THE SATURATION CONJECTURE (AFTER A. KNUTSON AND T. TAO) WITH AN APPENDIX BY WILLIAM FULTON

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The purpose of this exposition[1] is to give a simple treatment of Knutson and Tao's recent proof of the saturation conjecture [10].

A finite dimensional irreducible polynomial representation of $\mathrm{GL}_n(\mathbb{C})$ is determined by its highest weight, which is a weakly decreasing sequence of n non-negative integers, also called a partition [5, §8]. The irreducible representation with highest weight λ is denoted V_λ. The Littlewood-Richardson coefficient $c^\nu_{\lambda\mu}$ is defined to be the multiplicity of V_ν in the decomposition of $V_\lambda \otimes V_\mu$ into irreducibles. Define

$$T_n = \{(\lambda, \mu, \nu) \mid c^\nu_{\lambda\mu} \neq 0\}\,.$$

This set is important in numerous areas besides representation theory. In Schubert calculus it describes when an intersection of Schubert cells must be non-empty [5, §9.4]. In combinatorics, a triple is in T_n if and only if there exists a Littlewood-Richardson skew tableau with shape ν/λ and content μ [5, §5.2].

It is well known that $T_n \subset \mathbb{Z}^{3n}$ is a semi-group under addition, a fact which Zelevinsky attributes to Brion and Knop [12]. Klyachko has given [9] a nice description of the saturation

$$\bar{T}_n = \{(\lambda, \mu, \nu) \mid \exists N > 0 : (N\lambda, N\mu, N\nu) \in T_n\}\,.$$

A triple (λ, μ, ν) is in $\bar{T}_n$ if and only if the entries of λ, μ, and ν satisfy certain inequalities that come from Schubert calculus (see §5 and [6]). This made the following conjecture particularly important.

Saturation conjecture. *Let $(\lambda, \mu, \nu) \in \mathbb{Z}^{3n}$ and $N > 0$. Then $(\lambda, \mu, \nu) \in T_n$ if and only if $(N\lambda, N\mu, N\nu) \in T_n$.*

In other words T_n is saturated in $\mathbb{Z}^{3n}$. Note that the implication "only if" is a trivial consequence of the fact that T_n is a semi-group or of the original Littlewood-Richardson rule.

In July 1998, Knutson and Tao gave a proof of this conjecture, using two wonderful new constructions of polytopes, whose lattice points count Littlewood-Richardson coefficients. These constructions are called the *hive* and *honeycomb* models. Earlier Berenstein and Zelevinsky had defined equivalent polytopes, but with more complicated descriptions. In the first preprint of Knutson and Tao's paper, both hives and honeycombs were used. However, in their later version [10], hives were eliminated from the proof.

The goal of this exposition is to present a simple and complete proof using only the hive model. It is based on Knutson and Tao's first preprint, and most constructions used here come directly from this preprint. One innovation, in Section

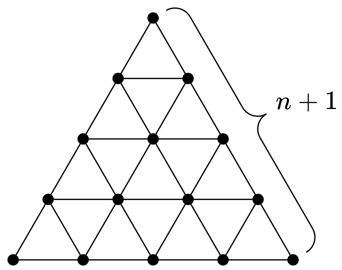
Date: October 18, 1999.

[1] given as a talk at UC Berkeley, September 1998

3, is the construction of a graph from a hive, which is used to simplify their argument. In an appendix of Fulton it is shown that the hive model is equivalent to the original Littlewood-Richardson rule. We thank W. Fulton, S. Hosten, F. Sottile, and B. Sturmfels for useful discussions, and Knutson and Tao for keeping us informed about their progress. We are also grateful to the referee for many useful suggestions.

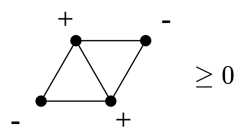
1. The hive model

We start with a triangular array of *hive vertices*, $n+1$ on each side.



This array is called the (big) *hive triangle*. When lines are drawn through the hive vertices as shown, the hive triangle is split up into n^2 *small triangles*. By a *rhombus* we mean the union of two small triangles next to each other.

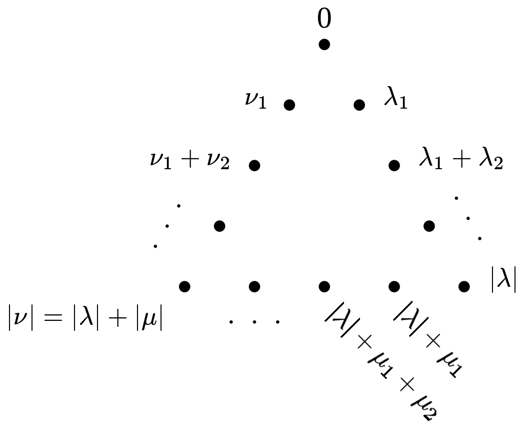
Let H be the set of hive vertices and $\mathbb{R}^H$ the labelings of these by real numbers. Each rhombus gives rise to an inequality on $\mathbb{R}^H$ saying that the sum of the labels at the obtuse vertices must be greater than or equal to the sum of the labels at the acute vertices:



A *hive* is a labeling that satisfies all rhombus inequalities. A hive is *integral* if all its labels are integers. We let $C \subset \mathbb{R}^H$ denote the convex polyhedral cone consisting of all hives.

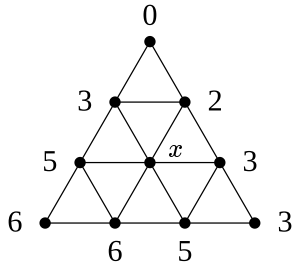
Denote by $|\lambda|$ the weight of the partition λ, which is the sum of its entries. The following theorem gives the relation between Littlewood-Richardson coefficients and hives.

Theorem 1. *Let λ, μ, and ν be partitions with $|\nu| = |\lambda| + |\mu|$. Then $c^{\nu}_{\lambda\mu}$ is the number of integral hives with border labels:*



Knutson and Tao prove this by translating hives with integer labels into tail-positive Berenstein-Zelevinsky patterns, which are known to count $c^{\nu}_{\lambda\mu}$ [1], [12]. An alternative direct proof of Fulton can be found in the appendix.

Example 1. To compute $c^{3\,2\,1}_{2\,1,\,2\,1}$ we can take $n = 3$ and border labels as in the picture.



Let x be the undetermined label of the middle hive vertex. Then the rhombus inequalities say that $4 \leq x \leq 5$. It follows that there are two integral hives with this border, so $c^{3\,2\,1}_{2\,1,\,2\,1} = 2$.

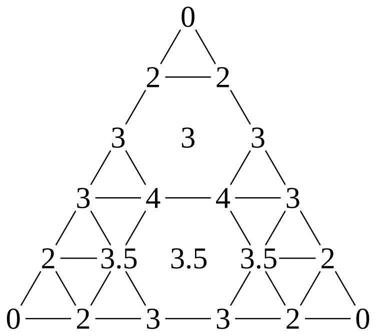
Let B be the set of border vertices, and $\rho : \mathbb{R}^H \to \mathbb{R}^B$ the restriction map. The restriction of a hive to the border vertices by ρ is called its *border*. For $b \in \mathbb{R}^B$, the fiber $\rho^{-1}(b) \cap C$ is easily seen to be a compact polytope, which we will call the *hive polytope* over b. If b comes from a triple of partitions as in Theorem 1, this is also called the hive polytope over the triple. We will call the vertices of a hive polytope its *corners*.

We can now describe the strategy of Knutson and Tao's proof. If $(N\lambda, N\mu, N\nu)$ is in T_n, then the hive polytope over this triple contains an integral hive. By scaling this polytope down by a factor N, it follows that the hive polytope over (λ, μ, ν) is not empty. Therefore it is enough to show that if $b \in \mathbb{Z}^B$ and $\rho^{-1}(b) \cap C \neq \emptyset$ then $\rho^{-1}(b) \cap C$ contains an integral hive.

Let ω be a functional on $\mathbb{R}^{H-B}$ which maps a hive to a linear combination of the labels at non-border vertices, with generic positive coefficients. Then for each

$b \in \rho(C)$, this ω takes its maximum at a unique hive in $\rho^{-1}(b) \cap C$. The strategy is to prove that this hive is integral if b is integral.

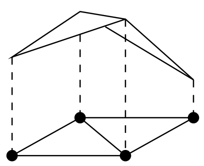
Example 2. Even though all rhombus inequalities are integrally defined, a hive polytope over an integral border can still have non-integral corners. The following hive is an example, and therefore it does not maximize any generic positive functional ω.



In the picture we have omitted the lines across rhombi where the rhombus inequality is satisfied with equality, which makes it easy to see that this hive is a corner of its hive polytope. In fact, it is not hard to show that for $n \leq 4$ and $b \in \mathbb{Z}^B$, all corners of $\rho^{-1}(b) \cap C$ are integral hives.

2. Flatspaces

We can consider a hive as a graph over the hive triangle. At each hive vertex we use the label as the height. We then extend these heights to a graph over the entire hive triangle by using linear interpolation over each small triangle. A rhombus inequality now says that the graph over the rhombus cannot bend up across the middle line.



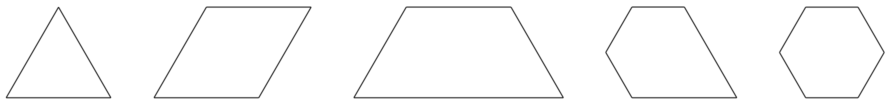
In this way the graph becomes the surface of a convex mountain. The graph is flat (but not necessarily horizontal) over a rhombus if and only if the rhombus inequality is satisfied with equality.

We define a *flatspace* to be a maximal connected union of small triangles such that any contained rhombus is satisfied with equality. The flatspaces split the hive triangle up in disjoint regions over which the mountain is flat. The flatspaces of the hive in Example 2 consist of two hexagons and 13 small triangles.

Flatspaces have a number of nice properties. We will list the ones we need below. Since all of these are straightforward to prove directly from the definitions, we will simply give intuitive reasons for them.

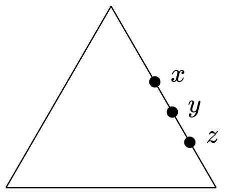
1. Flatspaces are convex. This is clear since they lie under intersections of a convex mountain with a (convex) plane.

2. All flatspaces have one of the following five shapes (up to rotations and different side lengths):



These are the only convex shapes that can be constructed from small triangles.
3. A side of a flatspace is either on the border of the big hive triangle, or it is also a side of a neighbor flatspace. In other words, a side of one flatspace can't be shared between several neighbor flatspaces. This again follows from the convexity of the mountain described above.

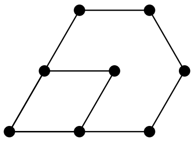
Given a labeling $b \in \mathbb{R}^B$, let x, y, z be labels of consecutive border vertices on the same side of the big hive triangle (in any direction).



If b is the border of a hive, then the rhombus inequalities imply that $y - x \geq z - y$, although more inequalities are needed to guarantee that $b \in \rho(C)$. We will say that b is *regular* if we always have $y - x > z - y$, when x, y, z are chosen in this way. When a border comes from a triple of partitions, it is regular exactly when each partition is strictly decreasing.
4. If the border of a hive is regular then no flatspace has a side of length ≥ 2 on the border of the big hive triangle. In fact, if the labels x, y, z above are on a flatspace side, then $y - x = z - y$.

Given a hive, a non-empty subset $S \subset H - B$ is called *increasable* if the same small positive amount can be added to the labels of all hive vertices in S, such that the labeling is still a hive.
5. The interior vertices of a hexagon-shaped flatspace form an increasable subset. Proving this is a matter of checking that each rhombus inequality still holds after adding a small enough amount to the labels of these vertices. Only rhombi that are already flat need to be considered, since for all others there is some "slack to cut".

Note that the corresponding statements for flatspaces of other shapes are false. The reason is that all other shapes have at least one sharp corner (with a 60° angle). Lifting the interior vertex closest to a sharp corner is prohibited by the inequality of the rhombus in that corner.



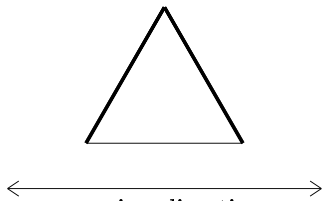
Note also that the sharp corners of a flatspace of any shape are endpoints of its longest sides.

Proposition 1. *If a hive with regular border has no increasable subsets, then its flatspaces consist of small triangles and small rhombi.*

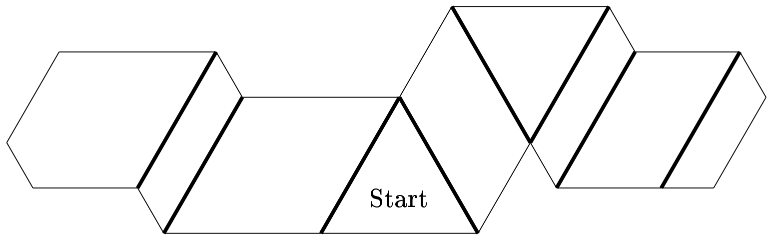
Proof. Otherwise some flatspace has a side of length ≥ 2. This follows because the only types of flatspaces that have all sides of length one are small triangles, rhombi, and small hexagons, and the later do not occur by property 5.

Let m be the maximal length among all sides of flatspaces. We will proceed by constructing a region consisting of flatspaces with a side of length m, such that the interior hive vertices of the region is an increasable subset. The crucial point is to avoid sharp corners pointing out from the region, since otherwise we would get the same problems as with the pentagon above. We need $m \geq 2$ to be sure that interior vertices exist.

Start by taking any flatspace having a side of length m, and mark this side. In the pictures this is shown by making the side thick. Then choose (and fix) a line crossing (the extension of) the marked side in an angle of 60° and call it the moving direction. If the flatspace is a triangle or a parallelogram, we furthermore mark an additional side. For a triangle, this is the other side not parallel to the moving direction, while for a parallelogram we mark the side opposite the one already marked.



We construct a region, starting with the chosen flatspace. This region will initially have one or two marked sides, depending on the shape of the chosen flatspace. As long as the region has a marked side on its outer border, the flatspace on the opposite side is added to the region. Note that this flatspace is well defined by property 3, since regularity prevents any marked edges from being on the border of the big hive triangle. If the new flatspace is a triangle, we mark its unmarked side which is not parallel to the moving direction. If the new flatspace is a parallelogram, we mark the side opposite the old marked side. If it is not a triangle or parallelogram, we don't mark any new sides.



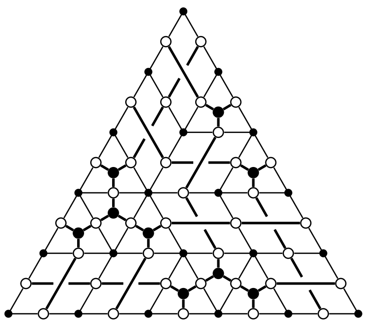
Since the region always grows along the moving direction, it will never go in loops. Now since no marked edges can ever reach the border of the big hive triangle, the described process will stop. Notice that by construction of the region, each included flatspace has enough of its longest sides marked, that all sharp corners are endpoints of marked sides. Since the final region has no marked sides on its outer border, this means that it can't have any sharp corners pointing out.

We claim that the inner vertices of the final region form an increasable subset. If not, some small rhombus satisfied with equality in the region has more obtuse than acute vertices on the region border. If any flat rhombus has both of its obtuse vertices on the border, then it follows, using convexity of the flatspace containing the rhombus, that one of the acute vertices is a sharp corner of the region. On the other hand, if a flat rhombus has one obtuse vertex and no acute vertices on the border, one can deduce, using property 3, that two marked sides must meet in a $120°$ angle at the obtuse vertex on the border. However, the construction never introduces marked sides that meet in this angle.

We have established that the presence of any flatspace which is not a small triangle or rhombus gives rise to an increasable subset. This finishes the proof. □

3. Small Flatspaces

Let h be a hive, all of whose flatspaces are small triangles or small rhombi. We construct a graph G from h as follows. G has one fat black vertex in the middle of each small triangle flatspace. In addition there is one circle vertex on every flatspace side. Each fat vertex is connected to the three vertices on the sides of its triangle, and the two circle vertices on opposite sides of a flat rhombus are connected. This graph is topologically equivalent to the reduced honeycomb tinkertoy of Knutson and Tao.

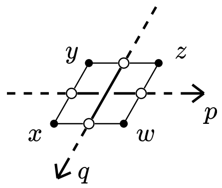


Lemma 1. *If h is a corner of its hive polytope $\rho^{-1}(\rho(h)) \cap C$, then G is acyclic.*

Proof. Suppose G has a non-trivial loop, and give this loop an orientation. Each hive vertex then has a well defined winding number, which is the number of times the loop goes around this vertex, counted positive in the counter clockwise direction. Note that the winding number is zero for each border vertex, and that some winding numbers are non-zero if the loop is not trivial.

For each $r \in \mathbb{R}$, let $h_r \in \mathbb{R}^H$ be the labeling which maps each hive vertex to the label of h at the vertex plus r times the winding number of the vertex. We will show that h_r is a hive for $r \in (-\epsilon, \epsilon)$, for a suitable $\epsilon > 0$. This implies that h is an interior point of a line segment contained in its hive polytope, which contradicts the assumption that h is a corner.

Choose any $\epsilon > 0$ such that each rhombus inequality that is strict for h is also satisfied for h_r when $|r| < \epsilon$. We claim that this ϵ will do. Consider any rhombus satisfied by h with equality:



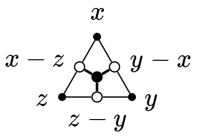
Suppose that the loop goes p times through horizontal edge in the indicated direction and q times down through the other edge. Let the vertex with label x have winding number t. Then going clockwise around the rhombus, the winding numbers of the three other vertices are $t+p$, $t+p+q$, and $t+q$. It follows that the labels of h_r are

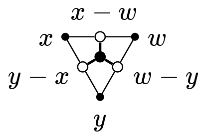
$$\begin{aligned} y' &= y + r(t+p) & \qquad z' &= z + r(t+p+q) \\ x' &= x + rt & \qquad w' &= w + r(t+q) \end{aligned}$$

Since the rhombus is flat for h, we have $x+z = y+w$. But this implies that $x'+z' = y'+w'$, and so the rhombus is also flat for h_r. $\square$

Proposition 2. *Let h be a hive which is a corner of its hive polytope $\rho^{-1}(\rho(h)) \cap C$. Suppose the flatspaces of h consist only of small triangles and small rhombi. Then the labels of h are integer linear combinations of the border labels.*

Proof. By Lemma 1, the graph G for h is acyclic. Label each circle vertex with the difference of the labels of the hive vertices on its side as shown below. A circle vertex on a horizontal side is always assigned the label of the left hive vertex minus the label of the right hive vertex on its side, etc.





By construction, the sum of the labels of three circle vertices surrounding any fat vertex is zero. Furthermore, if two circle vertices are connected by a single edge, then their labels are equal. This follows because the rhombus that separates them is satisfied with equality. We claim that all circle vertex labels are $\mathbb{Z}$-linear combinations of the border labels. Since this implies that also all labels of hive vertices are such linear combinations, this will finish the proof.

If the claim is false, let S be the non-empty set of circle vertices whose labels are not $\mathbb{Z}$-linear combinations of border labels, together with all fat vertices connected directly to one of these circle vertices. Since G is acyclic, some vertex $u \in S$ is connected to at most one other vertex in S.

Suppose u is a circle vertex. Then u can't be on the border of the big hive triangle, since its label would then be the difference of two border vertices, and so a $\mathbb{Z}$-linear combination of these. Therefore u is not an endpoint of G, so it is connected to a vertex v outside S. Since S contains all fat vertices connected to u by construction, v must be a circle vertex whose label is a $\mathbb{Z}$-linear combination of border labels. But u has the same label, a contradiction.

Therefore u must be a fat label, and exactly one of its three surrounding circle vertices is in S. This means that the labels of the other two circle neighbors are

$\mathbb{Z}$-linear combinations of border labels. But since the sum of the labels of all three circle vertices surrounding u is zero, all three labels must be $\mathbb{Z}$-linear combinations of the border labels. This contradiction shows that S is empty, which concludes the proof. □

4. Proof of the saturation conjecture

We will call a functional on $\mathbb{R}^{H-B}$ *generic* if it takes its maximum at a unique point in $\rho^{-1}(b) \cap C$ for each $b \in \rho(C)$. It follows from the existence of secondary fans in linear programming [11, §1] that the generic functionals form a dense open subset of $(\mathbb{R}^{H-B})^*$. We can now finish the proof of the saturation conjecture.

Theorem 2. *Let $(\lambda, \mu, \nu) \in \mathbb{Z}^{3n}$ and $N > 0$. Then $(\lambda, \mu, \nu) \in T_n$ if and only if $(N\lambda, N\mu, N\nu) \in T_n$.*

Proof. As already noted, it is enough to show that if $b \in \rho(C) \cap \mathbb{Z}^B$ then the fiber $\rho^{-1}(b) \cap C$ contains an integral hive.

Fix a generic functional ω on $\mathbb{R}^{H-B}$ which maps a hive to a linear combination with positive coefficients of the labels at non-border hive vertices. For each $b \in \rho(C)$, let $\ell(b)$ be the unique hive in $\rho^{-1}(b) \cap C$ where ω is maximal. Then $\ell : \rho(C) \to C$ is a continuous piece-wise linear map [11, §1]. Notice that since w has positive coefficients, $\ell(b)$ has no increasable subsets.

We want to prove that the labels of $\ell(b)$ are $\mathbb{Z}$-linear combinations of the labels of b. In particular $\ell(b)$ is an integral hive if b is integral. For a regular border $b \in \rho(C)$, Proposition 1 implies that the flatspaces of $\ell(b)$ consist of small triangles and rhombi; by Proposition 2 this implies that all labels of $\ell(b)$ are $\mathbb{Z}$-linear combinations of the labels of b. Finally, since the regular borders are dense in each maximal subcone of $\rho(C)$ where ℓ is linear, ℓ must be integrally defined everywhere. □

5. Remarks and questions

Knutson and Tao's proof of the saturation conjecture implies that Klyachko's inequalities for T_n can be produced by a simple recursive algorithm, which uses the inequalities for T_k, $1 \le k \le n-1$ ([9], [10], [12], [6]). A triple of partitions (λ, μ, ν) with $|\nu| = |\lambda| + |\mu|$ is in T_n if and only if

$$\sum_{i=1}^{k} \nu_{\gamma_i+k+1-i} \le \sum_{i=1}^{k} \lambda_{\alpha_i+k+1-i} + \sum_{i=1}^{k} \mu_{\beta_i+k+1-i}$$

for all triples $(\alpha, \beta, \gamma) \in T_k$ with $\gamma_1 \le n-k$. Another important consequence is Horn's conjecture, which says that the same inequalities describe which sets of eigenvalues can arise from two Hermitian matrices and their sum [8].

P. Belkale has shown that the inequality produced by a triple (α, β, γ) with Littlewood-Richardson coefficient $c^{\gamma}_{\alpha\beta} \ge 2$ follows from the other inequalities. Knutson, Tao, and Woodward have announced a proof that the remaining inequalities are independent, i.e. they describe the facets of the cone $\rho(C)$. Their proof uses an interesting operation of overlaying two hives, which is defined in terms of Knutson and Tao's honeycomb model [10].

These results have made it very interesting to determine which triples (λ, μ, ν) have coefficient $c^{\nu}_{\lambda\mu}$ equal to one. Fulton has conjectured that this is equivalent to $c^{N\nu}_{N\lambda, N\mu}$ being one for any $N \in \mathbb{N}$. This has been verified in all cases with $N|\nu| \le 68$. (Recently Knutson and Tao have reported that they can prove this as well.)

For $n = 3$ it is easy to show that a triple of partitions has Littlewood-Richardson coefficient one if and only if it corresponds to a point on the boundary of the cone $\rho(C)$. In general, Fulton's conjecture implies that the triples with coefficient one are exactly those corresponding to points in a collection of faces of $\rho(C)$. For $n \geq 3$ this means that all triples corresponding to interior points in $\rho(C)$ have coefficient at least two.

One approach for proving Fulton's conjecture is to show that if $b \in \rho(C) \cap \mathbb{Z}^B$, then any generic positive functional ω on $\mathbb{R}^{H-B}$ must be minimized (as well as maximized) at an integral hive in $\rho^{-1}(b) \cap C$. In fact, by Proposition 2 it is enough to prove:

If $b \in \rho(C)$ is a generic border and if a generic positive functional ω is minimized at $h \in \rho^{-1}(b) \cap C$, then the flatspaces of h consist of small triangles and rhombi.

Part of proving this is to specify when a border b is generic. We believe the statement is true if b avoids finitely many hyperplanes in $\mathbb{R}^B$.

The Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$ have the following natural generalization. Given decreasing sequences of integers ν, and $\lambda(1), \dots, \lambda(r)$, let $c^{\nu}_{\lambda(1),\dots,\lambda(r)}$ denote the multiplicity of V_ν in the holomorphic representation $V_{\lambda(1)} \otimes \cdots \otimes V_{\lambda(r)}$. When $\nu = (0, \dots, 0)$, this specializes to the symmetric Littlewood-Richardson coefficient $c_{\lambda(1),\dots,\lambda(r)}$ which is the dimension of the $\mathrm{GL}_n(\mathbb{C})$-invariant subspace of $V_{\lambda(1)} \otimes \cdots \otimes V_{\lambda(r)}$. Postnikov and Zelevinsky have pointed out that the saturation conjecture as stated in the introduction implies a similar result for these generalized coefficients, i.e.

$$c^{\nu}_{\lambda(1),\dots,\lambda(r)} \neq 0 \iff c^{N\nu}_{N\lambda(1),\dots,N\lambda(r)} \neq 0\,. \tag{5.1}$$

Knutson has shown us that by combining several hive triangles, one obtains a polytope whose integral points count these more general coefficients. This gives rise to another proof of (5.1).

In [3] other generalized Littlewood-Richardson coefficients related to quiver varieties are described. A different generalization related to Hecke algebras is defined in [7], and quantum Littlewood-Richardson coefficients are studied in [2]. It would be very interesting if these coefficients can be realized as the number of integral points in some polytopes.

Appendix A. A bijection between hives and Littlewood-Richardson skew tableaux (by William Fulton)

The aim of this appendix is to give a simple and direct bijection between the hives with given boundary (given by a triple of partitions), and the set of Littlewood-Richardson skew tableaux for the given triple. In principle one could construct such a mapping from [4], but it is simpler to do it directly from hives; in the description we give here, it is easy to see that the map is a bijection, without knowing that the two sets have the same cardinality. As in [4], we produce contratableaux, but there is a standard bijection between these and the original Littlewood-Richardson skew tableaux.

Consider an integral hive, with sides having $n + 1$ entries, corresponding to partitions λ, μ, and ν, with $|\nu| = |\lambda| + |\mu|$. The differences down the northwest to southeast border give the partition λ, the differences across the bottom border from right to left give μ, and the differences down the northeast to southwest border give

ν (see Theorem 1). The main idea for constructing a skew tableau with a reverse-lattice word is to use the other northwest to southeast rows of entries to construct a chain of subpartitions of λ.

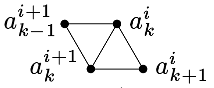
The entries of the hive will be denoted a^i_k, with $1 \leq i \leq n{+}1$ and $0 \leq k \leq n{+}1{-}i$. Here the superscript denotes the northwest to southeast row of the entry, with the first row being the long row on the boundary, and the others in order below that; the subscripts number the entries along the rows, from northwest to southeast.

$$\begin{array}{ccccccc} & & & a^1_0 & & & \\ & & a^2_0 & & a^1_1 & & \\ & a^3_0 & & a^2_1 & & a^1_2 & \\ a^4_0 & & a^3_1 & & a^2_2 & & a^1_3 \end{array}$$

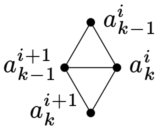
Note that $a^1_0 = 0$, and that $\lambda_k = a^1_k - a^1_{k-1}$ for $1 \leq k \leq n$.

For $1 \leq i \leq n$ define a sequence $\lambda^{(i)} = (\lambda^{(i)}_1, \ldots, \lambda^{(i)}_{n+1-i})$ by setting $\lambda^{(i)}_k = a^i_k - a^i_{k-1}$. Note that $\lambda^{(1)} = \lambda$.

There are three types of rhombus inequalities, depending on the orientation of the rhombus. We first consider two of them:

(1) 

This says that $\lambda^{(i+1)}_k \geq \lambda^{(i)}_{k+1}$.

(2) 

This says that $\lambda^{(i)}_k \geq \lambda^{(i+1)}_k$.

Together, (1) and (2) say that $\lambda^{(i)}_k \geq \lambda^{(i+1)}_k \geq \lambda^{(i)}_{k+1}$. In particular, each sequence $\lambda^{(i)}$ is weakly decreasing, and we have a nested sequence of partitions: $\lambda^{(1)} \supset \lambda^{(2)} \supset \cdots \supset \lambda^{(n)} \supset \lambda^{(n+1)} = \emptyset$.

For example, the hive

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 10 & & 6 & & & \\ & & 17 & & 14 & & 10 & & \\ & 24 & & 21 & & 18 & & 14 & \\ 28 & & 26 & & 23 & & 19 & & 15 \end{array}$$

gives the chain of partitions $(6,4,4,1) \supset (4,4,1) \supset (4,2) \supset (2) \supset \emptyset$.

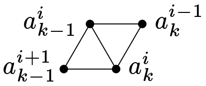
We identify partitions with Young diagrams, but rotated by 180 degrees, so the diagram for a partition λ has λ_k boxes in the k^{th} row from the bottom, and the rows are lined up on the right. Fill the boxes by putting the integer i in each box of $\lambda^{(i)} - \lambda^{(i+1)}$. The conditions (1) and (2) say exactly that the result T is a skew tableau on this shape, that is, it is weakly increasing across rows and strictly increasing down columns. Such a T is often called a contratableau of shape λ. In our example, T is

$$\begin{array}{|c|c|c|c|c|c|}
\cline{6-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 \\
\cline{3-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 1 & 1 & 2 \\
\cline{3-6}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & 2 & 3 & 3 \\
\hline
1 & 1 & 3 & 3 & 4 & 4 \\
\hline
\end{array}$$

The word $w(T)$ is obtained by reading from left to right in rows, from bottom to top. In the example, $w(T) = 1\,1\,3\,3\,4\,4\,2\,2\,3\,3\,1\,1\,1\,2\,1$.

Let $U(\mu)$ be the tableau of shape μ whose i^{th} row has μ_i entries, all equal to i. The word $w(U(\mu))$ is similarly read from left to right, bottom to top. In our example, $\mu = (4,4,3,2)$, and $w(U(\mu)) = 4\,4\,3\,3\,3\,2\,2\,2\,2\,1\,1\,1\,1$.

Now we consider the last rhombus inequalities:

(3) 

These say that $a^{i+1}_{k-1} - a^i_{k-1} \leq a^i_k - a^{i-1}_k$. We claim that this is equivalent to the condition that $w(T)\cdot w(U(\mu))$ is a reverse lattice word [5, §5.2].

This asserts that, if we divide this word at any point, the number of times that i occurs to the right of this point does not exceed the number of times that $i-1$ occurs to the right of this point. We only need to check this at a division corresponding to the place in the k^{th} row from the bottom of T that divides elements strictly smaller than i from elements greater than or equal to i. The number of times that i occurs here is

$$\begin{aligned}
&(\lambda^{(i)}_k - \lambda^{(i+1)}_k) + (\lambda^{(i)}_{k+1} - \lambda^{(i+1)}_{k+1}) + \cdots + (\lambda^{(i)}_{n+1-i} - 0) + \mu_i \\
&= (\lambda^{(i)}_k + \lambda^{(i)}_{k+1} + \cdots + \lambda^{(i)}_{n+1-i}) - (\lambda^{(i+1)}_k + \lambda^{(i+1)}_{k+1} + \cdots + \lambda^{(i+1)}_{n-i}) + \mu_i \\
&= (a^i_{n+1-i} - a^i_{k-1}) - (a^{i+1}_{n-i} - a^{i+1}_{k-1}) + (a^{i+1}_{n-i} - a^i_{n+1-i}) \\
&= a^{i+1}_{k-1} - a^i_{k-1}\,.
\end{aligned}$$

Similarly, the number of times that $i-1$ occurs is

$$(\lambda^{(i-1)}_{k+1} - \lambda^{(i)}_{k+1}) + (\lambda^{(i-1)}_{k+2} - \lambda^{(i)}_{k+2}) + \cdots + (\lambda^{(i-1)}_{n+2-i} - 0) + \mu_{i-1} = a^i_k - a^{i-1}_k\,.$$

Note that the number of times i occurs in all of T is $a^{i+1}_0 - a^i_0 - \mu_i = \nu_i - \mu_i$.

This process is reversible. Given any contratableau T of shape λ such that $w(T)\cdot w(U(\mu))$ is a reverse lattice word, T determines the chain $\lambda^{(1)} \supset \lambda^{(2)} \supset \cdots \supset \lambda^{(n)} \supset \emptyset$, and from these partitions one successively fills in the entries in the northwest to southeast diagonal rows of the hive; the rhombus inequalities (1)–(3) are automatically satisfied.

To make the story complete, we recall why such contratableaux correspond to Littlewood-Richardson skew tableaux, using standard results about tableaux, as in [5]. However, it may be pointed out that these contratableaux are at least as easy to produce and enumerate as the more classical skew tableaux. First, the condition that $w(T)\cdot w(U(\mu))$ is a reverse lattice word, given that the number of times i occurs in T is $\nu_i - \mu_i$, is equivalent to asserting that $w(T)\cdot w(U(\mu))$ is Knuth equivalent to $w(U(\nu))$ [5, §5.2]. The rectification R of a contratableau T of shape λ is easily seen to be a tableau of shape λ, and with the same property that $w(R)\cdot w(U(\mu))$ is Knuth equivalent to $w(U(\nu))$. The correspondence between tableaux and contratableaux of shape λ is a bijection, by reversing the rectification process.

Now the condition that $w(R) \cdot w(U(\mu))$ be Knuth equivalent to $w(U(\nu))$ is equivalent to the condition that $R \cdot U(\mu) = U(\nu)$ in the plactic monoid of tableaux [5, §2.1]. It is easy to see, from the definition of multiplying tableaux by column bumping entries of the first tableau into the second [5, §A.2], that if R and S are tableaux with $R \cdot S = U(\beta)$, then S must be equal to $U(\alpha)$ for some partition α. This gives a correspondence between the set of tableaux R that we are looking at and the set of pairs (R, S) with R of shape λ, S of shape μ, whose product is the tableau $U(\nu)$. There is a standard construction [5, §5.1] between these pairs and the set of skew tableau on the shape ν/λ of content μ whose word is a reverse-lattice word.

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