Oral Qual Transcript

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My examiners arrived in the order **K**→**S**→**R**→**Z**, with **Z** (to my surprise) arriving last but still on time. The exam commenced at 10AM.

Disclaimer: These are, for the most part, not the examiners' exact words.

S : So, is there anything you'd like to start with?

ME: Not particularly. There are some topics I'd rather *not* be asked about, but I probably shouldn't mention those.[*pause*] Ok, how about experimental math?

Z : Some ODEs, for example $y' = y$, are easy to solve exactly. But in many cases this is not possible. What are some ways to solve an ODE numerically?

ME: Well, there's Euler and Runge-Kutta...

Z : Good! Implement Euler's method in Maple code. [*I write the first few lines.*] Ok, good enough!

K : So can you draw a picture of what's going on? You know, there is a nicer way to write Euler's method. Why don't you integrate the ODE to represent $y(x+h)$?

ME: $y(x+h) = y_0 + \int_0^{x+h} f(t, y(t))\, dt$.

K : So now, what does Euler's method do? How could we get R-K from this integral equation?

ME: It approximates $\int_x^{x+h} f(t,y)\, dt$ by $f(x, y(x))h$. Would R-K come from using Simpson's rule on the integral? [*trails off...*]

Z : So why is R-K usually better than Euler?

ME: Euler has local truncation error $O(h^2)$ and global truncation error $O(h)$, while R-K has l.t.e. $O(h^5)$ and g.t.e. $O(h^4)$.

Z : Write down the R-K method.

ME: The standard one? Ok. [*starts to write*]

Z : Good enough! Now, how could you derive this from scratch experimentally? [*I start to explain the Butcher tableau for a general R-K method, symbolically solving in h and getting coefficients up to some order, etc...*] Ok good enough!

R : So suppose y is a vector and f is a matrix. Can you generalize these methods to solve $y' = f$? [*I mumble something about Euler being straightforward to generalize and being unsure about R-K.*] Ok, I was just curious!

Z : Now, what is a stock option? [*I define it.*] How is the fair price of an option defined? [*I talk about no arbitrage and a one-period binary model.*] How could we derive the Black-Scholes formula from this? [*I talk about looking at an n-stage binary model and letting $n \to \infty$.*] Ok, I'm finished!

R : [*mumbles something about Bernie Sanders and stock markets*]

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S : I see Markov chains are on your syllabus. Define a Markov chain.

ME: [*taken off guard because I'd expected **S** to ask about the combinatorics portion*] Well... it's a stochastic process. Discrete in time. And space. It has a transition matrix...

S : Ok. What can you say about the entries of this matrix? What is a steady state π? When might $\lim_n A^n p$ not approach π? [*This is the question that gave me the most trouble and embarrassment—not because it was hard, but because I hadn't prepared for this. After lots of awkward silences and ample hints from **S** and **K** , I arrived at a somewhat acceptable answer. After this, **S** asked a question regarding the digraph corresponding to the transition matrix of such a Markov chain, but I forget the details. I was relieved when **S** said*] Ok, enough of that. Let's move on to combinatorics.

Suppose I have a k-uniform hypergraph with $\leq m$ edges. Derive a condition on k and m for there to exist a 2-coloring on the vertices with no edge monochromatic.

ME: Consider a random coloring of the vertices... [*the rest goes smoothly*]

S : Ok good. Now suppose we replace the restriction on the size of the hypergraph with the condition that each vertex belongs to $\leq r$ edges. Derive an analogous condition.

ME: Again, we choose a random coloring. This time we use the Lovász local lemma. . . [*again, no problems here*]

S : Good. I have one more. Suppose $P_1 = \{A_1, \dots A_m\}$ and $P_2 = \{B_1, \cdots B_m\}$ are k-uniform partitions of $[n]$. Is there a permutation σ of $[n]$ s.t. $A_i \cap B_{\sigma(i)} \neq \emptyset$ $\forall i \in [n]$?

ME: Let's define a bipartite graph from $P_1 \to P_2$ with edges joining intersecting sets. We want a perfect matching. By König's theorem, if we can show $\tau = m$, we're good. Assume $\tau < m$ . . . [*With some help, I eventually get* $> m$ *disjoint* k*-sets in* $[mk]$*, a contradiction.*]

S : Ok, I'm done. It looks like the time is almost up.

Z : I have one more question! Suppose I am a writer of stories. I have ten stories, and I want to publish them in as many collections as possible. However, I don't want any collection to be contained in any other one. What is the most collections I can publish? [*This was probably the second most embarrassing part of the exam. I spent a long time muttering, even mentioning inclusion-exclusion.*] Ok maybe it's my fault. This uses a famous theorem that I see on your syllabus. But this guy has many theorems; maybe this is a different one.

S : No, no! **Z** is right. This is a good problem. Why don't you suppose there are four stories? Write the subsets in lexicographical order.

ME: [*embarrassed*] Oh! Sperner's theorem! The answer is $\binom{10}{5}$.

Z : Very good!

S : And can you prove Sperner's theorem?

ME: Can I assume the LYM inequality?

S : Ok, how would you prove it from that? [*I show it in one line.*] All right, now see if you can prove LYM.

ME: [*rushing because time is nearly up*] Consider a random maximal antichain. . . [*standard proof*]

S : Ok good enough! And you did it in three minutes. [*to the others*] Anything else?

K : One quick question. Going back to Markov chains—can you show that every Markov matrix has 1 as an eigenvalue?

ME: ...

K : You can solve this algebraically, but I think **Z** would have a different approach...

Z : Can you find one vector such that $Ax = x$?

ME: $(1, \dots, 1)^T$.

K : Good enough!

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Dr. Z. asked me to leave the room. After a minute or two, I was invited back in by Dr. Z., who told me I'd passed. I shook hands with my committee, and they signed the paperwork.

"Now you can relax," Dr. Saks said. I agreed!