QUIVER COEFFICIENTS ARE SCHUBERT STRUCTURE CONSTANTS

ANDERS SKOVSTED BUCH, FRANK SOTTILE, AND ALEXANDER YONG

1. Introduction

Buch and Fulton established a formula for the cohomology class of a quiver variety [8], which Buch later extended to K-theory [6]. The K-theory formula expresses a quiver class as an integral linear combination of products of stable Grothendieck polynomials. The *quiver coefficients* of this linear combination were conjectured to be nonnegative in cohomology and to alternate in sign in K-theory. These conjectures were recently proved by Knutson, Miller, and Shimozono [13], Buch [5], and Miller [18].

Buch, Kresch, Tamvakis, and Yong [9, 10] gave combinatorial formulas for the *decomposition coefficients* expressing a Grothendieck polynomial as an integer linear combination of products of stable Grothendieck polynomials. In particular, it was proved that the decomposition coefficients alternate in sign.

Alternation in sign also occurs in the Schubert calculus of the flag variety; Brion [4] proved that the K-theory Schubert structure constants alternate in sign.

We give natural and explicit equalities between the three aforementioned integers. Our argument uses results of Bergeron and Sottile [1] and Lenart, Robinson, and Sottile [17] who earlier established a connection between the decomposition coefficients and the Schubert structure constants. The other main ingredient is the ratio formula of Knutson, Miller, and Shimozono [13], as well as some combinatorial properties of this formula [5, 13]. We also give a direct argument that the decomposition coefficients have alternating signs, based on Brion's theorem, which then implies that quiver coefficients have alternating signs. A consequence of our theorem is that formulas for the other numbers give formulas for the quiver coefficients; we give examples in the last section.

2. The Main Result

Quiver coefficients $c_{\underline{\mu}}(r)$ are defined for a set of rank conditions $r = \{r_{i,j}\}$ for $0 \leq i \leq j \leq n$ and a sequence of partitions $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$, where μ_i fits in a $r_{i-1} \times r_i$ rectangle (for convenience, set $r_i := r_{i,i}$). The Grothendieck polynomial $\mathfrak{G}_w$ for a permutation w represents the class of the structure sheaf of the corresponding Schubert variety in the Grothendieck ring of the flag variety [16]. These form a $\mathbb{Z}$-linear basis for the Grothendieck ring. The *Schubert structure constants* are the

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unique integers $C_{u,v}^{w}$ such that

$$\mathfrak{G}_u \cdot \mathfrak{G}_v \;=\; \sum_{w} C_{u,v}^{w}\; \mathfrak{G}_w\,.$$

Brion's theorem [4, Thm. 1] proves that $(-1)^{\ell(wuv)} C_{u,v}^{w} \geq 0$.

For a partition λ, let $w(\lambda,k)$ denote the Grassmannian permutation for λ with descent at k. It is given by $w(\lambda,k)(i) = i + \lambda_{k+1-i}$ for $1 \leq i \leq k$ and $w(\lambda,k)(i) < w(\lambda,k)(i+1)$ for $i \neq k$. The Grothendieck polynomial $\mathfrak{G}_{w(\lambda,k)}(y_1,y_2,\dots)$ is symmetric in the variables $y_1,\dots,y_k$ and is independent of y_i for $i > k$. Thus we can write $G_\lambda(y_1,\dots,y_k)$ for this symmetric Grothendieck polynomial, without ambiguity.

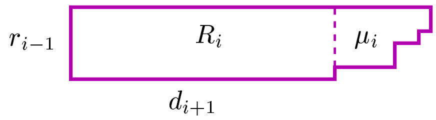
Suppose that v is a permutation whose descents occur at positions in $\{r_0, r_0+r_1,\dots,r_0+\cdots+r_{n-1}\}$. If $x^i = (x^i_1, x^i_2,\dots,x^i_{r_i})$ is a set of r_i variables, then the Grothendieck polynomial $\mathfrak{G}_v(x^0,x^1,\dots,x^n)$ is separately symmetric in each set of variables x^i and is independent of x^n. As the G_λ form a basis for all symmetric polynomials, there are integer *decomposition coefficients* $b_{\underline{\mu}}(v)$ defined by

$$\mathfrak{G}_v(x^0,x^1,\dots,x^n) \;=\; \sum_{\underline{\mu}} b_{\underline{\mu}}(v)\, G_{\mu_1}(x^0)\, G_{\mu_2}(x^1)\cdots G_{\mu_n}(x^{n-1})\,. \tag{1}$$

Formulas for these coefficients given in [10] show that $(-1)^{\sum|\mu_i|-\ell(v)} b_{\underline{\mu}}(v) \geq 0$. In Remark 3 below, we give a simple geometric argument that accounts for this via Brion's theorem.

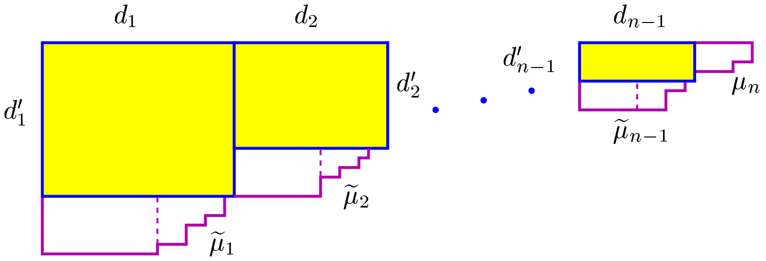
In order to state our main result, we will need some notation and terminology. For a set of rank conditions r, let $d'_i := r_i + \cdots + r_{n-1}$ and $d_i := d'_i + r_n$. Also let $R_i = (d_{i+1})^{r_{i-1}}$ be the rectangular partition with r_{i-1} rows and d_{i+1} columns. For a sequence of partitions $\underline{\mu} = (\mu_1,\mu_2,\dots,\mu_n)$ let $\widetilde{\mu}_i$ be the result of attaching μ_i to the right side of R_i, and let $\underline{\widetilde{\mu}} = (\widetilde{\mu}_1,\dots,\widetilde{\mu}_n)$ denote the sequence of these partitions.

(2) $\widetilde{\mu}_i \;:=$



Let ρ be a concatenation of rectangles of sizes $d'_i \times d_i$ for $i = 1,2,\dots,n-1$. This is the shaded part of the partition shown in (3). Finally, we let $\rho(\underline{\mu})$ be the partition obtained when $\widetilde{\mu}_i$ is placed under the ith rectangle of ρ and the result is concatenated with μ_n. This has $d := d'_0$ rows and is shown below.

(3)



The Zelevinsky permutation $v(r)$ encodes the rank conditions r; we refer to [13, Def. 1.7] for the precise definition. We note that the descents of $v(r)$ occur at positions in $\{r_0, r_0+r_1,\dots,r_0+\cdots+r_{n-1}\}$.

Theorem 1. *Let r be a set of rank conditions and $\underline{\mu} = (\mu_1, \dots, \mu_n)$ be a sequence of partitions with μ_i a subset of the $r_{i-1} \times r_i$ rectangle. Then the following numbers are equal:*

(I) *the quiver coefficient $c_{\underline{\mu}}(r)$;*
(II) *the decomposition coefficient $b_{\underline{\widetilde{\mu}}}(v(r))$;*
(III) *the Schubert structure constant $C^{w(\rho(\underline{\mu}),d)}_{v(r),\,w(\rho,d)}$.*

We remark that we follow the notation for quiver coefficients used in [13], which conjugates all partitions compared to the notation used in [8, 6, 5].

Proof. The ratio formula of Knutson, Miller, and Shimozono expresses a quiver class as a quotient of two Grothendieck polynomials [13, Thm. 2.7]. We need the following identity which is derived from the ratio formula in [5, §7]:

$$\frac{\mathfrak{G}_{v(r)}(x^0, x^1, \dots, x^n)}{\mathfrak{G}_{v(\varepsilon)}(x^0, x^1, \dots, x^n)} \;=\; \sum_{\underline{\mu}} c_{\underline{\mu}}(r)\, G_{\mu_1}(x^0)\, G_{\mu_2}(x^1) \cdots G_{\mu_n}(x^{n-1})\,. \tag{4}$$

Here ϵ denotes the maximal rank conditions given by $\epsilon_{ij} = \min\{r_i, \dots, r_j\}$. Notice that the cohomology version of (4) follows from Thm. 7.10 and Prop. 7.13 of [13]. The denominator of (4) is the monomial

$$\mathfrak{G}_{v(\epsilon)}(x^0, x^1, \dots, x^n) \;=\; \prod_{i=1}^{n-1} G_{R_i}(x^{i-1}) \;=\; \prod_{i=1}^{n-1} (x_1^{i-1} x_2^{i-1} \cdots x_{r_{i-1}}^{i-1})^{d_{i+1}}\,.$$

By using that $G_{R_i}(x^{i-1})\, G_{\mu_i}(x^{i-1}) = G_{\widetilde{\mu}_i}(x^{i-1})$ [7, Cor. 6.5] we deduce that

$$\mathfrak{G}_{v(r)}(x^0, x^1, \dots, x^n) = \sum_{\underline{\mu}} c_{\underline{\mu}}(r)\, G_{\widetilde{\mu}_1}(x^0)\, G_{\widetilde{\mu}_2}(x^1) \cdots G_{\widetilde{\mu}_n}(x^{n-1})\,.$$

This proves the equivalence of (I) and (II). Now since the last descent of $v(r)$ occurs before position d, it follows from [17, Thm. 9.7] that

$$\mathfrak{G}_{v(r)}(x^0, x^1, \dots, x^n) \;=\; \sum_{u_i \in \mathcal{S}_{d_{i-1}}} C^{(u_1 \times \cdots \times u_n)\cdot w(\rho,d)}_{v,\,w(\rho,d)}\, \mathfrak{G}_{u_1}(x^0) \cdots \mathfrak{G}_{u_n}(x^{n-1})\,, \tag{5}$$

where $\mathcal{S}_{d_{i-1}}$ is the symmetric group on d_{i-1} elements, and $u_1 \times \cdots \times u_n \in \mathcal{S}_{d_0 + \cdots + d_{i-1}}$ is the Cartesian product of the permutations u_i. Because the left hand side is symmetric in each set of variables x^i, it follows that all permutations u_i that occur with non-zero coefficient must be Grassmannian with descent at position r_{i-1}, so $u_i = w(\lambda_i, r_{i-1})$ for a partition λ_i. Furthermore, since $G_{R_i}(x^{i-1})$ divides the left hand side of (5), each partition λ_i must have the form $\widetilde{\mu}_i$ for a partition μ_i. Using that

$$w(\rho(\underline{\mu}), d) \;=\; \big(w(\widetilde{\mu}_1, r_0) \times \cdots \times w(\widetilde{\mu}_n, r_{n-1})\big) \cdot w(\rho, d)\,,$$

we deduce that

$$\mathfrak{G}_{v(r)}(x^0, x^1, \dots, x^n) \;=\; \sum_{\underline{\mu}} C^{w(\rho(\underline{\mu}),d)}_{v(r),w(\rho,d)}\, G_{\widetilde{\mu}_1}(x^0)\, G_{\widetilde{\mu}_2}(x^1) \cdots G_{\widetilde{\mu}_n}(x^{n-1})\,. \tag{6}$$

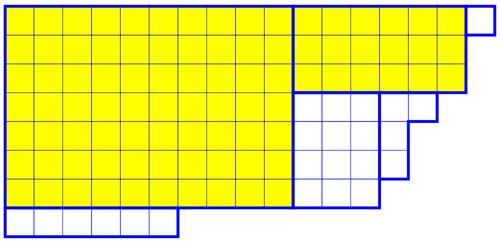
This proves the equivalence of (II) with (III). ∎

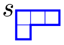
Example 2. Suppose that $n = 3$ and let r be the set of rank conditions

$$\begin{array}{ccccccc} r_{00} & & r_{11} & & r_{22} & & r_{33} \\ & r_{01} & & r_{12} & & r_{23} & \\ & & r_{02} & & r_{13} & & \\ & & & r_{03} & & & \end{array} \quad = \quad \begin{array}{ccccccc} 1 & & 4 & & 3 & & 3 \\ & 1 & & 2 & & 2 & \\ & & 1 & & 1 & & \\ & & & 0 & & & \end{array}$$

Here, $(d'_0, d'_1, d'_2) = (8, 7, 3)$, $(d_0, d_1, d_2, d_3) = (11, 10, 6, 3)$, and the Zelevinsky permutation $v(r)$ is $(7,\ 4, 5, 8, 9,\ 1, 2, 11,\ 3, 6, 10)$. The partition ρ is the concatenation of a 7×10 rectangle with a 3×6 rectangle, and so equals $(16, 16, 16, 10, 10, 10, 10)$. This is the shaded part of the partition shown in (7).

Let $\underline{\mu} = (\emptyset,\ (2, 1, 1),\ (1))$ be a sequence of partitions. Then $\underline{\widetilde{\mu}} = ((6),\ (5, 4, 4, 3),\ (1))$ and the partition $\rho(\underline{\mu})$ is $(17, 16, 16, 15, 14, 14, 13, 6)$, which is illustrated below.

(7) 

In Example 1 of [6], the quiver coefficient $c_{\underline{\mu}}(r)$ was computed to be 1. Due to our different conventions, the corresponding term there is $1 \otimes s$ $\otimes\ s$, which is indexed by the sequence of partitions $(\emptyset, (3, 1), (1))$. Later, we will use formulas from [9] and [14] to calculate that the corresponding decomposition coefficients and Schubert structure constants are both 1.

Remark 3. The alternating signs of the decomposition coefficients can be explained geometrically using Brion's theorem as follows. Let v be a permutation whose descents occur at positions in $\{r_0, r_0 + r_1, \ldots, r_0 + \cdots + r_{n-1}\}$. Fix a large integer N and let $Y = \mathrm{Fl}(r_0, r_0 + r_1, \ldots, r_0 + \cdots + r_{n-1}; \mathbb{C}^{nN})$ be the variety of partial flags of the indicated type in $\mathbb{C}^{nN}$. Then $\mathfrak{G}_v$ represents the class $[\mathcal{O}_{Y_v}]$ of the structure sheaf of the Schubert variety Y_v of Y [16]. The product of Grassmannians

$$X = \mathrm{Gr}(r_0, N) \times \mathrm{Gr}(r_1, N) \times \cdots \times \mathrm{Gr}(r_{n-1}, N)$$

can be identified with a subvariety of Y by mapping each point $(V_1, \ldots, V_n) \in X$ to the partial flag $V_1 \subset V_1 \oplus V_2 \subset \cdots \subset V_1 \oplus \cdots \oplus V_n$ in Y. If we let $x^i = \{x^i_1, \ldots, x^i_{r_i}\}$ denote the K-theoretic Chern roots of the dual of the tautological subbundle corresponding to the ith factor of X, then the specialization $\mathfrak{G}_v(x^0, x^1, ..., x^n)$ is obtained by restricting the class $[\mathcal{O}_{Y_v}]$ to the Grothendieck ring of X. When the Schubert variety Y_v is in general position, it follows from [4, Lemma 2] that the subvariety $X_v := X \cap Y_v$ of X has rational singularities. We furthermore obtain that

$$[\mathcal{O}_{X_v}] = \mathfrak{G}_v(x^0, x^1, \ldots, x^n) = \sum b_{\underline{\mu}}(v)\, G_{\mu_1}(x^0)\, G_{\mu_2}(x^1) \cdots G_{\mu_n}(x^{n-1})$$

in the Grothendieck ring of X. Now since the right hand side of this identity is a linear combination of Schubert structure sheaves on the generalized flag variety X, we conclude from [4, Thm. 1] that $(-1)^{\sum |\mu_i| - \ell(v)} b_{\underline{\mu}}(v) \geq 0$, as required.

In [1] similar geometry is used to study the restriction of Schubert classes $[Y_v]$ in a full flag manifold Y to products X of flag manifolds embedded in Y in a similar fashion to that given here. There, the subvariety X_v is identified as an intersection of Schubert varieties in Y. This is used to identify the decomposition coefficients as particular Schubert structure constants, for cohomology. The analogous identification of K-theoretic decomposition coefficients with K-theoretic Schubert structure constants is accomplished in [17].

3. Alternative formulas for quiver coefficients

By Theorem 1, formulas for the decomposition coefficients and Schubert structure constants give alternative formulas for the quiver coefficients. We give two examples of this for the cohomology quiver coefficients (which are indexed by sequences of partitions such that $\sum |\mu_i|$ equals the expected codimension of the quiver variety, see [8]).

The formulas of [9, 10] for the decomposition coefficients give formulas for quiver coefficients in cohomology and in K-theory. We state this formula for the cohomology decomposition coefficients. Suppose that v is a permutation whose descents occur at positions in $\{r_0, r_0+r_1, \ldots, r_0+\cdots+r_{n-1}\}$. The decomposition coefficient $b_{\underline{\mu}}(v)$ is equal to the number of sequences of semistandard tableaux $(T_1, \ldots, T_n)$ such that

(i) The shape of T_i is the conjugate (matrix transpose) of the partition μ_i,
(ii) The entries of T_i are strictly larger than $r_0 + \cdots + r_{i-2}$, and
(iii) Concatenating the bottom-up, left-to-right column reading words of the tableaux $T_1, T_2, \ldots, T_n$ gives a reduced word for v.

The quiver coefficient computed in Example 2 corresponds to the sequence of tableaux

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline 6\\\hline\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 2&3&4&5\\\hline 3&4&5&6\\\hline 4&5&6&7\\\hline 7&8&9\\\cline{1-3} 10\\\cline{1-1}\end{array}\qquad \begin{array}{|c|}\hline 8\\\hline\end{array}$$

which encodes the following reduced word:

$$v(r) = s_6 s_5 s_4 s_3 s_2 s_1 s_{10} s_7 s_4 s_3 s_2 s_8 s_5 s_4 s_3 s_9 s_6 s_5 s_4 s_7 s_6 s_5 s_8\,.$$

Second, Kogan [14] gives a generalization of the Littlewood-Richardson rule for the cohomology Schubert structure constants $C^u_{v,w(\lambda,d)}$, when v has no descents after position d. We give a mild reformulation of his formula in terms of chains in the Bruhat order from v to u that give valid reading words for tableaux of shape λ.

A *saturated chain* γ in the d-Bruhat order is a sequence of permutations

$$\gamma\,:\; v \;=\; v_0 \;\longrightarrow\; v_1 \;\longrightarrow\; v_2 \;\longrightarrow\; \cdots \;\longrightarrow\; v_t \;=\; u\,,$$

where $\ell(v_i) = \ell(v)+i$ and $v_{i-1}^{-1}v_i$ is a transposition (j_i, k_i) with $j_i \leq d < k_i$ for each $i = 1, \ldots, t$. The *word* of such a chain γ is the sequence of integers

$$v_1(k_1),\, v_2(k_2),\, \ldots\,,\, v_t(k_t)\,.$$

Kogan's formula [14, Theorem 2.4] asserts that if v has no descents after position d, then $C^u_{v,w(\lambda,d)}$ is equal to the number of saturated chains in the d-Bruhat order from v to u whose word is the left-to-right, bottom-up row reading word of a tableau of shape λ.

For the quiver coefficient of Example 2, we have $d = 8$ and there is exactly one saturated chain in the d-Bruhat order that goes from $v(r)$ to $w(\rho(\underline{\mu}))$ and whose word is a reading word for a semistandard tableau of shape ρ.

To describe it, we define a chain in the d-Bruhat order to be *increasing* if its word is an increasing sequence. If there is an increasing chain from v to u, then it is unique, the permutation vu^{-1} is the product of disjoint cycles where the numbers decrease in each cycle, and the partition of the numbers by the cycles they lie in is non-crossing [2, p. 655]. The desired chain has length 88 and it is the concatenation of increasing chains of lengths $10, 10, 10, 10, 16, 16, 16$, respectively. Below, we display each increasing chain on a separate line. Each product of cycles on a given line is $v_j v_i^{-1}$, where the increasing subchain for that line is from v_i to v_j.

$$
\begin{array}{l}
(21,20,19,18,17,16,15,14,13,12,11) \\
(19,18,17,16,15,14,13,12,11,10,9) \\
(18,17,16,15,14,13,12,11,10,9,8) \\
(16,15,14,13,12,11,10,9,8,6,5) \\
(25,24,23,22,21)(20,19)(17,16)(15,14,13,12,11,10,9,8,6,5,4) \\
(23,22,21,19,16,14,13,12,11,10,9,8,6,5,4,3,3,2) \\
(22,21,19,16,14,13,12,11,10,9,8,6,5,4,3,2,1)
\end{array}
$$

This chain corresponds to the following semistandard tableau of shape ρ.

1	2	3	4	5	6	8	9	10	11	12	13	14	16	19	21
2	3	4	5	6	8	9	10	11	12	13	14	16	19	21	22
4	5	6	8	9	10	11	12	13	14	16	19	21	22	23	24
5	6	8	9	10	11	12	13	14	15						
8	9	10	11	12	13	14	15	16	17						
9	10	11	12	13	14	15	16	17	18						
11	12	13	14	15	16	17	18	19	20						

Remark 4. It would be interesting to give bijections between the formulas discussed here for the quiver coefficients (in cohomology) with those given in [13].

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Matematisk Institut, Aarhus Universitet, Ny Munkegade, 8000 Århus C, Denmark
E-mail address: abuch@imf.au.dk

Department of Mathematics, University of Massachusetts, Amherst, MA 01003, USA
E-mail address: sottile@math.umass.edu

Department of Mathematics, University of California, Berkeley, CA, 94720, USA
E-mail address: ayong@math.berkeley.edu