# ON CALABI-YAU THREEFOLDS WITH LARGE NONABELIAN FUNDAMENTAL GROUPS

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Abstract. In this short note we construct Calabi-Yau threefolds with nonabelian fundamental groups of order 64 as quotients of the small resolutions of certain complete intersections of quadrics in $\mathbb{P}^7$ that were first considered by M. Gross and S. Popescu.

## 1. Introduction

In [GPo], M. Gross and S. Popescu constructed Calabi-Yau varieties that admit a free action of the group $(\mathbb{Z}/8\mathbb{Z})^2$. We observe that these varieties admit a larger automorphism group, and that two other subgroups of order 64 of this group act freely. Our construction is very simple, modulo the nontrivial calculations of [GPo]. The essential idea is that in the presence of only two, nonisomorphic, minimal models every automorphism of the singular variety $V_{8,y}$ naturally extends to the models. We then construct the additional automorphisms explicitly as permutations of coordinates.

In the last section we very briefly describe what led to the discovery of these examples and how they fit into a more general question of finding free group actions on complete intersections of quadrics.

**Acknowledgements.** We have used extensively GAP software package in our project, although no computer calculations are necessary to check the results of this paper.

## 2. Construction

We recall the definition of the Calabi-Yau varieties of Gross and Popescu, [GPo, GPa]. Consider the projective space $\mathbb{CP}^7$ with coordinates $(x_0 : \ldots : x_7)$. For any $(y_0 : y_1 : y_2) \in \mathbb{CP}^2$ in general position,

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consider the intersection $V_{8,y}$ of four quadrics

$$\begin{array}{l} y_1y_3(x_0^2+x_4^2)-y_2^2(x_1x_7+x_3x_5)+(y_1^2+y_3^2)x_2x_6=0\\ y_1y_3(x_1^2+x_5^2)-y_2^2(x_2x_0+x_4x_6)+(y_1^2+y_3^2)x_3x_7=0\\ y_1y_3(x_2^2+x_6^2)-y_2^2(x_3x_1+x_5x_7)+(y_1^2+y_3^2)x_4x_0=0\\ y_1y_3(x_3^2+x_7^2)-y_2^2(x_4x_2+x_6x_0)+(y_1^2+y_3^2)x_5x_1=0. \end{array}$$

This variety has 64 ODP singular points which are the orbit of $(0:y_1:y_2:y_3:0:-y_3:-y_2:-y_1)$ under the action of the group $G=(\mathbb{Z}/8\mathbb{Z})^2$ generated by $\tau$ and $\sigma$ where $\tau(x_i)=\zeta_8^{-i}x_i$ and $\sigma$ is a cyclic permutation of the variables given by the cycle (01234567). This group $G$ acts freely on $V_{8,y}$.

The variety $V_{8,y}$ admits two small resolutions $V_{8,y}^1$ and $V_{8,y}^2$ both of which are Calabi-Yau threefolds with $h^{1,1}=h^{1,2}=2$. These resolutions are related by a flop and are obtained from $X$ by blowups of abelian surfaces in $X$ of degrees 32 and 16 respectively.

The key to the current note is the following lemma that relies heavily on the results of [GPo].

**Lemma 2.1.** *Every automorphism $\gamma$ of $V_{8,y}$ lifts to automorphisms of $V_{8,y}^1$ and $V_{8,y}^2$. If $\gamma$ acts freely on $V_{8,y}$, then its lifts act freely on the small resolutions.*

*Proof.* We will show that $\gamma$ lifts to $V_{8,y}^2$, with the other case being completely analogous.

As in [GPo], we can think of $V_{8,y}^2$ as the blowup of $V_{8,y}$ by the ideal of some Weil divisor $A$. Then there is a natural isomorphism $\hat\gamma$ from $V_{8,y}^2$ to the blowup of $V_{8,y}$ by the ideal of $\gamma(A)$. Denote the latter blowup by $Z$. It is clearly a minimal model of $V_{8,y}$, so by the observation of [GPo], $Z$ is isomorphic to either $V_{8,y}^2$ or $V_{8,y}^1$ (over $V_{8,y}$). To prove the first assertion of the lemma, it remains to show that $V_{8,y}^2$ and $V_{8,y}^1$ are not isomorphic, since we could then compose $\hat\gamma$ with an isomorphism $Z\to V_{8,y}^2$ over $V_{8,y}$. From the description of their Kähler cones we can see that any such isomorphism between $V_{8,y}^i$ would have to respect their structures as fibrations over $\mathbb{P}^1$ as well as their maps to $\mathbb{P}^7$. However, the degrees of the fibers are different in these two cases, hence $V_{8,y}^2\not\cong V_{8,y}^1$.

To show the last assertion, observe that if the lift of $\gamma$ had a fixed point, then the image of that point in $V_{8,y}$ would be fixed by $\gamma$. □

We can now describe our construction.

**Proposition 2.2.** *We define the group $G_1$ generated by $\tau$ and the permutation $\sigma_1=(07214365)$ of the coordinates $x_i$. Then $G_1$ is a nonabelian group isomorphic to a semidirect product of two copies of $\mathbb{Z}/8\mathbb{Z}$. We define the group $G_2$ generated by $\tau$ and the permutations*

$\sigma_2 = (0246)(1357)$ *and* $\sigma_3 = (0145)(3276)$. *Then* $G_2$ *is a nonabelian group isomorphic to a semidirect product of normal subgroup* $\mathbb{Z}/8\mathbb{Z}$ *generated by* $\tau$ *and the quaternion group* $H$ *generated by* $\sigma_2$ *and* $\sigma_3$. *Both* $G_1$ *and* $G_2$ *act freely on* $V_{8,y}$.

*Proof.* The structure of $G_1$ and $G_2$ is immediate from their definition, and it is straightforward to see that $\sigma_i$ acts on $V_{8,y}$. To show that they act freely, it is enough to check the action of all involutions in $G_i$. In view of $1 \to \langle\tau\rangle \to G_1 \to \mathbb{Z}/8\mathbb{Z} \to 1$, all involutions of $G_1$ lie in the subgroup generated by $\tau$ and $\sigma_1^4$. Similarly, all involutions of $G_2$ lie in the subgroup generated by $\tau$ and $\sigma_2^2$. It remains to observe that these subgroups are contained in $G$, which is known to act freely [GPo]. □

**Corollary 2.3.** *For* $i = 1, 2$, $j = 1, 2$, *the quotients of* $V^i_{8,y}$ *by (the lift of) the group* $G_j$ *are smooth Calabi-Yau threefolds with fundamental groups* $G_j$ *of order* 64.

*Proof.* By [GPa], $V^i_{8,y}$ are simply connected. It remains to combine Lemma 2.1 and Proposition 2.2. □

We remark that $G_2$ contains the quaternion group in its regular representation, and $V_{8,y}$ can be thought of as a singular member of the family constructed by Beauville in [B]. Also, the quotients $V^i_{8,y}/G_j$ have Hodge numbers $h^{1,1} = h^{1,2} = 2$ and the structures of fibration with abelian surface fibers.

## 3. Comments

We have stumbled upon these examples largely by chance. We originally set out to investigate free actions of finite groups on complete intersections of four quadrics in hopes of extending the construction of [B]. Every such group action naturally leads to a projective representation of dimension eight, which can be then thought of as a linear representation of a Schur cover $S$ of the finite group. If the action is free, the holomorphic Lefschetz formula essentially dictates what character of $S$ one needs to consider, moreover, it leads to strong restrictions on possible group actions. We have used the GAP software package extensively in our search.

It turns out that the maximum possible order of the group (if one allows the variety to have ODP singularities) is 64. We have found five possible groups of order 64. Upon closer consideration, it turned out that three of them act on the same family of varieties and they are precisely the groups $G$, $G_1$ and $G_2$ that appear in this note. The other two, together with the classification of groups of smaller order will be addressed by the second author in [H].

## References


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