tag{3.3}$$

(Strictly speaking, we also need a trivialization of $\Lambda^{n,0}$ that removes the distinction between $\Lambda^{p,0}$ and $\Lambda^{0,n-p}$.) Rather than adopt an overtly invariant approach, we shall merely use these decompositions to motivate Cartan's technique.

First, the $\xi_{i_1\ldots i_p}$ represent the components of a spinor $\boldsymbol{\xi} \in \Delta_+$ relative to a basis compatible with (3.2). We arrange them into groups

$$\xi_0, \quad \xi_{i_1 i_2}\ (i_1 < i_2), \quad \xi_{i_1 i_2 i_3 i_4}\ (i_1 < i_2 < i_3 < i_4), \quad \cdots \tag{3.4}$$

and order them lexicographically within each group, to give a total of

$$N = \sum_{p \text{ even}} \binom{n}{p} = 2^{n-1}$$

scalars. For example, ξ_0 (here $p = 0$ so logically the subscript is $\emptyset$) represents the component of $\boldsymbol{\xi}$ in the trivial summand $\Lambda^{0,0}$.

Next, we use (2.4) and a compatible basis (δ_ℓ) of (3.3) to convert $\boldsymbol{\xi} = \phi$ into 1-forms α_ℓ for $\ell = 1, \ldots, N$. The η-s are precisely these 1-forms, but rearranged to respect (3.3). The summand $\Lambda^{1,0}$ gives us the first n of them, namely

$$\eta_i = \xi_0 dz^i - \sum_{k=1}^{n} \xi_{ik} d\overline{z}^k, \qquad i = 1, \ldots, n. \tag{3.5}$$

This formula reflects the fact that Clifford multiplication by a vector $\partial/\partial z^i$ or $\partial/\partial \overline{z}^j$ acts on (3.2) as the sum of an exterior and interior product respectively. (To make sense of this, it is easiest to regard the summands of $\Delta_\pm$ as exterior powers of *vectors*.) The skew-symmetry guarantees that the η_i span an *isotropic* subspace in (3.1). If $\xi_0 \neq 0$, this subspace will be maximal and thereby define $J \in Z_n^+$.

In general, the matrix (ξ_{ij}) defines an element of the *tangent space* to Z_n^+ at J, a $\mathrm{U}(n)$-module identified with $\Lambda^{2,0}$; cf. (2.9). The point of Z_n^+ determined by (3.5) with $\xi_0 = 1$ is parametrized by an affine space, and it is evident that it cannot cover all of Z_n^+. If $\xi_0 = 0$ and $n \geqslant 3$, the η_i are not even linearly independent, and we need to add more forms. We are therefore forced to consider 1-forms arising from further summands in (3.3). Applying interior and exterior products, we obtain the additional elements

$$\eta_{i_1\ldots i_q} = \sum_{k=1}^{q}(-1)^{k-1}\xi_{i_1\ldots\hat{i_k}\ldots i_q}dz^{i_k} - \sum_{m=1}^{n}\xi_{i_1\ldots i_q m}d\overline{z}^m, \qquad q \geqslant 3 \text{ odd}, \tag{3.6}$$

where the notation $\hat{i_k}$ means to omit this index.

In order that $\boldsymbol{\xi}$ be a pure spinor, the η-s (recall these are the α_ℓ-s in (2.4)) must span an *isotropic* subspace. In particular, the forms (3.5) and (3.6) must be mutually isotropic. This provides us with the scheme of equations

$$\xi_0\xi_{i_1\ldots i_p} \;=\; \sum_{k=1}^{p-1}(-1)^{k-1}\xi_{i_k i_p}\xi_{i_1\ldots\hat{i_k}\ldots i_{p-1}}, \qquad p \geqslant 4 \text{ even}. \tag{3.7}$$

(The last equation is a tautology if $p = 2$, which helps to check the signs.) If $\xi_0 = 1$, we have

$$\eta_{i_1\ldots i_q} = \sum_{k=1}^{q}(-1)^{k-1}\xi_{i_1\ldots\hat{i_k}\ldots i_q}\eta_{i_k},$$

and in this case isotropy is manifest.

Remark 3.1. If $\xi_0 \neq 0$, then the remaining components of $\boldsymbol{\xi}$ in (3.3) are determined by its projection β to $\Lambda^{2,0}$. Indeed, it follows from (3.7) that $\boldsymbol{\xi}$ can be identified with

$$\xi_0 \exp\beta = \xi_0\big(1 + \beta + \tfrac{1}{2}\beta\wedge\beta + \cdots\big).$$

(This fact is well-known in the context of generalized complex structures; see for example [Gua].) More generally, $\boldsymbol{\xi}$ will have the form $\gamma \wedge \exp\beta$ for some $\gamma \in \Lambda^{2k}$.

Let $Y_n \subset \mathbb{P}^{N-1}$ be the intersection of quadrics defined by the equations (3.7) with $p \geqslant 4$. One can show that these

$$\tilde{N} \;=\; \sum_{p\text{ even}>2}\binom{n}{p} = N - \binom{n}{2} - 1$$

equations are independent. We now define

$$f : Y_n \cap \{\xi_0 \neq 0\} \to Z_n^+, \tag{3.8}$$

by mapping $[\boldsymbol{\xi}]$ to the maximal isotropic subspace $(V_{\boldsymbol{\xi}})^\circ$ of $\mathbb{C}^{2n}$. More explicitly,

$$\begin{aligned}
[\xi_0, \xi_{12}, \ldots, \xi_{1\cdots n}] &\;\mapsto\; \mathrm{span}\{\eta_1, \ldots, \eta_{123}, \ldots, \eta_{2\cdots n}\} \qquad \text{if } n \text{ is even},\\
[\xi_0, \xi_{12}, \ldots, \xi_{2\cdots n}] &\;\mapsto\; \mathrm{span}\{\eta_1, \ldots, \eta_{123}, \ldots, \eta_{1\cdots n}\} \qquad \text{if } n \text{ is odd}.
\end{aligned}$$

There are $\tilde{N}$ equations in $\mathbb{P}^{N-1}$ defining Y_n, so Y_n is a real $n(n-1)$-dimensional manifold. On the other hand, the real dimension of Z_n^+ equals

$$\dim \mathrm{SO}(2n) - \dim \mathrm{U}(n) = n(2n-1) - n^2 = n(n-1).$$

This implies that the image of (3.8) is an open subset of Z_n^+.

We can remove the restriction $\xi_0 \neq 0$ as follows. In the general case, Y_n will have several irreducible components. We need to add all of the quadratic relations from Theorem 2.1; these additional relations will specify a unique irreducible component of Y_n that we will call Y_n^+. This is analogous to the well-known Plücker relations for the orthogonal Grassmannians. In general, the map

$$f : Y_n^+ \to Z_n^+, \quad [\boldsymbol{\xi}] \mapsto (V_{\boldsymbol{\xi}})^\circ \tag{3.9}$$

is well-defined, which allows the possibility that $\xi_0 = 0$.

Theorem 3.2. *The map* (3.9) *is a biholomorphism, where Z_n^+ has the complex structure as a Hermitian symmetric space, and Y_n^+ has the induced complex structure as a complex submanifold of $\mathbb{P}^{N-1}$.*

Proof. This is well-known, see [Ino92]. □

We will henceforth use the map f to identify Y_n^+ and Z_n^+.

3.1. **Low-dimensional cases.** This subsection summarizes the situation for n equal in succession to 2, 3, 4, as this helps to clarify the above discussion. In each case, we identify points of Z_n^+ with positively-oriented maximal isotropic subspaces of $\mathbb{C}^{2n}$.

For $n = 2$, we have

$$\eta_1 = \xi_0 dz^1 - \xi_{12} d\overline{z}^2, \qquad \eta_2 = \xi_0 dz^2 + \xi_{12} d\overline{z}^1. \tag{3.10}$$

The biholomorphism $f : Y_2^+ \to Z_2^+ \cong \mathbb{P}^1$ is given by

$$[\xi_0, \xi_{12}] \mapsto \mathrm{span}\{\eta_1, \eta_2\},$$

and there is no relation between ξ_0, ξ_{12}.

Now suppose that $n = 3$. In addition to

$$\begin{aligned} \eta_1 &= \xi_0 dz^1 - \xi_{12} d\overline{z}^2 - \xi_{13} d\overline{z}^3, \\ \eta_2 &= \xi_0 dz^2 - \xi_{23} d\overline{z}^3 + \xi_{12} d\overline{z}^1, \\ \eta_3 &= \xi_0 dz^3 + \xi_{13} d\overline{z}^1 + \xi_{23} d\overline{z}^2, \end{aligned} \tag{3.11}$$

defined by (3.5), we have

$$\eta_{123} = \xi_{23} dz^1 - \xi_{13} dz^2 + \xi_{12} dz^3, \tag{3.12}$$

since $\xi_{ijkl} = 0$ here. The biholomorphism $f : Y_3^+ \to Z_3^+ \cong \mathbb{P}^3$ is given by

$$[\xi_0, \xi_{12}, \xi_{13}, \xi_{23}] \mapsto \mathrm{span}\{\eta_1, \eta_2, \eta_3, \eta_{123}\},$$

and again there is no constraint.

Finally, suppose that $n = 4$, so that (3.12) is upgraded to

$$\eta_{ijk} = \xi_{jk}dz^i - \xi_{ik}dz^j + \xi_{ij}dz^k + \sum_m \xi_{ijkm}d\overline{z}^m.$$

The biholomorphism $f : Y_4^+ \to Z_4^+$ is given by

$$[\xi_0, \xi_{12}, \xi_{13}, \xi_{14}, \xi_{23}, \xi_{24}, \xi_{34}, \xi_{1234}] \mapsto \mathrm{span}\{\eta_1, \eta_2, \eta_3, \eta_4, \eta_{123}, \eta_{124}, \eta_{134}, \eta_{234}\},$$

and there is now a single quadratic relation,

$$\xi_0\xi_{1234} = \xi_{12}\xi_{34} - \xi_{13}\xi_{24} + \xi_{14}\xi_{23} \tag{3.13}$$

confirming that Z_4^+ is a nonsingular quadric hypersurface in $\mathbb{P}^7$. However, if $\xi_0 = 0$ then the ξ_{ijkm} are independent of the ξ_{ij}.

3.2. **Skew-symmetric orthogonal matrices.** In this subsection, we make explicit the map taking a pure spinor ϕ to a skew-symmetric matrix J_ϕ with Pfaffian 1, discussed at the start of Subsection 2.1.

For the most part, and for the sake of simplicity, we explain the construction in dimension 6. For any $[\xi_0, \xi_{12}, \xi_{13}, \xi_{23}] \in \mathbb{P}^3 = Z_3^+$, the associated maximal isotropic space of $(1,0)$-forms is spanned by (3.11) and (3.12). Its annihilator $T^{0,1}$ is spanned by

$$\begin{aligned} v_1 &= \xi_0\overline{\partial}_1 - \xi_{12}\partial_2 - \xi_{13}\partial_3 \\ v_2 &= \xi_0\overline{\partial}_2 - \xi_{23}\partial_3 + \xi_{12}\partial_1 \\ v_3 &= \xi_0\overline{\partial}_3 + \xi_{13}\partial_1 + \xi_{23}\partial_2 \end{aligned} \tag{3.14}$$

where $\partial_i = \partial/\partial z^i$ and $\overline{\partial}_j = \partial/\partial\overline{z}_j$, together with

$$v_{123} = \xi_{23}\overline{\partial}_1 - \xi_{13}\overline{\partial}_2 + \xi_{12}\overline{\partial}_3.$$

If $\xi_0 \neq 0$, then v_1, v_2, v_3 suffice. On the other hand, if (for example) $\xi_{12} \neq 0$, $T^{0,1}$ is spanned by

$$\begin{aligned} v_2 &= \xi_{12}\partial_1 + \xi_0\overline{\partial}_2 - \xi_{23}\partial_3 \\ v_1 &= -\xi_{12}\partial_2 - \xi_{13}\partial_3 + \xi_0\partial_1 \\ v_{123} &= \xi_{12}\overline{\partial}_3 + \xi_{23}\overline{\partial}_1 - \xi_{13}\overline{\partial}_2 \end{aligned} \tag{3.15}$$

We have arranged (3.15) so that the basis vectors coincide with those in (3.14) after swapping $\overline{\partial}_i \leftrightarrow \partial_i$ for $i = 1, 2$. An inspection of the new coefficients yields

Proposition 3.3. *Let $J = J_\phi$ be the skew-symmetric orthogonal matrix obtained from a projective spinor $\phi = [\xi_0, \xi_{12}, \xi_{13}, \xi_{23}]$, and $J' = J_{\phi'}$ the matrix obtained from $\phi' = [\xi_{12}, -\xi_0, \xi_{23}, \xi_{13}]$. Then $J' = AJA^{-1}$, where $A \in \mathrm{SO}(2n)$ is the matrix corresponding to the transformation $z^1 \mapsto \overline{z}^1$ and $z^2 \mapsto \overline{z}^2$.*

Given this proposition (and analogues for other permutations of the spinor coordinates), we can now concentrate on the case in which $T^{0,1}$ is spanned by (3.14) and we can take $\xi_0 = 1$. We may write

$$v_i = w_i + \sqrt{-1}Jw_i,$$

where w_i are *real* vectors. In this basis, the OCS is represented by the block diagonal matrix

$$\mathbb{J} = \operatorname{diag}(J_0, \dots, J_0), \tag{3.16}$$

where

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.17}$$

represents the standard complex structure on $\mathbb{R}^2$.

We let A denote the real matrix corresponding to the basis change

$$(w_1, Jw_1, w_2, Jw_2, w_3, Jw_3)^\top = A\Big(\tfrac{\partial}{\partial x_1}, \tfrac{\partial}{\partial y_1}, \tfrac{\partial}{\partial x_2}, \tfrac{\partial}{\partial y_2}, \tfrac{\partial}{\partial x_3}, \tfrac{\partial}{\partial y_3}\Big)^\top.$$

A computation shows that

$$A = \begin{pmatrix} -1 & 0 & f_{12} & g_{12} & f_{13} & g_{13} \\ 0 & -1 & g_{12} & -f_{12} & g_{13} & -f_{13} \\ -f_{12} & -g_{12} & -1 & 0 & f_{23} & g_{23} \\ -g_{12} & f_{12} & 0 & -1 & g_{23} & -f_{23} \\ -f_{13} & -g_{13} & -f_{23} & -g_{23} & -1 & 0 \\ -g_{13} & f_{13} & -g_{23} & f_{23} & 0 & -1 \end{pmatrix},$$

where $\xi_{ij} = f_{ij} + \sqrt{-1}g_{ij}$. This discussion yields the

Proposition 3.4. *The skew-symmetric orthogonal matrix corresponding to the projective spinor $\phi = [1, \xi_{12}, \xi_{13}, \xi_{23}]$ is*

$$J_\phi = A\mathbb{J}A^{-1}.$$

We will not write out the entire formula here, but we note that the matrix

$$(1 + |\xi_{12}|^2 + |\xi_{13}|^2 + |\xi_{23}|^2)J_\phi$$

has quadratic entries in the f_{ij} and g_{ij}, which is straightforward to verify. We next consider a special case.

Proposition 3.5. *If $\phi = [1, \xi_{12}, 0, 0]$ then J_ϕ is a product OCS of the form*

$$J = J(\xi_{12}) \oplus J_0,$$

where $J(\xi_{12})$ is the linear OCS on $\mathbb{R}^4 = \{(z^1, z^2, 0) : z^i \in \mathbb{C}\}$ corresponding to $[1, \xi_{12}]$, and J_0 acts on $\mathbb{R}^2 = \{(0, 0, z^3) : z^3 \in \mathbb{C}\}$ as in (3.17).

Proof. A computation shows that as a matrix, J_ϕ equals

$$\frac{1}{1 + |\xi_{12}|^2} \begin{pmatrix} 0 & |\xi_{12}|^2 - 1 & -2g_{12} & 2f_{12} & 0 & 0 \\ 1 - |\xi_{12}|^2 & 0 & 2f_{12} & 2g_{12} & 0 & 0 \\ 2g_{12} & -2f_{12} & 0 & |\xi_{12}|^2 - 1 & 0 & 0 \\ -2f_{12} & -2g_{12} & 1 - |\xi_{12}|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -(1 + |\xi_{12}|^2) \\ 0 & 0 & 0 & 0 & 1 + |\xi_{12}|^2 & 0 \end{pmatrix},$$

which implies the proposition. □

In higher dimensions we have the following. Let

$$\boldsymbol{\xi} = (\xi_0, \xi_{12}, \ldots, \xi_{1234}, \ldots, \xi_{\cdots n}) \in \Delta^+ \tag{3.18}$$

be a pure spinor, with skew-symmetric components (3.4).

Proposition 3.6. *Suppose that all the components of* (3.18) *that contain an index n vanish. Then the associated OCS has the form*

$$J_{\boldsymbol{\xi}} = J_{2n-2} \oplus J_0,$$

where J_{2n-2} is the linear OCS on $\mathbb{R}^{2n-2} = \{(z^1, \ldots z^{n-1}, 0) : z^i \in \mathbb{C}\}$, corresponding to the spinor with components $\xi_{i_i \cdots i_k}$ with $1 \leqslant i_1 < \cdots < i_k \leqslant n-1$.

Proof. The proof is a computation, analogous to the previous proposition. □

4. Integrability of the twistor space

We next explain how to construct a complete set of holomorphic coordinates on the twistor spaces

$$\mathscr{Z} = \mathscr{Z}(\mathbb{R}^{2n}) \subset \mathscr{Z}(S^{2n}) = Z^+_{n+1}$$

discussed in Subsection 2.2. In a sense, we have already done this for Z^+_{n+1}, but we must now base this construction on the twistor fiber Z^+_n so that it is compatible with the fibration (2.12).

Let $\Delta_\pm$, $\tilde{\Delta}_\pm$ denote the spinor bundles for $\mathrm{Spin}(2n)$, $\mathrm{Spin}(2n+2)$ respectively. When we reduce to the former group, it is well-known that $\tilde{\Delta}_+$ restricts to the total spin representation, so that

$$Z^+_{n+1} \subset \mathbb{P}(\tilde{\Delta}_+) = \mathbb{P}(\Delta_+ \oplus \Delta_-). \tag{4.1}$$

The idea is to characterize elements of Z^+_{n+1} in these terms, and then translate (3.2) and (3.3) into explicit formulae.

Theorem 4.1. *Suppose that $0 \neq \phi \in \Delta_+$ and $\psi \in \Delta_-$. Then $[\phi, \psi] \in Z^+_{n+1}$ if and only if $[\phi] \in Z^+_n$ and $\psi = v \cdot \phi$ for some* unique *$v \in \mathbb{R}^{2n}$.*

Recall that $\cdot$ denotes Clifford multiplication, and that the assertion $[\phi] \in Z^+_n$ means that ϕ is a *pure* spinor.

Proof. First observe that if $v \cdot \phi = v' \cdot \phi$ with $v, v' \in \mathbb{R}^{2n}$ then $v - v' \in V_\phi$ which forces $v = v'$ (recall (2.3)). So uniqueness is immediate.

We next prove that if $[\phi] \in Z_n$ then (relative to (4.1)) $[\phi,\, v \cdot \phi] \in Z_{n+1}$. Fix a pure spinor ϕ, and $v \in \mathbb{R}^{2n}$. We may choose a basis (e_i) of $\mathbb{R}^{2n}$ such that $v = e_{2n}$ and the annihilator of ϕ is spanned by $\alpha_k = e_{2k-1} + i e_{2k}$ for $k = 1, \ldots, n$. A quick calculation reveals that the annihilator of the Clifford product $v \cdot \phi$ is spanned by $\alpha_1, \ldots, \alpha_{n-1}, \overline{\alpha}_n$, and it follows that $v \cdot \phi$ is a pure spinor in Δ_-.

Using tildes to refer to $\mathbb{R}^{2n+2}$, we want to show that the "paired spinor"

$$\widetilde{\phi} = (\phi,\ v \cdot \phi) \in \widetilde{\Delta}_+$$

is also pure. To do this, we shall apply Theorem 2.1 to $\widetilde{\phi}\otimes\widetilde{\phi}$, which we need to show belongs to the underlined summand in

$$S^2(\widetilde{\Delta}_+) \;\cong\; \underline{\widetilde{\Lambda}_+^{n+1}} \oplus \widetilde{\Lambda}^{n-3} \oplus \widetilde{\Lambda}^{n-7} \oplus \cdots \tag{4.2}$$

(where $\widetilde{\Lambda}^\ell$ is absent if $\ell < 0$). We shall do this by considering equivariant mappings between irreducible G-modules, where $G = \mathrm{Spin}(2n)$. Schur's lemma is the assertion that any such mapping f is either zero or an isomorphism (and is obvious since the kernel and image of f are G-invariant subspaces).

Now $\widetilde{\phi}\otimes\widetilde{\phi}$ is a sum of

$$\phi\otimes\phi \in \Lambda^n_+, \qquad (v\cdot\phi)\otimes(v\cdot\phi) \in \Lambda^n_-, \tag{4.3}$$

and the "mixed" product, itself a contraction of

$$v\otimes(\phi\otimes\phi) \in \mathbb{R}^{2n}\otimes\Lambda^n_+. \tag{4.4}$$

Using an algorithm to commute the irreducible summands of a tensor product as in [Feg76], we see that the last space contains only one G-summand isomorphic to an exterior power, namely $\Lambda^{n+1} \cong \Lambda^{n-1}$. The exterior powers $\Lambda^k = \bigwedge^k(\mathbb{R}^{2n})$ for $0 \leqslant k \leqslant n-1$, together with Λ^n_+ and Λ^n_-, are of course distinct irreducible G-modules.

Each right-hand summand of (4.2), other than the first, decomposes as

$$\widetilde{\Lambda}^k \;=\; \textstyle\bigwedge^k(\mathbb{R}^{2n}\oplus\mathbb{R}^2) \;\cong\; \Lambda^k \oplus (\Lambda^{k-1}\otimes\mathbb{R}^2) \oplus \Lambda^{k-2},$$

and is the sum of at most four exterior powers, each of degree no greater than $n-3$. It follows that the G-equivariant projections of the elements (4.3) and (4.4) in (4.2) do indeed all lie in the first summand $\widetilde{\Lambda}^{n+1}_+$. The same is therefore true of $\widetilde{\phi}\otimes\widetilde{\phi}$.

To sum up, $(v, [\phi]) \mapsto [\phi,\, v\cdot\phi]$ defines a smooth injective mapping

$$\mathbb{R}^{2n}\times Z_n \to Z_{n+1}.$$

We leave the reader to check that the differential of this mapping has full rank, so that its image is open. A comparison with (2.12) shows that we may identify this image with $\pi^{-1}(\mathbb{R}^{2n})$, in which case the missing fibre $\pi^{-1}(\infty)$ consists of those points $[0, \psi]$ for which ϕ vanishes. □

We can now use the mapping

$$Z^+_{n+1} \setminus Z^+_n \to \mathbb{R}^{2n}, \qquad [\phi,\, v\cdot\phi] \mapsto v$$

to realize the twistor projection. We shall make it explicit in order to parametrize Z^+_{n+1} as a *complex analytic* manifold.

To this aim, we first introduce quantities

$$W_i = \xi_0 z^i - \sum_{k=1}^{n} \xi_{im}\overline{z}^m, \qquad i = 1, \ldots, n \tag{4.5}$$

This definition is an exact parallel of (3.5), and each W_i is merely the contraction of η_i with an arbitrary vector $v \in \mathbb{R}^{2n}$ expressed with coordinates $z^1, \ldots, \overline{z}^n$. It follows that the W_i can be regarded as the components of the Clifford product $v\cdot\boldsymbol{\xi}$ in the first summand $\Lambda^{1,0}$ of (3.3). Next, we treat W_i as a function of both $\boldsymbol{\xi}$ and v.

A crucial observation is that

$$dW_i = \eta_i + z^i d\xi_0 - \sum_{k=1}^{n} \overline{z}^m d\xi_{im}.$$

is a $(1,0)$-form relative to the complex structure on $\mathbb{R}^{2n}$ defined by any point $J \in Z_n^+$ whose homogeneous coordinates in $\mathbb{P}^{N-1}$ start with ξ_0 and the ξ_{ij}. It follows that each W_i is a *holomorphic* function on the twistor space $(\mathscr{Z}(\mathbb{R}^{2n}), \mathscr{J})$; this is because the value $\mathscr{J}$ at $(J, \mathbf{x})$ is defined relative to the complex structure that J itself induces on $\mathbb{R}^{2n}$; recall (2.10).

More generally, and in parallel to (3.6), we define functions

$$W_{i_1\ldots i_q} = \sum_{k=1}^{q} (-1)^{k-1} \xi_{i_1\ldots \hat{i_k} \ldots i_q} z^{i_k} - \sum_{m=1}^{n} \xi_{i_1\ldots i_q m} \overline{z}^m, \qquad q \geqslant 3 \text{ odd}. \tag{4.6}$$

These are effectively the components of $v \cdot \boldsymbol{\xi}$ in $\Lambda^{q,0}$ in (3.3), and the above considerations apply. Representing a point of $\mathbb{C}^n = (\mathbb{R}^{2n}, \mathbb{J})$ by $\mathbf{z} = (z^1, \ldots, z^n)$, we are now in a position to define a map

$$F : Z_n^+ \times \mathbb{R}^{2n} \to Z_{n+1}^+ \subset \mathbb{P}^{2N-1}, \tag{4.7}$$

by

$$([\xi_0, \xi_{12}, \ldots, \xi_{1\cdots n}],\, \mathbf{z}) \mapsto [\xi_0, \xi_{12}, \ldots, \xi_{1\cdots n}, W_1, \ldots, W_{123}, \ldots, W_{2\cdots n}],$$
$$([\xi_0, \xi_{12}, \ldots, \xi_{2\cdots n}],\, \mathbf{z}) \mapsto [\xi_0, \xi_{12}, \ldots, \xi_{2\cdots n}, W_1, \ldots, W_{123}, \ldots, W_{1\cdots n}],$$

according as n is even or odd, respectively.

Example 4.2. When $n = 3$, we are mapping $([\boldsymbol{\xi}], \mathbf{z})$ to

$$[\xi_0, \xi_{12}, \xi_{13}, \xi_{23}, W_1, W_2, W_3, W_{123}], \tag{4.8}$$

where

$$\begin{aligned} W_1 &= \xi_0 z^1 - \xi_{12}\overline{z}^2 - \xi_{13}\overline{z}^3 \\ W_2 &= \xi_0 z^2 - \xi_{23}\overline{z}^3 + \xi_{12}\overline{z}^1 \\ W_3 &= \xi_0 z^3 + \xi_{13}\overline{z}^1 + \xi_{23}\overline{z}^2, \end{aligned} \tag{4.9}$$

(cf. (3.11)), and

$$W_{123} = z^1\xi_{23} - z^2\xi_{13} + z^3\xi_{12}. \tag{4.10}$$

It follows that

$$\xi_0 W_{123} = \xi_{12}W_3 - \xi_{13}W_2 + \xi_{23}W_1, \tag{4.11}$$

which is a reincarnation of (3.13) in the twistor context. Slightly different versions of this quadratic equation will recur repeatedly in the remainder of this paper.

In conclusion,

Theorem 4.3. *The map F is a biholomorphism from $Z_n^+ \times \mathbb{R}^{2n}$ to $Z_{n+1}^+ \setminus Z_n^+$, where $Z_n^+ \times \mathbb{R}^{2n}$ has the complex structure $\mathscr{J}$, and Z_{n+1}^+ is a complex submanifold of $\mathbb{P}^{2N-1}$. The missing Z_n^+ is given by points with the first $N = 2^{n-1}$ coordinates equal to zero, i.e., points of the form $[0, \ldots, 0, W_1, \ldots, W_{\cdots n}]$.*

By adding the missing twistor fiber over the point at infinity, we obtain

Corollary 4.4. *The map F can be extended to a biholomorphism*

$$\hat{F} : \mathscr{Z}(S^{2n}) \to Z^+_{n+1}.$$

The map $\hat{F}$ is then another realization of the fibration (2.12).

At this juncture, as an application, we state and prove a general integrability result. Although this is fairly well-known, and was proved by the second author in [Sal85], it is readily formulated in the language of this section.

Proposition 4.5. *Let J be an orthogonal almost complex structure defined on an open set $\Omega \subset S^{2n}$. Then J is integrable if and only if the graph $J(\Omega)$ is a holomorphic n-fold in $(\pi^{-1}(\Omega), \mathscr{J})$.*

Proof. The invariant nature of the statement of the proposition makes it sufficient for us to prove it locally. We may therefore assume that the space of $(1,0)$-forms for J is generated by the 1-forms η_i defined in (3.5) with $\xi_0 = 1$. In this way, J is entirely determined by a skew-symmetric matrix (ξ_{ij}) of smooth functions on Ω.

The graph of J will be holomorphic if and only if the ξ_{ij} depend holomorphically on the W_s-s, i.e.

$$0 = \frac{\partial \xi_{ij}}{\partial \overline{W}_s} = \sum_l \frac{\partial z^l}{\partial \overline{W}_s} \xi_{ij,l} + \sum_m \frac{\partial \overline{z}^m}{\partial \overline{W}_s} \xi_{ij,\overline{m}}, \qquad s = 1, \dots, n.$$

We can use (4.5) to find the Jacobian matrix

$$\frac{\partial (W_r, \overline{W}_s)}{\partial (z^l, \overline{z}^m)} = \begin{pmatrix} I & -(\xi_{rm}) \\ -(\xi_{sl}) & I \end{pmatrix}.$$

Inverting this produces (up to a determinant) the "same" matrix with no minus signs. Thus $\partial z^l / \partial \overline{W}_s = \xi_{sl}$ and the condition is $\Xi_{ijk} = 0$ where

$$\Xi_{ijs} = \xi_{ij,\overline{s}} + \sum_l \xi_{ij,l} \xi_{sl}, \tag{4.12}$$

and the commas indicate "Euclidean" partial differentiation.

We next compute

$$d\eta_i = -(\xi_{ij,l} dz^l + \xi_{ij,\overline{l}} d\overline{z}^l) \wedge d\overline{z}^j.$$

For integrability, we need $d\eta_i(v_r, v_s) = 0$ for all i, r, s, where the vectors

$$v_r = \overline{\partial}_r - \sum_{k=1}^n \xi_{rk} \partial_k, \tag{4.13}$$

span $T^{0,1}$, as in (3.14). A computation shows that integrability of J is then equivalent to the condition that

$$\Xi_{ijk} = \Xi_{ikj}.$$

But since Ξ_{ijk} is skew-symmetric i, j, we obtain

$$\Xi_{ijk} = -\Xi_{jik} = -\Xi_{jki} = \Xi_{kji} = \Xi_{kij} = -\Xi_{ikj} = -\Xi_{ijk},$$

which implies that $\Xi_{ijk} = 0$. (The parallel to the proof of the fundamental theorem of Riemannian geometry arises from the fact that (4.13) can be viewed as a covariant derivative of the form $\nabla_{\overline{i}}$.) This completes the proof. □

Remark 4.6. A direct corollary is the non-existence of a global OCS on S^6; see [LeB87]. Our work does not shed further light on the question of whether S^6 admits a complex structure, although it is conceivable that generalizations of the twistor approach might be relevant to this problem. For some intriguing properties of a hypothetical complex structure on S^6, we refer the reader to [HKP00].

5. Warped product structures

Consider $\mathbb{R}^{2n}$ with coordinates $(z^1, \dots, z^n)$, and consider a smooth orthogonal almost complex structure of the form

$$J = J_1 \oplus J_0, \tag{5.1}$$

where $J_1 = J_1(z^1, \overline{z}^1, \dots z^n, \overline{z}^n)$ is an OCS (and so an *integrable* complex structure) defined on

$$\mathbb{R}^{2n-2} = \{z^n = \text{constant}\},$$

and J_0 is the standard OCS on the complementary $\mathbb{R}^2$ spanned by the real and imaginary parts x^n and y^n of z^n. If J is itself integrable, we shall call it a *warped product orthogonal complex structure*. Because there is only one OCS on $\mathbb{R}^2$ up to sign, an equivalent way of saying this is the following:

Definition 5.1. A *warped product OCS* on $\mathbb{R}^{2n}$ is an orthogonal complex structure J on $\mathbb{R}^{2n}$ preserving an orthogonal decomposition $\mathbb{R}^{2n} = \mathbb{R}^{2n-2} \oplus \mathbb{R}^2$ pointwise.

We shall tacitly assume that J_0, J_1 (and so J) are consistent with fixed orientations of the respective Euclidean spaces. Note that a warped product OCS is constant in dimension 4 since there is only one oriented OCS on $\mathbb{R}^2$.

Given a warped product OCS J, the orthogonal projection

$$\pi \colon (\mathbb{R}^{2n}, J) \to (\mathbb{R}^2, J_0)$$

is a *holomorphic* mapping between Hermitian manifolds since $\pi_* \circ J = J_0 \circ \pi_*$. In particular, J_1 defines an integrable complex structure on each fibre $\pi^{-1}(v) \cong \mathbb{R}^{2n-2}$ with $v \in \mathbb{R}^2$. In this way, π determines a complex analytic *deformation* of $(\pi^{-1}(0), J_1)$ in the orthogonal category. If we want the base parametrizing the deformations to be a Euclidean space $\mathbb{R}^{2m} \cong \mathbb{C}^m$ with a constant OCS, there is no restriction in assuming that $m = 1$ as we do, since the general case falls within the scope of Definition 5.1.

Proposition 5.2. *Let J be an almost complex structure of the form* (5.1) *with each J_1 an OCS. Then J is integrable if and only if the mapping*

$$(\mathbb{R}^2, J_0) \to Z^+_{n-1}, \qquad v \mapsto J_1(u, v) \tag{5.2}$$

is holomorphic for each fixed $u \in \mathbb{R}^{2n-2}$.

Proof. The value of J at each point of $\mathbb{R}^{2n}$ corresponds to a pure even spinor as in (3.18). We can ensure that this has the form (5.1) by requiring that all components that contain an index n vanish, as stated in Proposition 3.6. There remain 2^{n-1} (potentially non-zero) components ξ_{ij}, which rightly define an OCS on $\mathbb{R}^{2n-2}$. Let us assume that $\xi_0 \neq 0$, and then scale so that $\xi_0 \equiv 1$. Setting $s = n$ in the integrability equations (4.12), we have

$$\xi_{ij,\overline{n}} + \sum_l \xi_{ij,l}\xi_{nl} = 0.$$

Thus $\xi_{ij,\overline{n}} = 0$, which says that ξ_{ij} is holomorphic in the z^n coordinate, which amounts to the holomorphicity of (5.2) for fixed u. The remaining integrability equations just say that for each z^n fixed, J_1 is integrable as an OCS in $\mathbb{R}^{2n-2}$, which is what we are in any case assuming. □

The importance of the preceding proposition is that it enables one to construct warped product structures with relative ease. The simplest way of doing this is to choose a holomorphic function $f\colon \mathbb{C} \to Z^+_{n-1}$ and then define $J_1(z^n)$ to be the constant OCS on $\pi^{-1}(z) = \mathbb{R}^{2n-2}$ corresponding to $f(z^n)$. We shall do this in some examples below, but first we place our definitions in a more general context.

Let (M, J, g) be a Hermitian manifold of real dimension $2n$. This means that J is a complex structure defining a transformation at each point that is orthogonal relative to the Riemannian metric g, and there is an associated 2-form ω by

$$\omega(X, Y) = g(JX, Y).$$

The Hermitian manifold is *locally conformally Kähler* if there exists a positive function λ in a neighborhood of each point such that $d(\lambda\omega) = 0$. A complementary condition is that J be *cosymplectic*, meaning that

$$*\omega = \frac{1}{n!}\omega^{n-1} \tag{5.3}$$

is closed, $*$ being the Hodge operator. This concept is only useful if $n \geqslant 3$ since "cosymplectic" is equivalent to "Kähler" on a Hermitian manifold of real dimension 4. If general, if M is both locally conformally Kähler and cosymplectic then

$$0 = d(\lambda\omega) \wedge \omega^{n-2} = d\lambda \wedge \omega^{n-1}, \tag{5.4}$$

so λ is constant and M is Kähler.

These properties are easily assessed for the structures of Definition 5.1. Take M to be $\mathbb{R}^{2n}$ with the Euclidean metric $g = g_E$ and let J be a warped product orthogonal complex structure. The Kähler condition $d\omega = 0$ implies that $\nabla J = 0$ where ∇ denotes the Levi Civita connection for g_E. Thus, $\nabla_X J = 0$ for all X, where ∇_X is the Euclidean directional derivative, and any Kähler OCS is necessarily *constant* on $\mathbb{R}^{2n}$. Modulo orientation, such a J defines a *horizontal* section of the twistor space $\mathscr{Z}$ in (2.7), and is effectively an element of Z^+_n. The next result shows that, of the weaker conditions considered above, the cosymplectic one is more relevant to the warped product situation.

Proposition 5.3. *Let J be a warped product OCS on $\mathbb{R}^{2n}$ with $n \geqslant 3$.*

(i) *J is locally conformally Kähler if and only if it is constant on $\mathbb{R}^{2n}$.*
(ii) *J is cosymplectic if and only if J_1 is cosymplectic on each $\mathbb{R}^{2n-2}$.*

Proof. We first prove (ii). Write $\omega = \omega_1 + \omega_0$, where $\omega_0 = dx^n \wedge dy^n$ is closed, so

$$\omega^{n-1} = \omega_1^{n-1} + (n-1)\omega_0 \wedge \omega_1^{n-2}. \tag{5.5}$$

The first term on the right is a volume form on $\mathbb{R}^{2n-2}$, and so globally constant. Equation (5.3) implies that

$$n!\, d(*\omega) = (n-1)(n-2)\omega_0 \wedge (d\omega_1 \wedge \omega_1^{n-3}). \tag{5.6}$$

The vanishing of the right-hand side is equivalent to asserting that J_1 is cosymplectic.

In (i) the "if" statement is trivial. So suppose that J is locally conformally Kähler. Since $H^1(\mathbb{R}^{2n}) = 0$, the qualification "locally" can be dropped and we may suppose that $d(\lambda\omega) = 0$, in which the conformal factor λ is defined globally. It follows that $d\lambda \wedge \omega_0 = 0$ since this is the only term in the exterior derivative of $\lambda\omega$ divisible by $dx^n \wedge dy^n$. Hence $\lambda = \lambda(x^n, y^n)$ is constant on each $\mathbb{R}^{2n-2}$. It follows that J_1 is Kähler and constant on each $\mathbb{R}^{2n-2}$. By (ii), J is cosymplectic and from the argument (5.4), we conclude that J is also Kähler and constant on $\mathbb{R}^{2n}$. □

Remark 5.4. When $n = 2$, both statements are trivial since, as previously remarked, a warped product OCS must be constant in this case. When $n = 3$, J is always cosymplectic since the 4-dimensional Liouville theorem in [SV09] implies that J_1 must be constant. Part (ii) is motivated by an example in [BW03].

Part (i) is also a consequence of a more general but well-known result: a conformally flat Kähler metric in dimension $2n \geqslant 6$ is flat [YM55, Theorem 4.1], [Bes87, Proposition 2.68]. This implies that a conformally Kähler J will necessarily be Kähler relative to g_E, and therefore constant by the discussion preceding Proposition 5.3. By contrast, there *do* exist non-flat conformally flat Kähler metrics in dimension 4 including, for example, a complete product one on $S^2 \times H^2 \cong \mathbb{R}^4 \setminus \mathbb{R}$ that plays a key role in the classification of OCSes in dimension 4 [Pon92, SV09].

We next look at some non-trivial examples of warped product structures. In dimension 6, a warped product structure is obtained from the spinor

$$\phi = [\xi_0, \xi_{12}, \xi_{13}, \xi_{23}] = [\xi_0, \xi_{12}, 0, 0]. \tag{5.7}$$

This gives an OCS on $\mathbb{R}^4 = \{z^3 = \text{constant}\}$ and, as remarked above, this must be a *constant* orthogonal complex structure. Assuming $\xi_0 \neq 0$, Proposition 5.2 tells us that a warped product OCS arises from

$$\phi = [1, \xi_{12}(z^3),\, 0,\, 0], \tag{5.8}$$

with $\xi_{12}(z^3)$ a *holomorphic* function of z^3.

Remark 5.5. If we simply take $\xi_{12}(z^3) = z^3$ in (5.8), then we can identify

$$\mathbb{R}^6 = \mathbb{R}^4 \times \mathbb{C} \subset \mathbb{R}^4 \times \mathbb{P}^1 = \mathscr{Z}$$

as a complex submanifold of the twistor space of $\mathbb{R}^4$ (see (2.7)). In this case, by rescaling vertically, (5.1) extends to a Hermitian structure on $\mathscr{Z}$, the fiber $\mathbb{P}^1$ being a conformal compactification of $\mathbb{C}$. If we also rescale the resulting metric horizontally, it extends further to the standard Hermitian structure of $Z_3^+ = \mathbb{P}^3$.

Combining a similar argument with Theorem 4.3 in higher dimensions, we see that the choice of a rational curve $\mathbb{P}^1 \subset Z_{n-1}^+$ (and a marked point to remove) exhibits a warped product $(\mathbb{R}^{2n}, J)$ as a complex submanifold of Z_n^+. The warped product construction can be viewed as a generalization of these examples.

In dimension 8, a pure spinor class

$$\phi = [\xi_0, \xi_{12}, \xi_{13}, \xi_{14}, \xi_{23}, \xi_{24}, \xi_{34}, \xi_{1234}],$$

must lie on the quadric (3.13). A warped product arises from

$$\phi = [\xi_0, \xi_{12}, \xi_{13}, 0, \xi_{23}, 0, 0, 0]$$

Restricted to a hyperplane $\{z^4 = \text{constant}\}$, this will give an OCS on $\mathbb{R}^6$, with each component depending holomorphically on the coordinate z^4.

A special case will occur when the induced OCSes on the hyperplanes $z^4 = \text{constant}$ are warped products in the same direction, that is $\xi_{13} = \xi_{23} = 0$, and $\xi_{12} = \xi_{12}(z^3)$. This will look like

$$[\xi_0(z^3, z^4), \xi_{12}(z^3, z^4), 0, 0, 0, 0, 0, 0].$$

Writing $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$, these are warped products of the form

$$J = J_1 \oplus \widetilde{J}_0,$$

where J_1 is a constant OCS on each 4-dimensional hyperplane for which both z^3, z^4 are constant and $\widetilde{J}_0$ is the standard OCS on each complementary $\mathbb{R}^4$. By Proposition 5.2, it now suffices to take the map $(\mathbb{R}^4, \widetilde{J}_0) \to \mathbb{P}^1$ determined by J_1 to be holomorphic.

Question 5.6. *If J is an finite energy OCS on $\mathbb{R}^{2n}$, then is $\pm J$ conformally equivalent to a warped product OCS of the form* (5.1)*?*

In dimension 4, such an OCS is constant [Woo92, SV09], and no finite energy assumption is necessary. The main result of this paper is that the answer is *yes* in dimension 6. There does not seem to be any other obvious way to manufacture examples in higher dimensions other than the warped product construction, which would lead one to conjecture the answer might be *yes* in general. However, the twistor spaces become increasingly more complicated as the dimension grows, giving more room for the possibility of complicated subvarieties which could be graphs over $\mathbb{R}^{2n}$.

For example, if J_{2n} is an OCS defined globally on $\mathbb{R}^{2n}$, then the graph of J_{2n} lies in $Z_{n+1}^+ \setminus Z_n^+$, where the Z_n^+ is the twistor fiber over the point at infinity. The closure will add some subvariety of Z_n^+ of complex dimension $n-1$. But since Z_n^+ is the twistor space of $\mathbb{R}^{2n-2}$, this will correspond to some OCS J_{2n-2} on some subset of $\mathbb{R}^{2n-2}$. It can happen that J_{2n} is a warped product involving a deformation of J_{2n-2}, but it is possible that there are examples in higher dimensions where this fails.

5.1. **Warped product structures on tori.** In this subsection, we discuss the construction of the examples in Theorem 1.3.

As mentioned in the Introduction, if J is an OCS on a flat 4-torus (T^4, g_4), then J lifts to a constant OCS on $(\mathbb{R}^4, g_E)$, where g_E is the Euclidean metric. Consequently, the OCSes on (T^4, g_4) compatible with a fixed orientation are parametrized by $Z_2^+ = \mathbb{P}^1$. Take an elliptic curve (T^2, J_2) with a compatible flat metric g_2, and consider the product torus $(T^6, g_6) = (T^4 \times T^2, g_4 \oplus g_2)$.

We endow T_6 with a warped product OCS

$$J_6 = J_4 \oplus J_2,$$

where J_4 is determined by a holomorphic map $f : (T^2, J_2) \to Z_2^+ \cong \mathbb{P}^1$, which is the same as a meromorphic function on $\mathbb{C}$ invariant under the corresponding lattice. Such functions are exactly quotients of translated σ-functions, see [Ahl78, Chapter 7], and are non-algebraic if not constant. Thus, if non-constant, these structures must have infinite energy when lifted to $\mathbb{R}^6$, since Bishop's Theorem says that finite energy implies algebraic, see Section 8.

In dimension 8, we can perform the following construction. We can consider $(T^8, g_8) = (T^6 \times T^2, g_6 \oplus g_2)$, and endow this with a warped product structure

$$J_8 = J_6 \oplus J_2.$$

If we take J_6 to be a constant OCS on T^6, then it is determined by a holomorphic mapping $(T^2, J_2) \to Z_3^+ \cong \mathbb{P}^3$. However, this need not be so; for example, J_6 could itself be a warped product OCS on T^6. In this case, Proposition 5.2 tells us that J_8 arises from a holomorphic mapping from (T^2, J_2) into the space of meromorphic functions of fixed degree on another elliptic curve.

In a similar fashion, one can construct increasingly complicated warped product structures on tori in all higher even dimensions.

6. Asymptotically constant structures

In this section we prove Theorem 1.5. We first make some remarks in the case of dimension 6, and then give the general argument.

Let J be an OCS defined globally on $\mathbb{R}^6$ with the correct orientation. Then $J(\mathbb{R}^6)$ is a smooth variety inside $Z_4^+ = Q^6$. Using the notation of Example 4.2, the graph $J(\mathbb{R}^6)$ is given by the expression (4.8). Its closure is found by taking all limits of sequences (z_j^1, z_j^2, z_j^3), where at least one of the sequences z_j^i approaches infinity as $j \to \infty$. This will of course only add points of the form

$$[0, 0, 0, 0, W_1, W_2, W_3, W_{123}],$$

which are points in $\mathbb{P}^3_\infty = \pi^{-1}(\infty)$.

We next consider some special cases. First, if $\xi_{12}, \xi_{13}, \xi_{23}$ are constant, by a conformal transformation, we may assume that $\xi_{12} = \xi_{13} = \xi_{23} = 0$. The graph is then simply $[\xi_0, 0, 0, 0, \xi_0 z^1, \xi_0 z^2, \xi_0 z^3, 0]$. Clearly, the closure adds the $\mathbb{P}^2$ given by

$$[0, 0, 0, 0, W_1, W_2, W_3, 0],$$

whereas the $\mathbb{P}^3$ given by

$$[\xi_0, 0, 0, 0, W_1, W_2, W_3, 0] \tag{6.1}$$

is exactly the closure of the graph of the constant OCS $\mathbb{J}$ on $\mathbb{R}^6$ corresponding to $\phi = [1, 0, 0, 0]$.

Proposition 6.1. *If J is an OCS on $\mathbb{R}^6$ which is asymptotic to the one defined by $\phi = [1, 0, 0, 0]$, the closure of the graph of J adds the $\mathbb{P}^2$ given by*

$$[0, 0, 0, 0, W_1, W_2, W_3, 0]. \tag{6.2}$$

Moreover the closure of the graph of J is homeomorphic to $\mathbb{P}^3$.

Proof. The asymptotically constant condition means that the distance (in a suitable metric) between the graph of J and the graph of the constant OCS goes to zero as $z \to \infty$, so the closure must add the same points. There is then an obvious homeomorphism between the graph of J and the $\mathbb{P}^3$ given in (6.1). □

The same idea also works in higher dimensions.

Proposition 6.2. *Let J be a constant OCS on $\mathbb{R}^{2n}$. Then $\pm J$ is isometrically equivalent to the OCS $\mathbb{J}$ corresponding to $[1, 0, \ldots, 0] \in Z_n^+$. Furthermore, the closure of the graph of $\mathbb{J}$ adds the $\mathbb{P}^{n-1}$ in the fiber over infinity given by*

$$[W_1, \ldots, W_n, 0, \ldots, 0], \tag{6.3}$$

that is, all $W_* = 0$ if the multi-index $*$ is of length 3 or greater.

Proof. The orthogonal group $\mathrm{SO}(2n)$ acts transitively on Z_n^+ [Car81], so we can simply rotate to arrange that $J = \mathbb{J}$. Next, consider the $\mathbb{P}^n \subset Z_{n+1}^+$, call it P, defined by $\xi_* = 0$ for any multi-index $*$ of length 2 or greater, and $W_* = 0$ if the multi-index $*$ is of length 3 or greater. That is, in the $[\xi, W]$ coordinates on Z_{n+1}^+, this is given by

$$[(\xi_0, 0, \ldots, 0), (W_1, \ldots W_n), 0, \ldots, 0].$$

We claim that P is the closure of the graph of $\mathbb{J}$. To see this, using our twistor coordinates, the graph of $\mathbb{J}$ over $\mathbb{R}^{2n}$ is given by

$$[(\xi_0, 0, \ldots, 0), (\xi_0 z^1, \ldots, \xi_0 z^n, 0, \ldots, 0)].$$

To find the closure, we take all possible limits as $z \to \infty$, and this clearly adds all points in (6.3). □

Without loss of generality we may therefore assume that J is asymptotic to $\mathbb{J}$.

Proposition 6.3. *If J is an OCS on $\mathbb{R}^{2n}$ which is asymptotic to $\mathbb{J}$, then the closure of the graph of J adds the same $\mathbb{P}^{n-1}$ to the fiber over infinity as does $\mathbb{J}$. Moreover, the closure of the graph of J is homeomorphic to $\mathbb{P}^n$.*

Proof. Exactly as in the six-dimensional case, the asymptotically constant condition means that the distance (in a suitable metric) between the graph of J and the graph of the constant OCS goes to zero as $z \to \infty$, so the closure must add the same points as does $\mathbb{J}$. Since both the graphs of J and $\mathbb{J}$ hit every other twistor fiber in a single

point, there is then an obvious homeomorphism between the graph of J and the $\mathbb{P}^n$ given in (6.3) corresponding to $\mathbb{J}$. □

Proposition 6.4. *If J is an OCS on $\mathbb{R}^{2n}$ which is asymptotic to $\mathbb{J}$, then the closure of the graph of J is a linear $\mathbb{P}^n$.*

Proof. From the previous proposition, the assumption implies that $X = \overline{J(\mathbb{R}^{2n})}$ is homeomorphic to $\mathbb{P}^n$, which is contained inside the twistor space $Z^+_{n+1} \subset \mathbb{P}^{2N-1}$ where $N = 2^{n-1}$. Since X is homeomorphic to a manifold, it satisfies Poincaré duality. Also, taking the closure adds $X_0 = \mathbb{P}^{n-1}$ inside the fiber at infinity, so by the Thullen–Remmert–Stein Theorem, X is necessarily a variety [Thu35, RS53].

We denote by H the hyperplane section class on $\mathbb{P}^{2N-1}$, and also its pullbacks to subvarieties. Then the cup product $H^{n-1} \cup X_0$ equals 1 on X. Since $H^2(X)$ has rank 1 with generator the (Poincaré dual of) X_0, we have $H = kX_0$ in cohomology, which implies that $k^{n-1}X_0^n = 1$. In view of the integrality of intersection numbers, $k = 1$ so $H = X_0$. This means that $H^n = 1$ on X, so the degree of X is 1, which implies that X is a linear $\mathbb{P}^n$ in $\mathbb{P}^{2N-1}$; see [GH94, page 174]. □

Proposition 6.5. *If J is an OCS on $\mathbb{R}^{2n}$ which is asymptotic to $\mathbb{J}$, then the closure of the graph of J is the same linear $\mathbb{P}^n$ as that which corresponds to $\mathbb{J}$.*

Proof. From the previous proposition, X is a linear $\mathbb{P}^n \subset Z^+_{n+1} \subset \mathbb{P}^{2N-1}$ where $N = 2^{n-1}$. Therefore, there are exactly $2N - 1 - n$ linear equations defining X, which we write as

$$\mathbf{a}_j \cdot \boldsymbol{\xi} + \mathbf{b}_j \cdot \mathbf{W} = 0, \quad j = 1, 2, \ldots, 2^n - 1 - n$$

(with slight abuse of notation). If we restrict these equations to the fiber over infinity given by $\xi_* = 0$, we have

$$\mathbf{b}_j \cdot \mathbf{W} = 0, \quad j = 1, 2, \ldots, 2^n - 1 - n.$$

However, we know that these equations must define the $\mathbb{P}^{n-1}$ from (6.3), which is the condition that $W_* = 0$ if the multi-index $*$ is of length 3 or greater. Consequently, by taking linear combinations, we may regroup the defining equations as

$$\mathbf{a}'_j \cdot \boldsymbol{\xi} + W_{*_j} = 0, \quad j = 1, 2, \ldots, 2^{n-1} - n, \tag{6.4}$$

where $*_j$ is a multi-index of length 3 or greater (and j now labels these multi-indices), together with

$$\mathbf{a}''_j \cdot \boldsymbol{\xi} = 0, \quad j = 1, 2, \ldots, 2^{n-1} - 1. \tag{6.5}$$

All these equations must be linearly independent (else our equations would define a subspace $\mathbb{P}^l$, $l > n$). The collection (6.5) therefore specifies a single point in the fiber over the origin. When we restrict (6.4) to the origin (so $W_{*_j} = 0$) we cannot have $\boldsymbol{\xi} = 0$. Consequently, so we must be able to use (6.5) to rid (6.4) of all ξ terms. We have therefore reduced the equations to the form (6.5) and

$$W_{*_j} = 0, \quad *_j \text{ is a multi-index of length 3 or greater}, \tag{6.6}$$

Recall from (4.6) that if $*$ is a multi-index of length $q \geqslant 3$ then

$$W_* = \xi_- \cdot z + \xi_+ \cdot \overline{z} \tag{6.7}$$

where $-, +$ represent multi-indices of lengths $q-1, q+1$. But ξ is determined by (6.5), and the only way that (6.7) can vanish is if all of the coefficients of z and $\overline{z}$ are zero. From (6.6), this is true for all odd multi-indices of length 3 or greater, which shows that $\xi_* = 0$ for all even multi-indices of length 2 or greater. Therefore, the $\mathbb{P}^n$ is question is the same as that corresponding to $\mathbb{J}$. □

This completes the proof of Theorem 1.5.

Remark 6.6. We shall show in Section 8 that the closure of $J(\mathbb{R}^6)$ for any globally defined J adds a $\mathbb{P}^2$ in the twistor fiber over infinity. However, $X = \overline{J(\mathbb{R}^6)}$ can have singularities, and will not necessarily be homeomorphic to a manifold. The above proof will not work since X will then not necessarily satisfy Poincaré duality.

7. The twistor fibration to S^6

In this section, we shall provide an explicit matrix representation of the Clifford multiplication

$$\mathbb{R}^6 \otimes \Delta_\pm \to \Delta_\mp. \tag{7.1}$$

We shall then use this, in accordance with Theorem 4.1, to describe elements of the twistor space Q^6 of S^6. The results of this section can also be interpreted in terms of Cayley numbers, but the approach we adopt will provide an effective description of the action of the conformal group.

To emphasize the symmetry underlying the definitions of the functions W_1, W_2, W_3 and W_{123}, we introduce the following 4-vectors for exclusive use in this section:

$$\begin{aligned} \boldsymbol{\xi} &= (\xi_0,\, \xi_{12}, -\xi_{13},\, \xi_{23}) \in \Delta_+ \\ \mathbf{W} &= (-W_{123}, W_3, W_2, W_1) \in \Delta_-. \end{aligned} \tag{7.2}$$

They will be regarded as rows or columns, according to context. The choice of order and signs here represents a compromise between our previous ordering and a suitable canonical form for the matrix that follows.

The four equations (4.9), (4.10) can now be combined into the form $\mathbf{W} = M(\mathbf{z})\boldsymbol{\xi}$, where

$$M(\mathbf{z}) = \begin{pmatrix} 0 & -z^3 & -z^2 & -z^1 \\ z^3 & 0 & -\overline{z}^1 & \overline{z}^2 \\ z^2 & \overline{z}^1 & 0 & -\overline{z}^3 \\ z^1 & -\overline{z}^2 & \overline{z}^3 & 0 \end{pmatrix} \tag{7.3}$$

parametrizes a point in the Euclidean space $\mathbb{C}^3 = \mathbb{R}^6$. To emphasize this, we let $\mathscr{E}$ denote the set of all such matrices $M(\mathbf{z})$ with $\mathbf{z} = (z^1, z^2, z^3) \in \mathbb{C}^3$. Note that $\mathscr{E}$ is a *linear* subspace of the Lie algebra $\mathfrak{so}(4, \mathbb{C})$ of skew-symmetric complex 4×4 matrices.

With the adjusted conventions (7.2), the biholomorphism $F : Z_3^+ \times \mathbb{R}^6 \to Z_4^+ \setminus Z_3^+$ defined by (4.7) is given by mapping $([\boldsymbol{\xi}], \mathbf{z})$ to

$$[\boldsymbol{\xi},\, M(\mathbf{z})\boldsymbol{\xi}] = [\xi_0,\, \xi_{12}, -\xi_{13}, \xi_{23},\, -W_{123}, W_3, W_2, W_1], \tag{7.4}$$

rather than (4.8). The equation (4.11) characterizing Z_4^+ can be neatly written

$$\boldsymbol{\xi} \cdot \mathbf{W} = 0, \tag{7.5}$$

relative to the standard complex bilinear pairing, and the fact that $[\boldsymbol{\xi},\, \mathbf{W}] \in Z_4^+$ is now a consequence of the skew symmetry of $M(\mathbf{z})$.

The link with Clifford algebras is provided by the following result, whose proof is a direct calculation.

Lemma 7.1. *Let* $\mathbf{y}, \mathbf{z} \in \mathbb{C}^3$. *Then*

$$\overline{M(\mathbf{y})}M(\mathbf{z}) + \overline{M(\mathbf{z})}M(\mathbf{y}) = -2\,\mathfrak{Re}\langle \mathbf{y}, \mathbf{z}\rangle I, \tag{7.6}$$

where I *is the identity matrix, and* $\langle \mathbf{y}, \mathbf{z}\rangle = \sum_{i=1}^3 y^i \overline{z}^i$.

It will be convenient to denote by $\mathscr{E}^*$ the subset $\mathscr{E} \setminus \{0\}$, and set

$$\hat{\mathscr{E}} = \{M(\mathbf{z}) \in \mathscr{E}^* : \|\mathbf{z}\| = 1\},$$

where $\|\mathbf{z}\|^2 = \sum_{i=1}^3 |z^i|^2$. If $M = M(\mathbf{z}) \in \hat{\mathscr{E}}$ then its columns are orthonormal in the Hermitian sense, and $M \in U(4)$. Indeed, the lemma implies that $M\overline{M}^\top = \|\mathbf{z}\|^2 I$, and a direct calculation confirms that

$$\det M(\mathbf{z}) = \|\mathbf{z}\|^4. \tag{7.7}$$

The next result establishes a curious link with the way in which linear OCSes are themselves represented by matrices via (2.1).

Lemma 7.2. $\hat{\mathscr{E}} = \{M \in \mathrm{SU}(4) \cap \mathfrak{so}(4, \mathbb{C}) : \mathrm{Pf}\, M = 1\}$.

Proof. It is already clear that (7.3) belongs to both $\mathrm{SU}(4)$ and $\mathfrak{so}(4, \mathbb{C})$, and a standard formula for the Pfaffian shows that $\mathrm{Pf}\, M(\mathbf{z}) = 1$. Suppose that $M \in \mathrm{SU}(4) \cap \mathfrak{so}(4, \mathbb{C})$. Take the first column of M as in (7.3). The second column must then coincide with that of (7.3), except that $\overline{z}^1$ and $-\overline{z}^2$ are possibility mutliplied by a complex number λ of modulus 1. Analogous statements hold for columns 3 and 4 with the *same* λ that must satisfy $\lambda^2 = 1$. The only choice is to change all signs in the lower 3×3 block, and this reverses the sign of the Pfaffian. □

To sum up, $\mathscr{E}$ is a cone over one component of the intersection $\mathrm{SU}(4) \cap \mathfrak{so}(4, \mathbb{C})$.

Lemma 7.1 tells us that Clifford multiplication by $\mathbf{z}$ in (7.1) is represented by $M(\mathbf{z})$ on Δ_+ and by $\overline{M(\mathbf{z})}$ on Δ_-. In this light, the next result is a special case of Theorem 4.1, and consolidates various arguments in the previous section.

Theorem 7.3. *If* $[\boldsymbol{\xi}, \mathbf{W}] \in Q^6$ *and* $\boldsymbol{\xi} \neq 0$, *then* $\mathbf{W} = M(\mathbf{z})\boldsymbol{\xi}$ *for some* unique $\mathbf{z} \in \mathbb{C}^3$. *The twistor projection* $\pi : Q^6 \setminus \mathbb{P}^3_\infty \to \mathbb{R}^6$ *is given by* $\pi([\boldsymbol{\xi},\, \mathbf{W}]) = \mathbf{z}$, *where*

$$
\begin{aligned}
z^1 &= |\boldsymbol{\xi}|^{-2} \left(\overline{\xi}_0 W_1 + \overline{\xi}_{23} W_{123} + \xi_{13}\overline{W}_3 + \xi_{12}\overline{W}_2\right)\\
z^2 &= |\boldsymbol{\xi}|^{-2} \left(\overline{\xi}_0 W_2 - \overline{\xi}_{13} W_{123} - \xi_{12}\overline{W}_1 + \xi_{23}\overline{W}_3\right)\\
z^3 &= |\boldsymbol{\xi}|^{-2} \left(\overline{\xi}_0 W_3 + \overline{\xi}_{12} W_{123} - \xi_{23}\overline{W}_2 - \xi_{13}\overline{W}_1\right).
\end{aligned}
\tag{7.8}
$$

Proof. Uniqueness essentially follows from Lemma 7.1. More directly, if $M(\mathbf{z})\boldsymbol{\xi} = M(\mathbf{z}')\boldsymbol{\xi}$, then $(M(\mathbf{z}) - M(\mathbf{z}'))\boldsymbol{\xi} = 0$. Assuming $\boldsymbol{\xi} \neq 0$, we obtain

$$
\det(M(\mathbf{z}) - M(\mathbf{z})') = \|\mathbf{z} - \mathbf{z}'\|^4,
\tag{7.9}
$$

from (7.7).

We have already observed that $[\boldsymbol{\xi}, M(\mathbf{z})\boldsymbol{\xi}] \in Q^6$ provided $\boldsymbol{\xi} \neq 0$, and it induces an injective mapping $f : \mathbb{P}^3 \times \mathscr{E} \to Q^6$. As in the proof of Theorem 4.1, we may conclude that the closure of the image of f is obtained by adding a copy of $\mathbb{P}^3$ corresponding to points $[0, \mathbf{W}]$ generating the fiber $\pi^{-1}(\infty) = \mathbb{P}^3_\infty$. In any case, given $[\boldsymbol{\xi},\, \mathbf{W}] \in Q^6$ with $\boldsymbol{\xi} \neq 0$, the existence of $M(\mathbf{z})$ is now guaranteed.

To establish the first equation of (7.8), we proceed as follows. Multiplying the equations in (4.9) and (4.10) by the appropriate coefficients, we obtain

$$
\begin{aligned}
\xi_{12}\overline{W}_2 &= \overline{\xi}_0 \xi_{12}\overline{z}^2 + |\xi_{12}|^2 z^1 - \xi_{12}\overline{\xi}_{23} z^3\\
\xi_{13}\overline{W}_3 &= \overline{\xi}_0 \xi_{13}\overline{z}^3 + |\xi_{13}|^2 z^1 + \xi_{13}\overline{\xi}_{23} z^2\\
\overline{\xi}_0 W_1 &= |\xi_0|^2 z^1 - \overline{\xi}_0 \xi_{12}\overline{z}^2 - \overline{\xi_0}\xi_{13}\overline{z}^3\\
\overline{\xi}_{23} W_{123} &= |\xi_{23}|^2 z^1 - \xi_{13}\overline{\xi}_{23} z^2 + \xi_{12}\overline{\xi}_{23} z^3.
\end{aligned}
$$

Adding these four equations gives the required result:

$$
\xi_{12}\overline{W}_2 + \xi_{13}\overline{W}_3 + \overline{\xi}_0 W_1 + \overline{\xi}_{23} W_{123} = |\boldsymbol{\xi}|^2 z^1.
$$

Given the cyclic symmetry in the components of $\boldsymbol{\xi}$ and $\mathbf{W}$ in (7.8), the second and third equations must also hold.

The fact that the $\mathbf{z}$ defines the twistor projection is a consequence of the theory developed in the previous section. We leave the reader to double check that, having defined $\mathbf{z} = (z^1, z^2, z^3)$ by (7.8), it is indeed true that $\mathbf{W} = M(\mathbf{z})\boldsymbol{\xi}$. □

Remark 7.4. The component $\mathscr{E}^-$ of matrices in $\mathbb{R}^+ \times (\mathrm{SU}(4) \cap \mathfrak{so}(4, \mathbb{C}))$ with negative Pfaffian parametrizes a different set of vertical $\mathbb{P}^3$-s described in Subsection 2.3. If $L(\mathbf{y}) \in \mathscr{E}^-$ is the matrix with first row $(0, -y^3, -y^2, -y^1)$ then

$$
[\boldsymbol{\zeta},\, L(\mathbf{y})\boldsymbol{\zeta}] = [\boldsymbol{\xi},\, M(\mathbf{z})\boldsymbol{\xi}] \;\Rightarrow\; (L(\mathbf{y}) - M(\mathbf{z}))\boldsymbol{\xi} = 0,
$$

which implies that

$$
0 = \det(L(\mathbf{y}) - M(\mathbf{z})) = (\|\mathbf{y}\|^2 - \|\mathbf{z}\|^2) + 2i\, \mathfrak{Im}\, \langle \mathbf{y}, \mathbf{z} \rangle .
$$

If we fix a non-zero vector $\mathbf{y} \in \mathbb{C}^3$ then the set of $\mathbf{z}$ solving this equation is the intersection of an S^5 with a real hyperplane. The corresponding $\mathbb{P}^3$ is the twistor space (with fiber $\mathbb{P}^1$) of this S^4 consisting of a collection of points equidistant from the chosen origin in S^6.

7.1. **The conformal group.** We continue to use the coordinates (7.4) on $\mathbb{P}^7$, and consider first the automorphism group of the quadric Q^6, as defined by (7.5). Up to a finite ambiguity, this is isomorphic to the matrix group $\mathrm{SO}(8,\mathbb{C})$, but is defined in our context by

$$\left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,:\, X^\top \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} X = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right\}, \tag{7.10}$$

which amounts to asserting that

$$A^\top C + C^\top A = 0, \qquad B^\top D + D^\top B = 0, \tag{7.11}$$

and

$$A^\top D + C^\top B = I. \tag{7.12}$$

We shall use these relations shortly.

The double cover of the orientation-preserving conformal group, $\mathrm{Spin}_\circ(1,7)$ (identity component), turns out to be exactly the subgroup of (7.10) consisting of matrices that preserve the twistor fibration. Fix a matrix M in this subgroup, built up from the 4×4 blocks A, B, C, D. Given $M = M(\mathbf{z}) \in \mathscr{E}$, we therefore require that there exists a corresponding $M' = M(\mathbf{z}')$ with the following property. For each $\boldsymbol{\xi} \neq 0$, there exists $\boldsymbol{\xi}'$ such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \boldsymbol{\xi} \\ M\boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\xi}' \\ M'\boldsymbol{\xi}' \end{pmatrix}. \tag{7.13}$$

It follows that

$$C + DM = M'(A + BM), \tag{7.14}$$

provided that $A + BM \neq 0$.

We now list some special subgroups.

(i) If we take $B = C = 0$ then $A^\top D = I$, and (7.14) implies that

$$M \in \mathscr{E} \;\Rightarrow\; DMD^\top \in \mathscr{E}.$$

In particular, we can take $D = \overline{A} \in \mathrm{SU}(4)$, in which case

$$X = \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix}. \tag{7.15}$$

(ii) Again with $B = C = 0$, we can take $D = rI$ with $r \in \mathbb{R}^+$, so that

$$X = \begin{pmatrix} r^{-1}I & 0 \\ 0 & rI \end{pmatrix}. \tag{7.16}$$

(iii) Now suppose that $B = 0$ and $A = I$. Then $D = I$ and

$$X = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}, \tag{7.17}$$

with $C \in \mathscr{E}$.

(iv) Consider a special case in which $A = 0 = D$, namely

$$X = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{7.18}$$

This is the symmetric matrix that defines the quadric itself.

We bring these example together with

Proposition 7.5. *The orientation-preserving conformal group* $\mathrm{SO}_\circ(1,7)$ *is generated by matrices from the previous four cases. Moreover,*

(i) *corresponds to the group* $\mathrm{SO}(6)$ *of rotations fixing* 0 *and* ∞*,*
(ii) *arises from the scaling* $(z^1, z^2, z^3) \mapsto (r^2 z^1, r^2 z^2, r^2 z^3)$*,*
(iii) *corresponds to the translation* $(z^1, z^2, z^3) \mapsto (z^1 - c^1, z^2 - c^2, z^3 - c^3)$*,*
(iv) *arises from inversion in the unit sphere and minus conjugation:*

$$(z^1, z^2, z^3) \to -\|\mathbf{z}\|^{-2}(\overline{z}^1, \overline{z}^2, \overline{z}^3). \tag{7.19}$$

Proof. Take a point $[\boldsymbol{\xi}] \in \mathbb{P}^3$ giving rise to the linear OCS on $\mathbb{R}^6$ whose $(1,0)$ forms are spanned by (3.11) and (3.12). These four equations translate into the formula

$$\boldsymbol{\eta} = M(d\mathbf{z})\,\boldsymbol{\xi}$$

for the vector-valued 1-form $\boldsymbol{\eta}$. To find the action of a conformal map $\mathbf{z} \mapsto \mathbf{z}' = T \circ \mathbf{z}$ we therefore need to compute $M(d\mathbf{z}') = M(T^* d\mathbf{z})$ and define $\boldsymbol{\xi}'$ accordingly. The induced action

$$([\boldsymbol{\xi}],\, \mathbf{z}) \mapsto ([\boldsymbol{\xi}'],\, \mathbf{z}') \tag{7.20}$$

on twistor space can then be converted into a matrix relative to the coordinate system (7.4) using the diffeomorphism F. Roughly speaking, the latter replaces $\mathbf{z}$ in (7.20) by $\mathbf{W} = M(\mathbf{z})\boldsymbol{\xi}$, so that

$$(F \circ T \circ F^{-1})[\boldsymbol{\xi},\, \mathbf{W}] = F([\boldsymbol{\xi}'],\, \mathbf{z}') = [\boldsymbol{\xi}',\, \mathbf{W}],$$

where $\mathbf{W}' = M(\mathbf{z}')\boldsymbol{\xi}'$.

To tackle case (i), we work backwards. Let $A = \overline{D} \in \mathrm{SU}(4)$, and define $\mathbf{z}'$ by

$$M(\mathbf{z}') = DM(\mathbf{z})D^\top. \tag{7.21}$$

It follows from (7.7) that $\mathbf{z} \mapsto \mathbf{z}'$ is an *orthogonal* transformation of $\mathbb{R}^6$, and it must lie in $\mathrm{SO}(6)$ since $\mathrm{SU}(4)$ is connected. In this way, (7.21) neatly expresses the double covering

$$\mathrm{SU}(4) \cong \mathrm{Spin}(6) \to \mathrm{SO}(6).$$

It is appropriate here to set $\boldsymbol{\xi}' = A\boldsymbol{\xi}$, so that

$$\boldsymbol{\eta}' = M(d\mathbf{z}')\boldsymbol{\xi}' = DM(\mathbf{z})\boldsymbol{\xi} = D\boldsymbol{\eta},$$

ensuring that the new 1-forms are linear combinations of the old ones. Then

$$(F \circ T \circ F^{-1})[\boldsymbol{\xi},\, \mathbf{W}] = [A\boldsymbol{\xi},\, M(\mathbf{z}')A\boldsymbol{\xi}] = [A\boldsymbol{\xi},\, \overline{A}\mathbf{W}],$$

and the matrix is (7.15).

Next consider the dilation $z^i \mapsto r^2 z^i$, for $r \in \mathbb{R}_+$. The 1-forms simply scale, so we may take $\boldsymbol{\eta}' = \boldsymbol{\eta}$. Thus,

$$(F \circ T \circ F^{-1})[\boldsymbol{\xi},\, \mathbf{W}] = F([\boldsymbol{\xi}],\, r^2\mathbf{z}) = [\boldsymbol{\xi},\, r^2\boldsymbol{\xi}] = [r^{-1}\boldsymbol{\xi},\, r\mathbf{W}],$$

and after the projective scaling, we recover the matrix (7.16) in $\mathrm{SO}(8,\mathbb{C})$.

For translations, consider $z^i \mapsto z^i - c^i$ for $c^i \in \mathbb{C}$. Clearly the 1-forms are unchanged, so $\boldsymbol{\xi}' = \boldsymbol{\xi}$ whereas

$$\mathbf{z}' = \mathbf{z} - \mathbf{c} \tag{7.22}$$

withe $\mathbf{c} = (c^1, c^2, c^3)$. Moreover,

$$(F \circ T \circ F^{-1})[\boldsymbol{\xi},\, \mathbf{W}] = [\boldsymbol{\xi},\, M(\mathbf{z}-\mathbf{c})\boldsymbol{\xi}] = [\boldsymbol{\xi},\, \mathbf{W} - C\boldsymbol{\xi}],$$

where $C \in \mathscr{E}$. Thus, the lift is given by (7.17) with $-C$ in place of C.

Inversion is defined by the mapping

$$\mathbf{z}' = -\|\mathbf{z}\|^{-2}\overline{\mathbf{z}}.$$

This is faithfully reflected (up to sign) by the inverse of the matrix (7.3), since

$$M(\mathbf{z})^{-1} = -\|\mathbf{z}\|^{-2} M(\overline{\mathbf{z}}) = M(\mathbf{z}'),$$

using the various properties of $\mathscr{E}$. We may now define $\boldsymbol{\xi}' = M(\mathbf{z})\boldsymbol{\xi} = \mathbf{W}$ so that

$$(F \circ T \circ F^{-1})[\boldsymbol{\xi},\, \mathbf{W}] = F([\mathbf{W}],\, \mathbf{z}'] = [\mathbf{W},\, M(\mathbf{z})^{-1}\mathbf{W}] = [\mathbf{W},\, \boldsymbol{\xi}],$$

and we obtain (7.18).

It is well-known that the conformal group is generated by rotations, dilations, translations and inversions [SY94]. Thus, the proof of the proposition is complete. □

8. Threefolds in the 6-quadric

We assume J is an OCS defined on $S^6 \setminus K$, where K is a finite non-empty set of points. By the integrability assumption on J, the graph $J(S^6 \setminus K)$ is a *complex* 3-dimensional submanifold of Q^6. Moreover, it is a submanifold of $\mathbb{P}^7$ minus a subvariety consisting of finitely many vertical $\mathbb{P}^3$-s. We may therefore apply Bishop's theorem to conclude that if the graph of J has finite area then the closure is also a variety [Bis64]. The finite area condition is that

$$\mathcal{H}^6(J(\mathbb{R}^6 \setminus K)) < \infty, \tag{8.1}$$

where $\mathcal{H}^6$ denotes real 6-dimensional Hausdorff measure.

We shall see that (8.1) is in fact implied by the finite energy assumption (1.2). In order to compute the latter, we endow Q^6 with a "twistor" metric as follows. In accordance with (2.8), the tangent space

$$T_q(Q^6) = V_q \oplus H_q \tag{8.2}$$

splits into the tangent space of the fiber over $z = \pi(q)$ and the horizontal subspace determined by the Levi-Civita connection. Give V_q the Fubini–Study metric of the fiber $\mathbb{P}^3$, and H_q the round metric from S^6. It is known that, for an appropriate choice of scaling, this is exactly the Hermitian symmetric metric on Q^6 [Bar75]. We will use

this metric on Q^6 to compute areas, though any Riemannian metric constructed in a similar way would suffice to prove the next result.

Proposition 8.1. *Let J be an OCS on $S^6 \setminus K$, where K is a finite set of points. If J satisfies* (1.2)*, then* (8.1) *is satisfied.*

Proof. Fix $x \in S^6$, and set $q = J(x) \in Q^6$. Let $(J_*)_x$ denote the differential of the smooth mapping $J\colon S^6 \setminus K \to Q^6$ at x. When we identify H_q with T_xS^6 and V_q with a subspace of $\bigwedge^2 T_xS^6$ (as in (2.9)), we may write

$$(J_*)_x(v) = (\nabla_v J,\, v), \qquad v \in T_xS^6.$$

This is because, by its very definition, the vertical component of (8.2) can be identified with the covariant derivative of J. The linear mapping $(J_*)_x$ is now represented by a 12×6 matrix of the form

$$D_x = \begin{pmatrix} \nabla J \\ I \end{pmatrix},$$

where I is the 6×6 identity.

If we set $\Omega = S^6 \setminus K$, it follows that

$$\mathcal{H}^6(J(\Omega)) = \int_\Omega \sqrt{\det D^\top D}\, dx.$$

This is essentially a version of the area formula [Fed69, EG92], also familiar from the classical theory of surfaces in $\mathbb{R}^n$ in which the role of D is played by the $n \times 2$ matrix whose columns are the partial derivatives $\mathbf{x}_u, \mathbf{x}_v$ and $\det(D^\top D) = EG - F^2$.

The present result now follows from the estimate

$$\det(D^\top D) = \det\big(I + (\nabla J)^\top (\nabla J)\big) \leqslant c(1 + \|\nabla J\|^{12}),$$

where c is a universal constant. At each point $x \in S^6$, the transpose $(\nabla J)^\top$ can be interpreted as the adjoint of a linear transformation $T_xS^6 \to \bigwedge^2 T_xS^6$ with respect to the natural inner products on these spaces, which are also used to define the norm $\|\nabla J\|$. This interpretation is consistent with the choice of Riemannian metric on Q^6 defined via (8.2). □

In attempting to prove Theorem 1.2, we may now assume that the closure of the graph of J, namely $\overline{J(\mathbb{R}^6 \setminus K)}$, is an analytic variety. By Chow's Theorem, we may further assume that it is an *algebraic* subvariety of the quadric Q^6 [Cho49]; it has complex dimension 3 and bidegree $(1, p)$, for some integer $p \geqslant 0$.

Now suppose that X is an arbitrary algebraic threefold (an algebraic 3-dimensional subvariety) of Q^6 of bidegree $(1, p)$. We will call a twistor fiber *exceptional* if the intersection of the fiber with X consists of more than one point.

Remark 8.2. Given a twistor fiber F, suppose that the intersection $X \cap F$ is a finite number of points. Then we have

$$1 = X \cdot F = \sum_{z \in X \cap F} \mathrm{mult}_z(X, F).$$

This implies that $X \cap F$ must consist of exactly one point z. Furthermore, X must be smooth at z and X intersects F transversely at z [Ful98, Proposition 8.2 (a),(c)].

A point $p \in S^6$ will be called *exceptional* if the fiber over p is exceptional. It now remains to prove the following result that is a re-statement of Theorem 1.6.

Theorem 8.3. *The space of exceptional fibers of $X \to S^6$ has real dimension at least two, unless X is conformally equivalent to the closure of the graph of a warped product structure defined on $\mathbb{R}^6$.*

Remark 8.4. The space of exceptional points in S^6, call it E, has the structure of a real algebraic variety, and may have several components. Since an algebraic variety has integral dimension, the above theorem can be rephrased to say that there are no algebraic examples with E of dimension one, and if E has dimension zero, it must be a single point, and moreover in this case X corresponds to a warped product. It is clear that X will only define an OCS away from the set of exceptional points. This is because J cannot be defined continuously in a neighborhood of an exceptional point since there are at least two directions which have a different limit.

8.1. **Explicit examples and the classification.** We shall prove Theorem 8.3 by applying the main result of the paper [BV08]. In order for our notation to be consistent with that, we fix attention on the non-degenerate quadric

$$Q^6 = \{[x_1, \dots, x_8] \in \mathbb{P}^7 : x_1x_8 - x_2x_7 + x_3x_6 - x_4x_5 = 0\}. \tag{8.3}$$

In identifying Q^6 with the twistor space of S^6, we shall further suppose that the linear subspace

$$P_0 = \{[x_1, x_2, x_3, x_4, 0, 0, 0, 0]\} \cong \mathbb{P}^3 \tag{8.4}$$

of Q^6 is *vertical* in the sense of Subsection 2.3. Only later will we be able to relate the homogeneous coordinates x^i more explicitly to those in the previous section.

A key role will be played by the singular quadric Q^4_s given by the intersection of (8.3) and $\mathbb{P}^5$ defined by $x_7 = x_8 = 0$. Thus

$$Q^4_s = \{[x_1, \dots, x_6, 0, 0] \in \mathbb{P}^7 : x_3x_6 = x_4x_5\}. \tag{8.5}$$

is defined by a quadratic form of rank 4. We may regard Q^4_s as the union of the subspaces

$$P_\lambda = \{[u_0, u_1, au_2, au_3, bu_2, bu_3, 0, 0]\} \cong \mathbb{P}^3, \tag{8.6}$$

as $\lambda = b/a$ ranges over $\mathbb{P}^1$. Their common intersection is the line

$$L = \{[u_0, u_1, 0, 0, 0, 0, 0, 0]\} \cong \mathbb{P}^1 \tag{8.7}$$

on which $Q^4_s = \mathrm{Ann}(L)$ is singular. Since the family (8.6) includes the space (8.4), it follows that every P_λ is a vertical $\mathbb{P}^3$.

Recalling (2.15), we also need to mention a special irreducible threefold of bidegree $(1, 3)$, constructed as a cone over the Veronese embedding of $\mathbb{P}^2$ in the Grassmannian $\mathbb{G}\mathrm{r}(2, 4)$. The latter is identified with the *smooth* 4-quadric $Q^6 \cap \{x_1 = x_7 = 0\}$ via the Plücker embedding, and the threefold is the image of the weighted projective

space with homogeneous coordinates u_0, u_1, u_2, u_3 and weights $(1, 1, 1, 2)$ under the map given by

$$[u_0, u_1, u_2, u_3] \mapsto [u_3,\, u_0^2,\, u_0u_1,\, u_0u_2,\, u_1^2 - u_0u_2,\, u_1u_2,\, u_2^2,\, 0]. \tag{8.8}$$

It features in the main classification result from [BV08]:

Theorem 8.5 ([BV08, Theorem 2.7])**.** *Every irreducible threefold X of bidegree $(1, p)$ in Q^6 is given by one of the following, up to the action of $\mathrm{Aut}(Q^6) = \mathrm{PSO}(8, \mathbb{C})$:*

(i) *$p = 0$ and X is a horizontal $\mathbb{P}^3$,*
(ii) *$p = 1$ and X is a smooth quadric in $\mathbb{P}^4 \subset \mathbb{P}^7$,*
(iii) *$p = 3$ and X is the cone over the Veronese surface given by* (8.8)*,*
(iv) *$p \geqslant 1$ and X is a Weil divisor in the quadric* (8.5)*.*

Remark 8.6. Different notions of divisor crop up in the study of the singular spaces that play an essential role in our twistor theory. A smooth example of (iv) arises as follows. The zero set in Q^4_s of the irreducible polynomial $x_1x_6 - x_2x_4$ is a union $P_0 \cup D$ where P_0 is defined (as in (8.6)) by $x_3 = x_4 = 0$ and

$$D = \{[au_1, bu_1, au_2, au_3, bu_2, bu_3, 0, 0]\} \cong \mathbb{P}^1 \times \mathbb{P}^2,$$

is the image of the Segre embedding with $[a, b] \in \mathbb{P}^1$ and $[u_1, u_2, u_3] \in \mathbb{P}^3$. In fact, every Weil divisor of degree $(1, p)$ on Q^4_s can be written as the divisor of a polynomial $f(x_1, \ldots, x_6)$ of degree p minus $(p-1)P_0$ [BV08]. This example is generalized by the next proposition.

In [BV08, Section 4], it is shown that in case (iv), we have that

$$Q_\lambda = \overline{X \cap (P_\lambda \setminus L)} \tag{8.9}$$

is a $\mathbb{P}^2$ contained in X. We also need to know

Proposition 8.7 ([BV08, Proposition 4.4])**.** *If X is in case (iv) of Theorem 8.5, then X contains L and is equal to the union of the Q_λ over all $\lambda \in \mathbb{P}^1$. This is almost a disjoint union, in the sense that the only intersections occur at points of L. Furthermore, X is smooth away from L.*

Remark 8.8. A special sub-case of case (iv) is when every Q_λ contains L. In this case, X will be a *double cone* over a $(1, p)$ curve C in a smooth 2-quadric. That is, X consists of all lines through points on C and points of L. This case will play a crucial role in the proof of our main theorem.

We emphasize that the classification in Theorem 8.5 is modulo the action of the automorphism group $\mathrm{PSO}(8, \mathbb{C})$ of Q^6. To prove our main theorem, we need to take into account special properties of the twistor fibration and the conformal group studied in Section 7. Another imported result gives a strong restriction on the intersection of X with a twistor fiber:

Lemma 8.9 ([BV08, Proposition 3.6])**.** *If the intersection of X with a vertical $\mathbb{P}^3$ has a component of complex dimension 2 or more, than this component is unique and is a $\mathbb{P}^2$.*

With these tools, we can now prove our main result.

Proof (of Theorems 8.3 and 1.2). Suppose that we have X such that $X \to S^6$ has at most a real dimension one set of exceptional fibers. Then X must intersect at least one of the twistor fibers in a set of complex dimension at least two, else X and S^6 are isomorphic in real codimension 2. But the square of the hyperplane class is non-zero on X, and $H^4(S^6, \mathbb{Z}) = 0$, so this is not possible.

We identify $\mathbb{R}^6$ with $S^6 \setminus \{\infty\}$. By performing a conformal transformation, we may assume without loss of generality that X intersects the fiber $F_\infty \cong \mathbb{P}^3$ over infinity in a set of complex dimension 2. By Lemma 8.9, this intersection consists of a $\mathbb{P}^2$ and (perhaps) some lower dimensional components. We will now invoke Theorem 8.5.

If $p = 0$ and X is a horizontal $\mathbb{P}^3$, then this corresponds to a constant OCS. Indeed, X hits F_∞ in a $\mathbb{P}^2$, and hits every other fiber in a single point. Using SU(4), we can arrange so that this $\mathbb{P}^2$ is the same as that corresponding to $\mathbb{J}$ in (6.2). Since there is a unique horizontal $\mathbb{P}^3 \subset Q^6$ containing this $\mathbb{P}^2$, this proves that J is conformally equivalent to $\mathbb{J}$.

If $p = 1$ and we have a smooth quadric Q^3, then $X \cap F_\infty$ cannot be a $\mathbb{P}^2$ because the maximal linear subspace contained in Q^3 is a $\mathbb{P}^1$.

If $p = 3$ and we have a cone over the Veronese surface, then this X does not contain any $\mathbb{P}^2$-s either. To see this, the image of a $\mathbb{P}^2$ in the Veronese surface would be a linear subspace of complex dimension one or two. Of course, it is not a $\mathbb{P}^2$, since the Veronese image is not linear. It is not a $\mathbb{P}^1$ either, because all curves in the Veronese surface have even degree. Indeed, there are no lines inside the Veronese surface in $\mathbb{P}^5$.

It remains to consider the case (iv), in which X lies in the image $\tilde{Q}^4_s$ of the quadric (8.5) under a projective transformation of Q^6. We let $\tilde{L}$ denote the singular line of $\tilde{Q}^4_s$, and we continue to denote the fiber $\mathbb{P}^3$ at infinity by F_∞.

Claim 8.10. *The line $\tilde{L}$ must be contained in F_∞.*

Proof. First, we rule out that $\tilde{L}$ sits completely in some fiber F disjoint from F_∞. To see this, if it sits completely in some other fiber F then the span of $\tilde{L}$ and the $\mathbb{P}^2 \subset F_\infty \cap X$ would be an isotropic $\mathbb{P}^4$. Indeed, $\tilde{L}$ and $\mathbb{P}^2$ are both isotropic and orthogonal to each other, because $\tilde{Q}^4_s = \mathrm{Ann}(\tilde{L}) \cap Q^6 \subset \mathrm{Ann}(\tilde{L})$.

Second, if $\tilde{L}$ has a transversal intersection point with a fiber $F \cong \mathbb{P}^3$ different from F_∞ this forces F to be an exceptional fiber. Suppose, on the contrary, that F is not exceptional, and that $z \in F \cap \tilde{L}$ is a transversal intersection point and that z is the unique point of $F \cap X$. Now fix $\lambda \in \mathbb{P}^1$. Since both F and P_λ are vertical, the intersection $F \cap P_\lambda$ must be a $\mathbb{P}^1$. Indeed, two vertical $\mathbb{P}^3$-s are either disjoint, intersect in a $\mathbb{P}^1$, or are the same. Of course, F intersects each P_λ in the point z (since each P_λ contains $\tilde{L}$, and $z \in \tilde{L}$), so the first possibility does not occur. If $F = P_\lambda$, then F would contain Q_λ, and would thus be exceptional, contrary to assumption. Therefore $F \cap P_\lambda = \mathbb{P}^1$. Also, $X \cap P_\lambda$ contains $Q_\lambda = \mathbb{P}^2$. Since any $\mathbb{P}^1 \subset P_\lambda$ and any $\mathbb{P}^2 \subset P_\lambda = \mathbb{P}^3$ must intersect, $F \cap X$ must therefore contain a point of Q_λ. By assumption, this point must be z. This shows that z is contained in *every* Q_λ.

Proposition 8.7 now implies that X is a cone with vertex z, so it is singular at z, since $\deg(X) > 1$. Since it is singular at z, the intersection of X with F at z cannot be transversal (see Remark 8.2), which is a contradiction.

The previous paragraph implies that $\tilde{L}$ cannot have a transversal intersection point with *any* fiber. Indeed, since different points of $\tilde{L}$ belong to different F-s, we would have a real dimension 2 set of exceptional fibers, which contradicts the assumption on the dimension of the exceptional set. Together with the first paragraph, this proves the claim. □

We may now assume that the fiber F_∞ contains $\tilde{L}$. Under this assumption we have the following.

Claim 8.11. *For any twistor fiber F other than F_∞, $\tilde{Q}^4_s \cap F$ is a $\mathbb{P}^1$.*

Proof. To see this, recall that $\tilde{Q}^4_s = Q^6 \cap \mathrm{Ann}(\tilde{L})$, so $\tilde{Q}^4_s \cap F = \mathrm{Ann}(\tilde{L}) \cap F$. This space is orthogonal to $\tilde{L}$, so if this were a $\mathbb{P}^3$ or $\mathbb{P}^2$, we would have too large an isotropic subspace. Finally, $\dim \mathrm{Ann}(\tilde{L}) = 5$ and $\dim(F) = 3$, so the intersection must be at least a line. □

Next, if the projective plane Q_λ does not contain $\tilde{L}$ then it cannot lie entirely in a fiber and by Lemma 2.2 and Claim 8.11, there is a unique exceptional fiber F_λ for which

$$F_\lambda \cap Q_\lambda = F_\lambda \cap \tilde{Q}^4_s \cong \mathbb{P}^1. \tag{8.10}$$

Next, we notice that, among those λ for which Q_λ does not contain $\tilde{L}$, the correspondence $\lambda \mapsto F_\lambda$ is injective. To see this, if $F_{\lambda_1} = F_{\lambda_2} = F$, then (8.10) implies that $F \cap Q_{\lambda_1} = F \cap Q_{\lambda_2}$ is the same $\mathbb{P}^1$. Since distinct Q_λ-s can only intersect at points of $\tilde{L}$ (Proposition 8.7), we must have $Q_{\lambda_1} = Q_{\lambda_2}$, and thus $\lambda_1 = \lambda_2$. Consider the subset of $\mathbb{P}^1$ given by the λ-s for which Q_λ contains $\tilde{L}$. This is an algebraic set, so it either consists of a finite number of points, or is the entire $\mathbb{P}^1$. If it is finite, then there is a real 2-dimensional set of exceptional fibers, which is contrary to assumption. So it must be the entire $\mathbb{P}^1$, and X is a double cone (see Remark 8.8).

Claim 8.12. *If X is a double cone over $\tilde{L} \subset F_\infty$, then for any other twistor fiber F, $X \cap F$ is a point.*

Proof. Otherwise, $X \cap F$ would have to be at least one-dimensional and thus would equal the whole $\mathbb{P}^1 = \tilde{Q}^4_s \cap F$ from Claim 8.11. In this case, X would then contain the span of $\tilde{L}$ and this $\mathbb{P}^1$, which is an isotropic $\mathbb{P}^3$. This contradicts the fact that X is irreducible and $p > 0$. □

This claim implies that X is a graph over $\mathbb{R}^6$, so yields an globally defined OCS on $\mathbb{R}^6$. To see that it is a warped product, we argue as follows. From Proposition 7.5, SO(6) lifts to an action of SU(4) on F_∞. Since U(4) acts freely and transitively on unitary bases of $\mathbb{C}^4$, it follows that SU(4) acts transitively on full flags. Furthermore,

$-I \in \mathrm{SU}(4)$ only changes all signs of the basis elements, so $\mathrm{SO}(6)$ also acts transitively on full flags. Consequently, we may assume, after reverting to our previous coordinates

$$x_1 = \xi_0,\ x_2 = \xi_{12},\ x_3 = \xi_{13},\ x_4 = \xi_{23},\ x_5 = W_1,\ x_6 = W_2,\ x_7 = W_3,\ x_8 = W_{123},$$

that

$$X \cap F_\infty = \{[0,0,0,0,W_1,W_2,W_3,0]\} \cong \mathbb{P}^2,$$

and that $\tilde{L} = L$ is given by $W_3 = 0$ (and thus $\tilde{Q}^4_s = Q^4_s$). From [BV08, Lemma 3.5], X must lie in the annihilator of L, which forces $\xi_{13} = \xi_{23} = 0$. This implies that X corresponds to a warped product OCS described by (5.7). This completes the proof of Theorem 8.3 and Theorem 1.2 (i).

In twistor coordinates, we can describe X as a graph over $\mathbb{R}^6 = \mathbb{C}^3$ by

$$[\xi_0,\ \xi_{12},\ 0,\ 0,\ \xi_0 z^1 - \xi_{12}\overline{z}^2,\ \xi_0 z^2 + \xi_{12}\overline{z}^1,\ \xi_0 z^3,\ \xi_{12} z^3], \tag{8.11}$$

where the meromorphic function $f : \mathbb{C} \to \mathbb{P}^1$ is given by $f(z^3) = [\xi_0(z^3), \xi_{12}(z^3)]$. From this expression, one sees again directly that X is a double cone over a $(1,p)$-curve $\Sigma \subset \mathbb{P}^1 \times \mathbb{P}^1$. This $(1,p)$-curve is given by the closure in the smooth 2-quadric of the graph

$$[\xi_0(z^3),\ \xi_{12}(z^3),\ 0,\ 0,\ 0,\ 0,\ \xi_0(z^3)z^3,\ \xi_{12}(z^3)z^3] \tag{8.12}$$

as $z^3 \to \infty$. Note that since X is algebraic, up to a constant multiple ξ_0 and ξ_{12} can be chosen as polynomial functions of z_3 which do not vanish simultaneously. We then clearly have

$$\deg(f) = \max\{\deg(\xi_0), \deg(\xi_{12})\} = p = \deg_{\mathbb{P}^7} X - 1.$$

Theorem 1.2 (ii) follows easily since the action of the conformal group is linear (so preserves the degree), and $\deg_{\mathbb{P}^7} X = 1$ if and only if f is a constant if and only if J is conformally equivalent to the standard OCS $\mathbb{J}$ on $\mathbb{R}^6$ (from work in Sections 5 and 6). Theorem 1.2 (iii) follows from Proposition 5.3 above and Corollary 8.13 below. □

8.2. **Final Remarks.** We conclude this section with a few observations. The OCSes under consideration, while not conformally equivalent to $\mathbb{C}^3$ for $p > 0$, are in fact *biholomorphic* to $\mathbb{C}^3$.

Corollary 8.13. *Let J be an OCS on $\mathbb{R}^6$ with finite energy. Then there exists a harmonic biholomorphism $F\colon (\mathbb{R}^6, J) \to (\mathbb{C}^3, \mathbb{J})$.*

Proof. As seen above, finite energy implies that f is algebraic, so we can assume in (8.11) that ξ_0, ξ_{12} are polynomials in z^3 that do not vanish simultaneously. We specify the representative of the projective class by supposing that ξ_0 has a fixed non-zero value at some point. We claim that the three functions

$$\{W_1 = \xi_0(z^3)z^1 - \xi_{12}(z^3)\overline{z}^2,\ W_2 = \xi_0(z^3)z^2 + \xi_{12}(z^3)\overline{z}^1,\ z^3\}$$

are the components of a holomorphic mapping $F\colon (\mathbb{R}^6, J) \to (\mathbb{C}^3, \mathbb{J})$. This is best seen directly from the definition (5.1) of a warped product OCS J. Since ξ_0, ξ_{12} are themslves holomorphic functions, we may express the space of $(1,0)$ forms for J as

$$\langle \eta_1,\, \eta_2,\, dz^3 \rangle = \langle dW_1,\, dW_2,\, dz^3 \rangle$$

in the notation of (3.10).

It remains to check that F is bijective, but this is true because (z^1, z^2, z^3) can be recovered from (W^1, W^2, z^3) using the equations

$$(|\xi_0|^2 + |\xi_{12}|^2) z^1 = \overline{\xi}_0 W_1 + \xi_{12} \overline{W}_2,$$
$$(|\xi_0|^2 + |\xi_{12}|^2) z^2 = \overline{\xi}_0 W_2 - \xi_{12} \overline{W}_1,$$

that are special cases of (7.8).

The adjective "harmonic" refers to the Euclidean metric and the fact that

$$\Delta F = 4 \sum_{i=1}^{n} \frac{\partial^2 F}{\partial z^i \partial \overline{z}^i}$$

is zero because the variables are sufficiently separated. The result is also a consequence of Proposition 5.3(ii) and the fact that any holomorphic map from a cosymplectic manifold to $\mathbb{C}$ is necessarily harmonic [Lic70, Sal85]. □

Remark 8.14. It is also true that the variety X is *rational*, that is, X is birational to $\mathbb{P}^3$, though it is not biholomorphic to $\mathbb{P}^3$ unless f is constant. Moreover, in higher dimensions, it is true that the closure of the graph of any algebraic OCS J globally defined on $\mathbb{R}^{2n}$ is rational, but we omit the proof. However, we do not know whether $(\mathbb{R}^{2n}, J)$ is necessarily biholomorphic to $(\mathbb{C}^n, \mathbb{J})$ in higher dimensions.

By taking the meromorphic function f to be non-algebraic, we obtain the examples mentioned in the Introduction. In this case, the closure of (8.12) in $\mathbb{P}^1 \times \mathbb{P}^1$ will contain the entire missing $\mathbb{P}^1$ (except for possibly a finite set of points), so its preimage in Q^6 will essentially contain the whole twistor fiber over infinity. By Bishop's Theorem, these examples will necessarily have infinite energy.

Finally, we return to the examples on T^6 from Subsection 5.1. Such a structure will necessarily lift to a warped product structure $\tilde{J}_6$ on $\mathbb{R}^6$ with f a doubly-periodic meromorphic function on $\mathbb{C}$, invariant under a lattice, which is equivalent to a holomorphic function $f : (T^2, J_2) \to \mathbb{P}^1$. Since f is necessarily non-algebraic if non-constant, $\tilde{J}_6$ will also necessarily have infinite energy.

References

[Ahl78] Lars V. Ahlfors, *Complex analysis: An introduction to the theory of analytic functions of one complex variable*, third ed., McGraw-Hill Book Co., New York, 1978.

[AHS78] Michael Atiyah, Nigel Hitchin, and Isadore Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. Roy. Soc. London Ser. A **362** (1978), no. 1711, 425–461.

[Bar75] João Lucas Marquês Barbosa, *On minimal immersions of S^2 into S^{2m}*, Trans. Amer. Math. Soc. **210** (1975), 75–106.

[Bat90] Fiammetta Battaglia, *Linear almost complex structures on S^6*, Ann. Mat. Pura Appl. (4) **156** (1990), 181–193.

[Bes87] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 10, Springer-Verlag, Berlin, 1987.

[Bis64] Errett Bishop, *Conditions for the analyticity of certain sets*, Michigan Math. J. **11** (1964), 289–304.

[BV08] Lev Borisov and Jeff Viaclovsky, *Threefolds of order one in the six-quadric*, preprint, arXiv:0808.0343, to appear in Proceedings of the Steklov Institute of Mathematics, 2008.

[BW95] Paul Baird and John C. Wood, *Hermitian structures and harmonic morphisms in higher dimensional Euclidean spaces*, Internat. J. Math. **6** (1995), 161–192.

[BW03] ———, *Harmonic morphisms between Riemannian manifolds*, London Mathematical Society Monographs. New Series, vol. 29, Oxford Univ. Press, 2003.

[Cal67] Eugenio Calabi, *Minimal immersions of surfaces in Euclidean spheres*, J. Differential Geometry **1** (1967), 111–125.

[Car81] Élie Cartan, *The theory of spinors*, Dover Publications Inc., New York, 1981, A reprint of the 1966 English translation.

[Che97] Claude Chevalley, *The algebraic theory of spinors and clifford algebras*, Springer-Verlag, 1997, Collected works, Vol. 2.

[Cho49] Wei-Liang Chow, *On compact complex analytic varieties*, Amer. J. Math. **71** (1949), 893–914.

[dBN98] Paolo de Bartolomeis and Antonella Nannicini, *Introduction to the differential geometry of twistor spaces*, Geometric Theory of Singular Phenomena in P.D.E., Symposia Math., vol. 38, Cambidge University Press, 1998, pp. 91–160.

[EG92] Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[Fed69] Herbert Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153, Springer-Verlag New York Inc., New York, 1969.

[Feg76] Howard D. Fegan, *Conformally invariant first order differential operators*, Quart. J. Math. **27** (1976), 371–378.

[Ful98] William Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 2, Springer-Verlag, Berlin, 1998.

[GH94] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1994, Reprint of the 1978 original.

[Gua] Marco Gualtieri, *Generalized complex geometry*, arXiv:math/0703298.

[GW97] Sigmundur Gudmundsson and John C. Wood, *Harmonic morphisms between almost Hermitian manifolds*, Boll. Unione Mat. Ital. **11-B** (1997), suppl. fasc. 1, 30–58.

[HKP00] Alan Huckleberry, Stefan Kebekus, and Thomas Peternell, *Group actions on S^6 and complex structures on P^3*, Duke Math. J. **102** (2000), 101–124.

[Ino92] Yoshinari Inoue, *Twistor spaces of even-dimensional Riemannian manifolds*, J. Math. Kyoto Univ. **32** (1992), no. 1, 101–134.

[LeB87] Claude LeBrun, *Orthogonal complex structures on S^6*, Proc. Amer. Math. Soc. **101** (1987), no. 1, 136–138.

[Lic70] André Lichnerowicz, *Applications harmoniques et variétés kähleriennes*, Symposia Mathematica, vol. 3, Academic Press, London, 1970, pp. 341–402.

[LP93] Claude LeBrun and Yat Sun Poon, *Self-dual manifolds with symmetry*, Differential geometry: geometry in mathematical physics and related topics (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 365–377.

[OR85] Nigel O'Brian and John Rawnsley, *Twistor spaces*, Ann. Global Anal. Geom. **3** (1985), no. 1, 29–58.

[Pon92] Massimiliano Pontecorvo, *Uniformization of conformally flat Hermitian surfaces*, Differential Geom. Appl. **2** (1992), no. 3, 295–305.

[RS53] Reinhold Remmert and Karl Stein, *Über dei wesentlichen Singularitäten analytischer Mengen*, Math. Ann. **126** (1953), 263–306.

[Sal85] Simon Salamon, *Harmonic and holomorphic maps*, Geometry Seminar Luigi Bianchi II – 1984, Lec. Notes in Math., vol. 1164, Springer-Verlag, Berlin Heidelberg, 1985, pp. 161–224.

[Sal96] ———, *Orthogonal complex structures*, Differential geometry and applications (Brno, 1995), Masaryk Univ., Brno, 1996, pp. 103–117.

[Slu96] Marcus J. Slupinski, *The twistor space of the conformal six sphere and vector bundles on quadrics*, J. Geom. Phys. **19** (1996), no. 3, 246–266.

[SV09] Simon Salamon and Jeff Viaclovsky, *Orthogonal complex stuctures on domains in* $\mathbb{R}^4$, Math. Ann. **343** (2009), no. 4, 853–899.

[SY94] Richard Schoen and Shing-Tung Yau, *Lectures on differential geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994.

[Thu35] Peter Thullen, *Über die wesentlichen Singularitäten analytischer Funktionen und Flächen im Raume von n komplexen Veränderlichen*, Math. Ann. **111** (1935), no. 1, 137–157.

[Woo92] John C. Wood, *Harmonic morphisms and Hermitian structures on Einstein* 4*-manifolds*, Internat. J. Math. **3** (1992), no. 3, 415–439.

[YM55] Kentaro Yano and Isamu Mogi, *On real representations of Kaehlerian manifolds*, Ann. of Math. (2) **61** (1955), 170–189.

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