# QUIVER COEFFICIENTS OF DYNKIN TYPE

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## 1. Introduction

Let $Q = (Q_0, Q_1)$ be a quiver, consisting of a finite set of vertices $Q_0$ and a finite set of arrows $Q_1$. Each arrow $a \in Q_1$ has a head $h(a)$ and a tail $t(a)$ in $Q_0$. For convenience we will assume that the vertex set is an integer interval, $Q_0 = \{1, 2, \ldots, n\}$. Let $e = (e_1, \ldots, e_n) \in \mathbb{N}^n$ be a dimension vector, and fix vector spaces $E_i = \mathbb{K}^{e_i}$ for $i \in Q_0$ over a field $\mathbb{K}$. The representations of $Q$ on these vector spaces form the affine space $V = \bigoplus_{a \in Q_1} \mathrm{Hom}(E_{t(a)}, E_{h(a)})$, which has a natural action of the group $\mathbb{G} = \mathrm{GL}(E_1) \times \cdots \times \mathrm{GL}(E_n)$ given by $(g_1, \ldots, g_n).(\phi_a)_{a \in Q_1} = (g_{h(a)}\, \phi_a\, g_{t(a)}^{-1})_{a \in Q_1}$.

Define a *quiver cycle* to be any $\mathbb{G}$-stable closed irreducible subvariety $\Omega$ in $V$. A quiver cycle determines an equivariant (Chow) cohomology class $[\Omega] \in H^*_{\mathbb{G}}(V)$ and an equivariant Grothendieck class $[\mathcal{O}_\Omega] \in K_{\mathbb{G}}(V)$. These classes are well understood when the quiver $Q$ is *equioriented* of type A, that is, a sequence $\{1 \to 2 \to \cdots \to n\}$ of arrows in the same direction. In this case, a formula for the cohomology class $[\Omega]$ was given in joint work with Fulton [11], and this formula was generalized to $K$-theory in [8]. The $K$-theory formula states that the Grothendieck class $[\mathcal{O}_\Omega]$ is given by

$$[\mathcal{O}_\Omega] = \sum_{\mu} c_\mu(\Omega)\, \mathcal{G}_{\mu_1}(E_2 - E_1) \mathcal{G}_{\mu_2}(E_3 - E_2) \cdots \mathcal{G}_{\mu_{n-1}}(E_n - E_{n-1}) \ \in K_{\mathbb{G}}(V)$$

where the sum is over finitely many sequences $\mu = (\mu_1, \ldots, \mu_{n-1})$ of partitions $\mu_i$. Each factor $\mathcal{G}_{\mu_i}(E_{i+1} - E_i)$ is obtained by applying the stable Grothendieck polynomial for $\mu_i$ to the standard representations of $\mathbb{G}$ on $E_{i+1}$ and $E_i$. This notation will be explained in section 3.

The coefficients $c_\mu(\Omega)$ are interesting geometric and combinatorial invariants called (equioriented) *quiver coefficients*. They are integers and are non-zero only when the sum $\sum |\mu_i|$ of the weights of the partitions is greater than or equal to the codimension of $\Omega$. The coefficients for which this sum equals $\mathrm{codim}(\Omega)$ describe the cohomology class of $\Omega$ and are called *cohomological quiver coefficients*. It was proved in [25] that cohomological quiver coefficients are non-negative, and in [10, 29] that the more general $K$-theoretic quiver coefficients have *alternating signs*, in the sense that $(-1)^{\sum |\mu_i| - \mathrm{codim}(\Omega)} c_\mu(\Omega)$ is a non-negative integer. These properties had earlier been conjectured in [11, 8], and special cases had been proved in [6, 13, 14]. The equioriented quiver coefficients can furthermore be expressed in terms of counting factor sequences [11, 6, 25, 10, 12]. They are known to generalize Littlewood-Richardson coefficients [11], ($K$-theoretic) Stanley coefficients [7, 8], and the monomial coefficients of Schubert and Grothendieck polynomials [13, 14].

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The equioriented quiver coefficients are themselves special cases of the $K$-theoretic Schubert structure constants on flag manifolds [28, 10, 16].

The purpose of this paper is to introduce and study a more general notion of quiver coefficients, which can be defined for an arbitrary quiver $Q$ without oriented loops. For each vertex $i \in Q_0$, we define $M_i = \bigoplus_{a:h(a)=i} E_{t(a)}$ to be the direct sum of all vertex vector spaces at the tails of arrows pointing to $i$. (If there are two or more arrows to $i$ from a vertex $j$, then $E_j$ is included multiple times as a summand of $M_i$.) Given a quiver cycle $\Omega \subset V$, we show that there are unique coefficients $c_\mu(\Omega) \in \mathbb{Z}$, indexed by sequences $\mu = (\mu_1, \ldots, \mu_n)$ of partitions such that the length $\ell(\mu_i)$ is at most $e_i$, for which

$$[\mathcal{O}_\Omega] = \sum_\mu c_\mu(\Omega)\, \mathcal{G}_{\mu_1}(E_1 - M_1)\mathcal{G}_{\mu_2}(E_2 - M_2)\cdots\mathcal{G}_{\mu_n}(E_n - M_n)\,. \tag{1}$$

As in the equioriented case, a coefficient $c_\mu(\Omega)$ can be non-zero only if $\sum |\mu_i| \geq \operatorname{codim}(\Omega)$, and the lowest degree coefficients describe the cohomology class $[\Omega]$. However, the defining linear combination (1) might possibly be infinite, which makes sense modulo the gamma filtration on $K_{\mathbb{G}}(V)$. We pose the following.

**Conjecture 1.1.** *Let $Q$ be a quiver without oriented loops and $\Omega \subset V$ a quiver cycle.*

*(a) Only finitely many of the quiver coefficients $c_\mu(\Omega)$ for $\Omega$ are non-zero. In other words, the sum (1) is finite.*

*(b) All cohomological quiver coefficients $c_\mu(\Omega)$, with $\sum |\mu_i| = \operatorname{codim}(\Omega)$, are non-negative.*

*(c) If $\Omega$ has rational singularities, then the quiver coefficients for $\Omega$ have alternating signs, i.e. $(-1)^{\sum |\mu_i| - \operatorname{codim}(\Omega)} c_\mu(\Omega) \geq 0$.*

Our main result is a formula for the quiver coefficients when the quiver $Q$ is of Dynkin type and $\Omega$ has rational singularities. A quiver is of Dynkin type if the underlying (un-directed) graph is a simply-laced Dynkin diagram, i.e. a disjoint union of Dynkin diagrams of types A, D, and E. In this case, every quiver cycle is an orbit closure [22]. Bobiński and Zwara have proved that all orbit closures have rational singularities if $Q$ is a quiver of type A and $\mathbb{K}$ is an algebraically closed field [1], or if $Q$ is of type D and $\mathbb{K}$ is algebraically closed of characteristic zero [2] (see also [27] for the equioriented case). Our formula relies on an explicit desingularization of an orbit closure given by Reineke [31], as well as a list of geometric and combinatorial properties of stable Grothendieck polynomials established in [9, 8], and it proves the finiteness part (a) of Conjecture 1.1. Our new formula generalizes the formula for equioriented quiver coefficients proved in [8], but requires more operations on Grothendieck polynomials, including multiplication and Grothendieck polynomials indexed by sequences of negative integers. For quivers of type $A_3$, we prove the full statement of Conjecture 1.1, and we provide positive combinatorial formulas for the quiver coefficients in terms of counting set-valued tableaux.

We remark that the positivity properties of quiver cycles suggested by Conjecture 1.1 are analogous to positivity properties satisfied by a closed and irreducible subvariety $Y$ of a homogeneous space $G/P$. In fact, the cohomology class of $Y$ can be uniquely written as a positive linear combination of Schubert classes, where the coefficients count the intersection points of $Y$ with the dual Schubert varieties placed in general position. Furthermore, Brion has proved that if $Y$ has rational

singularities, then the Grothendieck class of $Y$ is an alternating linear combination of $K$-theoretic Schubert classes [5]. Aside from this analogy, our conjecture is supported by computer experiments.

Some other formulas for quiver cycles of Dynkin type have been given, which do not involve quiver coefficients. First of all, Fehér and Rimányi have proved that the cohomology class of an orbit closure of Dynkin type is uniquely determined, up to a constant, by the property that its restriction to any disjoint orbit vanishes [18]. Rimányi and the author have used this result to prove a positive combinatorial formula for the cohomology class of any orbit closure for a (non-equioriented) quiver of type A, which expresses this class as a sum of products of Schubert polynomials [15]. A conjectured $K$-theory version furthermore expresses the Grothendieck classes of such orbit closures as alternating sums of products of Grothendieck polynomials. These formulas generalize the (non-stable) component formulas for equioriented quivers proved by Knutson, Miller, and Shimozono in cohomology [25] and by the author in $K$-theory [10]. Despite the positivity displayed by the generalized component formulas, we have not been able to relate them to positivity properties of quiver coefficients in the non-equioriented cases. Finally, a recent preprint of Knutson and Shimozono [26] contains a formula for the Grothendieck class of any orbit closure of Dynkin type which has rational singularities. This formula is stated in terms of Demazure operators, but does not to our knowledge suggest any positivity properties of quiver cycles.

This paper is organized as follows. In section 2 we recall the definition and required properties of stable Grothendieck polynomials. Section 3 describes the equivariant Grothendieck class of a quiver cycle, defines the corresponding quiver coefficients, and discusses the available evidence for Conjecture 1.1. We also give an example of an orbit closure for which the associated quiver coefficients do not have alternating signs. This orbit closure was earlier studied by Zwara [35], who proved that it does not have rational singularities. In section 4 we interpret quiver coefficients in terms of formulas for degeneracy loci defined by a quiver of vector bundles over a base variety. In section 5 we describe Reineke's desingularization of orbit closures of Dynkin type. This desingularization is used in section 6 to prove a combinatorial formula for quiver coefficients of Dynkin type. The last section contains the proof of Conjecture 1.1 for quivers of type $A_3$.

Our formula for orbit closures of Dynkin type was proved at the time the preprint [26] became available. We do, however, thank Allen Knutson for earlier suggesting that resolutions that we used to compute quiver coefficients of types A and D might be special cases of Reineke's general construction. We have benefited from many discussions with Richárd Rimányi on this general subject, and from answers to questions and useful references provided by Wilbert van der Kallen and Michel Brion regarding group actions and equivariant $K$-theory. We also thank Johan de Jong, Friedrich Knop, Chris Woodward, and Bobiński Zwara for helpful comments and answers to questions.

## 2. Grothendieck polynomials

In this section we fix notation for stable Grothendieck polynomials and state the required properties. We refer to [9, 8] for more details.

A *partition* is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0)$. The *weight* of $\lambda$ is the sum $|\lambda| = \sum \lambda_i$ of its parts and the

*length* $\ell(\lambda)$ is the number of non-zero parts. We will identify the partition $\lambda$ with its *Young diagram*, which has $\lambda_1$ boxes in the top row, $\lambda_2$ boxes in the next row, etc. A *set-valued tableau* of shape $\lambda$ is a filling $T$ of the boxes of $\lambda$ with finite non-empty sets of positive integers, such that the largest integer in any box is smaller than or equal to the smallest integer in the box to the right of it, and strictly smaller than the smallest integer in the box below it. Given an infinite set of commuting variables $x = (x_1, x_2, \dots)$, we let $x^T$ denote the monomial in which the exponent of $x_i$ is the number of boxes of $T$ containing $i$, and we let $|T|$ be the (total) degree of $x^T$. For example, the set-valued tableau

$$T = \begin{array}{|c|c|c|}\hline 1,2 & 2 & 2,5,8 \\ \hline 4 & 7,8 \\ \cline{1-2}\end{array}$$

has shape $\lambda = (3,2)$ and gives $x^T = x_1 x_2^3 x_4 x_5 x_7 x_8^2$ and $|T| = 9$.

The *single stable Grothendieck polynomial* for the partition $\lambda$ is defined as the formal power series

$$\mathcal{G}_\lambda = \mathcal{G}_\lambda(x) = \sum_T (-1)^{|T|-|\lambda|}\, x^T\,,$$

where the sum is over all set-valued tableaux $T$ of shape $\lambda$. This power series is symmetric, and its term of lowest degree is the Schur function $s_\lambda$. It was proved in [9] to be a special case of the stable Grothendieck polynomials indexed by permutations of Fomin and Kirillov [19], which in turn were constructed as limits of Lascoux and Schützenberger's ordinary Grothendieck polynomials. By convention, a stable Grothendieck polynomial applied to a finite set of variables is defined by $\mathcal{G}_\lambda(x_1, \dots, x_p) = \mathcal{G}_\lambda(x_1, \dots, x_p, 0, 0, \dots)$.

Given a set-valued tableau $T$, define its *word* $w(T)$ to be the sequence of integers in its boxes when read one row at the time from left to right, with the rows ordered from bottom to top. Integers in the same box are arranged in increasing order. For example, the tableau displayed above gives $w(T) = (4, 7, 8, 1, 2, 2, 2, 5, 8)$. A word of positive integers is called a *reverse lattice word* if every occurrence of an integer $i \geq 2$ is followed by more occurrences of $i-1$ than of $i$. The *content* of a word is the sequence $\nu = (\nu_1, \nu_2, \dots)$ where $\nu_i$ is the number of occurrences of $i$ in the word. For any partition $\mu = (\mu_1, \dots, \mu_l)$, let $u(\mu) = (l^{\mu_l}, \dots, 2^{\mu_2}, 1^{\mu_1})$ be the word of the tableau of shape $\mu$ in which all boxes in row $i$ contains the single integer $i$. We need the following generalization of the classical Littlewood-Richardson rule from [9, Thm. 5.4] (an alternative proof can be found in [12, §3.5]).

**Theorem 2.1.** *The product of two stable Grothendieck polynomials is given by*

$$\mathcal{G}_\lambda \cdot \mathcal{G}_\mu = \sum_\nu c^\nu_{\lambda\mu}\, \mathcal{G}_\nu$$

*where the sum is over all partitions $\nu$, and $c^\nu_{\lambda\mu}$ is equal to $(-1)^{|\nu|-|\lambda|-|\mu|}$ times the number of set-valued tableaux $T$ of shape $\lambda$ for which the composition $w(T)u(\mu)$ is a reverse lattice word with content $\nu$.*

For example, the set-valued tableaux $\begin{array}{|c|}\hline 1 \\ \hline\end{array}$, $\begin{array}{|c|}\hline 2 \\ \hline\end{array}$, and $\begin{array}{|c|}\hline 1,2 \\ \hline\end{array}$ correspond to the terms of the product $\mathcal{G}_{\square} \cdot \mathcal{G}_{\square} = \mathcal{G}_{\square\square} + \mathcal{G}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}} - \mathcal{G}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}$. If a coefficient $c^\nu_{\lambda\mu}$ is non-zero, then $|\lambda| + |\mu| \leq |\nu|$ and (the Young diagrams of) $\lambda$ and $\mu$ can be contained in $\nu$.

Theorem 2.1 implies that the linear span $\Gamma = \bigoplus \mathbb{Z}\mathcal{G}_\lambda$ of all stable Grothendieck polynomials is a commutative ring. The stable Grothendieck polynomials are linearly independent since the term of lowest degree in $\mathcal{G}_\lambda$ is the Schur function $s_\lambda$.

If $\lambda$, $\mu$, and $\nu$ are partitions such that $\lambda$ and $\mu$ fit inside a rectangular partition $R$, we define

$$d^\nu_{\lambda\mu} = c^\rho_{R\,\nu} \quad , \quad \text{where} \quad \rho = (R+\mu, \lambda) = \begin{array}{|c|c|}\hline R & \mu \\ \hline \lambda \\ \cline{1-1}\end{array}$$

is the partition obtained by attaching $\lambda$ and $\mu$ to the bottom and right sides of $R$. This constant $d^\nu_{\lambda\mu}$ is independent of the choice of rectangle $R$, and it is non-zero only if $|\nu| \le |\lambda|+|\mu|$ and $\lambda, \mu \subset \nu$ [9, Thm. 6.6]. These constants define a coproduct $\Delta : \Gamma \to \Gamma \otimes \Gamma$ given by $\Delta(\mathcal{G}_\nu) = \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, \mathcal{G}_\lambda \otimes \mathcal{G}_\mu$, which gives $\Gamma$ a structure of commutative and cocommutative bialgebra with unit and counit [9, Cor. 6.7].

Given an additional set of commuting variables $y = (y_1, y_2, \dots)$, define the *double stable Grothendieck polynomial* for the partition $\nu$ by

$$\mathcal{G}_\nu(x; y) = \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, \mathcal{G}_\lambda(x) \cdot \mathcal{G}_{\mu'}(y)\,,$$

where $\mu'$ is the conjugate partition of $\mu$, obtained by interchanging the rows and columns of $\mu$. These power series are separately symmetric in each set of variables $x$ and $y$, and they satisfy the identities

$$\mathcal{G}_\nu(1-a^{-1}, x\ ;\ 1-a, y) = \mathcal{G}_\nu(x; y) \tag{2}$$

for any indeterminate $a$ [19], and

$$\mathcal{G}_\nu(x, z; y, w) = \sum_{\lambda,\mu} d^\nu_{\lambda,\mu}\, \mathcal{G}_\lambda(x; y)\, \mathcal{G}_\mu(z; w) \tag{3}$$

for arbitrary sets of variables $x$, $y$, $z$, and $w$ [9, (6.1)]. Another useful identity is the factorization formula [9, Cor. 6.3], which states that

$$\begin{aligned}\mathcal{G}_{R+\mu,\lambda}(x_1, \dots, x_p; y_1, \dots, y_q) = \\ \mathcal{G}_\lambda(0; y_1, \dots, y_q) \cdot \mathcal{G}_R(x_1, \dots, x_p; y_1, \dots, y_q) \cdot \mathcal{G}_\mu(x_1, \dots, x_p)\end{aligned} \tag{4}$$

whenever $\lambda$ and $\mu$ are partitions with $\lambda_1 \le q$ and $\ell(\mu) \le p$, and $R = (q^p)$ is the rectangular partition with $p$ rows and $q$ columns.

**Lemma 2.2.** *Let $R$ be a commutative ring that is complete with respect to the ideal $\mathfrak{m} \subset R$, $R = \lim R/\mathfrak{m}^i$, and let $y_1, \dots, y_q \in \mathfrak{m}$. Any symmetric formal power series $f \in R[\![x_1, \dots, x_p]\!]^{\Sigma_p}$ can be written uniquely as an (infinite) linear combination*

$$f = \sum_\lambda b_\lambda\, \mathcal{G}_\lambda(x_1, \dots, x_p\,;\, y_1, \dots, y_q) \tag{5}$$

*where the sum is over all partitions $\lambda$ with $\ell(\lambda) \le p$, and $b_\lambda \in R$.*

*Proof.* Write $x = (x_1, \dots, x_p)$ and $y = (y_1, \dots, y_q)$. Set $z = (z_1, \dots, z_q)$ where $z_i = 1 - (1-y_i)^{-1} = -\sum_{k\ge1} y_i^k \in R$, which is well defined because $y_i \in \mathfrak{m}$. If $y_1 = \cdots = y_q = 0$, then the lemma follows because the term of lowest degree in $\mathcal{G}_\lambda(x)$ is the Schur polynomial $s_\lambda(x)$. Given an expression

$$f = \sum_\lambda b'_\lambda\, \mathcal{G}_\lambda(x) \tag{6}$$

we can define coefficients $b_\lambda \in R$ by (*) $b_\lambda = \sum_{\nu,\mu} b'_\nu \, d^\nu_{\lambda\mu} \, \mathcal{G}_\mu(z)$. This infinite sum is well defined in $R$ because $z_i \in \mathfrak{m}$ and $d^\nu_{\lambda\mu}$ is non-zero only when $|\mu| \geq |\nu| - |\lambda|$. By (2) and (3) we furthermore have

$$f = \sum_\nu b'_\nu \, \mathcal{G}_\nu(z, x\,;\, y) = \sum_{\nu,\lambda,\mu} b'_\nu \, d^\nu_{\lambda\mu} \, \mathcal{G}_\mu(z) \, \mathcal{G}_\lambda(x; y) = \sum_\lambda b_\lambda \, \mathcal{G}_\lambda(x; y)\,.$$

Similarly, given coefficients $b_\lambda \in R$ such that (5) holds, we obtain coefficients $b'_\lambda \in R$ for which (6) holds by setting $b'_\lambda = \sum_{\nu,\mu} b_\nu \, d^\nu_{\lambda\mu} \, \mathcal{G}_{\mu'}(y)$. If $f = 0$ then all these coefficients $b'_\lambda$ must be zero. On the other hand, the coefficients $b_\lambda$ can be recovered from the $b'_\lambda$ by (*) since for any fixed partition $\lambda$ we have

$$\begin{aligned}\sum_{\nu,\mu} \left( \sum_{\sigma,\tau} b_\sigma \, d^\sigma_{\nu\tau} \, \mathcal{G}_{\tau'}(y) \right) d^\nu_{\lambda\mu} \, \mathcal{G}_\mu(z) &= \sum_{\sigma,\nu,\mu,\tau} b_\sigma \, d^\sigma_{\lambda\nu} d^\nu_{\mu\tau} \, \mathcal{G}_\mu(z) \mathcal{G}_{\tau'}(y) \\ &= \sum_{\sigma,\nu} b_\sigma \, d^\sigma_{\lambda\nu} \, \mathcal{G}_\nu(z; y) = b_\lambda\,.\end{aligned}$$

The first equality holds because $\Delta$ is a coproduct and the last follows from (2) because $\mathcal{G}_\nu(z; y)$ is equal to one if $\nu$ is the empty partition and is zero otherwise. □

The stable Grothendieck polynomials given by partitions can be generalized to stable polynomials $\mathcal{G}_I$ indexed by arbitrary finite sequences of integers. These can be defined by the recursive identities

$$\mathcal{G}_{I,p,q,J} = \sum_{k=p+1}^{q} \mathcal{G}_{I,q,k,J} - \sum_{k=p+1}^{q-1} \mathcal{G}_{I,q-1,k,J} \tag{7}$$

whenever $I$ and $J$ are integer sequences and $p < q$ are integers, as well as the identity $\mathcal{G}_{I,p} = \mathcal{G}_I$ for any integer sequence $I$ and negative integer $p$. Thus any finite integer sequence $I$ gives a well defined element $\mathcal{G}_I \in \Gamma$. This notation is required in our formula for quiver coefficients of Dynkin type given in section 6.

## 3. Quiver coefficients

In this section we define quiver coefficients and discuss their conjectured positivity properties. We start by giving an elementary construction of the Grothendieck class of an invariant closed subvariety in a representation.

3.1. **Grothendieck classes.** Let $G$ be a linear algebraic group over the field $\mathbb{K}$ and let $V$ be a rational representation of $G$, i.e. $V$ is a $\mathbb{K}$-vector space of finite dimension and the $G$-action is given by a map of varieties $G \to \mathrm{GL}(V)$. Then the coordinate ring $\mathbb{K}[V] = \mathrm{Sym}^\bullet(V^\vee)$ of polynomial functions on $V$ has a locally finite linear $G$-action, which in set-theoretic notation is given by $(g.f)(v) = f(g^{-1}.v)$ for $g \in G$, $f \in \mathbb{K}[V]$, and $v \in V$. Locally finite means that $\mathbb{K}[V]$ is a union of rational representations of $G$. Define a $(\mathbb{K}[V], G)$-*module* to be a module $M$ over $\mathbb{K}[V]$ together with a locally finite linear $G$-action on $M$ which satisfies that $g.(f\,m) = (g.f)\,(g.m)$ for $m \in M$. We will say that $M$ is *finitely generated* (resp. *free*) if this is true as a $\mathbb{K}[V]$-module. If $M$ is finitely generated, then there exists a finite dimensional $G$-stable vector subspace $U \subset M$ which contains a set of generators. Notice that $\mathbb{K}[V] \otimes_{\mathbb{K}} U$ has a natural structure of $(\mathbb{K}[V], G)$-module, where $\mathbb{K}[V]$ acts on the first factor and $G$ acts on both factors. The map $\mathbb{K}[V] \otimes U \to M$ given by $f \otimes u \mapsto fu$ is a surjective $G$-equivariant map. Since $M$ has finite projective

dimension as a module over the polynomial ring $\mathbb{K}[V]$, and all projective $\mathbb{K}[V]$-modules are free, it follows that $M$ has a finite equivariant resolution by finitely generated free $(\mathbb{K}[V], G)$-modules.

Let $\Omega \subset V$ be a $G$-stable closed subvariety. Then the coordinate ring $\mathcal{O}_\Omega = \mathbb{K}[V]/I(\Omega)$ is a finitely generated $(\mathbb{K}[V], G)$-module, so it has an equivariant resolution

$$0 \to F_p \to F_{p-1} \to \cdots \to F_0 \to \mathcal{O}_\Omega \to 0 \tag{8}$$

where $F_i$ is a finitely generated free $(\mathbb{K}[V], G)$-module. Notice that $F_i/\mathfrak{m}F_i$ is a rational representation of $G$ for each $i$, where $\mathfrak{m} = I(0) \subset \mathbb{K}[V]$ is the maximal ideal of functions vanishing at the origin of $V$.

Let $\mathcal{R}(G)$ be the ring of *virtual representations* of $G$, i.e. formal linear combinations of irreducible rational representations. Multiplication in this ring is defined by tensor products. We define the *$G$-equivariant Grothendieck class* of $\Omega$ to be the virtual representation

$$[\mathcal{O}_\Omega] \;=\; \sum_{i\geq 0} (-1)^i [F_i/\mathfrak{m}F_i] \;\in \mathcal{R}(G)\,.$$

It follows from results of Thomason [33] that this class can be identified with the class of the structure sheaf of $\Omega$ in the equivariant $K$-theory of $V$, see section 4.

3.2. **Classes of quiver cycles.** Let $V = \bigoplus_{a\in Q_1} \mathrm{Hom}(E_{t(a)}, E_{h(a)})$ be the vector space of representations of the quiver $Q$. Then $V$ is a rational representation of the group $\mathbb{G} = \prod_{i=1}^n \mathrm{GL}(E_i)$. It follows that any quiver cycle $\Omega \subset V$ defines a Grothendieck class $[\mathcal{O}_\Omega] \in \mathcal{R}(\mathbb{G})$.

Choose a decomposition of each vertex vector space as a sum of one dimensional vector spaces, $E_i = L_1^i \oplus \cdots \oplus L_{e_i}^i$, and let $\mathbb{T} \subset \mathbb{G}$ be the maximal torus that preserves these decompositions. Then the virtual representations of $\mathbb{T}$ form the Laurent polynomial ring $\mathcal{R}(\mathbb{T}) = \mathbb{Z}[\,[L_j^i]^{\pm 1}]$. It follows from [24, Cor. II.2.7] that the restriction map $\mathcal{R}(\mathbb{G}) \to \mathcal{R}(\mathbb{T})$ is injective, and the image must consist of Laurent polynomials that are simultaneously symmetric in each group of variables $\{[L_1^i], \ldots, [L_{e_i}^i]\}$. Since all such polynomials can be generated by the exterior powers $[\bigwedge^j E_i] \in \mathcal{R}(\mathbb{G})$, it follows that $\mathcal{R}(\mathbb{G}) \subset \mathcal{R}(\mathbb{T})$ is the subring of simultaneously symmetric Laurent polynomials.

Set $x_j^i = 1 - [L_j^i]^{-1}$ for $1 \leq i \leq n$ and $1 \leq j \leq e_i$, and let $\mathbb{Z}[\![x_j^i]\!]$ be the ring of formal power series in these variables. We will consider $\mathcal{R}(\mathbb{T})$ as a subring of $\mathbb{Z}[\![x_j^i]\!]$, with $[L_j^i] = \sum_{p\geq 0} (x_j^i)^p$. In particular, the Grothendieck class $[\mathcal{O}_\Omega]$ can be regarded as a power series in $\mathbb{Z}[\![x_j^i]\!]$. The $\mathbb{T}$-equivariant cohomology of $V$ can be identified with the polynomial ring $H^*_{\mathbb{T}}(V) = \mathbb{Z}[x_j^i]$, and $H^*_{\mathbb{G}}(V) \subset H^*_{\mathbb{T}}(V)$ is the subring of simultaneously symmetric polynomials. The power series $[\mathcal{O}_\Omega] \in \mathbb{Z}[\![x_j^i]\!]$ has no non-zero terms of total degree smaller than $d = \mathrm{codim}(\Omega; V)$, and the term of degree $d$ is the cohomology class $[\Omega] \in H^d_{\mathbb{G}}(V)$, see section 4.2.

If $U$ is any rational representation of $\mathbb{G}$, we can write it as a direct sum of one dimensional $\mathbb{T}$-representations, $U = L_1 \oplus \cdots \oplus L_u$. Given a partition $\nu$ we then define $\mathcal{G}_\nu(U) = \mathcal{G}_\nu(1 - [L_1]^{-1}, \ldots, 1 - [L_u]^{-1}) \in \mathcal{R}(\mathbb{G}) \subset \mathcal{R}(\mathbb{T})$. For example, $\mathcal{G}_\nu(E_i) = \mathcal{G}_\nu(x_1^i, \ldots, x_{e_i}^i)$. More generally, given two rational $\mathbb{G}$-representations $U_1$

and $U_2$ we define

$$\mathcal{G}_\nu(U_1 - U_2) \;=\; \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, \mathcal{G}_\lambda(U_1)\, \mathcal{G}_{\mu'}(U_2^\vee) \;\in \mathcal{R}(\mathbb{G}) \tag{9}$$

where $U_2^\vee$ is the dual representation of $U_2$. The Schur function $s_\nu(U_1 - U_2)$ is defined as the term of total (and lowest) degree $|\nu|$ in $\mathcal{G}_\nu(U_1 - U_2)$ when considered as a power series in $\mathbb{Z}[\![x^i_j]\!]$.

From now on we assume that $Q$ is a quiver without oriented loops. Our definition of quiver coefficients is based on the following proposition. Recall that we set $M_i = \bigoplus_{a:h(a)=i} E_{t(a)}$ for $i \in Q_0$.

**Proposition 3.1.** *Let $Q$ be a quiver without oriented loops. Every element of $\mathcal{R}(\mathbb{G})$ can be expressed uniquely as a (possibly infinite) $\mathbb{Z}$-linear combination of products*

$$\mathcal{G}_{\mu_1}(E_1 - M_1)\, \mathcal{G}_{\mu_2}(E_2 - M_2) \cdots \mathcal{G}_{\mu_n}(E_n - M_n)$$

*given by partitions $\mu_1, \ldots, \mu_n$ such that $\ell(\mu_i) \leq e_i$ for each $i$.*

*Proof.* Let $l \in Q_0$ be a vertex which is not the tail of any arrow in $Q$. Since every element of $\mathcal{R}(\mathbb{G}) \subset \mathbb{Z}[\![x^i_j]\!]$ is symmetric in the variables $x^l_1, \ldots, x^l_{e_l}$, we can use Lemma 2.2 to write it as an (infinite) linear combination of the elements $\mathcal{G}_{\mu_l}(E_l - M_l)$ given by partitions $\mu_l$ with at most $e_l$ rows, and with coefficients in the subring $R = \mathbb{Z}[\![x^i_j : i \neq l]\!]$. By induction on $n$, applied to the quiver obtained from $Q$ by removing the vertex $l$ and all arrows to it, it follows that each of the coefficients are unique $\mathbb{Z}$-linear combinations of the products $\prod_{i \neq l} \mathcal{G}_{\mu_i}(E_i - M_i)$. □

**Definition 3.2.** Let $\Omega \subset V$ be a quiver cycle for a quiver $Q$ without oriented loops. The *quiver coefficients* of $\Omega$ are the unique integers $c_\mu(\Omega) \in \mathbb{Z}$, indexed by sequences $\mu = (\mu_1, \ldots, \mu_n)$ of partitions $\mu_i$ with $\ell(\mu_i) \leq e_i$, such that

$$[\mathcal{O}_\Omega] \;=\; \sum_\mu c_\mu(\Omega)\, \mathcal{G}_{\mu_1}(E_1 - M_1)\, \mathcal{G}_{\mu_2}(E_2 - M_2) \cdots \mathcal{G}_{\mu_n}(E_n - M_n) \;\in \mathcal{R}(\mathbb{G})\,.$$

The *cohomological quiver coefficients* of $\Omega$ are the coefficients $c_\mu(\Omega)$ for which $\sum |\mu_i| = \operatorname{codim}(\Omega)$.

It follows from Corollary 4.3 below that these coefficients generalize the equioriented quiver coefficients from [11, 8]. The cohomological quiver coefficients determine the cohomology class of $\Omega$ as

$$[\Omega] = \sum_{\sum |\mu_i| = \operatorname{codim}(\Omega)} c_\mu(\Omega)\, s_{\mu_1}(E_1 - M_1)\, s_{\mu_2}(E_2 - M_2) \cdots s_{\mu_n}(E_n - M_n) \in H^*_{\mathbb{G}}(V)\,.$$

**Example 3.3.** Let $Q = \{1 \to 2\}$ be a quiver of type $\mathrm{A}_2$. Then any quiver cycle in $V = \operatorname{Hom}(E_1, E_2)$ has the form $\Omega_r = \{\phi \in V \mid \operatorname{rank}(\phi) \leq r\}$. It follows from the Thom-Porteous formula of [8, Thm. 2.3] and Corollary 4.3 that $[\mathcal{O}_{\Omega_r}] = \mathcal{G}_R(E_2 - E_1)$, where $R = (e_1 - r)^{e_2 - r}$ is the rectangular partition with $e_2 - r$ rows and $e_1 - r$ columns. We have $c_{(R)}(\Omega_r) = 1$, and all other quiver coefficients of $\Omega_r$ are zero.

3.3. **Properties of quiver coefficients.** We do not know a good reason why the quiver coefficients should satisfy the finiteness and positivity properties stated in Conjecture 1.1. In the case of equioriented quivers where this conjecture is known, these properties are consequences of explicit formulas for quiver coefficients that are proved with a combination of geometric and combinatorial methods. This is also

true for our proof of the finiteness part (a) for quivers of Dynkin type in section 6, and our proof of the full conjecture for quivers of type $A_3$ in section 7. However, if the full conjecture is true, then it is natural to expect that some underlying geometric principle is in play.

One might try to express the classes of quiver cycles as linear combinations of other products of Grothendieck polynomials than those used in Definition 3.2, but most choices do not lead to finiteness or positivity properties of the coefficients (or they lead to such properties that follow from Conjecture 1.1). The one interesting alternative choice that we know about is to define *dual quiver coefficients* $\widetilde{c}_\mu(\Omega)$ of a quiver cycle $\Omega$ by the identity

$$[\mathcal{O}_\Omega] = \sum_\mu \widetilde{c}_\mu(\Omega)\, \mathcal{G}_{\mu_1}(N_1 - E_1)\, \mathcal{G}_{\mu_2}(N_2 - E_2) \cdots \mathcal{G}_{\mu_n}(N_n - E_n)\,,$$

where the sum is over sequences $\mu = (\mu_1, \ldots, \mu_n)$ of partitions such that $\mu_i$ has at most $e_i$ columns for each $i$, and $N_i = \bigoplus_{a:t(a)=i} E_{h(a)}$. These dual coefficients are nothing but the ordinary quiver coefficients for $\Omega$ when considered as a cycle of quiver representations on the dual vector spaces $E_i^\vee$, for the quiver obtained from $Q$ by reversing all arrows. This follows from the identity $\mathcal{G}_\lambda(U_1 - U_2) = \mathcal{G}_{\lambda'}(U_2^\vee - U_1^\vee)$ which holds for arbitrary rational representations $U_1$ and $U_2$ of $\mathbb{G}$ [9, Lemma 3.4]. We note that for an equioriented quiver $Q = \{1 \to 2 \to \cdots \to n\}$, the two notions of quiver coefficients also coincide without modifying the quiver. In fact, an equioriented coefficient $c_{(\mu_1,\ldots,\mu_n)}(\Omega)$ is non-zero only if $\mu_1$ is the empty partition, in which case we have $c_{(\emptyset,\mu_2,\ldots,\mu_n)}(\Omega) = \widetilde{c}_{(\mu_2,\ldots,\mu_n,\emptyset)}(\Omega)$. On the other hand, for quivers that are not equioriented, it appears to be difficult to relate the properties of quiver coefficients and dual quiver coefficients of the same quiver cycle. For the simplest example, the reader is invited to compare the formulas for inbound and outbound $A_3$-quivers proved in section 7.

It is convenient to encode the quiver coefficients for $\Omega$ as a linear combination of tensors,

$$P_\Omega = \sum_\mu c_\mu(\Omega)\, \mathcal{G}_{\mu_1} \otimes \mathcal{G}_{\mu_2} \otimes \cdots \otimes \mathcal{G}_{\mu_n}\,. \tag{10}$$

If Conjecture 1.1 (a) is true, then this is an element of the tensor power $\Gamma^{\otimes n}$ of the ring of stable Grothendieck polynomials $\Gamma$; otherwise $P_\Omega$ lives in a completion of this ring. We will use the notation that for any linear combination $P = \sum_\mu c_\mu\, \mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_n}$ and classes $\alpha_1, \ldots, \alpha_n \in \mathcal{R}(\mathbb{G})$, we set $P(\alpha_1, \ldots, \alpha_n) = \sum_\mu c_\mu\, \mathcal{G}_{\mu_1}(\alpha_1) \cdots \mathcal{G}_{\mu_n}(\alpha_n)$. The definition of quiver coefficients then states that $[\mathcal{O}_\Omega] = P_\Omega(E_1 - M_1, \ldots, E_n - M_n) \in \mathcal{R}(\mathbb{G})$.

In addition to the evidence for Conjecture 1.1 mentioned above, we have used Macaulay 2 [23] and other software to compute the quiver coefficients of many quiver cycles, including some that are not orbit closures (and not of Dynkin type). In almost all cases where Macaulay 2 was able to produce a free resolution of the coordinate ring of a quiver cycle, we could convert the corresponding expression for its Grothendieck class into a *finite* linear combination of products of Grothendieck polynomials as in Definition 3.2. In a few cases we did not succeed in this, but expect that this was caused by lack of computing power. We have never encountered any negative cohomological quiver coefficients; and when the general quiver coefficients failed to have alternating signs, we could often show that the corresponding quiver

cycle did not have rational singularities, for example by using Brion's theorem described in the introduction [5].

**Example 3.4.** Let $Q = \{1 \rightrightarrows 2\}$ be the Kronecker quiver and fix the dimension vector $e = (3,3)$. Let $\Omega \subset V$ be the closure of the orbit through the point

$$\left( \left[\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] \right) .$$

Zwara has shown in [35] that this orbit closure has ugly singularities, in particular they are not rational. With help from Macaulay 2 [23], we have determined the quiver coefficients for $\Omega$. There are finitely many of them, and they are encoded in the following expression $P_\Omega$ satisfying that $P_\Omega(E_1\,, E_2 - E_1 \oplus E_1) = [\mathcal{O}_\Omega]$:

$$\begin{aligned}
P_\Omega = {} & 3 \otimes \mathcal{G}_{3,1} + 4\,\mathcal{G}_1 \otimes \mathcal{G}_3 + 1 \otimes \mathcal{G}_{2,2} + 2\,\mathcal{G}_1 \otimes \mathcal{G}_{2,1} + 3\,\mathcal{G}_2 \otimes \mathcal{G}_2 + \mathcal{G}_2 \otimes \mathcal{G}_{1,1} \\
& + 2\,\mathcal{G}_3 \otimes \mathcal{G}_1 + \mathcal{G}_4 \otimes 1 \\
& - 3 \otimes \mathcal{G}_{3,2} - 8\,\mathcal{G}_1 \otimes \mathcal{G}_{3,1} - 6\,\mathcal{G}_2 \otimes \mathcal{G}_3 - 2\,\mathcal{G}_1 \otimes \mathcal{G}_{2,2} - 5\,\mathcal{G}_2 \otimes \mathcal{G}_{2,1} - 4\,\mathcal{G}_3 \otimes \mathcal{G}_2 \\
& - 2\,\mathcal{G}_3 \otimes \mathcal{G}_{1,1} - 2\,\mathcal{G}_4 \otimes \mathcal{G}_1 \\
& - 1 \otimes \mathcal{G}_{4,2} - 3 \otimes \mathcal{G}_{4,1,1} - 6\,\mathcal{G}_1 \otimes \mathcal{G}_{4,1} - 3\,\mathcal{G}_2 \otimes \mathcal{G}_4 - 6\,\mathcal{G}_{1,1} \otimes \mathcal{G}_4 + 4\,\mathcal{G}_1 \otimes \mathcal{G}_{3,2} \\
& + 7\,\mathcal{G}_2 \otimes \mathcal{G}_{3,1} + 2\,\mathcal{G}_3 \otimes \mathcal{G}_3 + \mathcal{G}_2 \otimes \mathcal{G}_{2,2} + 4\,\mathcal{G}_3 \otimes \mathcal{G}_{2,1} + \mathcal{G}_4 \otimes \mathcal{G}_2 + \mathcal{G}_4 \otimes \mathcal{G}_{1,1} \\
& + 1 \otimes \mathcal{G}_{4,3} + 5 \otimes \mathcal{G}_{4,2,1} + 10\,\mathcal{G}_1 \otimes \mathcal{G}_{4,2} + 10\,\mathcal{G}_1 \otimes \mathcal{G}_{4,1,1} + 14\,\mathcal{G}_2 \otimes \mathcal{G}_{4,1} \\
& + 15\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,1} + 4\,\mathcal{G}_3 \otimes \mathcal{G}_4 + 12\,\mathcal{G}_{2,1} \otimes \mathcal{G}_4 - \mathcal{G}_2 \otimes \mathcal{G}_{3,2} \\
& - 2\,\mathcal{G}_3 \otimes \mathcal{G}_{3,1} - \mathcal{G}_4 \otimes \mathcal{G}_{2,1} \\
& - 2 \otimes \mathcal{G}_{4,3,1} - 4\,\mathcal{G}_1 \otimes \mathcal{G}_{4,3} - 1 \otimes \mathcal{G}_{4,2,2} - 16\,\mathcal{G}_1 \otimes \mathcal{G}_{4,2,1} - 16\,\mathcal{G}_2 \otimes \mathcal{G}_{4,2} \\
& - 12\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,2} - 12\,\mathcal{G}_2 \otimes \mathcal{G}_{4,1,1} - 10\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,1,1} - 10\,\mathcal{G}_3 \otimes \mathcal{G}_{4,1} \\
& - 29\,\mathcal{G}_{2,1} \otimes \mathcal{G}_{4,1} - \mathcal{G}_4 \otimes \mathcal{G}_4 - 7\,\mathcal{G}_{3,1} \otimes \mathcal{G}_4 - 3\,\mathcal{G}_{2,2} \otimes \mathcal{G}_4 \\
& + 1 \otimes \mathcal{G}_{4,3,2} + 6\,\mathcal{G}_1 \otimes \mathcal{G}_{4,3,1} + 5\,\mathcal{G}_2 \otimes \mathcal{G}_{4,3} + 3\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,3} + 2\,\mathcal{G}_1 \otimes \mathcal{G}_{4,2,2} \\
& + 18\,\mathcal{G}_2 \otimes \mathcal{G}_{4,2,1} + 14\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,2,1} + 8\,\mathcal{G}_3 \otimes \mathcal{G}_{4,2} + 22\,\mathcal{G}_{2,1} \otimes \mathcal{G}_{4,2} \\
& + 6\,\mathcal{G}_3 \otimes \mathcal{G}_{4,1,1} + 18\,\mathcal{G}_{2,1} \otimes \mathcal{G}_{4,1,1} + 2\,\mathcal{G}_4 \otimes \mathcal{G}_{4,1} + 16\,\mathcal{G}_{3,1} \otimes \mathcal{G}_{4,1} \\
& + 6\,\mathcal{G}_{2,2} \otimes \mathcal{G}_{4,1} + \mathcal{G}_{4,1} \otimes \mathcal{G}_4 + 3\,\mathcal{G}_{3,2} \otimes \mathcal{G}_4 \\
& - 2\,\mathcal{G}_1 \otimes \mathcal{G}_{4,3,2} - 6\,\mathcal{G}_2 \otimes \mathcal{G}_{4,3,1} - 4\,\mathcal{G}_{1,1} \otimes \mathcal{G}_{4,3,1} - 2\,\mathcal{G}_3 \otimes \mathcal{G}_{4,3} - 5\,\mathcal{G}_{2,1} \otimes \mathcal{G}_{4,3} \\
& - \mathcal{G}_2 \otimes \mathcal{G}_{4,2,2} - \mathcal{G}_{1,1} \otimes \mathcal{G}_{4,2,2} - 8\,\mathcal{G}_3 \otimes \mathcal{G}_{4,2,1} - 24\,\mathcal{G}_{2,1} \otimes \mathcal{G}_{4,2,1} - \mathcal{G}_4 \otimes \mathcal{G}_{4,2} \\
& - 11\,\mathcal{G}_{3,1} \otimes \mathcal{G}_{4,2} - 3\,\mathcal{G}_{2,2} \otimes \mathcal{G}_{4,2} - \mathcal{G}_4 \otimes \mathcal{G}_{4,1,1} - 9\,\mathcal{G}_{3,1} \otimes \mathcal{G}_{4,1,1} \\
& - 3\,\mathcal{G}_{2,2} \otimes \mathcal{G}_{4,1,1} - 2\,\mathcal{G}_{4,1} \otimes \mathcal{G}_{4,1} - 6\,\mathcal{G}_{3,2} \otimes \mathcal{G}_{4,1} \\
& + \mathcal{G}_2 \otimes \mathcal{G}_{4,3,2} + \mathcal{G}_{1,1} \otimes \mathcal{G}_{4,3,2} + 2\,\mathcal{G}_3 \otimes \mathcal{G}_{4,3,1} + 6\,\mathcal{G}_{2,1} \otimes \mathcal{G}_{4,3,1} + 2\,\mathcal{G}_{3,1} \otimes \mathcal{G}_{4,3} \\
& + \mathcal{G}_{2,1} \otimes \mathcal{G}_{4,2,2} + \mathcal{G}_4 \otimes \mathcal{G}_{4,2,1} + 11\,\mathcal{G}_{3,1} \otimes \mathcal{G}_{4,2,1} + 3\,\mathcal{G}_{2,2} \otimes \mathcal{G}_{4,2,1} \\
& + \mathcal{G}_{4,1} \otimes \mathcal{G}_{4,2} + 3\,\mathcal{G}_{3,2} \otimes \mathcal{G}_{4,2} + \mathcal{G}_{4,1} \otimes \mathcal{G}_{4,1,1} + 3\,\mathcal{G}_{3,2} \otimes \mathcal{G}_{4,1,1} \\
& - \mathcal{G}_{2,1} \otimes \mathcal{G}_{4,3,2} - 2\,\mathcal{G}_{3,1} \otimes \mathcal{G}_{4,3,1} - \mathcal{G}_{4,1} \otimes \mathcal{G}_{4,2,1} - 3\,\mathcal{G}_{3,2} \otimes \mathcal{G}_{4,2,1}
\end{aligned}$$

We note that while this expression fails to have alternating signs, the signs are still periodic in a curious way. In fact, the terms $\mathcal{G}_\lambda \otimes \mathcal{G}_\nu$ of $P_\Omega$ displayed above are sorted according to the lexicographic order on the partitions, with $\nu$ taking precedence over $\lambda$, which makes the periodicity readily visible. Furthermore, starting from the degree 8 term, the signs of the quiver coefficients are the opposite of the expected. We have also observed this phenomenon for other quiver cycles without rational singularities, but have no explanation for it.

Our calculation also shows that $\Omega$ is the cone over a subvariety of $\mathbb{P}^{17}$ with Grothendieck class equal to

$$51\,h^4 - 132\,h^5 + 70\,h^6 + 144\,h^7 - 261\,h^8 + 184\,h^9 - 66\,h^{10} + 12\,h^{11} - h^{12}$$

where $h$ is the class of a hyperplane. Using Brion's result [5], this gives an alternative proof that $\Omega$ lacks rational singularities.

Finally, if the cohomology class of $\Omega$ is expressed in the basis of products $s_{\mu_1}(E_1)\, s_{\mu_2}(E_2 - E_1)$, then we obtain

$$\begin{aligned}[\Omega] = {} & 3s_{3,1}(E_2-E_1) + s_1(E_1)s_3(E_2-E_1) + s_{2,2}(E_2-E_1) - 2s_1(E_1)s_{2,1}(E_2-E_1) \\ & - 2s_{1,1}(E_1)s_2(E_2-E_1) + s_{1,1}(E_1)s_{1,1}(E_2-E_1) + 3s_{1,1,1}(E_1)s_1(E_2-E_1)\,.\end{aligned}$$

This illustrates that our choice of basis is essential to the positivity conjecture. It is also essential to the finiteness conjecture, since in general it requires an infinite linear combination of products $\mathcal{G}_{\mu_1}(E_1)\mathcal{G}_{\mu_2}(E_2-E_1)$ to express a class $\mathcal{G}_\lambda(E_2 - E_1 \oplus E_1)$.

## 4. Degeneracy loci

This section interprets quiver coefficients as formulas for degeneracy loci defined by quivers of vector bundles over a base variety. We start by summarizing some facts about equivariant $K$-theory of schemes based on Thomason's paper [33].

4.1. **$K$-theory.** Let $G$ be an algebraic group over the field $\mathbb{K}$ and let $X$ be an algebraic $G$-scheme over $\mathbb{K}$. A *$G$-equivariant sheaf* on $X$ is a coherent $\mathcal{O}_X$-module $\mathcal{F}$ together with a given isomorphism $I : a^*\mathcal{F} \cong p_2^*\mathcal{F}$, where $a : G \times X \to X$ is the action and $p_2 : G \times X \to X$ is the projection. This isomorphism must satisfy that $(m \times \mathrm{id}_X)^* I = p_{23}^* I \circ (\mathrm{id}_G \times a)^* I$ as morphisms of sheaves on $G \times G \times X$, where $m$ is the group operation on $G$ and $p_{23}$ is the projection to the last two factors of $G \times G \times X$. A *$G$-equivariant vector bundle* on $X$ is a locally free $G$-equivariant sheaf of constant rank.

The *$G$-equivariant $K$-homology* of $X$ is the Grothendieck group $K_G(X)$ generated by isomorphism classes of $G$-equivariant sheaves, modulo relations saying that $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ if there exists a $G$-equivariant short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. The *$G$-equivariant $K$-cohomology* of $X$ is the Grothendieck ring $K^G(X)$ of $G$-equivariant vector bundles. The group $K_G(X)$ is a module over the ring $K^G(X)$; both the ring structure of $K^G(X)$ and its action on $K_G(X)$ are defined by tensor products. If $X$ is a non-singular variety and $G$ is a linear algebraic group, then the implicit map $K^G(X) \to K_G(X)$ that sends an equivariant vector bundle to its sheaf of sections is an isomorphism [33, Thm. 1.8]. The equivariant $K$-theory of a point is the ring $K^G(\mathrm{point}) = \mathcal{R}(G)$ of virtual representations of $G$. Any $G$-equivariant map $f : X \to Y$ defines a ring homomorphism $f^* : K^G(Y) \to K^G(X)$ given by pullback of vector bundles. If $f$ is flat then it also defines a pullback map $f^* : K_G(Y) \to K_G(X)$ on Grothendieck groups. The same is true if $f$ is a regular embedding, in which case the pullback is given by $f^*[\mathcal{F}] = \sum_{i\geq 0}(-1)^i[\mathrm{Tor}_i^Y(\mathcal{O}_X, \mathcal{F})]$. A proper equivariant map $f : X \to Y$ defines a pushforward map $f_* : K_G(X) \to K_G(Y)$ given by $f_*[\mathcal{F}] = \sum_{i\geq 0}(-1)^i[R^i f_*\mathcal{F}]$. This pushforward map is a homomorphism of $K^G(Y)$-modules by the projection formula. If $\pi : E \to X$ is (the total space of) a $G$-equivariant vector bundle, then $\pi^* : K_G(X) \to K_G(E)$ is an isomorphism [33, Thm. 1.7], and we will identify $K_G(E)$ with $K_G(X)$ using this map. The inverse map is pullback along any equivariant section $X \to E$. When $G = \{e\}$ is the trivial group, we will use the notation $K^\circ(X) = K^{\{e\}}(X)$ and $K_\circ(X) = K_{\{e\}}(X)$ for the ordinary $K$-theory groups of $X$.

Stable Grothendieck polynomials can be used to define $K$-theory classes as follows. Given a vector bundle over $X$ which can be written as a direct sum of line bundles $\mathcal{E} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$, and a partition $\nu$, we define

$$\mathcal{G}_\nu(\mathcal{E}) \;=\; \mathcal{G}_\nu(1 - \mathcal{L}_1^{-1}, \ldots, 1 - \mathcal{L}_r^{-1}) \;\in K^\circ(X)\,. \tag{11}$$

The symmetry of $\mathcal{G}_\nu$ implies that this class is a polynomial in the exterior powers of the dual bundle $\mathcal{E}^\vee$, so it is well defined even when $\mathcal{E}$ is not a direct sum of line bundles. Furthermore, if $X$ is a $G$-scheme and $\mathcal{E}$ is a $G$-equivariant vector bundle, then (11) defines a class $\mathcal{G}_\nu(\mathcal{E}) \in K^G(X)$. Given two $G$-vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ we define

$$\mathcal{G}_\nu(\mathcal{E}_1 - \mathcal{E}_2) \;=\; \sum_{\lambda,\mu} d^\nu_{\lambda\mu}\, \mathcal{G}_\lambda(\mathcal{E}_1)\, \mathcal{G}_{\mu'}(\mathcal{E}_2^\vee) \;\in K^G(X)\,. \tag{12}$$

This extends (9). The linear map $\Gamma \to K^G(X)$ given by $\mathcal{G}_\nu \mapsto \mathcal{G}_\nu(\mathcal{E}_1 - \mathcal{E}_2)$ is a ring homomorphism. The identity (2) implies that $\mathcal{G}_\nu(\mathcal{E}_1 \oplus \mathcal{E}_3 - \mathcal{E}_2 \oplus \mathcal{E}_3) = \mathcal{G}_\nu(\mathcal{E}_1 - \mathcal{E}_2)$ for any third $G$-vector bundle $\mathcal{E}_3$. Equivalently, the stable Grothendieck polynomial for $\nu$ defines a linear operator $\mathcal{G}_\nu : K^G(X) \to K^G(X)$. Equation (3) implies that $\mathcal{G}_\nu(\alpha+\beta) = \sum_{\lambda,\mu} d^\nu_{\lambda\mu} \mathcal{G}_\lambda(\alpha)\mathcal{G}_\mu(\beta)$ for all classes $\alpha, \beta \in K^G(X)$.

4.2. **Interpretations of Grothendieck classes.** Assume that $G$ is a connected reductive linear algebraic group containing a $\mathbb{K}$-split maximal torus $T \subset G$, i.e. $T \cong (\mathbb{G}_m)^r$ is defined over $\mathbb{K}$. Let $V$ be a rational representation of $G$ and let $\Omega \subset V$ be a $G$-stable closed subvariety. Then the structure sheaf $\mathcal{O}_\Omega$ is a $G$-equivariant sheaf on $V$, so it defines a class $[\mathcal{O}_\Omega] \in K_G(V)$. If we use the fact that $V$ is an equivariant vector bundle over a point to identify $K_G(V)$ with $\mathcal{R}(G)$, then this class agrees with the Grothendieck class of $\Omega$ defined in Section 3.1.

Let $X$ be an algebraic scheme equipped with a principal $G$-bundle $P \to X$, i.e. $G$ acts freely on $P$ and $X$ equals $P/G$ as a geometric quotient [30]. For a $G$-variety $Y$ we write $Y_G = P \times^G Y = (P \times Y)/G$. We will use this notation only when $Y$ is equivariantly embedded as a closed subvariety of a non-singular variety, in which case it follows from [17, Prop. 23] that $Y_G$ is defined as a scheme. Using that the category of $G$-equivariant sheaves on $P$ is equivalent to the category of coherent $\mathcal{O}_X$-modules [4, Thm. 6.1.4], it follows that $V_G$ is a vector bundle over $X$ with fibers isomorphic to $V$ [17, Lemma 1], and the closed subscheme $\Omega_G \subset V_G$ is a *translated degeneracy locus*, consisting of one copy of $\Omega$ in each fiber. It's structure sheaf defines a Grothendieck class $[\mathcal{O}_{\Omega_G}] \in K_\circ(V_G) = K_\circ(X)$.

More generally, let $H$ be a second algebraic group over $\mathbb{K}$, and assume that $P$ and $X$ are $H$-schemes so that the map $P \to X$ is equivariant and the $H$-action on $P$ commutes with the $G$-action. In this case $V_G$ is an $H$-vector bundle over $X$, and $\Omega_G$ defines an equivariant class $[\mathcal{O}_{\Omega_G}] \in K_H(V_G) = K_H(X)$. Let $\phi_G : \mathcal{R}(G) \to K^H(X)$ be the ring homomorphism defined by $\phi_G(U) = [U_G]$ for any rational $G$-representation $U$. The following lemma interprets the Grothendieck class $[\mathcal{O}_\Omega] \in \mathcal{R}(G)$ as a formula for degeneracy loci.

**Proposition 4.1.** *The $H$-equivariant Grothendieck class of $\Omega_G \subset V_G$ is given by $[\mathcal{O}_{\Omega_G}] = \varphi_G([\mathcal{O}_\Omega]) \in K_H(X)$.*

*Proof.* A finitely generated free $(\mathbb{K}[V], G)$-module $F$ corresponds to a $G$-equivariant vector bundle $\widetilde{F} = \mathrm{Spec}(\mathrm{Sym}^\bullet F^\vee)$ over $V$, which in turn defines the $H$-equivariant vector bundle $\widetilde{F}_G = P \times^G \widetilde{F}$ on $V_G$ [17, Lemma 1]. This construction applied to (8) produces an exact sequence

$$0 \to (\widetilde{F}_r)_G \to (\widetilde{F}_{r-1})_G \to \cdots \to (\widetilde{F}_0)_G \to \mathcal{O}_{\Omega_G} \to 0$$

of $H$-equivariant coherent sheaves on $V_G$. Let $s : X \to V_G$ be the zero section. Since the fiber of $\widetilde{F}_i$ over the origin of $V$ equals $F_i/\mathfrak{m}F_i$, it follows that $s^*(\widetilde{F}_i)_G =$

$(F_i/\mathfrak{m}F_i)_G$. We conclude that

$$[\mathcal{O}_{\Omega_G}] = \sum_{i\geq 0}(-1)^i\, s^*[(\widetilde{F}_i)_G] = \sum_{i\geq 0}(-1)^i[(F_i/\mathfrak{m}F_i)_G] = \varphi_G([\mathcal{O}_\Omega])$$

in $K_H(X)$, as required. □

Write $T = (\mathbb{G}_m)^r$ as a product of multiplicative groups, and define one-dimensional $T$-representations $L_1, \dots, L_r$ by $L_i = \mathbb{K}$ and $(t_1, \dots, t_r).v = t_i v$ for $v \in L_i$. Then we have $\mathcal{R}(T) = \mathbb{Z}[L_1^{\pm 1}, \dots, L_r^{\pm 1}] \subset \mathbb{Z}[\![x_1, \dots, x_r]\!]$ where $x_i = 1 - L_i^{-1}$. Since $\mathcal{R}(G) \subset \mathcal{R}(T)$ by [24, Cor. II.2.7], we may regard the class $[\mathcal{O}_\Omega]$ as a power series.

The variety $\Omega \subset V$ also defines a class $[\Omega]$ in the equivariant Chow cohomology ring $H_T^*(V)$. If we abuse notation and write $x_i$ also for the Chern root $c_1(L_i) \in H_T^*(\text{point}) = H_T^*(V)$, then this ring is the polynomial ring $H_T^*(V) = \mathbb{Z}[x_1, \dots, x_r]$ by [34, §15], and the class $[\Omega]$ coincides with the term of total degree $d = \operatorname{codim}(\Omega; V)$ in the power series $[\mathcal{O}_\Omega]$. To see this, we need Totaro's algebraic approximation of the classifying space for $T$ [34]. Set $P = \prod_{i=1}^r \left(L_i^{\oplus d+1} \smallsetminus \{0\}\right)$ and $X = P/T = \prod_{i=1}^r \mathbb{P}^d$. Then $H_T^i(V) = H^i(V_T) = H^i(X)$ for $i \leq d$ by [34, Thm. 1.1] or [17, Prop. 4], where $V_T = P \times^T V$ and $x_i \in H_T^i(V)$ corresponds to a hyperplane class in the $i$-th factor of $X$. The cohomology class of $\Omega$ is defined by $[\Omega] := [\Omega_T] \in H^d(V_T)$. Let $\operatorname{ch} : K^\circ(V_T) \to H^*(V_T) \otimes \mathbb{Q}$ be the Chern character, i.e. the ring homomorphism defined formally by $\operatorname{ch}(\mathcal{L}) = \exp(c_1(\mathcal{L}))$ for any line bundle $\mathcal{L}$ on $V_T$ [20, Ex. 3.2.3]. Then we have $\operatorname{ch}(\varphi_T(x_i)) = 1 - \exp(-x_i)$, so the lowest term of $[\mathcal{O}_\Omega]$ agrees with the lowest term of $\operatorname{ch}(\varphi_T([\mathcal{O}_\Omega]))$. Now Proposition 4.1 and [20, Ex. 15.2.16] imply that $\operatorname{ch}(\varphi_T([\mathcal{O}_\Omega])) = \operatorname{ch}([\mathcal{O}_{\Omega_T}]) = [\Omega_T] + \text{higher terms}$. This shows that $[\Omega]$ is the lowest term in $[\mathcal{O}_\Omega]$, and also that $[\mathcal{O}_\Omega]$ has no non-zero terms of degree smaller than $\operatorname{codim}(\Omega; V)$.

We finally prove that the Grothendieck class of $\Omega$ is uniquely determined by the formula it provides in ordinary $K$-theory.

**Proposition 4.2.** *The equivariant Grothendieck class of $\Omega$ is the unique virtual representation $[\mathcal{O}_\Omega] \in \mathcal{R}(G)$ for which $[\mathcal{O}_{\Omega_G}] = \varphi_G([\mathcal{O}_\Omega]) \in K_\circ(X)$ for every non-singular variety $X$ and principal $G$-bundle $P \to X$.*

*Proof.* In view of Proposition 4.1, it is enough to show that if $\alpha \neq 0 \in \mathcal{R}(G)$, then for some principal $G$-bundle $P \to X$ with $X$ non-singular we have $\varphi_G(\alpha) \neq 0 \in K_\circ(X)$.

Let $d$ be the degree of the lowest non-zero term of $\alpha \in \mathbb{Z}[\![x_1, \dots, x_r]\!]$. As in [17, Lemma 9] we embed $G$ in $\mathrm{GL}(m)$ for some $m$ and let $P$ be the set of all $m \times (m+d)$ matrices of full rank. Then $G$ acts freely on $P$, the quotients $X = P/G$ and $P/T$ are non-singular varieties, and since $P$ has codimension $d+1$ in the vector space of all $m \times (m+d)$ matrices, it follows from [34, Thm. 1.1] or [17, Prop. 4] that $H^i(P/T) = H_T^i(V)$ for $i \leq d$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{R}(G) & \longrightarrow & \mathcal{R}(T) & & \\ \downarrow{\scriptstyle \varphi_G} & & \downarrow{\scriptstyle \varphi_T} & & \\ K^\circ(X) & \longrightarrow & K^\circ(P/T) & \xrightarrow{\operatorname{ch}} & H^*(P/T) \otimes \mathbb{Q} \end{array}$$

where the bottom-left map is pullback along $P/T \to P/G = X$. Since the image of $\alpha$ in $H^d(P/T) \otimes \mathbb{Q} = H_T^d(V) \otimes \mathbb{Q}$ is non-zero, we conclude that $\varphi_G(\alpha) \in K^\circ(X) = K_\circ(X)$ is non-zero as well. □

4.3. **Degeneracy loci defined by quiver cycles.** Let $V$ and $\mathbb{G}$ be as in section 3.2, and let $\Omega \subset V$ be a quiver cycle. We will use the constructions given above to interpret the quiver coefficients of $\Omega$ in terms of formulas for degeneracy loci. Let $X$ be an algebraic scheme over $\mathbb{K}$ equipped with vector bundles $\mathcal{E}_1, \ldots, \mathcal{E}_n$ of ranks given by the dimension vector $e = (e_1, \ldots, e_n)$. Define the bundle $\mathcal{V} = \bigoplus_{a \in Q_1} \operatorname{Hom}(\mathcal{E}_{t(a)}, \mathcal{E}_{h(a)})$ over $X$. Since the fibers of $\mathcal{V}$ are isomorphic to the representation space $V$, any quiver cycle $\Omega \subset V$ defines a translated degeneracy locus $\widetilde{\Omega} \subset \mathcal{V}$. To be precise, let $\pi : P \to X$ be the principal $\mathbb{G}$-bundle such that $\mathcal{E}_i = (E_i)_{\mathbb{G}} = P \times^{\mathbb{G}} E_i$ for each $i$. This bundle can be constructed as a multi-frame bundle $P \subset \mathcal{E}_1^{\oplus e_1} \oplus \cdots \oplus \mathcal{E}_n^{\oplus e_n}$, with fibers $\pi^{-1}(x)$ consisting of lists of bases of the fibers $\mathcal{E}_i(x)$. Then we have $\mathcal{V} = V_{\mathbb{G}}$ and $\widetilde{\Omega} = \Omega_{\mathbb{G}} \subset \mathcal{V}$.

**Corollary 4.3.** *The Grothendieck class of the translated degeneracy locus $\widetilde{\Omega} \subset \mathcal{V}$ is given by*

$$[\mathcal{O}_{\widetilde{\Omega}}] = \sum_{\mu} c_\mu(\Omega)\, \mathcal{G}_{\mu_1}(\mathcal{E}_1 - \mathcal{M}_1) \cdots \mathcal{G}_{\mu_n}(\mathcal{E}_n - \mathcal{M}_n) \ \in K_\circ(\mathcal{V})\,,$$

*where $\mathcal{M}_i = \bigoplus_{a:h(a)=i} \mathcal{E}_{t(a)} = P \times^{\mathbb{G}} M_i$. Furthermore, the quiver coefficients for $\Omega$ are uniquely determined by the truth of this identity for all non-singular varieties $X$ and vector bundles $\mathcal{E}_1, \ldots, \mathcal{E}_n$.*

*Proof.* This follows from Proposition 4.2 and the definition of quiver coefficients, since $\varphi_{\mathbb{G}}(\mathcal{G}_{\mu_i}(E_i - M_i)) = \mathcal{G}_{\mu_i}(\mathcal{E}_i - \mathcal{M}_i)$. □

Define a *representation* $\mathcal{E}_\bullet$ of $Q$ on the vector bundles $\mathcal{E}_1, \ldots, \mathcal{E}_n$ over $X$ to be a collection of bundle maps $\mathcal{E}_{t(a)} \to \mathcal{E}_{h(a)}$ corresponding to the arrows $a \in Q_1$. Such a representation defines a section $s : X \to \mathcal{V}$. We define the degeneracy locus $\Omega(\mathcal{E}_\bullet)$ as the scheme-theoretic inverse image $\Omega(\mathcal{E}_\bullet) = s^{-1}(\widetilde{\Omega}) \subset X$. This degeneracy locus consists of all points in $X$ over which the bundle maps of $\mathcal{E}_\bullet$ degenerate to representations in $\Omega$. For example, if $\widetilde{\mathcal{E}}_\bullet$ denotes the tautological representation of $Q$ over $\mathcal{V}$, defined by the universal maps between the pullbacks of the vector bundles $\mathcal{E}_i$ to $\mathcal{V}$, then $\widetilde{\Omega} = \Omega(\widetilde{\mathcal{E}}_\bullet)$.

Assume that $X$ has an action of an algebraic group $H$ over $\mathbb{K}$ and the representation $\mathcal{E}_\bullet$ consists of $H$-equivariant vector bundles and bundle maps. Then $P$ has a commuting $H$-action as in section 4.2 and $\mathcal{V}$ is an $H$-vector bundle, so it follows from Proposition 4.1 that the identity of Corollary 4.3 holds in $K_H(\mathcal{V})$. It also follows that $s : X \to \mathcal{V}$ is an equivariant section.

We can define a localized class $\mathbf{\Omega}(\mathcal{E}_\bullet)$ in $K_H(\Omega(\mathcal{E}_\bullet))$ by

$$\mathbf{\Omega}(\mathcal{E}_\bullet) = s^!([\mathcal{O}_{\widetilde{\Omega}}]) = \sum_{j \geq 0} (-1)^j [\operatorname{Tor}_j^{\mathcal{V}}(\mathcal{O}_X, \mathcal{O}_{\widetilde{\Omega}})]\,.$$

This definition is compatible with ($H$-equivariant) flat or regular pullback and proper pushforward [21], and the image of $\mathbf{\Omega}(\mathcal{E}_\bullet)$ in $K_H(X)$ is given by

$$\mathbf{\Omega}(\mathcal{E}_\bullet) = s^*[\mathcal{O}_{\widetilde{\Omega}}] = \sum_{\mu} c_\mu(\Omega)\, \mathcal{G}_{\mu_1}(\mathcal{E}_1 - \mathcal{M}_1) \cdots \mathcal{G}_{\mu_n}(\mathcal{E}_n - \mathcal{M}_n)\,.$$

Furthermore, if $X$ and $\Omega$ are Cohen-Macaulay and the codimension of $\Omega(\mathcal{E}_\bullet)$ in $X$ is equal to the codimension of $\Omega$ in $V$, then we have $\mathbf{\Omega}(\mathcal{E}_\bullet) = [\mathcal{O}_{\Omega(\mathcal{E}_\bullet)}] \in K_H(\Omega(\mathcal{E}_\bullet))$. This is true because a local regular sequence generating the ideal of $X$ in $\mathcal{V}$ restricts to a local regular sequence defining the ideal of $\Omega(\mathcal{E}_\bullet)$ in $\widetilde{\Omega}$ [20, Lemma A.7.1].

This implies that $\mathrm{Tor}_j^{\mathcal{V}}(\mathcal{O}_X, \mathcal{O}_{\widetilde{\Omega}}) = 0$ for all $j > 0$, so $\mathbf{\Omega}(\mathcal{E}_\bullet) = [\mathcal{O}_X \otimes_{\mathcal{O}_\mathcal{V}} \mathcal{O}_{\widetilde{\Omega}}] = [\mathcal{O}_{\Omega(\mathcal{E}_\bullet)}]$. We note that if $Q$ is a Dynkin quiver of type A or D and $\mathbb{K}$ is algebraically closed, then any orbit closure $\Omega \subset V$ is Cohen-Macaulay [27, 1, 2]. The following corollary generalize all the above formulas involving quiver coefficients, including Definition 3.2.

**Corollary 4.4.** *Let $\mathcal{E}_\bullet$ be a representation of $Q$ consisting of $H$-equivariant vector bundles and bundle maps over $X$. Assume that both $X$ and $\Omega$ are Cohen-Macaulay and that the codimension of $\Omega(\mathcal{E}_\bullet)$ in $X$ is equal to the codimension of $\Omega$ in $V$. Then we have*

$$[\mathcal{O}_{\Omega(\mathcal{E}_\bullet)}] = \sum_\mu c_\mu(\Omega)\, \mathcal{G}_{\mu_1}(\mathcal{E}_1 - \mathcal{M}_1) \cdots \mathcal{G}_{\mu_n}(\mathcal{E}_n - \mathcal{M}_n) \in K_H(X)\,.$$

Let $X$ be a non-singular variety. Subject to mild conditions, corollaries 4.3 and 4.4 have cohomological analogues. For a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$ and vector bundles $\mathcal{A}$ and $\mathcal{B}$ over $X$, define $s_\lambda(\mathcal{A} - \mathcal{B}) = \det(h_{\lambda_i + j - i})_{l \times l} \in H^*(X)$, where the classes $h_i$ are defined by $\sum_{i \geq 0} h_i = c(\mathcal{B}^\vee)/c(\mathcal{A}^\vee)$, and $c(\mathcal{A}^\vee) = 1 - c_1(\mathcal{A}) + c_2(\mathcal{A}) - \cdots$ is the total Chern class of $\mathcal{A}^\vee$.

**Corollary 4.5.** *If $X$ admits an ample line bundle or if $Q$ is a quiver of Dynkin type, then the Chow class of the translated degeneracy locus $\widetilde{\Omega} \subset \mathcal{V}$ is given by*

$$[\widetilde{\Omega}] = \sum_{\sum |\mu_i| = \mathrm{codim}(\Omega)} c_\mu(\Omega)\, s_{\mu_1}(\mathcal{E}_1 - \mathcal{M}_1) \cdots s_{\mu_n}(\mathcal{E}_n - \mathcal{M}_n) \cap [\mathcal{V}] \ \in H_*(\mathcal{V})\,.$$

*Without these conditions, this identity holds in $H^*(\mathcal{V}) \otimes \mathbb{Q}$.*

If $X$ has an ample line bundle, then one can deduce this statement from the expression for $[\Omega] \in H^*_{\mathbb{G}}(V)$ along the lines of [15, §2.5], and if $Q$ is of Dynkin type, then one can replace Grothendieck polynomials with Schur polynomials in the proof of the formula for quiver coefficients given in section 6. The formula with rational coefficients follows from Corollary 4.3 by using the Chern character [20, Ex. 15.2.16]. If $H$ is a linear algebraic group, then a cohomological analogue of Corollary 4.4 can be proved from Corollary 4.5 by first replacing $X$ with the Borel construction $P \times^H X$, where $P/H$ is an algebraic approximation of the classifying space of $H$ [34, 17], and then applying [20, Prop. 7.1]. We leave the details to the reader. We expect that Corollary 4.5 is true without the assumptions, but have not found a proof.

## 5. Resolution of singularities

Our formula for quiver coefficients of Dynkin type is based on Reineke's resolution of the singularities of orbit closures for Dynkin quivers [31]. It will be convenient to formulate Reineke's construction for an arbitrary quiver $Q$, together with a representation of $Q$ on vector bundles over a base scheme $X$.

Let $X$ be an algebraic scheme over $\mathbb{K}$ equipped with a representation $\mathcal{E}_\bullet$ of $Q$ on vector bundles over $X$, with $\mathrm{rank}(\mathcal{E}_i) = e_i$. Let $i \in Q_0$ be a quiver vertex and let $r$ be an integer with $1 \leq r \leq e_i$. Let $\rho : Y = \mathrm{Gr}(e_i - r, \mathcal{E}_i) \to X$ be the Grassmann bundle of rank $r$ quotients of $\mathcal{E}_i$, with universal exact sequence $0 \to \mathcal{S} \to \mathcal{E}_i \to \mathcal{Q} \to 0$. (We will avoid explicit notation for pullback of vector bundles.) We define the scheme $X_{i,r} = X_{i,r}(\mathcal{E}_\bullet)$ to be the zero scheme $X_{i,r} = Z(\mathcal{M}_i \to \mathcal{Q}) \subset Y$, where $\mathcal{M}_i = \bigoplus_{a : h(a) = i} \mathcal{E}_{t(a)}$ and the map $\mathcal{M}_i \to \mathcal{Q}$ is obtained by composing the

projection $\mathcal{E}_i \to \mathcal{Q}$ with the sum of the bundle maps $\mathcal{E}_j \to \mathcal{E}_i$ of the representation $\mathcal{E}_\bullet$. This scheme has a natural projection $\rho : X_{i,r} \to X$. Notice that on $X_{i,r}$, all the maps $\mathcal{E}_j \to \mathcal{E}_i$ can be factored through the subbundle $\mathcal{S} \subset \mathcal{E}_i$. Using the factored maps, we obtain an *induced representation* $\mathcal{E}'_\bullet$ over $X_{i,r}$ on vector bundles given by $\mathcal{E}'_j = \mathcal{E}_j$ for $j \neq i$ and $\mathcal{E}'_i = \mathcal{S}$.

More generally, let $\mathbf{i} = (i_1, \ldots, i_m)$ be a sequence of quiver vertices and $\mathbf{r} = (r_1, \ldots, r_m)$ a sequence of positive integers, such that for each $i \in Q_0$ we have $e_i \geq \sum_{i_j = i} r_j$. We can iterate the above construction and define

$$X_{\mathbf{i},\mathbf{r}} = X_{\mathbf{i},\mathbf{r}}(\mathcal{E}_\bullet) = (\cdots((X_{i_1,r_1})_{i_2,r_2})\cdots)_{i_m,r_m}\,.$$

The variety $(X_{i_1,r_1})_{i_2,r_2}$ is constructed using the induced representation $\mathcal{E}'_\bullet$ on $X_{i_1,r_1}$, etc. Let $\pi : X_{\mathbf{i},\mathbf{r}} \to X$ denote the projection. In general, this map may have fibers of positive dimension.

Now let $Q$ be a quiver of Dynkin type and let $\Phi^+ \subset \mathbb{N}^n$ be the set of positive roots for the underlying Dynkin diagram. Here we identify the simple roots with the unit vectors $\epsilon_i \in \mathbb{N}^n$, $1 \leq i \leq n$. According to Gabriel's classification [22], there is a unique indecomposable representation of $Q$ with dimension vector $\alpha$ for every positive root $\alpha \in \Phi^+$, and all indecomposable representations have this form. This implies that the $\mathbb{G}$-orbits in $V$ correspond to sequences $(m_\alpha) \in \mathbb{N}^{\Phi^+}$ for which $\sum m_\alpha \alpha$ is equal to the dimension vector $e$. Furthermore, since the number of orbits is finite, it follows that every quiver cycle in $V$ is an orbit closure.

For dimension vectors $\alpha, \beta \in \mathbb{N}^n$, let $\langle \alpha, \beta \rangle = \sum_{i=1}^n \alpha_i \beta_i - \sum_{a \in Q_1} \alpha_{t(a)} \beta_{h(a)}$ denote the Euler form for $Q$. Let $\Phi' \subset \Phi^+$ be any subset of the positive roots. A partition $\Phi' = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_s$ of this set is called *directed* if $\langle \alpha, \beta \rangle \geq 0$ for all $\alpha, \beta \in \mathcal{I}_j$, $1 \leq j \leq s$, and $\langle \alpha, \beta \rangle \geq 0 \geq \langle \beta, \alpha \rangle$ for all $\alpha \in \mathcal{I}_i$ and $\beta \in \mathcal{I}_j$ with $i < j$. A directed partition always exists because the category of representations of $Q$ is representation-directed [32].

Let $(m_\alpha) \in \mathbb{N}^{\Phi^+}$ be a sequence representing an orbit closure $\Omega \subset V$, let $\Phi' \subset \Phi^+$ be a subset containing $\{\alpha : m_\alpha \neq 0\}$, and let $\Phi' = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_s$ be a directed partition. For each $j \in [1, s]$, write $\sum_{\alpha \in \mathcal{I}_j} m_\alpha \alpha = (p_1^j, \ldots, p_n^j) \in \mathbb{N}^n$. Then let $\mathbf{i}^j = (i_1, \ldots, i_l)$ be any sequence of the vertices $i \in Q_0$ for which $p_i^j \neq 0$, with no vertices repeated, and ordered so that the tail of any arrow of $Q$ comes before the head. Set $\mathbf{r}^j = (p_{i_1}^j, \ldots, p_{i_l}^j)$. Finally, let $\mathbf{i}$ and $\mathbf{r}$ be the concatenated sequences $\mathbf{i} = \mathbf{i}^1\mathbf{i}^2\cdots\mathbf{i}^s$ and $\mathbf{r} = \mathbf{r}^1\mathbf{r}^2\cdots\mathbf{r}^s$. We will call any pair of sequences $(\mathbf{i}, \mathbf{r})$ arising in this way for a *resolution pair* for $\Omega$.

Let $\widetilde{E}_\bullet$ denote the representation of $Q$ on the vector bundles $\widetilde{E}_i = V \times E_i$ over $V$, defined by the tautological maps $E_{t(a)} \to E_{h(a)}$, $(\phi, y) \mapsto (\phi, \phi_a(y))$, for $a \in Q_1$.

**Theorem 5.1** (Reineke)**.** *Let $Q$ be a quiver of Dynkin type, $\Omega \subset V$ an orbit closure, and $(\mathbf{i}, \mathbf{r})$ a resolution pair for $\Omega$. Then the map $\pi : V_{\mathbf{i},\mathbf{r}}(\widetilde{E}_\bullet) \to V$ has image $\Omega$ and is a birational isomorphism of $V_{\mathbf{i},\mathbf{r}}(\widetilde{E}_\bullet)$ with $\Omega$.*

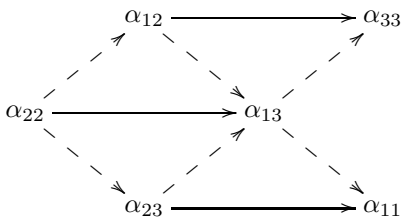
We note that Reineke's paper [31] states this theorem only in the case where the resolution pair $(\mathbf{i}, \mathbf{r})$ is constructed from a directed partition of the set of all positive roots $\Phi^+$, but the proof covers the more general statement.

Our formula for quiver coefficients given in the next section uses a resolution pair $(\mathbf{i}, \mathbf{r})$ and requires a number of steps proportional to the common length of $\mathbf{i}$ and $\mathbf{r}$. It is therefore desirable to make these sequences as short as possible.

One reasonable choice is to take the minimal set $\Phi' = \{\alpha : m_\alpha \neq 0\}$ and use the following 'greedy' algorithm to produce a shortest possible directed partition of $\Phi'$.

Define $\mathcal{I}(\Phi')$ to be the (unique) largest subset of $\Phi'$ for which every element $\alpha$ in $\mathcal{I}(\Phi')$ satisfies that $\langle\alpha,\beta\rangle \geq 0$ for all $\beta \in \Phi'$, and $\langle\beta,\alpha\rangle \leq 0$ for all $\beta \in \Phi' \smallsetminus \mathcal{I}(\Phi')$. This set can be constructed by starting with all roots $\alpha \in \Phi'$ for which the first inequality holds, and then discarding roots until the second inequality is satisfied. Since at least one directed partition for $\Phi'$ exists, it follows that $\mathcal{I}(\Phi') \neq \emptyset$. We now obtain a shortest possible directed partition of $\Phi'$ by setting $\mathcal{I}_1 = \mathcal{I}(\Phi')$, $\mathcal{I}_2 = \mathcal{I}(\Phi' \smallsetminus \mathcal{I}_1)$, $\mathcal{I}_3 = \mathcal{I}(\Phi' \smallsetminus (\mathcal{I}_1 \cup \mathcal{I}_2))$, etc.

**Example 5.2.** Let $Q = \{1 \to 2 \leftarrow 3\}$ be the quiver of type $\mathrm{A}_3$ in which both arrows point toward the center. The set of positive roots is $\Phi^+ = \{\alpha_{ij} \mid 1 \leq i < j \leq 3\}$, where $\alpha_{ij} = \sum_{p=i}^{j} \varepsilon_p$. Given an arbitrary partition $\Phi^+ = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_s$, we write $\eta(\alpha) = j$ for $\alpha \in \mathcal{I}_j$. The partition is directed if and only if $\eta(\alpha) \leq \eta(\beta)$ whenever the following graph has an arrow from $\alpha$ to $\beta$, and $\eta(\alpha) < \eta(\beta)$ when the graph has a solid arrow from $\alpha$ to $\beta$.




This graph is constructed by drawing a solid arrow from $\alpha$ to $\beta$ if $\langle\beta,\alpha\rangle < 0$, and a dotted arrow from $\alpha$ to $\beta$ if $\langle\beta,\alpha\rangle \geq 0$ and $\langle\alpha,\beta\rangle > 0$. The shortest directed partition of the positive roots is $\Phi^+ = \{\alpha_{22},\alpha_{12},\alpha_{23}\} \cup \{\alpha_{13},\alpha_{11},\alpha_{33}\}$.

Let $\Omega \subset V = \mathrm{Hom}(E_1,E_2) \oplus \mathrm{Hom}(E_3,E_2)$ be an orbit closure, corresponding to the integer sequence $(m_{ij}) \in \mathbb{N}^{\Phi^+}$ with $\sum m_{ij}\alpha_{ij} = e = (e_1,e_2,e_3)$. Then $\Omega$ is defined set-theoretically by

$$\Omega = \{(\phi_1,\phi_3) \in V \mid \mathrm{rank}(\phi_1) \leq m_{12}+m_{13} \text{ and } \mathrm{rank}(\phi_3) \leq m_{23}+m_{13} \\ \text{and } \mathrm{rank}(\phi_1+\phi_3 : E_1 \oplus E_3 \to E_2) \leq m_{12}+m_{23}+m_{13}\}\,.$$

As preparation for section 7, we will work out the desingularization of $\Omega$ obtained from the directed partition $\Phi^+ = \{\alpha_{22}\} \cup \{\alpha_{12},\alpha_{23},\alpha_{13}\} \cup \{\alpha_{11},\alpha_{33}\}$. The corresponding resolution pair $(\mathbf{i},\mathbf{r})$ is given by $\mathbf{i} = (2,1,3,2,1,3)$ and $\mathbf{r} = (m_{22}, m_{12}+m_{13}, m_{23}+m_{13}, e_2-m_{22}, m_{11}, m_{33})$. Form the product of Grassmann varieties $P = \mathrm{Gr}(m_{11},E_1) \times \mathrm{Gr}(e_2-m_{22},E_2) \times \mathrm{Gr}(m_{33},E_3)$. The desingularization of $\Omega$ defined by $(\mathbf{i},\mathbf{r})$ is the variety

$$V_{\mathbf{i},\mathbf{r}}(\widetilde{E}_\bullet) = \{(S_1,S_2,S_3,\phi_1,\phi_3) \in P \times V \mid \phi_i(E_i) \subset S_2 \text{ and } \phi_i(S_i) = 0 \text{ for } i = 1,3\}\,.$$

## 6. A formula for Quiver coefficients

Let $Q$ be an arbitrary quiver, and let $X$ be an algebraic scheme over $\mathbb{K}$ equipped with vector bundles $\mathcal{E}_1,\ldots,\mathcal{E}_n$ such that $\mathrm{rank}(\mathcal{E}_i) = e_i$ for each $i$. Over the scheme $\mathcal{V} = \bigoplus_{a\in Q_1} \mathrm{Hom}(\mathcal{E}_{t(a)},\mathcal{E}_{h(a)})$ we have a tautological representation $\widetilde{\mathcal{E}}_\bullet$ of $Q$ on (the pullbacks of) the bundles $\mathcal{E}_i$. Any pair of sequences $\mathbf{i} = (i_1,\ldots,i_m) \in Q_0^m$ and $\mathbf{r} = (r_1,\ldots,r_m) \in \mathbb{N}^m$, with $\sum_{i_j=i} r_j \leq e_i$ for each $i$, defines a map $\pi : \mathcal{V}_{\mathbf{i},\mathbf{r}}(\widetilde{\mathcal{E}}_\bullet) \to \mathcal{V}$.

In this section we give a formula for coefficients $c_\mu(\mathbf{i},\mathbf{r}) \in \mathbb{Z}$, indexed by sequences of partitions $\mu = (\mu_1, \ldots, \mu_n)$ with $\ell(\mu_i) \leq e_i$, such that

$$\pi_*[\mathcal{O}_{\mathcal{V}_{\mathbf{i},\mathbf{r}}}] = \sum_\mu c_\mu(\mathbf{i},\mathbf{r})\, \mathcal{G}_{\mu_1}(\mathcal{E}_1 - \mathcal{M}_1)\, \mathcal{G}_{\mu_2}(\mathcal{E}_2 - \mathcal{M}_2) \cdots \mathcal{G}_{\mu_n}(\mathcal{E}_n - \mathcal{M}_n) \ \in K_\circ(\mathcal{V})\,,$$

where $\pi_* : K_\circ(\mathcal{V}_{\mathbf{i},\mathbf{r}}) \to K_\circ(\mathcal{V})$ is the proper pushforward along $\pi$. If $Q$ is a quiver of Dynkin type and $(\mathbf{i},\mathbf{r})$ is a resolution pair for an orbit closure $\Omega \subset V$ with rational singularities, then $c_\mu(\Omega) = c_\mu(\mathbf{i},\mathbf{r})$. Our formula is stated in terms of operators on tensors of Grothendieck polynomials which we proceed to define.

Let $i \in Q_0$ be a quiver vertex. We let $\psi_i : \Gamma^{\otimes n+1} \to \Gamma^{\otimes n+1}$ denote the linear operator which applies the coproduct $\Delta$ to the $i$-th factor and multiplies one of the components of this coproduct to the last factor. More precisely, $\psi_i$ is defined by

$$\psi_i(\mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_n} \otimes \mathcal{G}_\lambda) = \\ \sum_{\sigma,\nu} \left( \sum_\tau d^{\mu_i}_{\sigma\tau}\, c^\nu_{\tau\lambda} \right) \mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_{i-1}} \otimes \mathcal{G}_\sigma \otimes \mathcal{G}_{\mu_{i+1}} \otimes \cdots \otimes \mathcal{G}_{\mu_n} \otimes \mathcal{G}_\nu\,,$$

where the sum is over all partitions $\sigma$, $\tau$, and $\nu$, and the constants $d^{\mu_i}_{\sigma\tau}$ and $c^\nu_{\tau\lambda}$ are defined in section 2.

For integers $r, c$ with $r \geq 0$, define the linear map $\mathcal{A}_{i,r\times c} : \Gamma^{\otimes n+1} \to \Gamma^{\otimes n}$ by

$$\mathcal{A}_{i,r\times c}(\mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_n} \otimes \mathcal{G}_\nu) = \mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_{i-1}} \otimes \mathcal{G}_{(c)^r+\nu,\mu_i} \otimes \mathcal{G}_{\mu_{i+1}} \otimes \cdots \otimes \mathcal{G}_{\mu_n}$$

if $\ell(\nu) \leq r$, and $\mathcal{A}_{i,r\times c}(\mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_n} \otimes \mathcal{G}_\nu) = 0$ otherwise. Here $(c)^r + \nu, \mu_i$ denotes the concatenation of the integer sequence $(c+\nu_1, \ldots, c+\nu_r)$ with the partition $\mu_i$. When this does not result in a partition, then the Grothendieck polynomial $\mathcal{G}_{(c)^r+\nu,\mu_i}$ is defined by equation (7). The operator $\mathcal{A}_{i,r\times c}$ will be applied with negative as well as positive integers $c$.

Let $a_1, \ldots, a_l \in Q_1$ be the arrows starting at $i$, i.e. $t(a_j) = i$ for each $j$. Define the linear map $\Phi^{Q,e}_{i,r} : \Gamma^{\otimes n} \to \Gamma^{\otimes n}$ by

$$\Phi^{Q,e}_{i,r}(P) = \mathcal{A}_{i,r\times c}\, \psi_{h(a_1)} \cdots \psi_{h(a_l)}(P \otimes 1)$$

where $c = \operatorname{rank}(\mathcal{M}_i) - e_i + r$.

Given sequences $\mathbf{i} = (i_1, \ldots, i_m) \in Q_0^m$ and $\mathbf{r} = (r_1, \ldots, r_m) \in \mathbb{N}^m$ as above, we define a tensor $P^{Q,e}_{\mathbf{i},\mathbf{r}} \in \Gamma^{\otimes n}$ as follows. If $m = 0$, then we set $P^{Q,e}_{\mathbf{i},\mathbf{r}} = 1 \otimes \cdots \otimes 1$. Otherwise we may assume by induction that $P^{Q,e'}_{\mathbf{i}',\mathbf{r}'} \in \Gamma^{\otimes n}$ has already been defined, where $\mathbf{i}' = (i_2, \ldots, i_m)$ and $\mathbf{r}' = (r_2, \ldots, r_m)$, and $e'$ is the dimension vector defined by $e'_j = e_j$ for $j \neq i_1$ and $e'_{i_1} = e_{i_1} - r_1$. In this case we set $P^{Q,e}_{\mathbf{i},\mathbf{r}} = \Phi^{Q,e}_{i_1,r_1}(P^{Q,e'}_{\mathbf{i}',\mathbf{r}'})$. We define the coefficients $c_\mu(\mathbf{i},\mathbf{r})$ as the coefficients in the expansion

$$P^{Q,e}_{\mathbf{i},\mathbf{r}} = \sum_\mu c_\mu(\mathbf{i},\mathbf{r})\, \mathcal{G}_{\mu_1} \otimes \mathcal{G}_{\mu_2} \otimes \cdots \otimes \mathcal{G}_{\mu_n}\,.$$

It follows from this definition that $c_\mu(\mathbf{i},\mathbf{r})$ is zero unless $\ell(\mu_i) \leq e_i$ for each $i$.

Given any element $P = \sum c_\mu\, \mathcal{G}_{\mu_1} \otimes \cdots \otimes \mathcal{G}_{\mu_n} \in \Gamma^{\otimes n}$ and $\alpha_1, \ldots, \alpha_n \in K^\circ(X)$, we set $P(\alpha_1, \ldots, \alpha_n) = \sum c_\mu\, \mathcal{G}_{\mu_1}(\alpha_1)\, \mathcal{G}_{\mu_2}(\alpha_2) \cdots \mathcal{G}_{\mu_n}(\alpha_n) \in K^\circ(X)$. The following theorem gives the geometric interpretation of the coefficients $c_\mu(\mathbf{i},\mathbf{r})$.

**Theorem 6.1.** *Let $\pi : \mathcal{V}_{\mathbf{i},\mathbf{r}}(\widetilde{\mathcal{E}}_\bullet) \to \mathcal{V}$ be the map associated to sequences $\mathbf{i},\mathbf{r}$. Then $\pi_*([\mathcal{O}_{\mathcal{V}_{\mathbf{i},\mathbf{r}}}]) = P^{Q,e}_{\mathbf{i},\mathbf{r}}(\mathcal{E}_1 - \mathcal{M}_1, \ldots, \mathcal{E}_n - \mathcal{M}_n) \in K^\circ(\mathcal{V})$.*

**Corollary 6.2.** *Let $Q$ be a quiver of Dynkin type, $\Omega \subset V$ an orbit closure, and $(\mathbf{i}, \mathbf{r})$ a resolution pair for $\Omega$. If $\Omega$ has rational singularities then $P_\Omega = P^{Q,e}_{\mathbf{i},\mathbf{r}}$, or equivalently, the quiver coefficients of $\Omega$ are given by $c_\mu(\Omega) = c_\mu(\mathbf{i}, \mathbf{r})$. Furthermore, this identity is true for all cohomological quiver coefficients, without the assumption about rational singularities.*

*Proof.* If $X$ is a non-singular variety, then it follows from Reineke's theorem that $\pi : \mathcal{V}_{\mathbf{i},\mathbf{r}}(\widetilde{\mathcal{E}}_\bullet) \to \widetilde{\Omega}$ is a desingularization of the translated degeneracy locus $\widetilde{\Omega} \subset \mathcal{V}$. If $\Omega$ has rational singularities, then $\pi_*([\mathcal{O}_{\mathcal{V}_{\mathbf{i},\mathbf{r}}}]) = [\mathcal{O}_{\widetilde{\Omega}}] \in K_\circ(\mathcal{V})$, so the corollary follows by comparing Theorem 6.1 to Corollary 4.3. Without this assumption, we still have $\pi_*[\mathcal{V}_{\mathbf{i},\mathbf{r}}] = [\widetilde{\Omega}]$ in the Chow ring of $\mathcal{V}$, which suffices to determine the cohomological quiver coefficients. □

**Remark 6.3.** If $\Omega \subset V$ is an orbit closure of Dynkin type, then the quiver coefficients for $\Omega$ are identical to the quiver coefficients for $\overline{\Omega} = \Omega \times_{\mathrm{Spec}(\mathbb{K})} \mathrm{Spec}(\overline{\mathbb{K}})$, where $\overline{\mathbb{K}}$ is an algebraic closure of $\mathbb{K}$. Corollary 6.2 therefore applies also if $\overline{\Omega}$ has rational singularities, which has been proved for quivers of type A in any characteristic and for quivers of type D in characteristic zero [27, 1, 2].

We have computed the coefficients $c_\mu(\mathbf{i}, \mathbf{r})$ for lots of randomly chosen quivers $Q$ and sequences $\mathbf{i}$ and $\mathbf{r}$, and in all cases they had alternating signs in the following sense.

**Conjecture 6.4.** *We have $(-1)^{\sum |\mu_i| + \sum |\mu'_i|}\, c_\mu(\mathbf{i}, \mathbf{r})\, c_{\mu'}(\mathbf{i}, \mathbf{r}) \geq 0$ for arbitrary sequences of partitions $\mu$ and $\mu'$.*

In almost all examples that we computed, the coefficients $c_\mu(\mathbf{i}, \mathbf{r})$ of lowest degree were positive. However, we also found examples where the lowest degree coefficients were negative, the next degree up were positive, etc. We speculate that in many examples, the class $\pi_*([\mathcal{O}_{\mathcal{V}_{\mathbf{i},\mathbf{r}}}])$ has been equal to the Grothendieck class of the image of $\pi$, which is always a quiver cycle in $V$. We therefore regard our verification of Conjecture 6.4 as additional evidence for Conjecture 1.1. For the proof of Theorem 6.1, we need the following Gysin formula from [8, Thm. 7.3].

**Theorem 6.5.** *Let $\mathcal{F}$ and $\mathcal{B}$ be vector bundles on $X$. Write $\mathrm{rank}(\mathcal{F}) = s + q$ and let $\rho : \mathrm{Gr}(s, \mathcal{F}) \to X$ be the Grassmann bundle of $s$-planes in $\mathcal{F}$ with universal exact sequence $0 \to \mathcal{S} \to \rho^*\mathcal{F} \to \mathcal{Q} \to 0$. Let $I = (I_1, \ldots, I_q)$ and $J = (J_1, J_2, \ldots)$ be finite sequences of integers such that $I_j \geq \mathrm{rank}(\mathcal{B})$ for all $j$. Then*

$$\rho_*(\mathcal{G}_I(\mathcal{Q} - \rho^*\mathcal{B}) \cdot \mathcal{G}_J(\mathcal{S} - \rho^*\mathcal{B})) = \mathcal{G}_{I-(s^q),J}(\mathcal{F} - \mathcal{B}) \ \in K_\circ(X)\,,$$

*where $I - (s^q), J = (I_1 - s, \ldots, I_q - s, J_1, J_2, \ldots)$.*

Consider a variety $\mathcal{V}_{i,r} = Z(\mathcal{M}_i \to \mathcal{Q}) \subset Y = \mathrm{Gr}(e_i - r, \mathcal{E}_i)$ as in the previous section, where $0 \to \mathcal{S} \to \mathcal{E}_i \to \mathcal{Q} \to 0$ is the universal exact sequence on $Y$. Let $\rho : \mathcal{V}_{i,r} \to \mathcal{V}$ be the projection and let $\mathcal{E}'_\bullet$ be the induced representation on $\mathcal{V}_{i,r}$.

**Lemma 6.6.** *Let $P' \in \Gamma^{\otimes n+1}$ and set $P = \psi_i(P')$. Then $P'(\alpha_1, \ldots, \alpha_n, \mathcal{Q}) = P(\alpha_1, \ldots, \alpha_{i-1}, \alpha_i - \mathcal{Q}, \alpha_{i+1}, \ldots, \alpha_n, \mathcal{Q})$ for any elements $\alpha_1, \ldots, \alpha_n \in K^\circ(\mathcal{V}_{i,r})$.*

*Proof.* For partitions $\mu_i$ and $\lambda$ we have $\mathcal{G}_{\mu_i}(\alpha_i) \cdot \mathcal{G}_\lambda(\mathcal{Q}) = \mathcal{G}_{\mu_i}(\alpha_i - \mathcal{Q} + \mathcal{Q}) \cdot \mathcal{G}_\lambda(\mathcal{Q}) = \sum_{\sigma,\tau} d^{\mu_i}_{\sigma\tau}\, \mathcal{G}_\sigma(\alpha_i - \mathcal{Q}) \cdot \mathcal{G}_\tau(\mathcal{Q}) \cdot \mathcal{G}_\lambda(\mathcal{Q}) = \sum_{\sigma,\tau} d^{\mu_i}_{\sigma\tau}\, \mathcal{G}_\sigma(\alpha_i - \mathcal{Q}) \sum_\nu c^\nu_{\tau\lambda}\, \mathcal{G}_\nu(\mathcal{Q})$. □

**Proposition 6.7.** *Let $P' \in \Gamma^{\otimes n}$ and set $P = \Phi^{Q,e}_{i,r}(P)$ and $\mathcal{M}'_i = \bigoplus_{h(a)=i} \mathcal{E}'_{t(a)}$. Then $\rho_*(P'(\mathcal{E}'_1 - \mathcal{M}'_1, \ldots, \mathcal{E}'_n - \mathcal{M}'_n)) = P(\mathcal{E}_1 - \mathcal{M}_1, \ldots, \mathcal{E}_n - \mathcal{M}_n)$ in $K_\circ(\mathcal{V})$.*

*Proof.* For each $j \in Q_0$ we have $[\mathcal{M}_j] = [\mathcal{M}'_j] + p[\mathcal{Q}] \in K^\circ(\mathcal{V}_{i,r})$, where $p$ is the number of arrows from $i$ to $j$. Lemma 6.6 therefore implies that $P'(\mathcal{E}'_1 - \mathcal{M}'_1, \ldots, \mathcal{E}'_n - \mathcal{M}'_n) = P''(\mathcal{E}'_1 - \mathcal{M}_1, \ldots, \mathcal{E}'_n - \mathcal{M}_n, \mathcal{Q})$ where $P'' = \psi_{h(a_1)} \cdots \psi_{h(a_l)}(P' \otimes 1)$.

It follows from Example 3.3 that $[\mathcal{O}_{\mathcal{V}_{i,r}}] = \mathcal{G}_R(\mathcal{Q} - \mathcal{M}_i)$ in $K_\circ(Y)$, where $R = (\mathrm{rank}(\mathcal{M}_i)^r)$. The pushforward of $P'(\mathcal{E}'_1 - \mathcal{M}'_1, \ldots, \mathcal{E}'_n - \mathcal{M}'_n)$ from $\mathcal{V}_{i,r}$ to $Y$ is therefore equal to $P''(\mathcal{E}'_1 - \mathcal{M}_1, \ldots, \mathcal{E}'_n - \mathcal{M}_n, \mathcal{Q}) \cdot \mathcal{G}_R(\mathcal{Q})$.

Let $\mu_i$ and $\nu$ be partitions. If $\ell(\nu) > r$ then $\mathcal{G}_\nu(\mathcal{Q}) = 0$. Otherwise it follows from the factorization formula (4) that $\mathcal{G}_\nu(\mathcal{Q})\mathcal{G}_R(\mathcal{Q} - \mathcal{M}_i) = \mathcal{G}_{R+\nu}(\mathcal{Q} - \mathcal{M}_i)$, and Theorem 6.5 implies that $\rho'_*(\mathcal{G}_{R+\nu}(\mathcal{Q} - \mathcal{M}_i) \cdot \mathcal{G}_{\mu_i}(\mathcal{S} - \mathcal{M}_i)) = \mathcal{G}_{(c)^r+\nu,\mu_i}(\mathcal{E}_i - \mathcal{M}_i)$, where $\rho' : Y \to \mathcal{V}$ is the projection and $c = \mathrm{rank}(\mathcal{M}_i) - e_i + r$. We conclude that $\rho_*(P'(\mathcal{E}'_1 - \mathcal{M}'_1, \ldots, \mathcal{E}'_n - \mathcal{M}'_n)) = P(\mathcal{E}_1 - \mathcal{M}_1, \ldots, \mathcal{E}_n - \mathcal{M}_n)$ where $P = \mathcal{A}_{i,r\times c}(P'') = \Phi^{Q,e}_{i,r}(P')$. □

*Proof of Theorem 6.1.* Let $X' = \mathrm{Gr}(e_{i_1} - r_1, \mathcal{E}_{i_1}) \to X$ be the Grassmann bundle of rank $r_1$ quotients of $\mathcal{E}_{i_1}$. Then the bundles $\mathcal{E}'_j$ are defined on $X'$, and $Y = \mathcal{V} \times_X X'$ can be constructed as the bundle $\bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{E}_{t(a)}, \mathcal{E}_{h(a)})$ over $X'$. It follows that $\mathcal{V}_{i_1,r_1} = Z(\mathcal{M}_i \to \mathcal{E}_i/\mathcal{E}'_i) \subset Y$ is isomorphic to the bundle $\bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{E}_{t(a)}, \mathcal{E}'_{h(a)})$, which implies that $\mathcal{V}_{i_1,r_1}$ is an affine bundle over $\mathcal{V}' = \bigoplus_{a\in Q_1} \mathrm{Hom}_{\mathcal{O}_{X'}}(\mathcal{E}'_{t(a)}, \mathcal{E}'_{h(a)})$. We furthermore have a fiber square:

$$\begin{array}{ccc} \mathcal{V}_{\mathbf{i},\mathbf{r}} & \longrightarrow & \mathcal{V}'_{\mathbf{i}',\mathbf{r}'}(\mathcal{E}'_\bullet) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} \\ \mathcal{V}_{i_1,r_1} & \longrightarrow & \mathcal{V}' \end{array}$$

By induction on $m$ we know that $\beta'_*(1) = P^{Q,e'}_{\mathbf{i}',\mathbf{r}'}(\mathcal{E}'_1 - \mathcal{M}'_1, \ldots, \mathcal{E}'_n - \mathcal{M}'_n) \in K_\circ(\mathcal{V}')$, and since the horizontal maps are flat, this implies that $\beta_*([\mathcal{O}_{\mathcal{V}_{\mathbf{i},\mathbf{r}}}]) = \beta_*(1) = P^{Q,e'}_{\mathbf{i}',\mathbf{r}'}(\mathcal{E}'_1 - \mathcal{M}'_1, \ldots, \mathcal{E}'_n - \mathcal{M}'_n) \in K_\circ(\mathcal{V}_{i_1,r_1})$. Proposition 6.7 finally shows that $\pi_*([\mathcal{O}_{\mathcal{V}_{\mathbf{i},\mathbf{r}}}]) = \rho_*(P^{Q,e'}_{\mathbf{i}',\mathbf{r}'}(\mathcal{E}'_1 - \mathcal{M}'_1, \ldots, \mathcal{E}'_n - \mathcal{M}'_n)) = P^{Q,e}_{\mathbf{i},\mathbf{r}}(\mathcal{E}_1 - \mathcal{M}_1, \ldots, \mathcal{E}_n - \mathcal{M}_n) \in K_\circ(\mathcal{V})$, as required. □

**Remark 6.8.** For applications of our formula, it would be useful to know the reduced equations generating the ideal of an orbit closure $\Omega \subset V$ for a quiver $Q$ of Dynkin type. For example, such equations will result in a more explicit construction of the degeneracy loci $\Omega(\mathcal{E}_\bullet)$ defined by $\Omega$.

Let $\phi \in V$ be a representation of $Q$ on the vector spaces $E_1, \ldots, E_n$, and fix another representation $\psi = (\psi_a)_{a\in Q_1}$ on vector spaces $F_1, \ldots, F_n$. A homomorphism from $\psi$ to $\phi$ is a collection $\beta$ of linear maps $\beta_i : F_i \to E_i$ such that $\phi_a\beta_{t(a)} = \beta_{h(a)}\psi_a$ as a map from $F_{t(a)}$ to $E_{h(a)}$ for all $a \in Q_1$. Let $\mathrm{Hom}(\psi, \phi)$ denote the vector space of all such homomorphisms. Bongartz has proved in [3, Prop. 3.2] that $\phi'$ belongs to the orbit closure $\Omega = \overline{\mathbb{G}.\phi}$ if and only if $\dim \mathrm{Hom}(\psi, \phi') \geq \dim \mathrm{Hom}(\psi, \phi)$ for all (indecomposable) representations $\psi$ of $Q$. Set $A = \bigoplus_{i\in Q_0} \mathrm{Hom}(F_i, E_i)$ and $B = \bigoplus_{a\in Q_1} \mathrm{Hom}(F_{t(a)}, E_{h(a)})$, and let $\gamma_{\psi,\phi} : A \to B$ be the linear map given by $\gamma_{\psi,\phi}(\beta) = (\beta_{h(a)}\psi_a - \phi_a\beta_{t(a)})_{a\in Q_1}$. Define $\mathrm{rank}_\psi(\phi) = \mathrm{rank}(\gamma_{\psi,\phi})$. We then have

$$\Omega = \{\phi' \in V \mid \mathrm{rank}_\psi(\phi') \leq \mathrm{rank}_\psi(\phi)\ \forall \text{ indecomp. representations } \psi \text{ of } Q\}\,. \tag{13}$$

This description of the orbit closure $\Omega$ gives rise to set-theoretic equations for $\Omega$ in terms of minors of the matrices $\gamma_{\psi,\phi}$. It is interesting to ask if these equations in

fact generate the ideal $I(\Omega) \subset k[V]$. This has been proved for equioriented quivers of type A by Lakshmibai and Magyar [27], but reduced equations for orbit closures appear to be unknown for quivers of other types. We have used Macaulay 2 [23] to check that minors of the matrices $\gamma_{\psi,\phi}$ in fact generate the ideal of the inbound $\mathrm{A}_3$-orbit closure given by $m_{ij} = 1$ for $1 \le i < j \le 3$ (see Example 5.2).

If $\mathcal{E}_\bullet$ is a representation of $Q$ on vector bundles over $X$, then each fixed representation $\psi$ of $Q$ defines a vector bundle map from $\mathcal{A} = \bigoplus_{i \in Q_0} \mathrm{Hom}(F_i \otimes \mathcal{O}_X, \mathcal{E}_i)$ to $\mathcal{B} = \bigoplus_{a \in Q_1} \mathrm{Hom}(F_{t(a)} \otimes \mathcal{O}_X, \mathcal{E}_{h(a)})$, and the degeneracy locus $\Omega(\mathcal{E}_\bullet)$ is the set of points $x \in X$ where the rank of this bundle map is at most $\mathrm{rank}_\psi(\phi)$ for all $\psi$. Assuming that (13) gives the reduced equations of $\Omega$, this description of $\Omega(\mathcal{E}_\bullet)$ also captures its scheme structure.

## 7. Quiver coefficients of type $\mathrm{A}_3$

In this section we prove combinatorial formulas for the (non-equioriented) quiver coefficients of type $\mathrm{A}_3$. These formulas are based on counting set-valued tableaux, and show that the coefficients have alternating signs.

7.1. **Inbound $\mathbf{A}_3$ quiver.** Let $Q = \{1 \to 2 \leftarrow 3\}$ be the inbound quiver of type $\mathrm{A}_3$ from Example 5.2, and let $\Omega \subset V$ be the orbit closure given by $(m_{ij}) \in \mathbb{N}^{\Phi^+}$. For partitions $\lambda$, $\mu$, and $\nu$, define the coefficient

$$c_{\lambda,\mu,\nu} = \sum_{\sigma,\tau} d_{\lambda,\sigma}^{(m_{33})^{m_{12}}} \, d_{\tau,\nu}^{(m_{11})^{m_{23}}} \, c_{\sigma\tau}^{\mu} \,,$$

where the sum is over all partitions $\sigma$ and $\tau$.

**Proposition 7.1.** *The coefficient $c_{\lambda,\mu,\nu}$ is equal to $(-1)^{|\lambda|+|\mu|+|\nu|-m_{33}m_{12}-m_{11}m_{23}}$ times the number of pairs $(\sigma, T)$ of a partition $\sigma$ contained in the rectangle $(m_{33})^{m_{12}}$ with $m_{12}$ rows and $m_{33}$ columns, and a set-valued tableau $T$ whose shape is a partition contained in $(m_{11})^{m_{23}}$, satisfying the following conditions.*

- (i) *If $\sigma$ is placed in the upper-left corner of the rectangle $(m_{33})^{m_{12}}$ and the 180 degree rotation of $\lambda$ is placed in the lower-right corner, then their union is the whole rectangle and their overlap is a rook-strip, i.e. the the overlap contains at most one box in any row or column.*
- (ii) *If $T$ is placed in the upper-left corner of the rectangle $(m_{11})^{m_{23}}$ and the 180 degree rotation of $\nu$ is placed in the lower-right corner, then their union is the whole rectangle and their overlap is a rook-strip.*
- (iii) *The composition $w(T)u(\sigma)$ is a reverse lattice word with content $\mu$ (with the terminology of Theorem 2.1.)*

*Proof.* This follows from Theorem 2.1 because $d_{\lambda,\sigma}^{(m_{33})^{m_{12}}}$ is non-zero exactly when the condition (i) is satisfied, in which case $d_{\lambda,\sigma}^{(m_{33})^{m_{12}}} = (-1)^{|\lambda|+|\sigma|-m_{33}m_{12}}$. Notice also that (i) and (ii) can only be satisfied if $\lambda \subset (m_{33})^{m_{12}}$ and $\nu \subset (m_{11})^{m_{23}}$. □

**Theorem 7.2.** *The quiver coefficients of the inbound quiver of type $A_3$ are given by*

$$P_\Omega = \sum_{\lambda,\mu,\nu} c_{\lambda,\mu,\nu} \, \mathcal{G}_\lambda \otimes \mathcal{G}_{(m_{11}+m_{13}+m_{33})^{m_{22}},\mu} \otimes \mathcal{G}_\nu \,.$$

**Lemma 7.3.** *In the situation of Theorem 6.5, let $\lambda$ be a partition such that $\lambda_1 = \lambda_b = s$, where $b = \mathrm{rank}(\mathcal{B})$. Then $\rho_*(\mathcal{G}_\lambda(\rho^*\mathcal{B} - \mathcal{S})) = \mathcal{G}_{(\lambda_{q+1},\lambda_{q+2},\dots)}(\mathcal{B} - \mathcal{F})$.*

*Proof.* The Grassmann bundle $\mathrm{Gr}(s,\mathcal{F})$ of $s$-planes in $\mathcal{F}$ is identical to the bundle $\mathrm{Gr}(q,\mathcal{F}^\vee)$ of $q$-planes in $\mathcal{F}^\vee$, with tautological exact sequence $0 \to \mathcal{Q}^\vee \to \rho^*\mathcal{F}^\vee \to \mathcal{S}^\vee \to 0$. The lemma follows from Theorem 6.5 by using the identity $\mathcal{G}_\lambda(\rho^*\mathcal{B}-\mathcal{S}) = \mathcal{G}_{\lambda'}(\mathcal{S}^\vee - \rho^*\mathcal{B}^\vee)$. □

*Proof of Theorem 7.2.* Let $X$ be a smooth variety with vector bundles $\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3$ of ranks $e_1,e_2,e_3$, and let $\widetilde{\Omega} \subset \mathcal{V} = \mathrm{Hom}(\mathcal{E}_1,\mathcal{E}_2)\oplus\mathrm{Hom}(\mathcal{E}_3,\mathcal{E}_2)$ be the translated degeneracy locus. Form the product of Grassmann bundles

$$P = \mathrm{Gr}(m_{11},\mathcal{E}_1)\times_{\mathcal{V}}\mathrm{Gr}(e_2-m_{22},\mathcal{E}_2)\times_{\mathcal{V}}\mathrm{Gr}(m_{33},\mathcal{E}_3)\xrightarrow{\pi}\mathcal{V}$$

with tautological subbundles $\mathcal{E}'_i \subset \mathcal{E}_i$, $1\le i\le 3$. The desingularization of $\widetilde{\Omega}$ is the iterated zero section $\mathcal{V}_{\mathbf{i},\mathbf{r}} = Z(\mathcal{E}'_1\oplus\mathcal{E}'_3 \to \mathcal{E}'_2) \subset Z(\mathcal{E}_1\oplus\mathcal{E}_3 \to \mathcal{E}_2/\mathcal{E}'_2) \subset P$. The Thom-Porteous formula (Example 3.3) implies that the Grothendieck class of this locus in $K_\circ(P)$ is given by

$$[\mathcal{O}_{\mathcal{V}_{\mathbf{i},\mathbf{r}}}] = \mathcal{G}_{(m_{11})^{e_2-m_{22}}}(\mathcal{E}'_2-\mathcal{E}'_1)\,\mathcal{G}_{(m_{33})^{e_2-m_{22}}}(\mathcal{E}'_2-\mathcal{E}'_3)\,\mathcal{G}_{(e_1+e_2)^{m_{22}}}(\mathcal{E}_2/\mathcal{E}'_2-\mathcal{E}_1\oplus\mathcal{E}_3)\,.$$

The pushforward of this class along the projection $P\to\mathrm{Gr}(e_2-m_{22},\mathcal{E}_2)$ is equal to $\mathcal{G}_{(m_{11})^{m_{23}}}(\mathcal{E}'_2-\mathcal{E}_1)\,\mathcal{G}_{(m_{33})^{m_{12}}}(\mathcal{E}'_2-\mathcal{E}_3)\,\mathcal{G}_{(e_1+e_3)^{m_{22}}}(\mathcal{E}_2/\mathcal{E}'_2-\mathcal{E}_1\oplus\mathcal{E}_3)$ by Lemma 7.3. The first two factors of this product can be rewritten as

$$\begin{aligned}
&\mathcal{G}_{(m_{11})^{m_{23}}}(\mathcal{E}'_2-\mathcal{E}_1)\,\mathcal{G}_{(m_{33})^{m_{12}}}(\mathcal{E}'_2-\mathcal{E}_3)\\
&=\sum_{\lambda,\sigma,\tau,\nu} d_{\lambda,\sigma}^{(m_{33})^{m_{12}}}\,d_{\tau,\nu}^{(m_{11})^{m_{23}}}\,\mathcal{G}_\lambda(\mathcal{E}_1)\,\mathcal{G}_\sigma(\mathcal{E}'_2-\mathcal{E}_1\oplus\mathcal{E}_3)\,\mathcal{G}_\tau(\mathcal{E}'_2-\mathcal{E}_1\oplus\mathcal{E}_3)\,\mathcal{G}_\nu(\mathcal{E}_3)\\
&=\sum_{\lambda,\mu,\nu} c_{\lambda,\mu,\nu}\,\mathcal{G}_\lambda(\mathcal{E}_1)\,\mathcal{G}_\mu(\mathcal{E}'_2-\mathcal{E}_1\oplus\mathcal{E}_3)\,\mathcal{G}_\nu(\mathcal{E}_3)\,.
\end{aligned}$$

Theorem 6.5 applied to the bundle $\mathrm{Gr}(e_2-m_{22},\mathcal{E}_2)\to\mathcal{V}$ therefore shows that

$$\pi_*([\mathcal{O}_{\mathcal{V}_{\mathbf{i},\mathbf{r}}}]) = \sum_{\lambda,\mu,\nu} c_{\lambda,\mu,\nu}\,\mathcal{G}_\lambda(\mathcal{E}_1)\,\mathcal{G}_{(m_{11}+m_{13}+m_{33})^{m_{22}},\mu}(\mathcal{E}_2-\mathcal{E}_1\oplus\mathcal{E}_3)\,\mathcal{G}_\nu(\mathcal{E}_3)$$

in $K_\circ(\mathcal{V})$, as required. □

7.2. **Outbound $\mathbf{A}_3$ quiver.** Now let $Q=\{1\leftarrow 2\rightarrow 3\}$ be the quiver of type $\mathrm{A}_3$ with both arrows pointing away from the center, and let $\Omega\subset V$ be the orbit closure corresponding to the sequence $(m_{ij})\in\mathbb{N}^{\Phi^+}$, where $\Phi^+=\{\alpha_{ij}\mid 1\le i<j\le 3\}$. Let $R=(m_{22})^{m_{13}}$ be the rectangle with $m_{13}$ rows and $m_{22}$ columns. For partitions $\lambda,\mu,\nu$, we let $d^R_{\lambda,\mu,\nu}$ denote the 2-fold coproduct coefficients defined by $\Delta^2(\mathcal{G}_R) = \sum_{\lambda,\mu,\nu} d^R_{\lambda,\mu,\nu}\,\mathcal{G}_\lambda\otimes\mathcal{G}_\mu\otimes\mathcal{G}_\nu$.

**Proposition 7.4.** *The coefficient $d^R_{\lambda,\mu,\nu}$ is zero unless $\lambda$, $\mu$, and $\nu$ are contained in $R$, in which case it is equal to $(-1)^{|\lambda|+|\mu|+|\nu|-m_{22}m_{13}}$ times the number of triples $(\sigma,\tau,T)$, where $\sigma$ and $\tau$ are partitions such that $\sigma\subset\tau\subset R$, and $T$ is a set-valued tableau of skew shape $\tau/\sigma$, satisfying the following conditions.*

(i) *The Young diagram $\sigma$ is contained in $\lambda$, and $\lambda/\sigma$ is a rook-strip.*
(ii) *If $\tau$ is placed in the upper-left corner of $R$ and the 180 degree rotation of $\nu$ is placed in the lower-right corner, then their union is $R$ and their overlap is a rook-strip.*
(iii) *The word $w(T)$ is a reverse lattice word with content $\mu$.*

*Proof.* It follows from [9, Lemma 6.1] that $\Delta^2(\mathcal{G}_R) = \sum (-1)^{|\lambda|+|\tau/\sigma|+|\nu|-|R|}\, \mathcal{G}_\lambda \otimes \mathcal{G}_{\tau/\sigma} \otimes \mathcal{G}_\nu$, where the sum is over all partitions $\lambda, \sigma, \tau, \mu \subset R$ satisfying (i) and (ii). The coefficient of $\mathcal{G}_\mu$ in $\mathcal{G}_{\tau/\sigma}$ is equal to $(-1)^{|\mu|-|\tau/\sigma|}$ times the number of set-valued tableaux $T$ of shape $\tau/\sigma$ satisfying (iii) by [9, Thm. 6.9]. □

**Theorem 7.5.** *The quiver coefficients of the outbound quiver of type $A_3$ are given by*

$$P_\Omega = \sum_{\lambda,\mu,\nu} d^R_{\lambda,\mu,\nu}\, \mathcal{G}_{(m_{22}+m_{23})^{m_{11}},\lambda} \otimes \mathcal{G}_\mu \otimes \mathcal{G}_{(m_{22}+m_{12})^{m_{33}},\nu}\,.$$

*Proof.* We use the directed partition $\Phi^+ = \{\alpha_{11}\} \cup \{\alpha_{33}, \alpha_{23}, \alpha_{13}\} \cup \{\alpha_{22}, \alpha_{12}\}$, and resolution pair $\mathbf{i} = (1,2,1,3,2,1)$ and $\mathbf{r} = (m_{11}, m_{23}+m_{13}, m_{13}, e_3, m_{22}+m_{12}, m_{12})$. Given a non-singular variety $X$ with vector bundles $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ of ranks $e_1, e_2, e_3$, form the product $P = \mathrm{Fl}(m_{12}, m_{12}+m_{13}; \mathcal{E}_1) \times_{\mathcal{V}} \mathrm{Gr}(m_{22}+m_{12}, \mathcal{E}_2) \to \mathcal{V}$, with universal subbundles $\mathcal{E}_1'' \subset \mathcal{E}_1' \subset \mathcal{E}_1$ and $\mathcal{E}_2' \subset \mathcal{E}_2$. The desingularization of $\widetilde{\Omega} \subset \mathcal{V}$ corresponding to $(\mathbf{i}, \mathbf{r})$ is the iterated zero section $\mathcal{V}_{\mathbf{i},\mathbf{r}} = Z(\mathcal{E}_2' \to \mathcal{E}_1'/\mathcal{E}_1'' \oplus \mathcal{E}_3) \subset Z(\mathcal{E}_2 \to \mathcal{E}_1/\mathcal{E}_1') \subset P$. The Grothendieck class of this locus in $K_\circ(P)$ is

$$[\mathcal{O}_{\mathcal{V}_{\mathbf{i},\mathbf{r}}}] = \mathcal{G}_{(e_2)^{m_{11}}}(\mathcal{E}_1/\mathcal{E}_1' - \mathcal{E}_2)\, \mathcal{G}_{(m_{22}+m_{12})^{m_{13}}}(\mathcal{E}_1'/\mathcal{E}_1'' - \mathcal{E}_2')\, \mathcal{G}_{(m_{22}+m_{12})^{e_3}}(\mathcal{E}_3 - \mathcal{E}_2')\,,$$

and by Theorem 6.5, the pushforward of this class along the projection $P \to P' = \mathrm{Gr}(m_{12}+m_{13}, \mathcal{E}_1) \times_{\mathcal{V}} \mathrm{Gr}(m_{22}+m_{12}, \mathcal{E}_2)$ is equal to

$$\mathcal{G}_{(e_2)^{m_{11}}}(\mathcal{E}_1/\mathcal{E}_1' - \mathcal{E}_2)\, \mathcal{G}_R(\mathcal{E}_1' - \mathcal{E}_2')\, \mathcal{G}_{(m_{22}+m_{12})^{e_3}}(\mathcal{E}_3 - \mathcal{E}_2')$$

in $K_\circ(P')$. After using the three-fold coproduct identity

$$\mathcal{G}_R(\mathcal{E}_1' - \mathcal{E}_2') = \sum d^R_{\lambda,\mu,\nu}\, \mathcal{G}_\lambda(\mathcal{E}_1' - \mathcal{E}_2')\, \mathcal{G}_\mu(\mathcal{E}_2)\, \mathcal{G}_\nu(-\mathcal{E}_2')\,,$$

as well as the factorization identity

$$\mathcal{G}_\nu(-\mathcal{E}_2')\, \mathcal{G}_{(m_{22}+m_{12})^{e_3}}(\mathcal{E}_3 - \mathcal{E}_2') = \mathcal{G}_{(m_{22}+m_{12})^{e_3},\nu}(\mathcal{E}_3 - \mathcal{E}_2')\,,$$

it follows from Theorem 6.5 and Lemma 7.3 that the pushforward of the class in $K_\circ(P')$ along $P' \to \mathcal{V}$ is equal to

$$\pi_*([\mathcal{O}_{\mathcal{V}_{\mathbf{i},\mathbf{r}}}]) = \sum_{\lambda,\mu,\nu} d^R_{\lambda,\mu,\nu}\, \mathcal{G}_{(m_{22}+m_{23})^{m_{11}},\lambda}(\mathcal{E}_1 - \mathcal{E}_2)\, \mathcal{G}_\mu(\mathcal{E}_2)\, \mathcal{G}_{(m_{22}+m_{12})^{m_{33}},\nu}(\mathcal{E}_3 - \mathcal{E}_2)\,,$$

as required. □

## References


[1] G. Bobiński and G. Zwara, *Normality of orbit closures for Dynkin quivers of type $\mathbb{A}_n$*, Manuscripta Math. **105** (2001), no. 1, 103–109. MR MR1885816 (2002k:14077)

[2] ———, *Schubert varieties and representations of Dynkin quivers*, Colloq. Math. **94** (2002), no. 2, 285–309. MR MR1967381 (2004a:16026)

[3] K. Bongartz, *On degenerations and extensions of finite-dimensional modules*, Adv. Math. **121** (1996), no. 2, 245–287. MR MR1402728 (98e:16012)

[4] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 21, Springer-Verlag, Berlin, 1990. MR MR1045822 (91i:14034)

[5] M. Brion, *Positivity in the Grothendieck group of complex flag varieties*, J. Algebra **258** (2002), no. 1, 137–159. MR MR1958901 (2003m:14017)

[6] A. S. Buch, *On a conjectured formula for quiver varieties*, J. Algebraic Combin. **13** (2001), no. 2, 151–172. MR MR1826950 (2002g:14074)

[7] ———, *Stanley symmetric functions and quiver varieties*, J. Algebra **235** (2001), no. 1, 243–260. MR MR1807664 (2001m:05257)

[8] ———, *Grothendieck classes of quiver varieties*, Duke Math. J. **115** (2002), no. 1, 75–103. MR MR1932326 (2003m:14018)

[9] ———, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), no. 1, 37–78. MR MR1946917 (2003j:14062)
[10] ———, *Alternating signs of quiver coefficients*, J. Amer. Math. Soc. **18** (2005), no. 1, 217–237. MR MR2114821 (2006d:14052)
[11] A. S. Buch and W. Fulton, *Chern class formulas for quiver varieties*, Invent. Math. **135** (1999), no. 3, 665–687. MR MR1669280 (2000f:14087)
[12] A. S. Buch, A. Kresch, M. Shimozono, H. Tamvakis, and A. Yong, *Stable Grothendieck polynomials and K-theoretic factor sequences*, preprint, 2006.
[13] A. S. Buch, A. Kresch, H. Tamvakis, and A. Yong, *Schubert polynomials and quiver formulas*, Duke Math. J. **122** (2004), no. 1, 125–143. MR MR2046809 (2005b:05218)
[14] ———, *Grothendieck polynomials and quiver formulas*, Amer. J. Math. **127** (2005), no. 3, 551–567. MR MR2141644 (2007d:14018)
[15] A. S. Buch and R. Rimányi, *A formula for non-equioriented quiver orbits of type A*, J. Algebraic Geom. **16** (2007), no. 3, 531–546. MR MR2306279
[16] A. S. Buch, F. Sottile, and A. Yong, *Quiver coefficients are Schubert structure constants*, Math. Res. Lett. **12** (2005), no. 4, 567–574. MR MR2155232 (2006g:14082)
[17] D. Edidin and W. Graham, *Equivariant intersection theory*, Invent. Math. **131** (1998), no. 3, 595–634. MR MR1614555 (99j:14003a)
[18] L. Fehér and R. Rimányi, *Classes of degeneracy loci for quivers: the Thom polynomial point of view*, Duke Math. J. **114** (2002), no. 2, 193–213. MR MR1920187 (2003j:14005)
[19] S. Fomin and A. N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, Proc. Formal Power Series and Alg. Comb. (1994), 183–190.
[20] W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 2, Springer-Verlag, Berlin, 1998. MR MR1644323 (99d:14003)
[21] W. Fulton and R. MacPherson, *Categorical framework for the study of singular spaces*, Mem. Amer. Math. Soc. **31** (1981), no. 243, vi+165. MR MR609831 (83a:55015)
[22] P. Gabriel, *Unzerlegbare Darstellungen. I*, Manuscripta Math. **6** (1972), 71–103; correction, ibid. 6 (1972), 309. MR MR0332887 (48 #11212)
[23] D. R. Grayson and M. E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, Available at http://www.math.uiuc.edu/Macaulay2/.
[24] J. C. Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, AMS, Providence, RI, 2003. MR MR2015057 (2004h:20061)
[25] A. Knutson, E. Miller, and M. Shimozono, *Four positive formulae for type A quiver polynomials*, Invent. Math. **166** (2006), no. 2, 229–325. MR MR2249801
[26] A. Knutson and M. Shimozono, *Kempf collapsing and quiver loci*, preprint, 2006.
[27] V. Lakshmibai and P. Magyar, *Degeneracy schemes, quiver schemes, and Schubert varieties*, Internat. Math. Res. Notices (1998), no. 12, 627–640. MR MR1635873 (99g:14065)
[28] C. Lenart, S. Robinson, and F. Sottile, *Grothendieck polynomials via permutation patterns and chains in the Bruhat order*, Amer. J. Math. **128** (2006), no. 4, 805–848. MR MR2251587
[29] E. Miller, *Alternating formulas for K-theoretic quiver polynomials*, Duke Math. J. **128** (2005), no. 1, 1–17. MR MR2137947 (2006e:05181)
[30] D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34, Springer-Verlag, Berlin, 1965. MR MR0214602 (35 #5451)
[31] M. Reineke, *Quivers, desingularizations and canonical bases*, Progr. Math., vol. 210, Birkhäuser Boston, Boston, MA, 2003, pp. 325–344. MR MR1985731 (2004j:16017)
[32] C. M. Ringel, *Tame algebras and integral quadratic forms*, Lecture Notes in Mathematics, vol. 1099, Springer-Verlag, Berlin, 1984. MR MR774589 (87f:16027)
[33] R. W. Thomason, *Equivariant algebraic vs. topological K-homology Atiyah-Segal-style*, Duke Math. J. **56** (1988), no. 3, 589–636. MR MR948534 (89f:14015)
[34] B. Totaro, *The Chow ring of a classifying space*, Algebraic K-theory (Seattle, WA, 1997), Proc. Sympos. Pure Math., vol. 67, Amer. Math. Soc., Providence, RI, 1999, pp. 249–281. MR MR1743244 (2001f:14011)
[35] G. Zwara, *An orbit closure for a representation of the Kronecker quiver with bad singularities*, Colloq. Math. **97** (2003), no. 1, 81–86. MR MR2010544 (2004j:16018)

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