Solutions to Problems from Math 135 Recitation on 4/25/15

For the problems see
`http://math.rutgers.edu/~az202/teaching/`.

1. Substituting in $x = 0$ gives $0/1 = \boxed{0}$, and we're done, since it's not indeterminate.

2. This time substituting gives $0/0$, so we factor and simplify:

$$\lim_{x\to 2} \frac{x-2}{x^2+x-6} = \lim_{x\to 2} \frac{x-2}{(x+3)(x-2)} = \lim_{x\to 2} \frac{1}{(x+3)} = \boxed{\frac{1}{5}.}$$

3. We have a rational function as $x \to -\infty$, so we divide the top and bottom by the highest power of x, which is x^{10} in this case:

$$\lim_{x\to -\infty} \frac{5x^{10}+6x+1}{9x^{10}-42x} = \lim_{x\to -\infty} \frac{5+6x^{-9}+x^{-10}}{9-42x^{-9}} = \boxed{\frac{5}{9}.}$$

4. Substituting gives $0/0$. Since there's a square root, let's multiply the top and bottom by the conjugate:

$$\begin{aligned}\lim_{x\to 2^+} \frac{\sqrt{x-1}-1}{x^2-x-2} &= \lim_{x\to 2^+} \frac{(\sqrt{x-1}-1)(\sqrt{x-1}+1)}{(x^2-x-2)(\sqrt{x-1}+1)} \\ &= \lim_{x\to 2^+} \frac{(x-2)}{(x^2-x-2)(\sqrt{x-1}+1)}.\end{aligned}$$

Substituting *still* gives $0/0$, but we can factor x^2-x-2 and cancel:

$$\lim_{x\to 2^+} \frac{(x-2)}{(x-2)(x+1)(\sqrt{x-1}+1)} = \lim_{x\to 2^+} \frac{1}{(x+1)(\sqrt{x-1}+1)} = \boxed{\frac{1}{6}.}$$

5. As x gets large, e^{-x} approaches zero, and $\cos x$ oscillates between -1 and 1 (see Figure 1). So we expect the limit to be 0. To prove this, note that $e^{-x}\cos x$ is always between $-e^{-x}$ and e^{-x}, which both approach 0 as $x \to \infty$. So by the Squeeze Theorem, the original limit is also $\boxed{0}$.

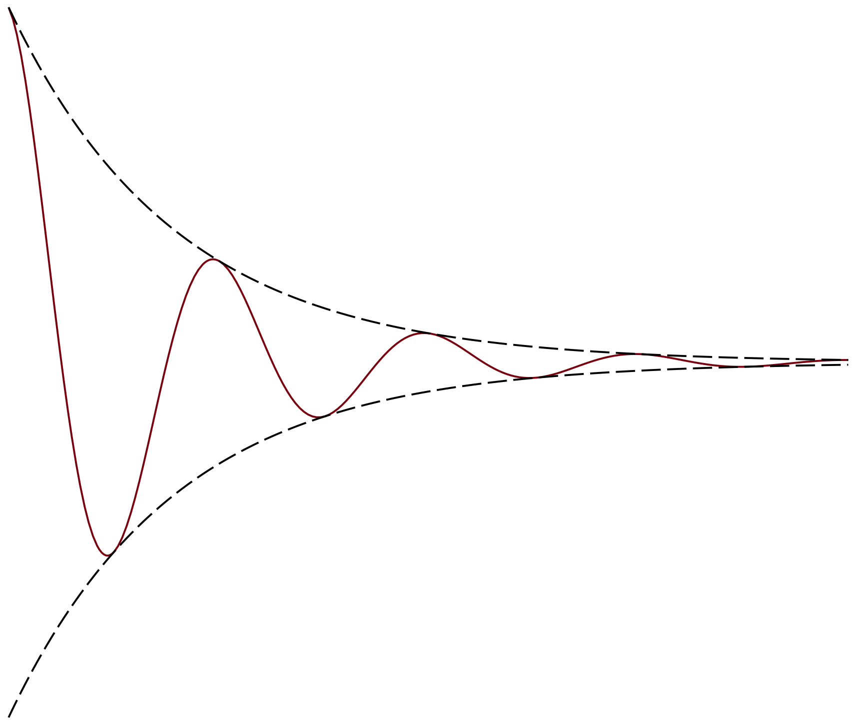

Figure 1: The function $y = e^{-x}\cos x$ (solid red) is squeezed between $-e^{-x}$ and e^{-x} (dashed black). Not to scale.

6. The only place where $f(x)$ might *not* be continuous is $x = 1$. To make f continuous at 1, we find the left/right limits and function value at 1:

$$\begin{aligned} \lim_{x\to 1^-} f(x) &= 1 + A \\ \lim_{x\to 1^+} f(x) &= e^B \\ f(1) &= 3. \end{aligned}$$

Next, we set all three equal to each other: $1 + A = e^B = 3$. Solving gives $\boxed{A = 2}$ and $\boxed{B = \ln 3}$.

7. Using the quotient rule,

$$\left(\frac{x+1}{\sin x}\right)' = \boxed{\frac{\sin x - (x+1)\cos x}{\sin^2 x}.}$$

8. By the chain rule and power rule,

$$\left((x^3+x)^{10}\right)' = \boxed{10(x^3+x)^9(x^3+x)' = 10(x^3+x)^9(3x^2+1).}$$

9. We rewrite y and use the power rule:

$$\begin{aligned} y &= x^{-1} + x^{-1/2} \\ \implies y' &= \boxed{-x^{-2} - \frac{1}{2}x^{-3/2}.} \end{aligned}$$

10. We can either simplify y first, or use the chain rule:

$$y = 8\ln x \implies y' = \boxed{\frac{8}{x}}$$

or

$$(\ln x^8)' = \frac{8x^7}{x^8} = \boxed{\frac{8}{x}.}$$

11.

$$\begin{aligned} (\cos^3 x \sin(x^5))' &= \cos^3 x(\sin(x^5))' + \sin(x^5)(\cos^3 x)' \\ &= \cos^3 x\cos(x^5)(x^5)' + \sin(x^5)\cdot 3\cos^2 x(\cos x)' \\ &= \boxed{5x^4\cos^3 x\cos(x^5) - \sin(x^5)\cdot 3\cos^2 x\sin x.} \end{aligned}$$

12.

$$\begin{aligned} (e^{x^2}\sqrt{1+x^2})' &= e^{x^2}(\sqrt{1+x^2})' + \sqrt{1+x^2}\left(e^{x^2}\right)' \\ &= e^{x^2}\frac{(1+x^2)'}{2\sqrt{1+x^2}} + \sqrt{1+x^2}e^{x^2}(x^2)' \\ &= \boxed{e^{x^2}\frac{x}{\sqrt{1+x^2}} + 2xe^{x^2}\sqrt{1+x^2}.} \end{aligned}$$

13. Since both the base and exponent depend on x, this is a job for logarithmic differentiation. We take the ln of BOTH sides and differentiate implicitly:

$$\begin{aligned} \ln y &= \tan x\ln(\sin x) \\ \implies \frac{y'}{y} &= \tan x\frac{\cos x}{\sin x} + \ln(\sin x)\sec^2 x = 1 + \ln(\sin x)\sec^2 x \\ \implies y' &= y(1+\ln(\sin x)\sec^2 x) = \boxed{(\sin x)^{\tan x}(1+\ln(\sin x)\sec^2 x).} \end{aligned}$$

14.

Differentiate implicitly:

$$x^3 + xy + y^2 = 7 \implies 3x^2 + xy' + y + 2yy' = 0$$
$$\text{Plug in } (1,2) \text{ and solve for } y' :$$
$$3 + y' + 2 + 4y' = 0 \implies 5y' = -5 \implies y' = -1.$$

The tangent line has slope -1 and passes through $(1, 2)$, so its equation is $y - 2 = -(x - 1)$, or $\boxed{y = -x + 3.}$

15. This time we just differentiate and solve for y':

$$x^3 + 3x^2y - 4y^2 = 16 \implies 3x^2 + 3x^2y' + 6xy - 8yy' = 0$$
$$\implies y'(3x^2 - 8y) = -3x^2 - 6xy \implies y' = \boxed{\frac{-3x^2 - 6xy}{3x^2 - 8y}.}$$

16. The trick here is to simplify the left-hand side: $e^{\sin^2 y}(1 + e^{\cos^2 y}) = e^{\sin^2 y} + e^{\sin^2 y + \cos^2 y} = e^{\sin^2 y} + e$. So we can write the original equation as

$$e^{\sin^2 y} + e = xy \implies 2 \sin y \cos y \, e^{\sin^2 y} y' = xy' + y$$
$$\implies y' = \boxed{\frac{y}{2 \sin y \cos y \, e^{\sin^2 y} - x}.}$$

17. (a) We differentiate the formula for A with respect to t and plug in:

$$A = xy \implies \frac{dA}{dt} = \frac{dx}{dt} y + x \frac{dy}{dt} = 2 \cdot 3 + 4 \cdot 3 = \boxed{18.}$$

(b) By the Pythagorean Theorem,

$$z^2 = x^2 + y^2. \qquad (1)$$

So when $x = 4$ and $y = 3$, we have $z = \sqrt{16 + 9} = 5$. Differentiating (1) gives

$$2z \frac{dz}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$
$$\implies 2 \cdot 5 \cdot \frac{dz}{dt} = 2 \cdot 4 \cdot 2 + 2 \cdot 3 \cdot 3$$
$$\implies \frac{dz}{dt} = \frac{34}{10} = \boxed{\frac{17}{5}.}$$

18. Let r be the radius, V be the volume, and A be the surface area. We know that $A = 4\pi r^2$. Differentiating with respect to t gives

$$\frac{dA}{dt} = 8\pi r \frac{dr}{dt} = 8\pi \frac{dr}{dt}. \tag{2}$$

Now all we need is dr/dt. If we differentiate the formula $V = (4/3)\pi r^3$, we get

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \implies \frac{dr}{dt} = \frac{dV/dt}{4\pi r^2} = \frac{4\pi}{4\pi} = 1.$$

Substituting $dr/dt = 1$ back into (2) gives $dA/dt = \boxed{8\pi\ in^2/sec.}$

19. The formula for the linearization of $f(x)$ about $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In our case, $f(x) = e^x$ and $a = 0$, so $L(x) = e^0 + e^0(x - 0) = \boxed{1 + x}$. For the second part, we plug in the specific value $x = .1$ to get $e^{.1} \approx \boxed{1.1.}$ (The exact value is $1.1051\ldots$.)

20. Doing the same thing here gives $\boxed{L(x) = 2 + \frac{1}{4}(x - 4)}$ and $\sqrt{4.2} \approx L(4.2) = 2 + .2/4 = \boxed{2.05.}$ (The exact value is $2.0493\ldots$.)

21. First, we can simplify:

$$(\ln 5 + \ln 11 - \ln 50)\sin(1.1\pi) = \ln(5 \cdot 11/50)\sin(1.1\pi) = \ln(1.1)\sin(1.1\pi).$$

So to rephrase the problem, we must approximate $f(1.1)$, where $f(x) = \ln x \sin(\pi x)$. Let's linearize about the point $a = 1$ since that's close to 1.1. First, we have $f'(x) = \sin(\pi x)/x + \pi\cos(\pi x)\ln x$. Thus, $f(a) = f(1) = 0$ and $f'(a) = f'(1) = 0$, since $\sin\pi$ and $\ln 1$ are both zero! So $L(x) = f(a) + f'(a)(x - a) = 0 + 0(x - 1) = 0$. Finally, the number we started with is $f(1.1) \approx L(1.1) = \boxed{0.}$ A lot of work for a very simple answer! (The exact value is $-.0294\ldots$.)