Exercises are from Haberman [Hab] and Strauss [Str]. Be sure to justify your answer completely, stating definitions and theorems used. Show all calculations.

- **Qualitative Properties of the Heat Equation II: Maximum Principle, Uniqueness & Stability [Str 2.3.5, 2.3.6, 2.3.8]**

  1. We shall show the Maximum Principle does not hold for the variable coefficient equation $u_t = xu_{xx}$.

     (a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\mathcal{R} = \{(x,t) : -2 \leq x \leq 2, \ 0 \leq t \leq 1\}$.

     (b) Where does the proof of the maximum principle fail for this equation?

  2. Prove the **Comparison Principle** for the diffusion equation: If $w$ and $v$ both solve the equation $u_t = ku_{xx}$ and $w \leq v$ for $t = 0$, for $x = 0$ and for $x = L$, then $w \leq v$ for $0 \leq x \leq L, \ 0 \leq t < \infty$.

  3. (Optional) Consider the diffusion equation on $(0, L)$ with Robin boundary conditions $u_x(0,t) - a_0u(0,t) = 0$ and $u_x(L,t) + a_Lu(L,t) = 0$. If $a_0 > 0$ and $a_L > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^L u(x,t)^2 dx$. This is interpreted to mean that part of the “energy” is lost at the boundary, so we call such boundary conditions **dissipative**.

- **Fundamental solution of the Heat Equation [Str 2.4.1, 2.4.9]**

  4. Solve the diffusion equation $u_t = ku_{xx}$ with initial condition $\varphi(x) = 1$ for $|x| < L$ and $\varphi(x) = 0$ for $|x| > L$. Write your answer in terms of $\text{Erf}(x)$.

  5. Solve the diffusion equation with initial condition $u(x, 0) = x^2$ by using the invariance properties: Show that $u_{xxx}$ satisfies the PDE with zero initial condition. Conclude by uniqueness that $u_{xxx} \equiv 0$. Integrate the result thrice to obtain $u(x,t)$ and solve for the integration coefficients $A(t)$, $B(t)$, and $C(t)$, by plugging into the original problem.

- **Fourier Cosine & Sine Series [Hab, Sections 3.1-3.3, Str]**

  6. Haberman, pp. 95-96, #3.2.1e, f, 3.2.2c, e, 3.2.3

  7. Haberman, pp. 113-116 #3.3.1c, e, 3.3.5b, 3.3.8, 3.3.10, 3.3.13, 3.3.17