SOME IMPLICATIONS OF THE GENERALIZED GAUSS-BONNET THEOREM

BY

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1. **Introduction.** Perhaps the most significant aspect of differential geometry is that which deals with the relationship between the curvature properties of a Riemannian manifold M and its topological structure. One of the beautiful results in this connection is the (generalized) Gauss-Bonnet theorem which relates the curvature of compact and oriented even-dimensional manifolds with an important topological invariant, viz., the Euler-Poincaré characteristic $\chi(M)$ of M. In the 2-dimensional case, the sign of the Gaussian curvature determines the sign of $\chi(M)$. Moreover, if the Gaussian curvature vanishes identically, so does $\chi(M)$. In higher dimensions, the Gauss-Bonnet formula (cf. §3) is not so simple, and one is led to the following important

QUESTION. *Does a compact and oriented Riemannian manifold of even dimension $n=2m$ whose sectional curvatures are all non-negative have non-negative Euler-Poincaré characteristic, and if the sectional curvatures are nonpositive is $(-1)^m \chi(M) \geq 0$?*

H. Samelson [7] has verified this for homogeneous spaces of compact Lie groups with the bi-invariant metric. Unfortunately, however, a proof employing the Gauss-Bonnet formula is lacking. An examination of the Gauss-Bonnet integrand at one point of M leads one to an extremely difficult algebraic problem which has been resolved in dimension 4 by J. Milnor:

THEOREM 1.1. *A compact and oriented Riemannian manifold of dimension 4 whose sectional curvatures are non-negative or nonpositive has non-negative Euler-Poincaré characteristic. If the sectional curvatures are always positive or always negative, the Euler-Poincaré characteristic is positive.*

A subsequent proof was provided by S. Chern [3]. A new and perhaps clearer version indicating some promise for the higher dimensional cases is given in §4. This proof is not essentially different from the one given in [3]. An application of our method yields[3]

THEOREM 1.2. *In order that a 4-dimensional compact and orientable manifold M carry an Einstein metric, i.e., a Riemannian metric of constant Ricci or mean curvature R, it is necessary that its Euler-Poincaré characteristic be non-negative.*

COROLLARY 1.2. *If V is the volume of M,*

$$\chi(M) \geqq \frac{VR^2}{12\pi^2},$$

equality holding if and only if M has constant curvature.

Theorem 1.2 may be improved by relaxing the restriction on the Ricci curvature (cf. §5).

As a first step to the general case, it is natural to consider manifolds with specific curvature properties. A large class of such spaces is afforded by those complex manifolds having the Kaehler property. For this reason, the curvature properties of Kaehler manifolds are examined. We are especially interested in the relationship between the holomorphic and non holomorphic sectional curvatures. In particular, with the aid of Lemma 4.1, *sharper bounds on curvature than those given by M. Berger* [1] *are obtained.* (The right-hand inequality in 4.2 of [1] is incorrect as was pointed out to us by the author; see [1'].) Milnor's result is also partially improved by restricting the hypothesis to the holomorphic sectional curvatures. Indeed, the following theorem is proved:

THEOREM 1.3. *A compact Kaehler manifold of dimension 4 whose holomorphic sectional curvatures are non-negative or nonpositive has non-negative Euler-Poincaré characteristic. If the holomorphic sectional curvatures are always positive or always negative, the Euler-Poincaré characteristic is positive.*

An upper bound for $\chi(M)$ is obtained in terms of the volume and the maximum absolute value of holomorphic curvature of M. More important, *an upper bound may be obtained in terms of curvature alone when holomorphic curvature is strictly positive* (see Theorem 10.2). The technique employed to yield this bound also gives a known bound for the diameter of M [1;9].

Let M be a Kaehler manifold with almost complex structure tensor J. Let $G_{n,P}^2$ denote the Grassmann manifold of 2-dimensional subspaces of T_P (the tangent space at $P \in M$) and consider the subset

$$H_{n,P}^2 = \{\sigma \in G_{n,P}^2 \mid J\sigma = \sigma \text{ or } J\sigma \perp \sigma\}.$$

The plane section σ is called *holomorphic* if $J\sigma = \sigma$, and *anti-holomorphic* if $J\sigma \perp \sigma$, i.e., if it has a basis X, Y where X is perpendicular to both Y and JY. Let $R(\sigma)$ denote the curvature transformation (cf. §2) associated with an orthonormal basis of σ and $K(\sigma)$ the sectional curvature at $\sigma \in G_{n,P}^2$.

A Kaehler manifold is said to have the property (P) *if at each point of M*

*there exists an orthonormal holomorphic basis $\{X_\alpha\}$ of the tangent space with
respect to which*

$$(R_\sigma(\sigma))^2 = -(K(\sigma))^2 I$$

*for all sections $\sigma = \sigma(X_\alpha, X_\beta)$ where $R_\sigma(\sigma)$ denotes the restriction of $R(\sigma)$ to the
section σ, and I is the identity transformation.* (In other words, in the case where
$K(\sigma) \neq 0, R_\sigma(\sigma)$ defines a complex structure on σ.)

We shall prove

THEOREM 1.4. *Let M be a 6-dimensional compact Kaehler manifold having
the property* (P). *If for all $\sigma = \sigma(X_a, X_\beta)$, $K(\sigma) \geq 0$, then $\chi(M) \geq 0$, and if
$(\sigma)K \leq 0$, $\chi(M) \leq 0$. If the sectional curvatures are always positive (resp.,
negative), the Euler-Poincaré characteristic is positive (resp., negative).*

*A similar statement is valid for manifolds of dimension $4k$ (see Theorem 11.1).
A Kaehler manifold possessing the property* (P) *for all $\sigma \in H^2_{n,P}$ has constant
holomorphic curvature.*

T. Frankel has conjectured that the compact Kaehler manifolds of strictly
positive curvature are topologically, and even analytically, the same as the complex
projective spaces. A. Andreotti and Frankel have already established this in
dimension 4 [10]. In dimension 6, it is not yet known whether a compact Kaehler
manifold of positive curvature is homologically complex projective space.
However, we have recently shown that *the second betti number of a compact
Kaehler manifold of strictly positive curvature is* 1.

2. **Preliminary notions.** Let M be an $(n=2)$-m dimensional Riemannian mani-
fold with metric $\langle , \rangle$ and norm $\| \quad \| = \langle , \rangle^{1/2}$. Let $\sigma \in G^2_{n,P}$ be a plane section
at $P \in M$, and $X, Y \in T_P$ two vectors spanning σ. The Riemannian or sectional
curvature $K(\sigma)$ at σ is defined by

$$K(\sigma) = \frac{\langle R(X,Y)X,Y \rangle}{\| X \wedge Y \|^2}$$

where $R(X,Y)$ is the tensor of type $(1,1)$(associated with X and Y), called the
curvature transformation (cf. §6; $R(X,Y)$ is the negative of the classical curvature
transformation), and $\| X \wedge Y \|^2 = \| X \|^2 \| Y \|^2 - \langle X,Y \rangle^2$. The curvature
transformation is a skew-symmetric linear endomorphism of T_P. Note that K
is not a function on M but rather on $\bigcup_{P \in M} G^2_{n,P}$. It is continuous, and so if M
is compact, it is bounded.

LEMMA 2.1. *For any $X, Y, Z, W \in T_P$, the curvature transformation has
the properties*:

  (i) $R(X,Y) = -R(Y,X)$,
  (ii) $\langle R(X,Y)Z,W \rangle = -\langle R(X,Y)W,Z \rangle$,
  (iii) $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$,
  (iv) $\langle R(X,Y)Z,W \rangle = \langle R(Z,W)X,Y \rangle$.

If $\{X_1, \cdots, X_n\}$ is a basis of T_P, the classical components of the curvature tensor are given by

$$R_{ijkl} = \langle R(X_i, X_j)X_k, X_l \rangle.$$

The relations (i)–(iv) then correspond to the classical formulas

(i)′ $R_{ijkl} = -R_{jikl}$,

(ii)′ $R_{ijkl} = -R_{ijlk}$,

(iii)′ $R_{ijkl} + R_{jkil} + R_{kijl} = 0$,

(iv)′ $R_{ijkl} = R_{klij}$.

COROLLARY 2.1. $K(\sigma)$ is a well-defined function on $\bigcup_{P \in M} G_{n,P}^2$.

3. **The Gauss-Bonnet theorem** [3].   A convenient formulation is given in terms of orthonormal bases. Indeed, over a neighborhood of $P \in M$, there exists a family of orthonormal frames $P, X_1, \cdots, X_n$ and differential forms $\omega_1, \cdots, \omega_n$ such that the Riemannian metric may be written as

$$ds^2 = \sum_{i=1}^{n} \omega_i^2.$$

The equations of structure of (M, ds^2) are

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji}, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}$$

where the ω_{ij} are the connection forms and Ω_{ij} the curvature forms. Define the tensor field

$$\Omega = \sum_{ij} (X_i \wedge X_j) \otimes \Omega_{ij}$$

$$= \sum_{i,j,k,l} \langle R(X_i, X_j)X_k, X_l \rangle (X_i \wedge X_j) \otimes (\omega_k \wedge \omega_l).$$

It is of type $(2,2)$ and assigns to every $P \in M$ an element of $\bigwedge^2(T_P) \otimes \bigwedge^2(T_P^*)$ where $\bigwedge^p (V)$ is the vector space over V generated by all elements of the form $v_{i_1} \wedge \cdots \wedge v_{i_p}$, $v_{i_j} \in V$. It follows easily that Ω^m has as its skew-symmetric part

(3.1) $(X_1 \wedge \cdots \wedge X_{2m}) \otimes \sum \varepsilon_{i_1 \ldots i_{2m}} \varepsilon_{j_1 \ldots j_{2m}} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{2m-1} i_{2m} j_{2m-1} j_{2m}} \omega_1 \wedge \cdots \wedge \omega_{2m}.$

Let $X_1 \wedge \cdots \wedge X_{2m}$ be the generator of $\bigwedge^n(T_P)$ uniquely determined by the conditions $\| X_1 \wedge \cdots \wedge X_{2m} \| = 1$ and that its orientation is coherent with that of T_P. Tehn, the Euler-Poincaré characteristic is given by the *Gauss-Bonnet formula*

(3.2) $$\chi(M) = \frac{1}{2^{3m}\pi^m m!} \int_M \Theta$$

where the integrand is the $2m$-form defined by

$$\Omega^m = (X_1 \wedge \cdots \wedge X_{2m}) \otimes \Theta.$$

4. Normalization of curvature. One of the major obstacles in the way of resolving the *Question* raised in §1 is the presence of terms in (3.1) involving curvature components of the type $\langle R(X,Y)X,Z \rangle, Z \neq Y$. By choosing a basis of the tangent space T_P which bears a special relation to the curvatures of sections in T_P one is able to simplify the components of the curvature tensor. These simplifications are based on the following lemma.

LEMMA 4.1. *Let* X_i, X_j, X_k *be part of an orthonormal basis of* T_P. *If the section* (X_i, X_j) *is a critical point of the sectional curvature function* K *restricted to the submanifold of sections* $\{(X_i, X_j \cos \theta + X_k \sin \theta)\}$, *then the curvature component* R_{ijik} *vanishes.*

Proof. Set $f(\theta) = K(X_i, X_j \cos \theta + X_k \sin \theta)$. Then,

$$f(\theta) = \langle R(X_i, X_j \cos \theta + X_k \sin \theta) X_i, X_j \cos \theta + X_k \sin \theta \rangle$$

$$= \cos^2 \theta K_{ij} + \sin^2 \theta K_{ik} + \sin 2\theta \, R_{ijik}$$

where $K_{ij} = K(X_i, X_j)$. Since the derivative at $\theta = 0$ of $f(\theta)$ is $2R_{ijik}$, the result follows.

COROLLARY 4.1. *If* M *is a 4-dimensional Riemannian manifold, there exists an orthonormal basis* $\{X_1, X_2, X_3, X_4\}$ *of* T_P *such that the curvature components* $R_{1213}, R_{1214}, R_{1223}, R_{1224}, R_{1314}$ *and* R_{1323} *all vanish.*

Proof. Choose the plane $\sigma(X_1, X_2)$ so that $K(X_1, X_2)$ is the maximum curvature at P. Then, choose $X_1 \in \sigma(X_1, X_2)$ and X_3 in the orthogonal complement of $\sigma(X_1, X_2)$ so that $K(X_1, X_3)$ is a maximum of K restricted to $\{(X_1 \cos \theta + X_2 \sin \theta, X_3 \cos \phi + X_4 \sin \phi)\}$. Applications of Lemma 4.1 with various choices for i, j and k yield the result.

Proof of Theorem 1.1. The idea of the proof is to show that the integrand in the Gauss-Bonnet formula is a non-negative multiple of the volume element. For any basis, the integrand is a positive multiple of the volume element and the sum

$$\varepsilon_{i_1 i_2 i_3 i_4} \varepsilon_{j_1 j_2 j_3 j_4} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4}.$$

The terms for which $(i_1, i_2) = (j_1, j_2)$ are products of two curvatures. These terms are therefore non-negative. The terms for which (i_1, i_2, j_1, j_2) is a permutation of $(1, 2, 3, 4)$ are squares, hence non-negative. If we choose the basis to satisfy the conditions of Corollary 4.1, then all other terms vanish. Indeed, they are of the form $\pm R_{ijik} R_{iklj}$. If one of i or l is 1 or 2 and the other is not, then one of

R_{1213}, R_{1214}, R_{2123}, R_{2124} must occur. If i and l are 1 and 2 in some order, then R_{1314} occurs. If neither i nor l are 1 or 2, then R_{3132} occurs.

For later references it is important to know explicitly what the integrand reduces to after this choice of basis. A counting procedure yields

$$(4.1) \quad \frac{1}{4\pi^2} \left[K_{12}K_{34} + K_{13}K_{24} + K_{14}K_{23} + (R_{1234})^2 + (R_{1324})^2 + (R_{1423})^2 \right]\omega,$$

where ω is the Riemannian volume element.

5. **Mean curvature and Euler-Poincaré characteristic.**   The same conclusion is also valid for 4-dimensional Einstein spaces. An independent proof is given below.

Proof of Theorem 1.2.   Since the Ricci tensor R_{ij} is a multiple of the identity transformation δ_{ij}, i.e., $R_{ij} = R\delta_{ij}$,

$$K_{12} + K_{13} + K_{14} = K_{21} + K_{23} + K_{24}$$
$$= K_{31} + K_{32} + K_{34} = K_{41} + K_{42} + K_{43}.$$

(The symbol R employed here is $\frac{1}{4}$ of the Ricci scalar curvature.) It follows that

$$K_{12} = K_{34}, \quad K_{13} = K_{24}, \quad K_{14} = K_{23}.$$

Thus the terms in (3.1) which are products of two curvatures are squares. As before, so are the terms having four distinct indices in each factor. The remaining terms are all of the form

$$\varepsilon_{ijlk}\varepsilon_{iklj}R_{ijik}R_{lklj} = -R_{ijik}R_{lklj},$$

but since $R_{jk} = R_{ijik} + R_{lklj} = 0$, $j \neq k$, these terms are also squares.

Proof of Corollary 1.2.   If we set $x = K_{12} = K_{34}, y = K_{13} = K_{24}$, and $z = K_{14} = K_{23}$, the minimum of $x^2 + y^2 + z^2$ subject to the restriction $x + y + z = R$ is found to be $R^2/3$. We note that $x^2 + y^2 + z^2 = R^2/3$ only if $x = y = z$. The integral can attain the lower bound of $VR^2/12\pi^2$ only if the other terms all vanish, which implies that the sectional curvature is constant.

Theorem 1.2 generalizes a result due to H. Guggenheimer [5].

Since an irreducible symmetric space is an Einstein space, its Euler-Poincaré characteristic, in the compact case, is non-negative in dimension 4. This is, of course, true for all even dimensions [7].

The cases where curvature or mean curvature is strictly positive in Theorems 1.1 and 1.2, respectively, are consequences of Myers' theorem which says that the fundamental group is finite. Indeed, the hypothesis of compactness may be weakened to completeness in these cases, since compactness is what is first established.

In both Theorems 1.1 and 1.2, it is clear from the proof that $\chi(M) \neq 0$ unless M is locally flat.

EXAMPLE. Let $M = S^2 \times S^2$ be the product of two 2-dimensional unit spheres with metric tensor the sum of those for the 2-spheres: $ds^2 = ds_1^2 + ds_2^2$. The Riemannian manifold M is then an Einstein space with (constant) Ricci curvature 1. The sectional curvatures vary from 0 to 1 inclusive, and hence they are not bounded away from 0. However, both Theorems 1.1 and 1.2 imply that $\chi(M) > 0$. This follows from Theorem 1.1 since M is not locally flat, and from Theorem 1.2 since $R \neq 0$. Since M does not have constant curvature, $\chi(M) > V/12\pi^2 > 1$. Corollary 1.2 therefore yields information beyond Theorem 1.1 if the manifold carries an Einstein metric.

Theorem 1.2 may be improved by relaxing the restriction on mean curvature. Let M be any 4-dimensional compact and orientable Riemannian manifold, R_0 the maximum mean curvature, that is, the maximum of $R_{11} = K_{12} + K_{13} + K_{14}$ as a function of a point of M and an orthonormal basis at that point, and r the minimum mean curvature. The generalization of Theorem 1.2 will then take the form of finding a lower bound for $\chi(M)$ which is given in terms of R_0, r and V. In particular, we shall give conditions on R_0 and r in order that $\chi(M)$ be non-negative.

The problem reduces to that of minimizing the expression

$$K_{12}K_{34} + K_{13}K_{24} + K_{14}K_{23}$$

subject to the restrictions

$$r \leqq K_{12} + K_{13} + K_{14} \leqq R_0, \quad r \leqq K_{21} + K_{23} + K_{24} \leqq R_0,$$

$$r \leqq K_{31} + K_{32} + K_{34} \leqq R_0, \quad r \leqq K_{41} + K_{42} + K_{43} \leqq R_0.$$

As an outline of the technique used, a substitution $K_{12} = x - u$, $K_{13} = y - v$, $K_{14} = z - w$, $K_{34} = x + u$, $K_{24} = y + v$, $K_{23} = z + w$ will reduce $K_{12}K_{34} + K_{13}K_{24} + K_{14}K_{23}$ to normal form $x^2 + y^2 + z^2 - u^2 - v^2 - w^2$. The inequalities all involve $x + y + z$, so we may replace x, y and z by their mean $s = (x + y + z)/3$ without altering the validity of the inequalities but decreasing the quadratic expression. This reduces the quadratic form to four variables s, u, v, w and the inequalities describe a cube in this 4-space. The form is indefinite or negative definite on this cube and all its faces, so the minimum μ must occur on a corner. We summarize the results:

1. If $R_0 \leqq 2r$, $\mu = r^2/3$.
2. If $0 \leqq 2r \leqq R_0$, $\mu = R_0(3r - R_0)/6$.
3. If $r \leqq 0 \leqq R_0$, $\mu = -(R_0^2 - 4R_0 r + r^2)/6$.
4. If $r \leqq 2R_0 \leqq 0$, $\mu = r(3R_0 - r)/6$.
5. If $2R_0 \leqq r$, $\mu = R_0^2/3$.

The conclusions derived are

THEOREM 5.1. *If M is a 4-dimensional compact and orientable Riemannian manifold, R_0 the maximum mean curvature, r the minimum mean curvature, V the volume of M, and $\mu = \mu(R_0, r)$ as specified above, then $\mu V/4\pi^2$ is a lower bound for the Euler-Poincaré characteristic of M.*

COROLLARY 5.1.   *If $R_0 \leqq 3r$ or $3R_0 \leqq r$, the Euler-Poincaré characteristic is non-negative.*

The case $0 < R_0 \leqq 3r$ follows from Myers' theorem.

COROLLARY 5.2.   *If k is an absolute bound for mean curvature $(-k \leqq r, R_0 \leqq k)$, then $-k^2 V/4\pi^2$ is a lower bound for the Euler-Poincaré characteristic.*

We note that this method fails to yield an upper bound for $\chi(M)$ in terms of mean curvature. Moreover, it is not simple to extend these results to higher dimensions.

6. **Curvature and holomorphic curvature.**   It is well known that results on Riemannian curvature are sometimes valid for Kaehler manifolds when the hypothesis is restricted to holomorphic curvature alone. For example, J. L. Synge's theorem that a complete orientable even-dimensional Riemannian manifold of strictly positive curvature is simply connected [8] corresponds to Y. Tsukamoto's result that a complete Kaehler manifold of strictly positive holomorphic curvature is simply connected (cf. §10).

It suits our purposes well here to avoid complex vector spaces. Indeed, a Kaehler manifold is considered as a Riemannian manifold admitting a self-parallel skew-symmetric linear transformation field J such that $J^2 = -I$. The field J is usually called the almost complex structure tensor.

We shall require the following

LEMMA 6.1.   *The relationship between the curvature transformation $R(X, Y)$ and the metric is given by*

$$R(X, Y) = D_{[X, Y]} - [D_X, D_Y]$$

where D_X denotes the operation of covariant differentiation in the direction of X, and

$$2\langle X, D_Z Y \rangle = Z\langle X, Y \rangle - X\langle Y, Z \rangle + Y\langle Z, X \rangle$$
$$+ \langle Y, [X, Z] \rangle - \langle X, [Y, Z] \rangle - \langle Z, [Y, X] \rangle.$$

LEMMA 6.2.   *Let M be a Kaehler manifold with almost complex structure tensor J. Then, for any $X, Y \in T_P$,*
  (i) $R(JX, JY) = R(X, Y)$,
  (ii) $K(JX, JY) = K(X, Y)$,
and when X, Y, JX, JY are orthonormal,
  (iii) $\langle R(X, JX)Y, JY \rangle = K(X, Y) + K(JX, Y)$.

Formula (i) is a consequence of the fact that J is parallel. Indeed, J being parallel is equivalent to $D_X(JY) = JD_X Y$ for all X, Y. Applying Lemma 6.1, $R(X, Y)(JZ) = J(R(X, Y)Z)$. Since J is an isometry, $\langle R(X, Y)JZ, JW \rangle = \langle JR(X, Y)Z, JW \rangle$

$= \langle R(X, Y)Z, W \rangle$, so that $R(JZ, JW) = R(Z, W)$ now follows from (iv) of Lemma 2.1.

Replacing Y by JY and using the skew-symmetry of $R(X, Y)$ we get $R(X, JY) = R(Y, JX)$. For sectional curvature we have the corresponding relation $K(X, JY) = K(Y, JX)$.

A plane section is holomorphic if it has a basis $\{X, JX\}$ for some X. A plane section is anti-holomorphic if it has a basis $\{X, Y\}$ where X is perpendicular to both Y and JY. More generally, with each section we associate an acute angle θ which measures by how much the section fails to be holomorphic. If $\{X, Y\}$ is an orthonormal basis of the section then $\cos \theta = |\langle X, JY \rangle|$; it is readily verified that this is independent of the choice of X and Y. The following lemma is trivial.

LEMMA 6.3. *If X and Y are orthonormal vectors which do not span a holomorphic section, then X and JY span an anti-holomorphic section.*

The *holomorphic curvature $H(X)$* of a nonzero vector X is the curvature of the holomorphic section $\sigma(X, JX)$, i.e., $H(X) = K(X, JX)$.

In a Riemannian manifold it is well known that the curvature tensor is determined algebraically by the *biquadratic curvature form B*:

$$B(X, Y) = \langle R(X, Y)X, Y \rangle.$$

In fact,

$$6\langle R(X, Y)Z, W \rangle = \frac{\partial^2}{\partial s \partial t}(B(X + sZ, Y + tW) - B(X + sW, Y + tZ))\big|_{s=t=0}.$$

Since sectional curvature $K(X, Y)$ is the quotient of $B(X, Y)$ and $\| X \wedge Y \|^2$, it follows that the curvature tensor is determined algebraically by the functions K and $\langle , \rangle$.

If the manifold is Kaehlerian, we define the *quartic holomorphic curvature form Q*:

$$Q(X) = \langle R(X, JX)X, JX \rangle.$$

That the holomorphic sectional curvatures are of fundamental importance for Kaehler manifolds is given by

THEOREM 6.1. *B is determined algebraically by Q.*

Perhaps more interesting is the formula which reduces the proof to a verification:

(6.1)
$$B(X, Y) = \frac{1}{32}\left[3Q(X + JY) + 3Q(X - JY) - Q(X + Y)\right.$$
$$\left. - Q(X - Y) - 4Q(X) - 4Q(Y)\right].$$

As an immediate consequence of this formula we derive

COROLLARY 6.1. *Let X and Y be orthonormal vectors, and $\langle X, JY \rangle = \cos \theta > 0$. Then,*

$$K(X, Y) = \frac{1}{8} \left[3(1 + \cos\theta)^2 H(X + JY) + 3(1 - \cos\theta)^2 H(X - JY) \right.$$
$$\left. - H(X + Y) - H(X - Y) - H(X) - H(Y) \right].$$

(6.2)

Moreover, if $\langle X, JY \rangle = 0$, *then*

$$K(X, Y) + K(X, JY) = \frac{1}{4} \left[H(X + JY) + H(X - JY) + H(X + Y) \right.$$
$$\left. + H(X - Y) - H(X) - H(Y) \right],$$

(6.3)

and, more generally,

$$K(X, Y) + K(X, JY) \sin^2\theta = \frac{1}{4} \left[(1 + \cos\theta)^2 H(X + JY) \right.$$
$$+ (1 - \cos\theta)^2 H(X - JY) + H(X + Y)$$
$$\left. + H(X - Y) - H(X) - H(Y) \right].$$

(6.4)

As a consequence, we obtain a well-known result.

COROLLARY 6.2. *If holomorphic curvature is a constant H, then curvature is given by*

(6.5)
$$K(X, Y) = \frac{H}{4} (1 + 3\cos^2\theta).$$

In particular, if curvature is constant, the manifold is locally flat for $m \geq 2$.

Formulas (6.2)–(6.4) will be used in §8 to derive inequalities between curvature and holomorphic curvature.

7. Curvature as an average. When holomorphic curvature is constant, the anti-holomorphic curvature is also a constant $A = H/4$, and we may rewrite (6.5) as

$$K(X, Y) = H - 3A \sin^2\theta.$$

For any two orthonormal vectors X and Y such that $\langle X, JY \rangle > 0$, we say that the holomorphic sections generated by $X \cos\alpha + Y \sin\alpha$ are the *holomorphic sections associated* with the section spanned by the pair (X, Y), and the sections spanned by the vectors $X \cos\alpha + Y \sin\alpha$, $-JX \sin\alpha + JY \cos\alpha$ the *anti-holomorphic sections associated* with (X, Y). These 'circles' of sections depend only on the plane of X and Y, and not on the choice of the vectors X, Y. If the manifold has constant holomorphic curvature, then H may clearly be interpreted as the average associated holomorphic curvature, and A as the average associated anti-holomorphic curvature. Thus, the following result may be viewed as a generalization of formula (6.5).

THEOREM 7.1. *Let* $H(X, Y)$ *be the average associated holomorphic curvature and* $A(X, Y)$ *the average associated anti-holomorphic curvature to the plane of the vectors* X *and* Y, *i.e., when* X *and* Y *are orthonormal,*

$$H(X, Y) = \frac{1}{\pi} \int_0^\pi H(X \cos \alpha + Y \sin \alpha) \, d\alpha,$$

$$A(X, Y) = \frac{1}{\pi} \int_0^\pi K(X \cos \alpha + Y \sin \alpha, -JX \sin \alpha + JY \cos \alpha) \, d\alpha.$$

Then,

(7.1) $$K(X, Y) = H(X, Y) - 3 \, A(X, Y) \sin^2 \theta.$$

Since $H(X \cos \alpha + Y \sin \alpha)$ and $K(X \cos \alpha + Y \sin \alpha, -JX \sin \alpha + JY \cos \alpha)$ are quartic polynomials in $\cos \alpha$, $\sin \alpha$, indeed, quadratic polynomials in $\cos 2\alpha$, $\sin 2\alpha$, their average may be obtained by averaging any four equally spaced values:

$$H(X, Y) = \frac{1}{4} \left[H(X) + H(X + Y) + H(Y) + H(X - Y) \right],$$

$$A(X, Y) = \frac{1}{4} \left[K(X, JY) + K(X + Y, -JX + JY) + K(Y, JX) + K(X - Y, JX + JY) \right]$$

$$= \frac{1}{2} \left[K(X, JY) + K(X + Y, -JX + JY) \right].$$

8. Inequalities between holomorphic curvature and curvature. Throughout this section assume that the metric has been normalized so that every curvature $H(X)$ satisfies $\lambda \leq H(X) \leq 1$. The Kaehler manifold is then said to be λ-*holomorphically pinched* [1]. We shall derive inequalities between the curvatures of holomorphic and nonholomorphic sections.

To begin with, we consider anti-holomorphic curvature. By formula (6.2) with $\cos \theta = 0$, we obtain

LEMMA 8.1. *If X, Y span an anti-holomorphic section, then*

$$\frac{3\lambda - 2}{4} \leq K(X, Y) \leq \frac{3 - 2\lambda}{4}.$$

Similarly, by (6.3), we derive

LEMMA 8.2. *If X, Y and X, JY span anti-holomorphic sections, then*

$$\lambda - \frac{1 + H(X)}{4} \leq K(X, Y) + K(X, JY) \leq \frac{2 - \lambda}{2}.$$

Using these bounds one can obtain bounds on mean curvature. Let X_1 be any unit vector. Choose an orthonormal basis $\{X_i, JX_i\}$, $i = 1, \cdots, m$. Then, the mean curvature in the 'direction' of X_1 is

$$K(X_1, JX_1) + \sum_{i=2}^{m} [K(X_1, X_i) + K(X_1, JX_i)].$$

The first term is holomorphic and the remaining ones are anti-holomorphic in pairs. Thus, we obtain

THEOREM 8.1. *Let M be a λ-holomorphically pinched Kaehler manifold of complex dimension m. Then,*
(i) *if $m \leq 5$,*

$$r = \frac{(3m + 1)\lambda - (m - 1)}{4}, \qquad R_0 = \frac{3m + 1 - (m - 1)\lambda}{4}$$

and
(ii) *if $m > 5$,*

$$r = (m - 1)\lambda - \frac{m - 3}{2}, \qquad R_0 = m - 1 - \frac{m - 3}{2}\lambda$$

where r, R_0 are lower and upper bonds, respectively for mean curvature. In particular, for $m = 2$, mean curvature is non-negative if $\lambda \geq 1/7$. In every dimension, mean curvature is positive if $\lambda \geq 1/2$. Finally, for $m = 2$ and $\lambda \geq 0$ (resp., $\lambda \leq 0$), the Ricci scalar curvature is non-negative (resp., nonpositive).

To get an upper bound for an arbitrary sectional curvature, we eliminate the function $H(X, Y)$ which occurs in both formulas (6.2) and (7.1), thereby obtaining

$$(8.1) \qquad K(X, Y) = \frac{1}{4}[(1 + \cos\theta)^2 H(X + JY) + (1 - \cos\theta)^2 H(X - JY)] - \sin^2\theta\, A(X, Y).$$

Using the lower bound for $A(X, Y)$ obtained from Lemma 8.1 results in the inequality

$$(8.2) \qquad K(X, Y) \leq 1 - \frac{3\lambda\sin^2\theta}{4}.$$

This proves

THEOREM 8.2. *If the holomorphic sectional curvatures are non-negative, then a maximum curvature is holomorphic.*

To obtain a lower bound we apply formula (6.2) directly. Thus,

$$K(X, Y) \geq \frac{1}{8}[6(1 + \cos^2\theta)\lambda - 4].$$

Hence,

$$K(X, Y) \geq \frac{3\lambda - 2}{4}, \qquad \lambda \geq 0.$$

To obtain a better upper bound than (8.2) when $\lambda < 0$, we assume that $K(X, Y)$ is a maximum for all curvatures. Then, since $\langle X, JY \rangle = \cos\theta$, the derivative at $\alpha = 0$ of $K(X\cos\alpha + JY\sin\alpha, Y)$ is $-2(\langle R(X, Y)Y, JY \rangle + \cos\theta\, K(X, Y)) = 0$, and similarly with X and Y interchanged. Thus,

$$\langle R(X, Y)Y, JY \rangle = \langle R(X, Y)X, JX \rangle = -K(X, Y)\cos\theta.$$

We use this to expand $H(X + JY)(1 + \cos\theta)^2$ and $H(X - JY)(1 - \cos\theta)^2$. The result is

$$K(X, Y) = \frac{(1 + \cos\theta)^2 H(X + JY) - (1 - \cos\theta)^2 H(X - JY)}{4\cos\theta}.$$

Eliminating $H(X - JY)$ between this and (8.1) yields

$$K(X, Y) = \frac{1}{2}(1 + \cos\theta)H(X + JY) - (1 - \cos\theta)A(X, Y)$$

(8.3)
$$\leqq \frac{1}{2}(1 + \cos\theta) - (1 - \cos\theta)\frac{3\lambda - 2}{4}$$

$$= 1 - \frac{3}{4}(1 - \cos\theta)\lambda.$$

From (6.1) by inserting X, JY in place of X, Y we get

$$K(X, JY)\sin^2\theta = \frac{1}{8}[3H(X - Y) + 3H(X + Y) - (1 + \cos\theta)^2 H(X + JY)$$

(8.4)
$$- (1 - \cos\theta)^2 H(X - JY) - H(X) - H(Y)]$$

$$\geqq \frac{1}{8}[3H(X - Y) + 3H(X + Y) - H(X) - H(Y)] - \frac{1 + \cos^2\theta}{4}.$$

Averaging this as we did to get (7.1) we find

$$A(X, Y)\sin^2\theta \geqq \frac{1}{2}H(X, Y) - \frac{1}{4}(1 + \cos^2\theta)$$

(8.5)
$$\geqq \frac{\lambda}{2} - \frac{1}{4}(1 + \cos^2\theta).$$

Combining this with (8.3) gives

$$K(X, Y) \leqq \frac{1}{2}(1 + \cos\theta) - \frac{1 - \cos\theta}{\sin^2\theta}\left(\frac{\lambda}{2} - \frac{1 + \cos^2\theta}{4}\right)$$

(8.6)
$$= \frac{3 + 4\cos\theta + 3\cos^2\theta - 2\lambda}{4(1 + \cos\theta)}.$$

As a function of $\cos\theta$ this bound is either increasing as $\cos\theta$ increases from 0 to 1, or it has a minimum with larger values on the ends of the interval. The other bound, $1 - 3(1 - \cos\theta)\lambda/4$, is a decreasing function of $\cos\theta$. It follows that

$K(X, Y)$ is bounded by either the common value when the two bounds coincide, which occurs for $\cos \theta = 1/\sqrt{3}$, or the bound from (8.6) with $\cos \theta = 0$. The two numbers in question are $1 - (3 - \sqrt{3})\lambda/4$ and $(3 - 2\lambda)/4$, respectively. They coincide for $\lambda = -(1 + \sqrt{3})/2$. Hence,

$$K(X, Y) \leqq \begin{cases} 1 - \dfrac{3 - \sqrt{3}}{4}\lambda, & -\dfrac{1 + \sqrt{3}}{2} \leqq \lambda \leqq 0, \\[2ex] \dfrac{3 - 2\lambda}{4}, & \lambda \leqq -\dfrac{1 + \sqrt{3}}{2}. \end{cases}$$

It is not necessary to duplicate the above analysis to obtain lower bounds. Indeed, we can change all signs and directions of inequalities (making the minimum $H = -1$), then rescale the result so that the minimum H is again λ when $\lambda < 0$. We summarize the results as follows:

THEOREM 8.3. *Let M be a λ-holomorphically pinched Kaehler manifold. Then,*

(i) $\dfrac{3\lambda - 2}{4} \leqq K(X, Y) \leqq 1, \quad \lambda \geqq 0,$

(ii) $K(X, Y) \leqq 1 - \dfrac{(3 - \sqrt{3})\lambda}{4}, \quad -\dfrac{1 + \sqrt{3}}{2} \leqq \lambda \leqq 0,$

(iii) $K(X, Y) \leqq \dfrac{3 - 2\lambda}{4}, \quad \lambda \leqq -\dfrac{1 + \sqrt{3}}{2},$

(iv) $K(X, Y) \geqq \dfrac{3\lambda - 2}{4}, \quad -\sqrt{3} + 1 \leqq \lambda \leqq 0,$

(v) $K(X, Y) \geqq \lambda - \dfrac{3 - \sqrt{3}}{4}, \quad \lambda \leqq -\sqrt{3} + 1.$

Finally, if $-1 \leqq H(X) \leqq -\lambda$ for all X, then

(vi) $-1 \leqq K(X, Y) \leqq -\dfrac{3\lambda - 2}{4}.$

It is suspected that the bounds in cases (ii) and (v) can be improved, with corresponding alterations on the bounds on λ in (iii) and (iv):

CONJECTURE.

(ii)′ $K(X, Y) \leqq 1, \quad -\dfrac{1}{2} \leqq \lambda \leqq 0,$

(v)′ $K(X, Y) \geqq \lambda, \quad \lambda \leqq -2.$

Further improvement by the methods employed here (consideration of the curvature at one point) is precluded by the examples A and B below where the curvature components R_{ijkl} are taken with respect to an orthonormal holomorphic basis $X_1, X_2, X_3 = JX_1, X_4 = JX_2$. In each of these examples $\lambda \leqq H(X) \leqq 1$.

	A	B
R_{1212}	$\dfrac{2-\lambda}{4}$	$\dfrac{2\lambda-1}{4}$
R_{1213}	0	0
R_{1214}	0	0
R_{1224}	0	0
R_{1313}	1	λ
R_{1314}	0	0
R_{1424}	0	0
R_{2424}	1	λ
R_{1414}	$\dfrac{3\lambda-2}{4}$	$\dfrac{3-2\lambda}{4}$

The other curvature components are determined by Lemmas 2.1 and 6.2.

For example A we have $(3\lambda-2)/4 \leqq K(X,Y) \leqq 1$ if $-2 \leqq \lambda \leqq 1$; if $\lambda \leqq -2$, then $\lambda \leqq K(X,Y) \leqq (2-\lambda)/4$. For example B, $(2\lambda-1)/4 \leqq K(X,Y) \leqq 1$ if $-1/2 \leqq \lambda \leqq 1$; if $\lambda \leqq -1/2$, then $\lambda \leqq K(X,Y) \leqq (3-2\lambda)/4$.

It is noteworthy that in each of these examples the mean curvature is constant, viz., $1+\lambda/2$ for A and $\lambda+1/2$ for B.

9. Holomorphic curvature and Euler-Poincaré characteristic. The Gauss-Bonnet integral can also be simplified by a normalization of the basis depending on holomorphic curvature (cf. §6). Our considerations, as before, are restricted to the 4-dimensional case. Since only orthonormal holomorphic bases are considered we should expect fewer terms of the form R_{ijik}, $k \neq j$, to vanish. Fortunately, however, this is compensated for by virtue of the additional relations provided by Lemma 6.2. It is for this reason that the proof of Theorem 1.3 presents no essential difficulties. In fact, if $H(X_1)$ is taken to be the maximal holomorphic curvature, then, by evaluating the derivative of $H(X_1 \cos \alpha + X_2 \sin \alpha)$ at $\alpha = 0$, it follows that $R_{1314} = 0$ $(X_3 = JX_1, X_4 = JX_2)$. By taking the second derivative, the inequality

$$(9.1) \qquad\qquad K_{12} + 3K_{14} \leqq K_{13}$$

is obtained. By using X_4 in place of X_2, we get $R_{1213} = 0$ and

$$(9.2) \qquad\qquad 3K_{12} + K_{14} \leqq K_{13}.$$

(If $K_{13} = H(X_1)$ is a minimum rather than a maximum, the inequalities are reversed.)

There is still some choice possible after making $H(X_1)$ critical, since this only determines the plane of X_1 and X_3. For, X_1 and X_2 can be chosen in such a way that K_{12} will be a maximum (or minimum) among sections having a basis of the form $\{X_1 \cos \alpha + X_3 \sin \alpha, X_2 \cos \beta + X_4 \sin \beta\}$. Then, by differentiating $K(X_1 \cos \alpha + X_3 \sin \alpha, X_2)$ we find $R_{1214} = 0$.

The above technique clearly extends to higher dimensions. However, the Gauss-Bonnet integrand has so many terms, that this normalization does not clarify the relation between curvature and the Euler-Poincaré characteristic. This is not so for dimension 4, because the integrand with respect to this normalized basis is simply

$$(9.3) \qquad \frac{1}{4\pi^2} \left[2(K_{12}^2 + K_{14}^2) + (K_{12} + K_{14})^2 + K_{13}K_{24}\right]\omega$$

where ω is the volume element. This proves Theorem 1.3.

EXAMPLE. Let M be a 4-dimensional compact complex manifold on which there exist at least two closed (globally defined) holomorphic differentials $\alpha^r = a_i^{(r)}dz^i$, $r = 1, \cdots, N$, such that rank $(a_i^{(r)}) = 2$. We do not assume that M is parallelisable. Indeed, some or all of the α^r may have zeros on M. Topologically, M may be the Cartesian product of the Riemann sphere with a 2-sphere having N handles. The fundamental form $\sqrt{(-1)} \sum_r \alpha^r \wedge \bar{\alpha}^r$ of M is closed and of maximal rank. Hence, we have a globally defined Kaehler metric $g = 2 \sum_r \alpha^r \otimes \bar{\alpha}^r$. That this metric has nonpositive holomorphic curvature may be seen as follows. At the pole of a system of geodesic complex coordinates (z^1, z^2), the components of the curvature tensor are

$$\left\langle R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^l}\right\rangle = -\frac{\partial^2 g_{ij^*}}{\partial z^k \partial \bar{z}^l}$$

where

$$g_{ij^*} = \sum_r a_i^{(r)} \bar{a}_j^{(r)}.$$

Thus,

$$\left\langle R\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}\right)\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^i}\right\rangle = -\frac{\partial^2 g_{ii^*}}{\partial z^i \partial \bar{z}^i} = -\sum_r \left|\frac{\partial a_i^{(r)}}{\partial z^i}\right|^2 \leqq 0,$$

and so by Theorem 1.3, $\chi(M)$ is non-negative.

Note that since the first betti number $b_1 \geqq 4$, the second $b_2 \geqq 6$.

As a matter of fact, S. Bochner [2] has shown that the Euler-Poincaré characteristic of a compact complex manifold M of complex dimension m, on which there exists at least m closed holomorphic differentials $\alpha^r = a_i^{(r)}dz^i$ such that rank $(a_i^{(r)})$ is maximal at each point of M, is non-negative for m even and nonpositive for m odd. Since the holomorphic sectional curvatures are nonpositive we ask the following question:

Is the sign of the Euler-Poincaré characteristic of a compact Kaehler manifold of negative holomorphic sectional curvature given by $(-1)^m$?

The expression (9.3) is now used to obtain an upper bound for $\chi(M)$ in terms of volume and the bounds on holomorphic curvature. Suppose that M is λ-holomorphically pinched. Choose $H(X_1)$ to be minimum, so we may assume it is λ. Let $x = H(X_1 + X_4) + H(X_1 - X_4)$, $y = H(X_1 + X_2) + H(X_1 - X_2)$, $z = H(X_2)$ $= K_{24}$. Then, by (6.2)

$$K_{12} = \frac{1}{8}(3x - y - z - \lambda), \quad K_{14} = \frac{1}{8}(3y - x - z - \lambda),$$

and so by the inequalities (9.1) and (9.2), since $K_{12} \geqq K_{14}$,

$$(9.4) \qquad \frac{3\lambda + z}{2} \leqq y \leqq x \leqq 2, \quad \lambda \leqq z \leqq 1.$$

The integrand, except for the factor $\omega/4\pi^2$, is

$$f(x, y, z) = \frac{1}{8}\left[3(x^2 + y^2) + z^2 - 2xy - 2xz - 2yz - 2\lambda x - 2\lambda y + 10\lambda z + \lambda^2\right].$$

The maximum value of f on the region determined by the inequalities (9.4) is

$$(9.5) \qquad \frac{1}{2}(3\lambda^2 - 4\lambda + 4), \quad \lambda \geqq -1,$$

$$(9.6) \qquad \frac{1}{2}(4\lambda^2 - 4\lambda + 3), \quad \lambda \leqq -1.$$

That there are no inequalities superior to (9.4), in terms of which better bounds for f can be obtained, is a consequence of examples A and B, §8. For, example A yields (9.5) and B yields (9.6) as respective integrand factors.

Making use of the symmetry of (9.5) and (9.6), they may be combined to give

THEOREM 9.1. *Let M be a compact 4-dimensional Kaehler manifold, L the maximum absolute value of holomorphic curvature, $(1-\lambda)L$ the variation (maximum minus minimum) of holomorphic curvature, and V the volume of M. Then,*

$$(9.7) \qquad \chi(M) \leqq \frac{1}{8\pi^2}(3\lambda^2 - 4\lambda + 4)L^2 V.$$

Since $\lambda \geqq -1$, we always have

$$\chi(M) \leqq \frac{11L^2 V}{8\pi^2}.$$

Note that the bound (9.7) is achieved for the complex projective space P_2 but for $M = S^2 \times S^2$ the bound is $11\chi(M)/8$. (For $P_2: L = 1$, $\lambda = 1$, $V = 8\pi^2$, whereas for $S^2 \times S^2: L = 1$, $\lambda = 1/2$, $V = 16\pi^2$.)

10. **Curvature and volume.** In this section, we shall assume that M is a complete λ-holomorphically pinched Kaehler manifold with $\lambda > 0$. Our goal is to obtain an upper bound for the volume of M in terms of λ and the dimension of M. The ensuing technique also yields a well-known bound for the diameter, viz., $\pi / \sqrt{\lambda}$. The approach will be to obtain a bound B on the Jacobian of the exponential map. The bound on volume is then obtained by integrating B on the interior of a sphere of radius $\pi / \sqrt{\lambda}$ in the tangent space.

The following facts about the exponential map, Jacobi fields, and second variation of arc length are required. Let γ be a geodesic starting at $P \in M$, γ parametrized with respect to arc length, t a distance along γ such that there are no conjugate points of P between P and $\gamma(t)$. Let X_1 be the tangent field to γ and $X_2 = JX_1, X_3, X_4 = JX_3, \cdots, X_{2m} = JX_{2m-1}$ parallel fields along γ which together with X_1 form an orthonormal basis at every point of γ. Covariant differentiation with respect to X_1 will be denoted by a prime, so if $V = \sum g_i X_i$, then $D_{x_1} V = V' = \sum g_i' X_i$. A vector field V along γ is called a *Jacobi field* if $V'' + R(X_1, V)X_1 = 0$. The *second variation of arc length* along γ of a vector field X is the second derivative of the arc lengths of a one-parameter family of curves having X as the *associated transverse vector field*. (For example, $\gamma_s(\alpha) = \exp_{\gamma(\alpha)} sX(\alpha), \quad 0 \leqq \alpha \leqq t$.)

(a) If X is prependicular to X_1, then the first variation (defined similarly) is zero, so the second variation determines whether the neighboring curves are longer or shorter than γ.

(b) If V is a Jacobi field such that $V(0) = 0$, then $V(\alpha) = d \exp_P \alpha T$, where T is a constant vector in the tangent space T_P. If T_P is identified with its tangent spaces, then $T = V'(0)$.

(c) The second variation of a Jacobi field V (as in (b)) is $\langle V, V \rangle'(t)/2$.

(d) If W is a vector field along γ such that $W(0) = 0$, and W is perpendicular to X_1, then the second variation of W is

$$\int_0^t \left[\langle W', W' \rangle - \langle W, W \rangle K(X_1, W) \right] d\alpha.$$

(e) If V and W are as in (b) and (d), then the second variation of W is an upper bound for that of V, equality occurring if and only if $V = W$. In other words *second variation is minimized by Jacobi fields up to the first conjugate point.*

(f) The conjugate points of P are the points at which $\exp_P$ is singular.

(g) *Gauss' Lemma.* If T is perpendicular to $X_1(0)$ in T_P, then $d \exp_P T$ is perpendicular to X_1 in M.

Let $W_1, \cdots, W_k$ be vectors at a point of M. We denote by $W = \{W_1, \cdots, W_k\}$ the column of these vectors and by $\det W$ the volume of the parallelepiped they span, so $(\det W)^2 = \det(\langle W_i, W_j \rangle)$. Denote the Jacobian of $\exp_P$ at $\exp_P^{-1} \gamma(t)$ by $J(t)$. Choose a basis $T_1 = X_1(0), T_2, \cdots, T_n$ of T_P with T_i perpendicular to $T_1, i > 1$, and let V_i be the Jacobi field with $V_i(0) = 0$, $V_i'(0) = T_i$, $i > 1$, so that $V_i(\alpha) = d \exp_P \alpha T_i$.

Put $T = \{T_2, \cdots, T_n\}$ and $V = \{V_2, \cdots, V_n\}$. Then, by (g) and because $\exp_P$ preserves radial lengths

(10.1) $\det V(\alpha) = \alpha^{n-1} J(\alpha) \det T$.

Letting $X = \{X_2, \cdots, X_n\}$, we may write $V = FX$ where F is a nonsingular matrix function of α of order $n - 1$. Hence, $\det V = \det F$ since $\det X = 1$.

Let g and h be real-valued functions of α such that $g(0) = h(0) = 0$, $g(t) = h(t) = 1$, but otherwise unspecified as yet. They determine a column $W = \{gX_2, hX_3, hX_4, \cdots, hX_n\}$ which coincides at t with the column of Jacobi fields $U = (F(t))^{-1} V = \{U_2, \cdots, U_n\}$. Thus, we have

$$\det U(\alpha) = \det (F(t))^{-1} \det V(\alpha) = \frac{\alpha^{n-1} J(\alpha)}{t^{n-1} J(t)}.$$

By the rule for the derivative of a determinant and the fact that $U(t) = X(t)$ is an orthonormal column, we have

$$((\det U)^2)'(t) = \langle U_2, U_2 \rangle'(t) + \cdots + \langle U_n, U_n \rangle'(t)$$

$$= 2\left(\frac{n-1}{t} + \frac{J'(t)}{J(t)}\right).$$

But by (c) and (e), this is majorized by twice the sum of second variations of the W_i, that is, by (d),

(10.2)
$$\frac{n-1}{t} + \frac{J'(t)}{J(t)} \leqq \int_0^t \left[(g')^2 + (n-2)(h')^2 - g^2 H(X_1) \right.$$

$$\left. - \left(\sum_{i=3}^n K(X_1, X_i)\right) h^2\right] d\alpha.$$

However, by Lemma 8.2, we have for i odd,

$$K(X_1, X_i) + K(X_1, X_{i+1}) \geqq \lambda - \frac{1}{4}(1 + H(X_1)).$$

Letting $f = H(X_1)$, the problem of obtaining an upper bound for $J'(t)/J(t)$ has been reduced to the variational problem of minimizing

(10.3)
$$\int_0^t \left\{(g'(\alpha))^2 + (n-2)(h'(\alpha))^2 - \frac{1}{8}(4\lambda - 1)(n-2)(h(\alpha))^2 \right.$$

$$\left. - f(\alpha)\left[(g(\alpha))^2 - \frac{1}{8}(n-2)(h(\alpha))^2\right]\right\} d\alpha,$$

where f is an arbitrary function subject to the restrictions $\lambda \leqq f \leqq 1$, and g, h are functions subject to the restrictions $g(0) = h(0) = 0$, $g(t) = h(t) = 1$.

The Euler equations for this problem are

(10.4) $$g'' + fg = 0,$$

(10.5) $$8h'' + (4\lambda - 1 - f)h = 0.$$

Let G and H be the solutions of (10.4) and (10.5), respectively, such that $G(0) = H(0) = 0$, $G'(0) = H'(0) = 1$. Then, since f is an analytic function, so are G and H. Their power series therefore have the form $G(\alpha) = \alpha + \cdots$, $H(\alpha) = \alpha + \cdots$. Setting $g = G / G(t)$, $h = H / H(t)$ and integrating $(g'(\alpha))^2 \, d\alpha$, $(h'(\alpha))^2 d\alpha$ by parts, the integral (10.3) reduces to

(10.3)' $$g'(t)g(t) + (n - 2)h'(t)h(t) - g'(0)g(0) - (n - 2)h'(0)h(0)$$

plus an integral which is zero due to the fact that g and h satisfy (10.4) and (10.5). Since $g(t) = h(t) = 1$, $g(0) = h(0) = 0$ and $g'(t) = G'(t) / G(t)$, $h'(t) = H'(t) / H(t)$, we finally have

(10.6) $$\frac{J'(t)}{J(t)} \leqq \frac{G'(t)}{G(t)} + (n - 2)\frac{H'(t)}{H(t)} - \frac{n - 1}{t}.$$

Integrating both sides of this inequality from α to t, then taking the limit as $\alpha \to 0$ by using the facts that $G(\alpha) (H(\alpha))^{n-2}/ \alpha^{n-1} = 1 + \cdots$ and $J(0) = 1$, we derive

$$\log J(t) \leqq \log \frac{G(t)(H(t))^{n-2}}{t^{n-1}},$$

that is,

(10.7) $$J(t) \leqq \frac{G(t)(H(t))^{n-2}}{t^{n-1}}.$$

Since it follows from the Sturm comparison theorem that the solution G of (10.4) must have another zero in the interval $[0, \pi / \sqrt{\lambda}]$, the inequality (10.7) shows that $J(t)$ must also have a zero in $(0, \pi / \sqrt{\lambda}]$. Hence, there is a conjugate point to P along γ at a distance not greater than $\pi / \sqrt{\lambda}$.

THEOREM 10.1. *If M is a Kaehler manifold which is complete and λ-holomorphically pinched, $\lambda > 0$, then the diameter of M does not exceed $\pi / \sqrt{\lambda}$.*

COROLLARY 10.1. *A complete Kaehler manifold of strictly positive holomorphic curvature is compact.*

The integral of the bound on $J(t)$, given by (10.7), over the interior of the sphere of radius $\pi / \sqrt{\lambda}$ about 0 in T_P is thus a bound on the volume $v(M)$ of M. This integration is accomplished by multiplying by the volume of an $(n - 1)$-sphere of radius t, viz., $2t^{n-1}\pi^m /(m - 1)!$ where $m = n/2$, and integrating from 0 to r. Thus

THEOREM 10.2. *Let M be a complete λ-holomorphically pinched Kaehler manifold with $\lambda > 0$. Then*

$$(10.8) \qquad v(M) \leqq \frac{2\pi^m}{(m-1)!} \int_0^r G(t)(H(t))^{n-2}dt,$$

where r is the first zero of G beyond 0.

To realize an upper bound, consider the integral (10.3), where we note that $f = \lambda$ may be substituted for the coefficient of g^2 and $f = 1$ for the coefficient of h^2. The corresponding solutions of the Euler equations of G and H are

$$G(t) = \frac{1}{a} \sin at, \quad a = \sqrt{\lambda},$$

$$H(t) = \begin{cases} \dfrac{1}{b} \sin bt, & b = \sqrt{\dfrac{2\lambda - 1}{4}} & \text{if } \lambda > \dfrac{1}{2}, \\[3ex] t, & & \text{if } \lambda = \dfrac{1}{2}, \\[3ex] \dfrac{1}{b} \sinh bt, & b = \sqrt{\dfrac{1 - 2\lambda}{4}} & \text{if } \lambda < \dfrac{1}{2}. \end{cases}$$

When $\lambda = 1$, formula (10.8) reduces to an equation for the volume of complex projective space P_m.

Even better bounds can be obtained from (10.3) by a judicious choice of g and h, and by replacing f by λ or 1 depending on whether its coefficient is negative or positive, respectively. For example, if we take $g(\alpha) = \sin a t \alpha / \sin a, a = \sqrt{\lambda}$ and $h(\alpha) = \alpha / t$, we find that for $n \leqq 10$ the coefficient of $f(\alpha)$ is always nonpositive. The result is

$$v(M) \leqq \frac{2\pi^m}{(m-1)! \lambda^m} \int_0^\pi x^{n-2} \sin x \exp\left[-\frac{(n-2)(3\lambda-1)}{48\lambda} x^2 \right] dx, \quad n \leqq 10.$$

Applying Theorem 9.1, we find an upper bound for the Euler-Poincaré characteristic of a complete 4-dimensional λ-holomorphically pinched Kaehler manifold with $\lambda > 0$,

$$(10.9) \qquad \chi(M) \leqq \frac{3\lambda^2 - 4\lambda + 4}{4\lambda^2} \int_0^\pi x^2 \sin x \exp\left(-\frac{3\lambda - 1}{24\lambda} x^2 \right) dx.$$

For $M = S^2 \times S^2$, this bound is approximately $3.4\chi(M)$. Good bounds are obtained when $\lambda > 0.6$.

REMARKS. (a) *A complete Kaehler manifold M of strictly positive holomorphic curvature is simply connected.* For, if M is not simply connected, then in every nontrivial free homotopy class of closed curves of M there would be a

closed geodesic which is the shortest closed curve in the class. That this is impossible is seen by applying (a) and (d) above to the vector $W = JX_1$ along the geodesic γ. Indeed, its first variation is zero, and its second variation is negative.

(b) A 4-dimensional complete Kaehler manifold of strictly positive holomorphic curvature is compact, simply connected and has positive Euler-Poincaré characteristic bounded above by (10.9).

11. **The curvature transformation.** We have seen that one of the difficulties which arises when attempting to resolve the *Question* raised in §1 by considerations of (3.1) at one point is the presence of terms involving factors of the type $\langle R(X,Y)X,Z\rangle$, $Z \neq Y$. However, this is only part of the problem; for, one must still account for terms which are products of those of the form $\langle R(X,Y)Z,W\rangle$. Even in dimension 6 where there are 105 independent components of the curvature tensor, and indeed $(6!)^2$ terms to be summed in (3.1) the problem is formidable! For these reasons one is led to consider Kaehler manifolds where one may make essential use of the additional curvature properties provided by Lemma 6.2. The 1/4-pinched compact Kaehler manifolds are characteristic of the complex projective spaces [1]. If the curvatures are 0.24-pinched, W. Klingenberg proved that the manifold has the homotopy type of complex projective space, and hence, in particular, positive Euler-Poincaré characteristic [6].

We shall take a different point of view here. Indeed, the pinching hypothesis will be replaced by a normalization condition on the curvature transformation. The following lemma leads to the property (P) of Theorem 1.4.

LEMMA 11.1. *Let* $\{X_1,\cdots,X_n\}$ *be a basis of* T_P. *Then, a necessary and sufficient condition that* $\langle R(X_i,X_j)X_i,X_k\rangle = 0$, $k \neq j$, *is that the curvature transformation satisfy the relation*

$$(11.1) \qquad (R(X_i,X_j))^2 = -(K(X_i,X_j))^2 I$$

on $\sigma(X_i,X_j)$.

Proof. Set $K_{ij} = K(X_i,X_j)$ and let a,b be any real numbers. Then, for any $Z = aX_i + bX_j \in \sigma(X_i,X_j)$,

$$R(X_i,X_j)Z = a\sum_{k=1}^{n} R_{ijik}X_k - b\sum_{k=1}^{n} R_{ijkj}X_k$$

and

$$(R(X_i,X_j))^2 Z = -K_{ij}^2 Z - aK_{ij}\sum_{k \neq i} R_{ijkj}X_k - bK_{ij}\sum_{k \neq j} R_{ijik}X_k$$
$$+ a\sum_{k \neq j} R_{ijik}R(X_i,X_j)X_k - b\sum_{k \neq i} R_{ijkj}R(X_i,X_j)X_k.$$

Applying the condition (11.1), it follows that

(11.2) $$\sum_{k \neq j} R_{ijik} R(X_i, X_j) X_k = K_{ij} \sum_{k \neq i} R_{ijkj} X_j,$$

(11.2)′ $$\sum_{k \neq i} R_{ijkj} R(X_i, X_j) X_k = - K_{ij} \sum_{k \neq j} R_{ijik} X_k.$$

Taking the inner product of (11.2) with X_i and of (11.2)′ with X_j, we obtain

$$\sum_{k \neq j} (R_{ijik})^2 = 0, \qquad \sum_{k \neq i} (R_{ijkj})^2 = 0.$$

Hence, $R_{ijik} = 0$, $k \neq j$.

Conversely, if $R_{ijik} = 0$, $k \neq j$, $R(X_i, X_j) X_i = K_{ij} X_j$. Thus, $(R(X_i, X_j))^2 X_i = K_{ij} R(X_i, X_j) X_j = - K_{ij}^2 X_i$, and so by linear extension $(R(X_i, X_j))^2 Z = - K_{ij}^2 Z$ for any $Z \in \sigma(X_i, X_j)$.

COROLLARY 11.1. *Let $\{X_1, \cdots, X_n\}$ be an orthonormal basis of T_p. Then if $K(X_i, X_j) \neq 0$ is a minimum or maximum among all sectional curvatures on planes spanned by X_i and $X_j \cos \theta + X_k \sin \theta$, $k \neq i, j$, the curvature transformation $R(X_i, X_j)$ defines a complex structure on $\sigma(X_i, X_j)$.*

COROLLARY 11.2. *The curvature transformation $R(\sigma)$ of a manifold of constant nonzero curvature defines a complex structure on σ.*

REMARKS. (a) A proof of the following relevant result may be found in [4, p. 267]. Let A be a nonsingular linear transformation of the $2n$-dimensional vector space R^{2n} with a positive definite inner product. By means of the inner product, A may be identified with a bilinear form on R^{2n}. If this form is skew-symmetric, there is a unique decomposition of R^{2n} into subspaces $S_1, \cdots, S_r$ such that:

(i)   each S_i is invariant by the transformation A, and for $i \neq j$, $S_i \perp S_j$;

(ii) restricted to S_i, $A^2 = - a_i^2 I$, $a_i > 0$, and for $i \neq j$, $a_i \neq a_j$.

(b) A Kaehler manifold of constant mean curvature and of dimension > 4 does not in general have the property (P) although it does satisfy $\sum_{k \neq j} R_{ijik} = 0$ relative to an orthonormal basis.

Proof of Theorem 1.4. Let $\{X_1, \cdots, X\}$, $X_{3+i} = JX_i$, $i = 1, 2, 3$, be an orthonormal holomorphic basis of T_p with respect to which the curvature transformation satisfies the property (P). By Lemma 11.1, we need only consider those summands in (3.1) whose factors are of the form $\langle R(X, Y)Z, W \rangle$ where X, Y, Z, W are a part of the basis. Put $X_{i*} = JX_i$, $i = 1, 2, 3$, $i** = i$. By applying the identities (iii) of Lemma 2.1 and (i) of Lemma 6.2, $R_{\alpha\beta\gamma\delta} = 0$, $\alpha, \beta, \gamma, \delta = 1, \cdots, 6$, if either $\gamma = \alpha*$ and $\delta \neq \beta*$ or $\beta = \alpha*$ and $\delta \neq \gamma*$. Hence, the only nonvanishing terms are of the following types:

$$K_{I_1} K_{I_2} K_{I_3}, \quad (R_{I_1 J_1})^2 K_{I_2}, \quad R_{I_1 I_1 *} R_{I_2 I_2 *} R_{I_3 I_3 *},$$

$$R_{i_1 i_1 * j_1 j_1 *} R_{i_2 i_2 * j_2 j_2 *} R_{i_3 i_3 * j_3 j_3 *}, \quad i_r \neq j_r,$$

where I_1, I_2, I_3 are index pairs: $I = ij$, $i*j$ or $ij*$, and $I* = i*j*$, $ij*$ or $i*j$, resp. By Lemma 6.2 (i), we see that $R_{I_1 I_1*} R_{I_2 I_2*} R_{I_3 I_3*} = K_{I_1} K_{I_2} K_{I_3}$. On the other hand, by Lemma 6.2 (iii),

$$R_{i_1 i_1 * j_1 j_1 *} R_{i_2 i_2 * j_2 j_2 *} R_{i_3 i_3 * j_3 j_3 *} = (K_{i_1 j_1} + K_{i_1 j_1 *})(K_{i_2 j_2} + K_{i_2 j_2 *})(K_{i_3 j_3} + K_{i_3 j_3 *}).$$

Consequently, since

$$\varepsilon_{i_1 j_1 i_2 j_2 i_3 j_3} \varepsilon_{i_1 j_1 i_2 j_2 i_3 j_3} = +1,$$

$$\varepsilon_{i_1 j_1 k_1 l_1 i_2 j_2} \varepsilon_{k_1 l_1 i_1 j_1 i_2 j_2} = +1,$$

$$\varepsilon_{i_1 j_1 i_2 j_2 i_3 j_3} \varepsilon_{i_1 * j_1 * i_2 * j_2 * i_3 * j_3 *} = -1,$$

and

$$\varepsilon_{i_1 i_1 * i_2 i_2 * i_3 i_3 *} \varepsilon_{j_1 j_1 * j_2 j_2 * j_3 j_3 *} = \varepsilon_{i_1 i_2 i_3 i_1 * i_2 * i_3 *} \varepsilon_{j_1 j_2 j_3 j_1 * j_2 * j_3 *},$$

the various terms in the Gauss-Bonnet integrand are either all non-negative or all nonpositive depending on whether the sectional curvatures have the same property. Thus, if the holomorphic and anti-holomorphic sectional curvatures $K(\sigma)$ are non-negative (resp., nonpositive), $\chi(M) \geq 0$ (resp., $\chi(M) \leq 0$).

We now obtain a result valid for the dimensions $4k$, $k \geq 1$. We shall first require the following lemma.

LEMMA 11.2. *Let* $\{X_i, X_{i*}\}$, $i = 1, \cdots, m$, *be an orthonormal holomorphic basis of T_P. Then, a necessary and sufficient condition that $R_{ijkl*} = 0$, $(i,j) \neq (k,l)$, $i < j$, $k < l$, is that the curvature transformation have the property*

(Q): $$\| R(X_i, X_j) X_k \|^2 = \sum_l \langle R(X_i, X_j) X_k, X_l \rangle^2.$$

This is an immediate consequence of the fact that

$$R(X_i, X_j) X_k = \sum_l R_{ijkl} X_l + \sum_l R_{ijkl*} X_{l*}.$$

For,

$$\| R(X_i, X_j) X_k \|^2 = \sum_l (R_{ijkl})^2 + \sum_l (R_{ijkl*})^2.$$

REMARK. Property (Q) is implied by

(Q') $$(R(X_i, X_j))^2 X_k = - \sum_l (R_{ijkl})^2 X_k.$$

For, since the curvature transformation is a skew-symmetric transformation

$$\langle R(X_i, X_j) X_k, R(X_i, X_j) X_k \rangle = - \langle (R(X_i, X_j))^2 X_k, X_k \rangle = \sum_l (R_{ijkl})^2.$$

THEOREM 11.1. *Let M^{4k}, $k > 1$, be a compact Kaehler manifold, whose curvature transformation has the properties (P) and (Q), with respect to the orthonormal holomorphic basis $\{X_\alpha\}$. If for all $\sigma = \sigma(X_\alpha, X_\beta)$, $K(\sigma) \geq 0$, then $\chi(M^{4k}) \geq 0$, and if $K(\sigma) \leq 0$, $\chi(M^{4k}) \geq 0$. If the sectional curvatures are always positive or always negative, the Euler-Poincaré characteristic is positive.*

Proof.   As before, let $\{X_i, X_{i*}\}$, $i = 1, \cdots, 2k$, be an orthonormal holomorphic basis of T_P. Again, one need only consider those summands in (3.1) whose factors are of the form $\langle R(X, Y)Z, W \rangle$ where the vectors X, Y, Z and W are independent. Moreover, $R_{\alpha\beta\gamma\delta} = 0$, $\alpha, \beta, \gamma, \delta = 1, \cdots, 4k$, if either $\gamma = \alpha*$ and $\delta \neq \beta*$ or $\beta = \alpha*$ and $\delta \neq \gamma*$. Furthermore, by Lemma 11.2, R_{ijkl*} vanishes for all i, j, k, l. The nonvanishing terms may then be classified as before, viz.,

$$K_{I_1} \cdots K_{I_{2k}}, \quad (R_{I_1 J_1})^2 \cdots (R_{I_s J_s})^2 K_{I_{2s+1}} \cdots K_{I_{2k}}, \quad R_{I_1 I_1*} \cdots R_{I_{2k} I_{2k}*},$$

$$R_{i_1 i_1* j_1 j_1*} \cdots R_{i_{2k} i_{2k}* j_{2k} j_{2k}*},$$

and so, since

$$\varepsilon_{i_1 j_1 \cdots i_{2k} j_{2k}} \varepsilon_{i_1 j_1 \cdots i_{2k} j_{2k}} = +1,$$

$$\varepsilon_{i_1 j_1 k_1 l_1 \cdots i_s j_s k_s l_s i_{2s+1} j_2 s+1 \cdots i_{2k} j_{2k}} \varepsilon_{k_1 l_1 i_1 j_1 \cdots k_s l_s i_s j_s i_{2s+1} j_{2s+1} \cdots i_{2k} j_{2k}} = +1,$$

$$\varepsilon_{i_1 j_1 \cdots i_{2k} j_{2k}} \varepsilon_{i_1* j_1* \cdots i_{2k}* j_{2k}*} = +1,$$

$$\varepsilon_{i_1 i_1* \cdots i_{2k} i_{2k}*} \varepsilon_{j_1 j_1* \cdots j_{2k} j_{2k}*} = \varepsilon_{i_1 \cdots i_{2k} i_1* \cdots i_{2k}*} \varepsilon_{j_1 \cdots j_{2k} j_1* \cdots j_{2k}*} = +1,$$

the result follows.

The above proof breaks down in dimensions $4k + 2$. For example, if $k = 2$, the term $R_{1*2*34} R_{3*4*25} R_{123*5*} R_{352*4*} R_{5*411*}$ need not vanish on account of properties (P) and (Q).

REMARKS.   (a) The curvature tensor of a manifold M of constant holomorphic curvature 1 has the components

$$\langle R(X_i, X_j)X_k, X_l \rangle = \frac{1}{4} \big[(\delta_{jl}\delta_{ik} - \delta_{jk}\delta_{il}) + \langle X_j, JX_l \rangle \langle X_i, JX_k \rangle$$

$$- \langle X_j, JX_k \rangle \langle X_i, JX_l \rangle + 2\langle X_i, JX_j \rangle \langle X_k, JX_l \rangle \big]$$

relative to an orthonormal holomorphic basis. Hence, M has the properties (P) and (Q). Conversely, if a Kaehler manifold possesses the properties (P) (and (Q)) for all $\sigma \in H^2_{n,P}$, the space is of constant holomorphic curvature. For, let X, Y, JX, JY be part of an orthonormal basis of T_P. Then, $H(X) - H(Y) = \langle R(X + Y, JX + JY)(X + Y), JX - JY \rangle = 0$.

That a manifold with the properties (P) and (Q) (at one point) need not have constant holomorphic curvature is a consequence of either example A or B. (That such Kaehlerian manifolds actually exist is another matter.) It is not difficult to construct such examples in higher dimensions.

(b) The Kaehlerian product of m copies of S_2, with the canonical metric, satisfies the property (P) relative to the natural holomorphic basis.

(c) Let M be a 6-dimensional compact Kaehler manifold of strictly positive (or negative) curvature. At each point of M choose an orthonormal holomorphic basis $\{X_i, JX_i\}$, $i = 1, 2, 3$. Consider the sectional curvatures of planes spanned by the orthonormal vectors X_1 and $X_2 \cos \theta + X_i \sin \theta$, as well as those of the planes

spanned by X_1 and $X_2 \cos \theta + JX_i \sin \theta$. If K_{12} is a minimum for these curvatures $R_{121\alpha} = R_{212\alpha} = 0$ for $\alpha = 3, 1^*, 2^*, 3^*$. (The holomorphic sectional curvatures are greater than or equal to K_{12}.) Moreover, if K_{13} is a minimum for those sectional curvatures on planes spanned by vectors X_1 and $X_3 \cos \theta + JX_3 \sin \theta$, then $R_{1313^*} = 0$. Applying Lemma 6.2, we see that there are 28 vanishing curvature components. Of the remaining 77, 15 have the form $R_{ijij} = K_{ij}$, $R_{ij^*ij^*} = K_{ij^*}$, and 3 are of the form $R_{ii^*jj^*} = K_{ij} + K_{ij^*}$. The remaining 59 consist of essentially 19 distinct components by virtue of Lemma 6.2, and among these only two have the form R_{ijkl} with i, j, k, l all different. These cannot be coped with by the above methods unless further restrictions on M are imposed.

12. **Holomorphic pinching and Euler-Poincaré characteristic.** A procedure is now outlined by which a meaningful formula for the Gauss-Bonnet integrand G can be found when M is a 6-dimensional compact Kaehler manifold possesing the property (P). The formula obtained will then be used in two ways:

(1) To show that if M is λ-holomorphically pinched, $\lambda \geq 2 - 2^{2/3} \sim 0.42$, then $\chi(M) > 0$.

(2) To show that non-negative holomorphic curvature is not sufficient to make G non-negative. This will be accomplished by means of an example satisfying the condition (P).

In the following, a pair of indices (α, α^*) will be denoted by H or H', and a pair (α, β) where $\beta \neq \alpha^*$ by A. Then, condition (P) is equivalent to: The only nonzero curvature components are of the form $R_{HH'}$, R_{AA}, R_{AA^*}.

The nonzero terms of the integrand are now classified into three groups depending on the number of pairs of type H occurring in I_1, I_2, I_3.

(a) All I_j are of the type A. Then, if we require $\alpha < \beta$ in every pair (α, β), there are 12 possibilities for I_1, and once I_1 is chosen, 4 possibilities for I_2. This gives 48 possible choices for $I_1 I_2 I_3$. For each choice of $I_1 I_2 I_3$ we may choose $J_1 J_2 J_3$ in only 2 ways, equal to $I_1 I_2 I_3$ or $I_1^* I_2^* I_3^*$. The resulting product of curvature components is the same in either case, viz., $K_{I_1} K_{I_2} K_{I_3}$. Due to Lemma 6.2, there are only 4 essentially different terms, $K_{12} K_{16} K_{23}$, $K_{12} K_{13} K_{26}$, $K_{13} K_{15} K_{23}$ and $K_{15} K_{16} K_{26}$. Thus each will occur in the integrand with the factor $24 \cdot 2^6$. (The 2^6 accounts for the transpositions of each of the 6 pairs.)

(b) One I_j is of type H, two of type A. Hence, if $I_j = H$, $J_j = H$ also, so for each choice of $I_1 I_2 I_3$ there are again only two choices for $J_1 J_2 J_3$, each leading to a term $K_H K_A K_A$. The I_j which is of the type H may be chosen in any of the 3 positions and there are 3 type H pairs. Once it is chosen there are 4 possibilities for the other I's. This gives 72 terms divided among the 6 distinct possibilities $K_{14} K_{23}^2$, $K_{14} K_{26}^2$, $K_{25} K_{13}^2$, $K_{25} K_{16}^2$, $K_{36} K_{12}^2$, $K_{36} K_{15}^2$, so the sum of these is multiplied by $12 \cdot 2^6$.

(c) All I_j are of type H. Then, the J's may be any permutation of the I's, and the 3 distinct H's may be distributed among the I's arbitrarily, giving 6 terms for

each permutation of the J's. The identity permutation gives the term $K_{14}K_{25}K_{36}$. The other even permutations give the term $(K_{12} + K_{15})(K_{13} + K_{16})(K_{23} + K_{26})$. The 3 odd permutations give the 3 distinct terms $K_{14}(K_{23} + K_{26})^2$, $K_{25}(K_{13} + K_{16})^2$, $K_{36}(K_{12} + K_{15})^2$.

Finally, from the above classification, we see that G may be expressed in the form

$$G = \frac{1}{8\pi^3} \Big[4(K_{12}K_{16}K_{23} + K_{12}K_{13}K_{26} + K_{13}K_{15}K_{23} + K_{15}K_{16}K_{26})$$

$$+ K_{14}(3K_{23}^2 + 2K_{23}K_{26} + 3K_{26}^2) + K_{25}(3K_{13}^2 + 2K_{13}K_{16} + 3K_{16}^2)$$

$$+ K_{36}(3K_{12}^2 + 2K_{12}K_{15} + 3K_{15}^2) + K_{14}K_{25}K_{36}$$

$$+ 2(K_{12} + K_{15})(K_{13} + K_{16})(K_{23} + K_{26})\Big].$$

The first and last terms in this expression do not involve holomorphic curvatures, only anti-holomorphic ones, and these may be rewritten as

$$(xK_{12} + yK_{15})(xK_{13} + yK_{16})(xK_{26} + yK_{23})$$

$$+ (xK_{12} + yK_{15})(xK_{16} + yK_{13})(xK_{23} + yK_{26})$$

$$+ (xK_{15} + yK_{12})(xK_{13} + yK_{16})(xK_{23} + yK_{26})$$

$$+ (xK_{15} + yK_{12})(xK_{16} + yK_{13})(xK_{26} + yK_{23}).$$

Expanding, one finds that equality requires $(x + y)^3 = 8$ and $(x - y)^3 = 4$, so that $x = 1 + 2^{-1/3}$, $y = 1 - 2^{-1/3}$. The terms in question are products of the type $xK(X, Y) + yK(X, JY)$. Expressing the latter in terms of holomorphic curvatures, we obtain, by virtue of (6.2),

$$xK(X, Y) + yK(X, JY) = \frac{1}{8} \Big[(3x - y)(H(X + JY) + H(X - JY))$$

$$- (x - 3y)(H(X + Y) + H(X - Y)) - 2H(X) - 2H(Y)\Big].$$

Thus, if $\lambda \geqq 2 - 2^{2/3} = 2y$,

$$xK(X, Y) + yK(X, JY) \geqq - 2^{1/3}y,$$

and so

$$8\pi^3 G > 4(- 2^{1/3}y)^3 + K_{14}K_{25}K_{36} \geqq 0.$$

This proves

THEOREM 12.1. *A λ-holomorphically pinched 6-dimensional complete Kaehler manifold, $\lambda \geqq 2 - 2^{2/3} (\sim 0.42)$, having the property* (P), *has positive Euler-Poincaré characteristic.*

Note that the Ricci curvature is positive definite for this value of λ (cf. Theorem 8.1).

An obvious modification gives negative characteristic when holomorphic curvatures lie between $- 1$ and $- 2 + 2^{2/3}$.

If besides property (P), $K_{14} = K_{25} = K_{36} = 0$, $K_{12} = K_{26} = K_{13} = -1$, $K_{15} = K_{23} = K_{16} = 3$, then a computation shows that holomorphic curvature is non-negative and $G = -12/\pi^3$. Thus:

If M is a compact Kaehler manifold of dimension ≥ 6, it is not possible to prove by using only the algebra of the curvature tensor at a point that non-negative holomorphic curvature yields a non-negative Gauss-Bonnet integrand.

In fact, we are of the opinion that the *Question* (cf. §1) cannot be resolved in this manner.

REMARK. Conditions (P) and (Q) are preserved under Kaehlerian products. In particular, products of complex projective spaces satisfy these conditions.

Added in proof. (a) The technique employed in §10 for estimating volume may be applied to the Riemannian case thereby generalizing a result of Berger [*On the characteristic of positively-pinched Riemannian manifolds*, Proc. Nat. Acad. Sci. U.S.A. **48** (1962), 1915–1917]. The improvement comes from generalizing Rauch's theorem so as to estimate directly lengths of Grassman $(n-1)$-vectors mapped by *exp* rather than from using Rauch's estimate of lengths of vectors to estimate lengths of $(n-1)$-vectors as Berger does.

(b) Klingenberg's result (cf. §11, paragraph 1) has recently been improved by S. Kobayashi [*Topology of positively pinched Kaehler manifolds*, Tohoku Math. J. **15** (1963), 121–139] and a subsequent improvement is given in our paper [*On the topology of positively curved Kaehler manifolds*, ibid. **15** (1963), 359–364].

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