Hyperbolic Functions

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Recall: Trigonometric Functions.

The basic trigonometric functions are the sine and the cosine functions. We use them to get the other four trigonometric function:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}\,, \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}\,, \qquad \sec\theta = \frac{1}{\cos\theta}\,, \qquad \csc\theta = \frac{1}{\sin\theta}$$

One of the most important trigonometric identity is $\sin^2\theta + \cos^2\theta = 1$.

Divide both sides of the above equation by $\cos^2 x$ to get $\tan^2\theta + 1 = \sec^2\theta$.

The Hyperbolic Functions

As any point on the unit circle, $x^2 + y^2 = 1$, can be written as $(x = \cos\theta,\ y = \sin\theta)$ (when θ is the angle between the positive x-axis and the line that connects the point to the center of the circle), so does any point on the hyperbola $x^2 - y^2 = 1$, can be expressed as $(x = \cosh t,\ y = \sinh t)$ for some t.

We define: $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\qquad \cosh x = \dfrac{e^x + e^{-x}}{2}$

Recall that the sine function is an odd function; $\sin(-x) = -\sin x$ and the cosine is an even function: $\cos(-x) = \cos x$. When substitute $(-x)$ in the hyperbolic sine adn cosine, we get that the hyperbolic sine function is an odd function: $\sinh(-x) = -\sinh x$ and the hyperbolic cosine is an even function; $\cosh(-x) = \cosh x$.

The analog to $\sin^2\theta + \cos^2\theta = 1$ is $\cosh^2 x - \sinh^2 x = 1$.

We define the other four hyperbolic trigonometric functions from the $\sinh x$ and $\cosh x$ as in the classic trigonometric functions :

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}\,, \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

$$sechx = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}\,, \qquad cschx = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

Calculus of Hyperbolic Functions

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x + e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \sinh x$$

We get:

$$\frac{d}{dx}\sinh x = \cosh x\ , \qquad \int \cosh x dx = \sinh x + C$$

$$\frac{d}{dx}\cosh x = \sinh x\ , \qquad \int \sinh x = \cosh x + C$$

$$\frac{d}{dx}\tanh x = sech^2 x\ , \qquad \int sech^2 x dx = \tanh x + C$$

$$\frac{d}{dx} sechx = -sechx \tanh x\ , \qquad \int sechx \tanh x dx = -sechx + C$$

$$\frac{d}{dx}\coth x = -csch^2 x\ , \qquad \int csch^2 x = -\coth x + C$$

$$\frac{d}{dx} cschx = -cschx \coth x, \qquad \int cschx \coth x dx = -cschx + C$$

Note that except from the signs, the derivatives and the integrals of the classic trigonometric functions are analog to the derivatives and the integrals of the hyperbolic functions.

Calculus of Some Inverse Hyperbolic Functions

Like the trigonometric functions, the hyperbolic functions have inverses. The functions are NOT one to one so we have to restrict their domain as we do for the trigonometric functions.

Note: The domain of the inverse hyperbolic functions is different from function to function.

for all x: $\frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{x^2+1}}$, $\qquad \int \frac{1}{\sqrt{x^2+1}} dx = \sinh^{-1} x + C$

For $x \geq 1$: $\frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$, $\qquad \int \frac{1}{\sqrt{x^2-1}} dx = \cosh^{-1} x + C.$

For $|x| < 1$: : $\frac{d}{dx}\tanh^{-1} x = \frac{1}{x^2-1}$ $\qquad$ and for $|x| > 1$: $\frac{d}{dx}\coth^{-1} x = \frac{1}{x^2-1}$.

$$\int \frac{1}{x^2-1} dx == \begin{cases} \tanh^{-1} x + C & when\ \ |x| < 1 \\ \coth^{-1} x + C & when\ \ |x| > 1 \end{cases}$$

For a complete derivatives and integrals look at pages 186 and 422 of the book.