MATH 356 Homework Assignments

Fall 2006

• HW 1, Due Thursday, Sept. 21.
  – 1-1: 5
  – 1-2: 6
  – 2-1: 7
  – 3-1: 8

• HW 2, Due Thursday, Sept. 28.
  – 3-4: 2
  – 4-1: 6
  – The Pell sequence \( \{P_n\}_{n=0}^{\infty} \) is given by \( P_0 = 0, P_1 = 1 \), and
    \[
    P_{n+1} = 2P_n + P_{n-1},
    \]
    for \( n \geq 1 \).
    1. Express the generating function of \( \{P_n\}_{n=0}^{\infty} \) as a rational function.
    2. Prove that
    \[
    P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}}.
    \]

• HW 3, Due Thursday, Oct. 19.
  – 5-3: 1(b)
  – 6-1: 4, 6.
• HW 4, Due Thursday, Oct. 26.
  – 6-2: 2, 10, 11
  – 6-3: 1
  – 6-4: 11

• HW 5, Due Thursday, Nov. 2.
1. For each of the following partitions, draw the Ferrers graph and find the conjugate partition:
   (a) 5 + 3 + 2 + 1
   (b) 6 + 3 + 1
   (c) 7 + 6 + 4 + 3
2. Show that for all positive integers $n$, the number of partitions of $n$ into $m$ distinct parts equals the number of partitions of $n$ wherein $1, 2, 3, \ldots, m$ all appear at least once as a part, and no part is greater than $m$. **Hint: consider the Ferrers graph.**
3. Consider the following claim: the number of partitions of $n$ into nonmultiples of three equals the number of partitions of $n$ where no part may appear more than twice. Prove the claim
   (a) bijectively, and
   (b) using generating functions.
4. Prove that the number of partitions of $n$ into distinct parts congruent to 0, 2, or 3 modulo 4 equals the number of partitions of $n$ into parts congruent to 2, 3, or 7 modulo 8. **Hint: use generating functions.**

• HW 6, due Thursday, Nov. 30.
1. (a) Prove that the generating function for partitions with exactly $j$ parts is
   \[ q^j \over (1-q)(1-q^2) \cdots (1-q^j). \]
   (b) Give a combinatorial proof of the following series-product identity of Euler:
   \[ \sum_{j=0}^{\infty} {q^j \over (1-q)(1-q^2) \cdots (1-q^j)} = \prod_{k=1}^{\infty} {1 \over 1-q^k}. \]
2. The first Rogers-Ramanujan identity is given by

\[ \sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2) \cdots (1-q^j)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}. \quad (1) \]

Show that (1) is equivalent to the following partition theorem:

Let \( R(n) \) denote the number of partitions of \( n \) into parts which are distinct, nonconsecutive integers. Let \( S(n) \) denote the number of partitions of \( n \) into parts congruent to 1 or 4 modulo 5. Then \( R(n) = S(n) \) for all integers \( n \).

Suggested way to proceed:

(a) Show that

\[ \sum_{n=0}^{\infty} S(n)q^n = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}. \]

(b) Show that

\[ \frac{q^{j^2}}{(1-q)(1-q^2) \cdots (1-q^j)} \]

is the generating function for partitions of the type counted by \( R(n) \) which have exactly \( j \) parts.

(c) Use part (b) to show that

\[ \sum_{n=0}^{\infty} R(n)q^n = \sum_{j=0}^{\infty} \frac{q^{j^2}}{(1-q)(1-q^2) \cdots (1-q^j)}. \]

(d) Equate the generating functions and conclude that \( R(n) = S(n) \)

Note: You are not being asked to prove (1).