Solutions to Problems from Math 135 Recitation on 5/1/15

For the problems see
`http://math.rutgers.edu/~az202/teaching/`.

1. First, $f(x)$ is continuous on the closed, bounded interval $[1/2, 2]$, so the EVT applies. Let's find the critical points:

$$\begin{aligned} f'(x) &= 2 - 2/x^2 = 0 \\ &\implies 2x^2 - 2 = 0 \implies x = \cancel{-1}, 1, \end{aligned}$$

where we discard -1 since it's not in $[1/2, 2]$. If we plug the endpoints and critical points into $f(x)$, we get $f(1/2) = 5$, $f(1) = 4$, and $f(2) = 5$. So $\boxed{4}$ is the absolute minimum, and $\boxed{5}$ is the absolute maximum.

2. First,

$$f'(x) = \frac{-2x}{(x^2 - 1)^2},$$

so there are critical points at $x = 0$ and $x = \pm 1$. This number line shows the sign of $f'(x)$:

$$+ + + \, (-1) \, + + + \, (0) \, - - - \, (1) \, - - - \, .$$

Therefore, $f(x)$ increases on $(0, 1)$ and $(1, \infty)$, and decreases on $(-\infty, -1)$ and $(-1, 0)$. Further, $x = 0$ is a local maximum, and $f(0) = -1$.

To find the vertical asymptotes, we set the denominator of f equal to zero. So the vertical asymptotes are $x = -1$ and $x = 1$. To compute horizontal asymptotes, we find the following limits:

$$\begin{aligned} \lim_{x \to \infty} \frac{1}{x^2 - 1} &= 0 \\ \lim_{x \to -\infty} \frac{1}{x^2 - 1} &= 0. \end{aligned}$$

So the horizontal asymptote is $y = 0$.

Finally, we can sketch the graph; see Figure 1.

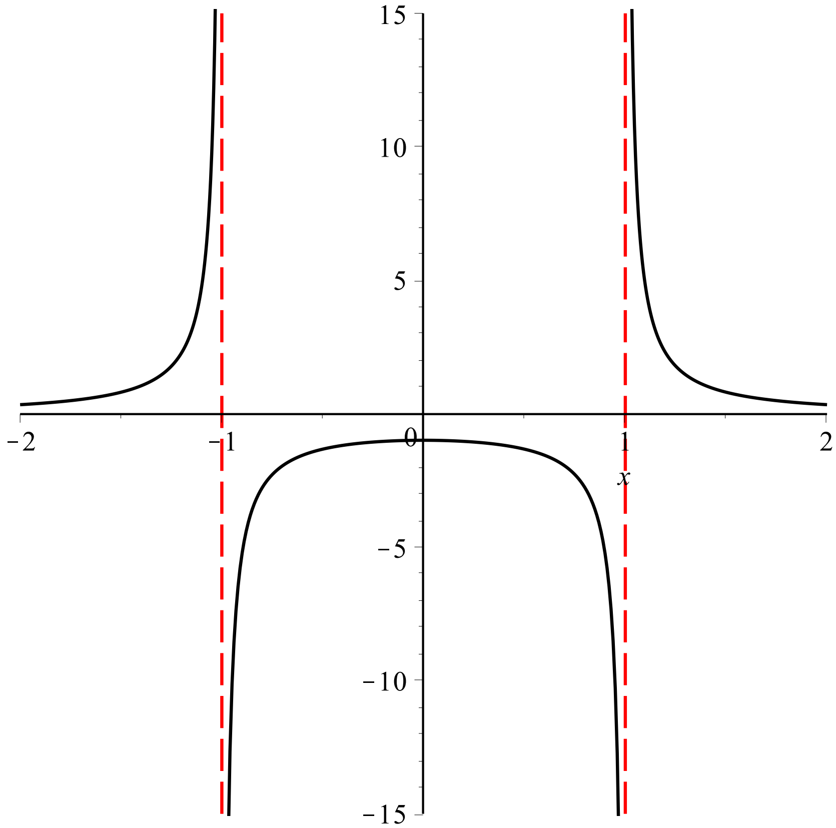

Figure 1: $f(x) = 1/(x^2 - 1)$

3. We have

$$f'(x) = -6\,\frac{x}{\left(x^2+3\right)^2},$$
$$f''(x) = 18\,\frac{x^2-1}{\left(x^2+3\right)^3}.$$

The "critical points" of the *second* derivative are -1 and 1, and its sign chart looks like this:

$$+++\,(-1)\ ---\ (1)\ +++\,.$$

So f is concave up on $(-\infty, -1)$ and $(1, \infty)$, and concave down on $(-1, 1)$. Because f'' changes sign *and* f is defined at $x = \pm 1$, the inflection points are

$\boxed{(-1, 3/4)}$ and $\boxed{(1, 3/4)}$. See Figure 2 for a graph.

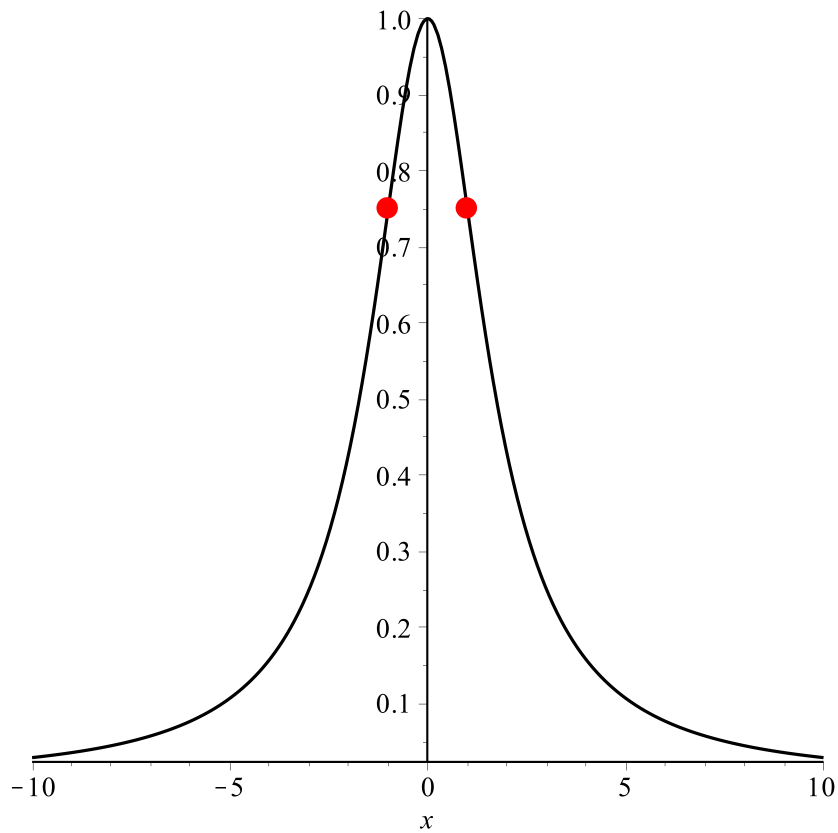

Figure 2: $f(x) = 3/(3 + x^2)$. (Red circles are inflection points.)

4. Substituting gives [0/0], and the numerator and denominator are differentiable, so L'Hôpital's Rule applies:

$$\lim_{x\to 0} \frac{x}{e^x - 1} = \lim_{x\to 0} \frac{x'}{(e^x - 1)'} = \lim_{x\to 0} \frac{1}{e^x} = \boxed{1.}$$

5. Substituting gives [0/0], and the numerator and denominator are differentiable near $x = 0$, so L'Hôpital's Rule applies:

$$\lim_{x\to 0} \frac{\tan(3x)}{\tan(4x)} = \lim_{x\to 0} \frac{(\tan(3x))'}{(\tan(4x))'} = \lim_{x\to 0} \frac{3\sec^2(3x)}{4\sec^2(4x)} = \boxed{\frac{3}{4}.}$$

6. Substituting gives $[e^{-\infty}\ln(\infty)] = [0 \cdot \infty]$, which is an indeterminate form, but not yet a job for L'H. So we first re-write the limit to include a fraction:

$$\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} = \left[\frac{\infty}{\infty}\right].$$

Now applying L'H gives

$$\lim_{x\to\infty} \frac{1/x}{e^x} = \boxed{0.}$$

7. This time, the indeterminate form is

$$\left[\frac{1}{\sin(0)} - \infty\right] = \left[\frac{1}{0} - \infty\right] = [\infty - \infty].$$

Again, L'H does not yet apply. That $\sin(1/x)$ is cumbersome; let's make the substitution $t = 1/x$. Then $x \to \infty \iff t \to 0^+$. So the original limit becomes

$$\lim_{t\to 0^+} \frac{1}{\sin t} - \frac{1}{t} = \lim_{t\to 0^+} \frac{t - \sin t}{t \sin t} = \left[\frac{0}{0}\right],$$

so, applying L'H,

$$\lim_{t\to 0^+} \frac{1 - \cos t}{t\cos t + \sin t} = \left[\frac{0}{0}\right],$$

so let's apply L'H again:

$$\lim_{t\to 0^+} \frac{\sin t}{-t\sin t + \cos t + \cos t} = \boxed{0.}$$

8. Let x and y be the dimensions of the rectangle, A be its area, and P be its perimeter. Then $A = xy = 1$, so $y = 1/x$. Next, $P = 2x + 2y = 2x + 2/x$. Finally, x must belong to the interval $[0, \infty)$ (there is no limit on how wide our rectangle can be). We have

$$P'(x) = 2 - \frac{2}{x^2} = 2\left(\frac{x^2 - 1}{x^2}\right).$$

The critical points in $[0, \infty)$ are $x = 0, 1$. Further, P' is negative when $0 < x < 1$ and positive when $x > 1$. So at $x = 1$, P attains its absolute min (see Figure 3). Thus the rectangle of area 1 with minimum perimeter has dimensions 1×1; in other words, it's a square.

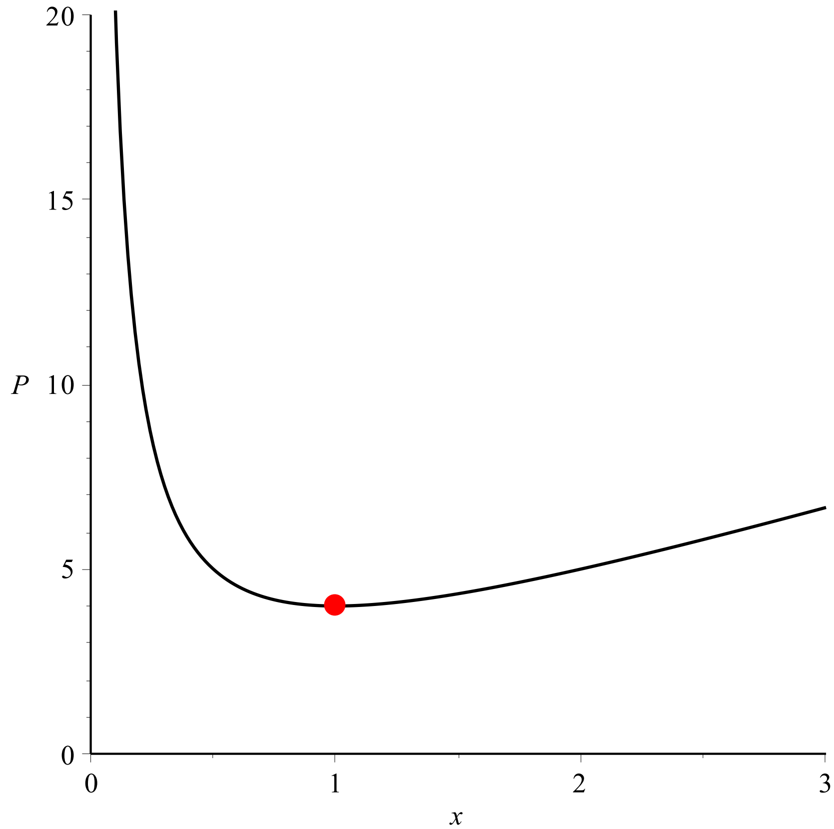

Figure 3: The graph of $P(x)$, showing the absolute minimum.

9. Let $D(x) = mx + b$ be the number of worms her friend buys if she prices them x cents each. Then $D(10) = 1000 \implies 10m + b = 1000$, and we also have $m = -10$. So $b = 1100$, and $D(x) = -10x + 1100$. Then the total profit is $P(x) = xD = x(-10x + 1100)$. This is a parabola opening down with vertex at $x = 55$, so selling worms at $\boxed{55}$ cents each maximizes the profit.

10. Let us measure prices in thousands of dollars. Let $A(t)$ be the difference between Bob's savings and the price of the car when he is $20 + t$ years old. Then $A(t) = 70e^{.04t} - (80 + 3t)$. We want to find out if $A(t)$ is ≥ 0 at some $t \in [0, 10]$. So let us find the absolute maximum of $A(t)$ on this interval. Let's find the

critical points:

$$\begin{aligned} A'(t) = 70(.04)e^{.04t} - 3 = 0 \\ \implies e^{.04t} &= \frac{3}{70(.04)} \\ \implies t &= \ln\left(\frac{3}{70(.04)}\right)\left(\frac{1}{.04}\right) \approx 1.725, \end{aligned}$$

which lies in $[0, 10]$. Next, plugging in the critical and end points gives $f(0) = -10$, $f(1.725) \approx -10.174$, and $f(10) = -5.572$. These are all negative! So he never has enough money to buy the car.

When $t = 10$ and he is 30 years old, he is closest to having enough. When $t \approx 1.725$ and he is around 21.725 years old, he is furthest from having enough.

11. (a) By the FTC, $\boxed{F'(x) = \cos(\ln x).}$

    (b) By the FTC and chain rule, $F'(x) = [e^{x^2}\sin(x^2)](x^2)' = \boxed{2xe^{x^2}\sin(x^2).}$

12. By the FTC, $F'(x) = \sin(x^2)$. Thus, $F'(\sqrt{\pi}) = \sin(\pi) = 0$. Further, at $\sqrt{\pi}$ the sign of $F'(x)$ changes from $+$ to $-$ (think of the sign change of $\sin x$ at $x = \pi$). Therefore, there is a local maximum at $x = \sqrt{\pi}$.