HW #3  Math 316  Key

4 pts each, 12 pts total

5.3

1(b) Solve the system

\[ x \equiv 1 \pmod{3} \]
\[ x \equiv 3 \pmod{5} \]
\[ x \equiv 5 \pmod{7} \]

\[ M = 3 \cdot 5 \cdot 7 = 105 \]

\[ n_1 = \frac{105}{3} = 35 \equiv 2 \pmod{5} \Rightarrow n_1^{-1} = 2 \]
\[ n_2 = \frac{105}{5} = 21 \equiv 1 \pmod{5} \Rightarrow n_2^{-1} = 1 \]
\[ n_3 = \frac{105}{7} = 15 \equiv 1 \pmod{7} \Rightarrow n_3^{-1} = 1 \]

\[ x_0 = (1)(35)(2) + (2)(21)(1) + (5)(15)(1) \]

\[ = 70 + 42 + 75 \]
\[ = 208 \]
\[ \equiv 103 \pmod{105} \]

General soln is any number of the form 105k + 103, k \in \mathbb{Z}.

6-1

4. Prove that \( \phi(m) \) is even if \( m > 2 \).

Case 1: \( m = 2^n \) for \( n > 1 \). Then \( \phi(m) = \phi(2^n) = 2^{n-1} \cdot 2 = 2^{n-1} \), which is clearly even for \( n > 1 \).

Case 2: \( m = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k} \) where at least one of the \( p_i \neq 2 \), and \( p_i \neq p_j \) if \( i \neq j \).

We may assume \( p_i \neq 2 \).

\[ \phi(m) = \phi(p_1^{a_1}) \cdot \phi\left( \frac{m}{p_1^{a_1}} \right) = \phi(p_1^{a_1}) \phi\left( \frac{m}{p_1^{a_1}} \right) = \left( p_1^{a_1} - p_1^{a_1-1} \right) \phi\left( \frac{m}{p_1^{a_1}} \right) \]

\[ = p_1^{a_1-1}(p-1) \phi\left( \frac{m}{p_1^{a_1}} \right) \]. Since \( p \) is odd, \( p-1 \) is even.

\[ \therefore \phi(m) \text{ is even} \]
Find all integers such that \( \phi(n) = 12 \)

\[
\begin{align*}
\phi(13) &= 12 \\
\phi(26) &= \phi(2) \phi(13) = 1 \cdot 12 = 12 \\
\phi(31) &= \phi(3) \phi(7) = 2 \cdot 6 = 12 \\
\phi(42) &= \phi(2) \phi(3) \phi(7) = 1 \cdot 2 \cdot 6 = 12 \\
\phi(58) &= \phi(4) \phi(7) = 2 \cdot 6 = 12 \\
\phi(36) &= \phi(4) \phi(9) = 2 \cdot 6 = 12
\end{align*}
\]
Let the prime factorization of $n$ be given by $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$.

Then $d \mid n \Rightarrow d = p_1^{\beta_1} p_2^{\beta_2} \ldots p_k^{\beta_k}$ where $0 \leq \beta_i \leq \alpha_i$ for each $i = 1, \ldots, k$.

Thus, by the general combinatorial principle,

$$\prod_{i=1}^k d = \prod_{i=1}^k p_i^{(\alpha_i + 1)(\alpha_i + 2) \ldots (\alpha_i + \beta_i)} = \prod_{i=1}^k p_i^{\frac{\alpha_i(\alpha_i + 1)}{2}} = (\prod_{i=1}^k p_i^{\alpha_i})^{\frac{\alpha_i}{2}} = n^{\frac{\alpha}{2}}$$

(See end for an alternate proof)

6.2 #10

$\sigma(210) = \sigma(2) \sigma(3) \sigma(5) \sigma(7) = 3 \times 4 \times 6 \times 8 = 576$

$n \sigma(100) = \sigma(2^2) \sigma(5^2) = (1+2+4)(1+5+25) = 7 \times 31 = 217$

$\sigma(999) = \sigma(3^3) \sigma(37) = (1+3+9+27)(1+37) = 40 \times 38 = 1520$

6.2 #11

$d(47) = 2$ (since 47 is prime)

$d(65) = d(7) d(9) = 2 \times 3 = 6$

$d(150) = d(2) d(3) d(5^2) = 2 \times 2 \times 3 = 12$

6.3 #1

Suppose $f \cdot g$ multiplicative and $f(p^r) = g(p^r)$ for all primes $p$ and any $r \in \mathbb{Z}_+$.

Prove $f(n) = g(n)$ for all $n \in \mathbb{Z}_+$

Let $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ with $p_i$'s all distinct. $f(n) = f(p_1^{\alpha_1} \ldots p_k^{\alpha_k})$

$= f(p_1^{\alpha_1}) \ldots f(p_k^{\alpha_k}) = g(p_1^{\alpha_1}) \ldots g(p_k^{\alpha_k}) = g(p_1^{\alpha_1} \ldots p_k^{\alpha_k}) = g(n)$.
Prove that if \( f(n) = \prod_{d \mid n} f(d)^{\mu(n/d)} \) then \( g(n) = \prod_{d \mid n} f(d)^{\mu(n/d)} \).

**Proof:** Suppose \( f(n) = \prod_{d \mid n} g(d) \). Thus \( \log f(n) = \log \prod_{d \mid n} g(d) \)

\[
= \sum_{d \mid n} \log g(d) \text{ (by law of logs)}
\]

Thus \( \log g(n) = \sum_{d \mid n} \mu(d) \log f(n/d) \) (by Thm 6-6).

\[
= \sum_{d \mid n} \log f(n/d)^{\mu(d)} \text{ (by law of logs)}
\]

\[
= \log \prod_{d \mid n} f(n/d)^{\mu(n/d)} \text{ (by law of logs)}
\]

\[
= \log \left( \frac{n}{\prod_{d \mid n} d} \right)^{\mu(n/d)} \text{ since } \{d : d \mid n \} = \{\frac{n}{d} : \frac{n}{d} \mid n \}
\]

\[
= \log \prod_{d \mid n} f(d)^{\mu(n/d)}
\]

Thus \( g(n) = \prod_{d \mid n} f(d)^{\mu(n/d)} \).

**6-2 #2 alternate proof of** \( \prod_{d \mid n} d = n^{d(n)/2} \)

**Case 1:** Suppose \( d(n) \) is even, so \( d(n) = 2k \) for some \( k \).

Let \( d_1, d_2, \ldots, d_k \) be the \( k \) smallest positive divisors of \( n \).

Then the set of all positive divisors is \( \{d_1, d_2, \ldots, d_k, n/d_1, \ldots, n/d_k\} \) and their product is \( n^k = \prod_{d \mid n} d \).

**Case 2:** Suppose \( d(n) \) is odd, so \( d(n) = 2k+1 \). By 6-2 #1, \( n \) is a perfect square.

Let \( d_1, d_2, \ldots, d_k \) be the \( k \) smallest positive divisors. The set of all positive divisors is \( \{d_1, d_2, \ldots, d_k\} \cup \{\sqrt{n}\} \cup \{n/d_1, n/d_2, \ldots, n/d_k\} \).

So \( \prod_{d \mid n} d = n^k \sqrt{n} = n^{d(n)/2} \sqrt{n} = n^{d(n)/2} \).

\[\square\]