A CHEVALLEY FORMULA FOR THE EQUIVARIANT QUANTUM K-THEORY OF COMINUSCULE VARIETIES

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Abstract. We prove a type-uniform Chevalley formula for multiplication with divisor classes in the equivariant quantum K-theory ring of any cominuscule flag variety G/P. We also prove that multiplication with divisor classes determines the equivariant quantum K-theory of arbitrary flag varieties. These results prove a conjecture of Gorbounov and Korff concerning the equivariant quantum K-theory of Grassmannians of Lie type A.

1. Introduction

Let $X = G/P$ be a flag variety defined by a semisimple complex Lie group G and a parabolic subgroup P. The (small) *equivariant quantum K-theory ring* $\mathrm{QK}_T(X)$ of Givental [26] is a common generalization of the main cohomology theories considered in Schubert calculus, including K-theory, equivariant cohomology, and quantum cohomology. Equivariant quantum K-theory is the most general theory for which the associated Schubert structure constants have positivity properties that are either known [30, 51, 6, 3, 2] or conjectured [48, 49, 13]. In this paper we prove a Chevalley formula that combinatorially determines the ring $\mathrm{QK}_T(X)$ when X is a *cominuscule variety*, that is, a Grassmann variety $\mathrm{Gr}(m, n)$ of type A, a Lagrangian Grassmannian $\mathrm{LG}(n, 2n)$, a maximal orthogonal Grassmannian $\mathrm{OG}(n, 2n)$, a quadric hypersurface Q^n, or one of two exceptional varieties called the Cayley plane E_6/P_6 and the Freudenthal variety E_7/P_7.

The Schubert structure constants of the ring $\mathrm{QK}_T(X)$ are defined as polynomial expressions in the (equivariant, K-theoretic) *Gromov-Witten invariants* of X. The non-equivariant Gromov-Witten invariants used to define the quantum K-theory ring $\mathrm{QK}(X)$ can be computed for some spaces using the reconstruction theorem of Lee and Pandharipande [47] together with the K-theoretic J-function of Taipale [62]. However, infinitely many Gromov-Witten invariants are required to evaluate the product of two Schubert classes in $\mathrm{QK}(X)$, and as a result we do not know much about the structure of this ring for general flag varieties. Iritani, Milanov,

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and Tonita have recently used an enhanced reconstruction theorem to compute the quantum K-theory of the variety $\mathrm{SL}(3)/B$ of complete flags in $\mathbb{C}^3$ [36].

When X is a Grassmann variety of type A, the non-equivariant ring $\mathrm{QK}(X)$ is determined by a Pieri formula of Buch and Mihalcea for products that involve special Schubert classes [13]. More generally, when the Picard group of X has rank one, it has been proved by the authors of the present paper that all products of Schubert classes in the equivariant ring $\mathrm{QK}_T(X)$ contain only finitely many non-zero terms [10, 11]. As a consequence, the multiplication table of $\mathrm{QK}_T(X)$ can be determined from finitely many Gromov-Witten invariants.

Our main result is a type-uniform Chevalley formula that describes the product of an arbitrary Schubert class with the divisor class in $\mathrm{QK}_T(X)$ when X is any cominuscule variety. The Schubert classes on a cominuscule variety can be indexed by diagrams of boxes that generalize the Young diagrams known from the Schubert calculus of classical Grassmannians. Our Chevalley formula can be formulated in terms of adding or removing boxes from these diagrams, and can equivalently be formulated in terms of operations on Weyl group elements. Chevalley formulas have previously been proved for the equivariant K-theory ring $K^T(X)$ by Lenart and Postnikov [49], and for the equivariant quantum cohomology ring $\mathrm{QH}_T(X)$ by Mihalcea [52]; these formulas are valid for arbitrary flag varieties.

The Gromov-Witten invariants of any cominuscule variety X can be computed using the '*quantum equals classical*' theorem [12, 17, 13, 19], which states that any (3 point, genus zero) Gromov-Witten invariant of X is equal to a K-theoretic triple intersection on a related space. When X is a Grassmannian of type A, this related space is a two-step partial flag variety of kernel-span pairs [7]. Our Chevalley formula is proved using an alternative version of the quantum equals classical theorem, which expresses the Gromov-Witten invariants of X in terms of *projected Gromov-Witten varieties* [39, 9]. Here a projected Gromov-Witten variety is the closure of the union of all rational curves of a fixed degree that pass through two opposite Schubert varieties in X. Our proof shows that the quantum terms in the Chevalley formula can be obtained by applying a linear operator to the classical terms. We also provide an alternative geometric proof of Lenart and Postnikov's Chevalley formula [49], specialized to the equivariant K-theory of cominuscule varieties. This proof is type-uniform for all minuscule varieties, but requires specialized arguments for Lagrangian Grassmannians and quadrics of odd dimension. It would be interesting to investigate whether similar geometric arguments extend beyond the case of cominuscule flag varieties.

Gorbounov and Korff have recently used ideas from integrable systems to define an algebra $qh^*(\mathrm{Gr}(m,n))$ that they conjecture is isomorphic to the equivariant quantum K-theory ring $\mathrm{QK}_T(\mathrm{Gr}(m,n))$ of a Grassmannian of type A [28]. In our last section we use Molev-Sagan equations [53, 41] to prove, for any flag variety X, that the structure of the ring $\mathrm{QK}_T(X)$, including its Schubert structure constants and the underlying Gromov-Witten invariants of X, is uniquely determined by products involving divisors. Since Gorbounov and Korff prove that that their algebra $qh^*(\mathrm{Gr}(m,n))$ satisfies the same Chevalley formula as the one proved for equivariant quantum K-theory in this paper, we obtain a proof of their conjecture. In particular, a presentation and a Giambelli formula from [28] for the algebra $qh^*(\mathrm{Gr}(m,n))$ are also valid in equivariant quantum K-theory.

Our paper is organized as follows. In Section 2 we recall the definition of quantum K-theory for arbitrary flag varieties. Section 3 covers quantum K-theory of cominuscule varieties and proves the quantum part of our Chevalley formula. Two equivalent versions of this formula are given in Theorem 3.9 and Corollary 3.10, which are followed by examples. Section 4 contains our geometric proof of the Chevalley formula for the equivariant K-theory of cominuscule varieties. Finally, Section 5 proves that the equivariant quantum K-theory of any flag variety is uniquely determined by products with divisor classes, see Proposition 5.13.

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2. Quantum K-theory of flag varieties

2.1. **Equivariant K-theory.** All algebraic varieties and schemes in this paper will be defined over the complex numbers $\mathbb{C}$. Given a variety X with an action of an algebraic torus $T = (\mathbb{C}^*)^n$, we let $K_T(X)$ denote the Grothendieck group of T-equivariant coherent $\mathcal{O}_X$-modules. This group is a module over the Grothendieck ring $K^T(X)$ of T-equivariant vector bundles over X. If X is non-singular, then the implicit map $K^T(X) \to K_T(X)$ that sends a vector bundle to its sheaf of sections is an isomorphism. Our main references for equivariant K-theory are [21, Ch. 5] and [24, §15.1]. A shorter summary of the main properties can also be found in [13, §3]. Pullback along the structure morphism $X \to \{\text{point}\}$ gives $K^T(X)$ a structure of algebra over the ring $\Gamma = K^T(\text{point})$. This ring Γ is simply the ring of virtual representations of T. Given any character $\alpha : T \to \mathbb{C}^*$ we let $\mathbb{C}_\alpha$ denote the one-dimensional representation of T defined by $t.z = \alpha(t)z$ for $t \in T$ and $z \in \mathbb{C}_\alpha$. The ring Γ has a $\mathbb{Z}$-basis consisting of the classes $[\mathbb{C}_\alpha]$ of these representations. When X is projective we let $\chi_X : K_T(X) \to \Gamma$ denote the pushforward along the structure morphism of X. This is a homomorphism of Γ-modules by the projection formula.

2.2. **Schubert varieties.** Let $X = G/P$ be a flag variety defined by a complex semisimple linear algebraic group G and a parabolic subgroup P. Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$, and let $B^- \subset G$ be the opposite Borel subgroup defined by $B \cap B^- = T$. Let $W = N_G(T)/T$ be the Weyl group of G, $W_P = N_P(T)/T$ the Weyl group of P, and let $W^P \subset W$ be the set of minimal representatives of the cosets in W/W_P. Each element $w \in W$ defines a B-stable Schubert variety $X_w = \overline{Bw.P}$ and an (opposite) B^--stable Schubert variety $X^w = \overline{B^-w.P}$ in X. When $w \in W^P$ is a minimal representative we have $\dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$, where $\ell(w)$ denotes the length of w. A product $w = u_1u_2 \cdot \ldots \cdot u_m$ of elements in W will be called *reduced* if $\ell(w) = \sum \ell(u_i)$. This implies that any consecutive subproduct $u_iu_{i+1} \cdot \ldots \cdot u_j$ is also reduced. The dual element of $u \in W^P$ is $u^\vee = w_0uw_P$, where w_0 is the longest element in W and w_P is the longest element in W_P. Notice that $1^\vee$ is the longest element in W^P. Let Φ be the root system of (G, T), interpreted as a set of characters of T. Let Φ^+ be the set of positive roots determined by B, let $\Delta \subset \Phi^+$ be the simple roots, and let $\Delta_P \subset \Delta$ be the set of simple roots β for which the associated reflection s_β belongs to W_P. We need the following topological fact.

Lemma 2.1. *Let $Z \subset X$ be a T-stable closed subvariety and let $x_0 \in X^T$ be a T-fixed point such that $Z \cap B.x_0 \neq \emptyset$. Then $x_0 \in Z$.*

Proof. Choose any group homomorphism $\phi : \mathbb{C}^* \to T$ such that, for each $\alpha \in \Phi^+$ the composition $\alpha\,\phi : \mathbb{C}^* \to \mathbb{C}^*$ is given by $\alpha(\phi(s)) = s^m$ for some $m > 0$. Let $U_\alpha \subset G$ be the T-stable subgroup with $\mathrm{Lie}(U_\alpha) = \mathrm{Lie}(G)_\alpha$. It follows from [34, Thm. 26.3] that $\lim_{s\to 0} \phi(s)b\phi(s)^{-1} = 1$ for any element $b \in U_\alpha$, $\alpha \in \Phi^+$. Since B is generated by T and the subgroups U_α for $\alpha \in \Phi^+$, we deduce that $\lim_{s\to 0} \phi(s)b\phi(s)^{-1} \in T$ for any $b \in B$. Now choose $b \in B$ such that $b.x_0 \in Z$. We then obtain $x_0 = \left(\lim_{s\to 0} \phi(s)b\phi(s)^{-1}\right).x_0 = \lim_{s\to 0} \phi(s)b.x_0 \in Z$, as required. □

2.3. **Schubert structure constants.** The equivariant K-theory ring $K^T(X)$ of the flag variety X has a basis over Γ consisting of the (opposite) *Schubert classes* $\mathcal{O}^w = [\mathcal{O}_{X^w}]$ for $w \in W^P$. The classes $\mathcal{O}_w = [\mathcal{O}_{X_w}]$ form an alternative basis. The *Schubert structure constants* of $K^T(X)$ are the classes $N_{u,v}^{w,0} \in \Gamma$ defined for $u, v, w \in W^P$ by the identity

$$\mathcal{O}^u \cdot \mathcal{O}^v = \sum_w N_{u,v}^{w,0}\, \mathcal{O}^w\,.$$

Let $\mathcal{O}_w^\vee \in K^T(X)$ denote the basis element dual to $\mathcal{O}^w$, defined by $\chi_X(\mathcal{O}^u \cdot \mathcal{O}_w^\vee) = \delta_{u,w}$ for $u, w \in W^P$. We then have $N_{u,v}^{w,0} = \chi_X(\mathcal{O}^u \cdot \mathcal{O}^v \cdot \mathcal{O}_w^\vee)$. The *boundary* of the Schubert variety X_w is the closed subvariety defined by $\partial X_w = X_w \smallsetminus Bw.P$. Brion has proved the identity

$$\mathcal{O}_w^\vee = [I_{\partial X_w}]\,, \tag{1}$$

where $I_{\partial X_w} \subset \mathcal{O}_{X_w}$ denotes the ideal sheaf of this boundary [6].

Calculations in the ring $K^T(X)$ may be carried out by utilizing that restriction to the set of T-fixed points $X^T = \{w.P \mid w \in W^P\}$ provides an injective ring homomorphism $K^T(X) \to K^T(X^T) = \prod_{w\in W^P} \Gamma$ [44] (see also [29]). The images of Schubert classes under this map are given by the restriction formulas in [1, 4, 31, 63] (see also [38]). The ring structure of $K^T(X)$ is also determined by the Chevalley formula of Lenart and Postnikov [49], which provides an explicit expression for the product of any Schubert class with a divisor in $K^T(X)$. A version of this formula for cominuscule varieties will be discussed in Section 3.4. Other useful formulas for structure constants in various special cases can be found in e.g. [16, 37, 8, 56] and the references therein.

2.4. **Quantum K-theory.** A homology class $d = \sum d_\beta [X_{s_\beta}] \in H_2(X;\mathbb{Z})$, with the sum over $\beta \in \Delta \smallsetminus \Delta_P$, is called an *effective degree* if $d_\beta \geq 0$ for each β. For $d, e \in H_2(X;\mathbb{Z})$ we write $e \leq d$ if and only if $d - e$ is effective. For any effective degree d we let $\overline{\mathcal{M}}_{0,n}(X,d)$ denote the Kontsevich moduli space of n-pointed stable maps to X of genus zero and degree d [25]. This space is equipped with evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X$ for $1 \leq i \leq n$. Given classes $\sigma_1, \sigma_2, \ldots, \sigma_n \in K^T(X)$, define a corresponding (equivariant K-theoretic) Gromov-Witten invariant of X by

$$I_d(\sigma_1, \sigma_2, \ldots, \sigma_n) \;=\; \chi_{\overline{\mathcal{M}}_{0,n}(X,d)}\left(\mathrm{ev}_1^*(\sigma_1) \cdot \mathrm{ev}_2^*(\sigma_2) \cdot \ldots \cdot \mathrm{ev}_n^*(\sigma_n)\right) \;\in \Gamma\,.$$

This invariant is Γ-linear in each argument σ_i.

The (small) T-equivariant quantum K-theory ring of X is an algebra $\mathrm{QK}_T(X)$ over the ring of formal power series $\Gamma[\![q]\!] = \Gamma[\![q_\beta : \beta \in \Delta \smallsetminus \Delta_P]\!]$, which has one variable q_β for each simple root $\beta \in \Delta \smallsetminus \Delta_P$. As a module over $\Gamma[\![q]\!]$ we have

$\mathrm{QK}_T(X) = K^T(X) \otimes_\Gamma \Gamma[\![q]\!]$. In other words, $\mathrm{QK}_T(X)$ is a free module over $\Gamma[\![q]\!]$ with a basis consisting of the Schubert classes $\mathcal{O}^w$ for $w \in W^P$. For $d = \sum d_\beta [X_{s_\beta}]$ we write $q^d = \prod_\beta q_\beta^{d_\beta}$. Multiplication in $\mathrm{QK}_T(X)$ is defined by

$$\mathcal{O}^u \star \mathcal{O}^v = \sum_{w,d\geq 0} N_{u,v}^{w,d}\, q^d\, \mathcal{O}^w$$

where the sum is over all effective degrees d and $w \in W^P$. The structure constants $N_{u,v}^{w,d}$ are defined recursively by

$$N_{u,v}^{w,d} \;=\; I_d(\mathcal{O}^u, \mathcal{O}^v, \mathcal{O}_w^\vee) \;-\; \sum_{\kappa,\, 0<e\leq d} N_{u,v}^{\kappa,d-e}\, I_e(\mathcal{O}^\kappa, \mathcal{O}_w^\vee) \tag{2}$$

where this sum is over all $\kappa \in W^P$ and degrees e for which $0 < e \leq d$. A theorem of Givental states that this defines an associative product [26] (see also [64, 60, 42, 46]).

The ring $\mathrm{QK}_T(X)$ is a formal deformation of the equivariant K-theory ring $K^T(X)$. It is best understood when X is a cominuscule variety. This case will be discussed in Section 3. When the Picard group of X has rank one, it has been proved in [10, 11] that the product $\mathcal{O}^u \star \mathcal{O}^v$ contains only finitely many non-zero terms. It is an open question if this holds for general flag varieties. Combinatorial models for the equivariant quantum K-theory of complete flag manifolds G/B have been defined by Lenart and Maeno [48] and by Lenart and Postnikov [49]. These models are correct for $\mathbb{P}^1$ [13] and consistent with the computation of the quantum K-theory of $\mathrm{SL}(3)/B$ [36], but for all other spaces it is an open problem to prove that the models agree with the geometric definition of $\mathrm{QK}_T(X)$. In addition, the lack of functoriality of quantum K-theory makes it unclear how these models relate to cominuscule varieties. Another model for the equivariant quantum K-theory of Grassmannians of type A has been conjectured by Gorbounov and Korff [28]. This model is a consequence of the results proved in this paper (see Remark 5.15). Finally it is expected that the structure constants of $\mathrm{QK}_T(X)$ satisfy the following positivity property [32, 48, 49, 13].

Conjecture 2.2. *Let $u, v, w \in W^P$, let $d \in H_2(X;\mathbb{Z})$ be an effective degree, and set $s = \ell(u) + \ell(v) + \ell(w) + \int_d c_1(T_X)$. Then we have*

$$(-1)^s\, N_{u,v}^{w,d} \;\in\; \mathbb{N}\big[\, [\mathbb{C}_{-\beta}] - 1 : \beta \in \Delta \,\big]\,.$$

In other words, the structure constant $N_{u,v}^{w,d}$ can be written, up to a sign, as a polynomial with non-negative integer coefficients in the classes $[\mathbb{C}_{-\beta}] - 1$ for $\beta \in \Delta$. This has been proved for the structure constants $N_{u,v}^{w,0}$ of $K^T(X)$ by Anderson, Griffeth, and Miller [3]. Earlier it was proved by Brion [6] that the structure constants of the ordinary K-theory ring $K(X)$ have signs that alternate with codimension.

2.5. **Curve neighborhoods.** Given a closed subvariety $\Omega \subset X$ and an effective degree $d \in H_2(X;\mathbb{Z})$, we let $\Gamma_d(\Omega)$ denote the closure of the union of all rational curves of degree d in X that pass through Ω. Equivalently we have $\Gamma_d(\Omega) = \mathrm{ev}_2(\mathrm{ev}_1^{-1}(\Omega))$, where $\mathrm{ev}_1, \mathrm{ev}_2 : \overline{\mathcal{M}}_{0,2}(X,d) \to X$ are the evaluation maps. It was proved in [10] that $\Gamma_d(\Omega)$ is irreducible whenever Ω is irreducible. In particular, if Ω is a B-stable Schubert variety in X then so is $\Gamma_d(\Omega)$. Given $w \in W^P$ we may therefore define related Weyl group elements $w(d)$ and $w(-d)$ in W^P by the identities $\Gamma_d(X_w) = X_{w(d)}$ and $\Gamma_d(X^w) = X^{w(-d)}$. We have $w(-d) \leq w \leq w(d)$ in

the Bruhat order of W^P, and since X^w is a translate of $X_{w^\vee}$, we obtain $w(-d)^\vee = w^\vee(d)$. A combinatorial description of $w(d)$ and $w(-d)$ can be found in [10] or Section 3.2 when X is cominuscule and in [14] for general flag varieties.

It was proved in [10] that $I_d(\mathcal{O}^w) = 1$, i.e. any single-pointed Gromov-Witten invariant of a Schubert class is equal to one. More generally, it follows from [10, Prop. 3.2] that two-pointed Gromov-Witten invariants are given by the identity

$$I_d(\mathcal{O}^u, \mathcal{O}^\vee_w) \;=\; \chi_X(\mathcal{O}^{u(-d)} \cdot \mathcal{O}^\vee_w) \;=\; \delta_{u(-d),w}\,.$$

As a consequence, the definition (2) of the structure constants of $\mathrm{QK}_T(X)$ is equivalent to the identity

$$I_d(\mathcal{O}^u, \mathcal{O}^v, \mathcal{O}^\vee_w) \;=\; \sum_{0\le e\le d}\;\sum_{\kappa\colon \kappa(-e)=w} N^{\kappa,d-e}_{u,v}\,. \tag{3}$$

For any classes $\sigma_1, \sigma_2 \in K^T(X)$ we define the power series

$$\sigma_1 \odot \sigma_2 \;=\; \sum_{w,d\ge 0} I_d(\sigma_1, \sigma_2, \mathcal{O}^\vee_w)\, q^d\, \mathcal{O}^w$$

in $\mathrm{QK}_T(X)$. We also define an automorphism $\Psi : \mathrm{QK}_T(X) \to \mathrm{QK}_T(X)$ of $\Gamma[\![q]\!]$-modules by

$$\Psi(\mathcal{O}^w) \;=\; \sum_{e\ge 0} q^e\, \mathcal{O}^{w(-e)}\,.$$

Equation (3) is equivalent to the following statement.

Proposition 2.3. *For $\sigma_1, \sigma_2 \in K^T(X)$ we have $\sigma_1 \odot \sigma_2 = \Psi(\sigma_1 \star \sigma_2)$ in $\mathrm{QK}_T(X)$.*

3. Cominuscule varieties

3.1. **Bruhat order.** A simple root $\gamma \in \Delta$ is called *cominuscule* if, when the highest root is written as a linear combination of simple roots, the coefficient of γ is one. The flag variety $X = G/P$ is called cominuscule if $\Delta \smallsetminus \Delta_P$ consists of a single cominuscule root γ. If in addition the root system Φ is simply laced, then X is also called *minuscule*. This holds for all the cominuscule varieties listed in the introduction, except for Lagrangian Grassmannians and quadrics of odd dimension. We will assume that X is cominuscule in this section. In this case it was proved by Proctor that the Bruhat order on W^P is a distributive lattice that agrees with the left weak Bruhat order [58]. Stembridge has proved that all elements of W^P are *fully commutative*, which means that any reduced expression for an element of W^P can be obtained from any other by interchanging commuting simple reflections. We proceed to summarize the facts we need in more detail. Proofs of our claims can be found in [58, 61, 57, 16].

The root lattice $\mathrm{Span}_{\mathbb{Z}}(\Delta)$ has a partial order defined by $\alpha' \le \alpha$ if and only if $\alpha - \alpha'$ can be written as a sum of positive roots. Let $\mathcal{P}_X = \{\alpha \in \Phi \mid \alpha \ge \gamma\}$ be the set of positive roots α for which the coefficient of the cominuscule root γ is one, with the induced partial order. For any element $u \in W$ we let $I(u) = \{\alpha \in \Phi^+ \mid u.\alpha < 0\}$ denote the inversion set of u. We then have $\ell(u) = |I(u)|$. The assignment $u \mapsto I(u)$ restricts to a bijection between the elements of W^P and the lower order ideals of $\mathcal{P}_X$. This assignment is order preserving in the sense that $u \le v$ if and only if $I(u) \subset I(v)$. Given a lower order ideal $\lambda \subset \mathcal{P}_X$, let $\lambda = \{\alpha_1, \alpha_2, \ldots, \alpha_{|\lambda|}\}$ be any ordering of its elements that is compatible with the partial order $\le$, i.e. $\alpha_i < \alpha_j$ implies $i < j$. Then the element of W^P corresponding to λ is the product of

reflections $w_\lambda = s_{\alpha_1} s_{\alpha_2} \dots s_{\alpha_{|\lambda|}}$. The order ideal λ is a generalization of the Young diagrams and shifted Young diagrams known from the Schubert calculus of the classical Grassmannians. For this reason the roots in $\mathcal{P}_X$ will sometimes be called *boxes*, and the order ideal $I(u)$ will be called the *shape* of u.

Given two elements $u, w \in W^P$ with $u \leq w$, we will use the notation $w/u = wu^{-1} \in W$. Since the Bruhat order on W^P agrees with the left weak Bruhat order, we have $\ell(w/u) = \ell(w) - \ell(u)$. For any root $\alpha \in \mathcal{P}_X$, consider the order ideal $\lambda = \{\alpha' \in \mathcal{P}_X \mid \alpha' < \alpha\}$ of roots that are smaller than α, and set $\delta(\alpha) = w_\lambda.\alpha$. Then $s_{\delta(\alpha)} = w_\lambda s_\alpha w_\lambda^{-1} = w_{\lambda\cup\alpha}/w_\lambda$ has length one. It follows that $\delta : \mathcal{P}_X \to \Delta$ is a labeling of the boxes in $\mathcal{P}_X$ by simple roots. In addition, the element w/u depends only on the *skew shape* $I(w) \smallsetminus I(u)$. More precisely, if $I(w) \smallsetminus I(u) = \{\alpha_1, \alpha_2, \dots, \alpha_\ell\}$ is any ordering compatible with $\leq$, then $w/u = s_{\delta(\alpha_\ell)} \cdots s_{\delta(\alpha_2)} s_{\delta(\alpha_1)}$ is a reduced expression for w/u. In the special case $u = 1$, every reduced expression for w can be obtained in this way.

We will say that w/u is a *rook strip* if this element of W is a product of commuting simple reflections. Equivalently, no pair of roots in $I(w) \smallsetminus I(u)$ are comparable by the order $\leq$. We call w/u a *short rook strip* if it is a product of commuting reflections defined by short simple roots, i.e. $I(w) \smallsetminus I(u)$ consists of incomparable short roots. Notice that if the root system Φ is simply laced, then all roots are long by convention, so w/u is a short rook strip if and only if $w = u$.

The Bruhat order on W^P is a distributive lattice with operations $u \cap v$ and $u \cup v$ for $u, v \in W^P$. These operations are defined combinatorially by $I(u \cap v) = I(u) \cap I(v)$ and $I(u \cup v) = I(u) \cup I(v)$, and geometrically by $X_{u\cap v} = X_u \cap X_v$ and $X^{u\cup v} = X^u \cap X^v$. Notice that for any $u, z \in W^P$ we have $(u \cup z)/z = u/(u \cap z)$. Since dualization of Weyl group elements is an order-reversing involution of W^P, we also have $(u \cap v)^\vee = u^\vee \cup v^\vee$ and $(u \cup v)^\vee = u^\vee \cap v^\vee$.

3.2. **Curve neighborhoods.** Since P is a maximal parabolic subgroup of G it follows that $H_2(X;\mathbb{Z}) = \mathbb{Z}$. For $u \in W^P$ and $d \in \mathbb{N}$ we let $u(d), u(-d) \in W^P$ be the unique Weyl group elements for which $\Gamma_d(X_u) = X_{u(d)}$ and $\Gamma_d(X^u) = X^{u(-d)}$. Since X is cominuscule, we have $u(d)(d') = u(d+d')$ and $u(-d)(-d') = u(-d-d')$ for any effective degrees $d, d' \in \mathbb{N}$, see [17] or [10, Lemma 4.2]. However, it is easy to find examples where both $u(d')(-d)$ and $u(-d)(d')$ are different from $u(d'-d)$. It was proved in [10, Lemma 4.4] that $u(1)$ is the minimal representative of the coset $uw_P s_\gamma W_P$ whenever $X_u \neq X$. Equivalently, $u(-1)$ is the minimal representative of $us_\gamma W_P$ whenever $u \neq 1$.

For $d \geq 0$ we let $z_d \in W^P$ be the unique element such that $X_{z_d} = \Gamma_d(1.P)$. This element satisfies that $z_d w_P$ is inverse to itself, where w_P denotes the longest element in W_P. In fact, if we let $C \subset X$ be any stable curve of degree d from $1.P$ to $z_d.P$, then $(z_d w_P)^{-1}.C$ is a curve of the same degree from $1.P$ to $(z_d w_P)^{-1}.P$, so we obtain $(z_d w_P)^{-1}.P \in \Gamma_d(1.P)$. This implies that $(z_d w_P)^{-1} \leq z_d w_P$ and therefore $(z_d w_P)^{-1} = z_d w_P$. It follows that $z_d w_P$ is the maximal representative of the *left* coset $W_P z_d w_P$. If $X_{z_d} \neq X$, we deduce that $\mathcal{P}_X \smallsetminus I(z_d)$ contains a unique minimal box α_{d+1}, and this box satisfies $\delta(\alpha_{d+1}) = \gamma$. In other words we have $I(z_d) = \{\alpha \in \mathcal{P}_X \mid \alpha \not\geq \alpha_{d+1}\}$. Notice also that since $z_{d+1} = z_d(1)$ is the minimal representative of the coset $z_d w_P s_\gamma W_P$, we may choose $x \in W_P$ such that $z_{d+1} w_P = z_d w_P s_\gamma x^{-1}$ holds as an identity of reduced products. This implies that $z_{d+1} w_P = (z_{d+1} w_P)^{-1} = x s_\gamma (z_d w_P)^{-1} = x s_\gamma z_d w_P$, hence $z_{d+1} = x s_\gamma z_d$ expresses z_{d+1} as a reduced product. Since every reduced expression of z_{d+1} corresponds

to an ordering of the boxes of $I(z_{d+1})$, this implies that α_{d+1} is the only box of $I(z_{d+1}) \smallsetminus I(z_d)$ that is sent to γ by the map δ. In fact, we have $\delta^{-1}(\gamma) = \{\alpha_1 < \alpha_2 < \cdots < \alpha_{d_X(2)}\}$ where $d_X(2)$ is the smallest degree of a rational curve joining two general points in X.

Lemma 3.1. *For $u \in W^P$ and $d \in \mathbb{N}$ we have $u(-d) = u/(u \cap z_d) = (u \cup z_d)/z_d$ and $u(d) = (u \cap z_d^\vee)z_d$.*

Proof. For $d = 0$ the lemma follows because $u(0) = u$ and $z_0 = 1$. Let $d > 0$ and assume by induction that $u(1-d) = u/(u \cap z_{d-1})$. Notice that, if $u(1-d) = 1$, then the lemma holds for d because $u \leq z_{d-1} \leq z_d$ and $u(-d) = 1$. We may therefore assume that $u(1-d) \neq 1$, or equivalently $\alpha_d \in I(u)$.

Set $v = u/(u \cap z_d) = (u \cup z_d)/z_d$. Then we have an identity of reduced products $vz_dw_P = (u \cup z_d)w_P$. Since z_dw_P is maximal in its left coset $W_Pz_dw_P$, it follows that v is minimal in its right coset vW_P, in other words $v \in W^P$. Since $I(u \cap z_d) \smallsetminus I(u \cap z_{d-1}) \subset I(z_d) \smallsetminus I(z_{d-1})$ and $\alpha_d \in I(u)$, it follows that $(u \cap z_d)/(u \cap z_{d-1}) = ys_\gamma$ for some $y \in W_P$. Since $u(1-d) = u/(u \cap z_{d-1}) = v\,(u \cap z_d)/(u \cap z_{d-1}) = vys_\gamma$, we obtain $v = u(1-d)s_\gamma y^{-1}$. It follows that $v \in W^P$ is the minimal representative of the coset $u(1-d)s_\gamma W_P$, as required.

The last identity now follows because $u^\vee(d) = u(-d)^\vee = w_0(u \cup z_d)z_d^{-1}w_P = (u \cup z_d)^\vee(z_dw_P)^{-1}w_P = (u^\vee \cap z_d^\vee)z_d$. □

Table 1 displays one cominuscule variety X from each family together with the associated Dynkin diagram and the partially ordered set $\mathcal{P}_X$. The marked node in the Dynkin diagram indicates the cominuscule simple root γ. The roots of $\mathcal{P}_X$ are represented as boxes, and the partial order is given by $\alpha' \leq \alpha$ if and only if α' is located north-west of α. Each box $\alpha \in \mathcal{P}_X$ is labeled by the number of the simple root $\delta(\alpha)$. In addition the shape $I(z_1)$ is marked. Lemma 3.1 implies that, given any element $u \in W^P$, one may obtain the shape of $u(-1)$ from the shape of u by first removing any boxes contained in $I(z_1)$, and then moving the remaining boxes to the upper-left corner of $\mathcal{P}_X$. More details about some of the cases can be found in Example 3.12, Example 3.13, and Section 4.3.

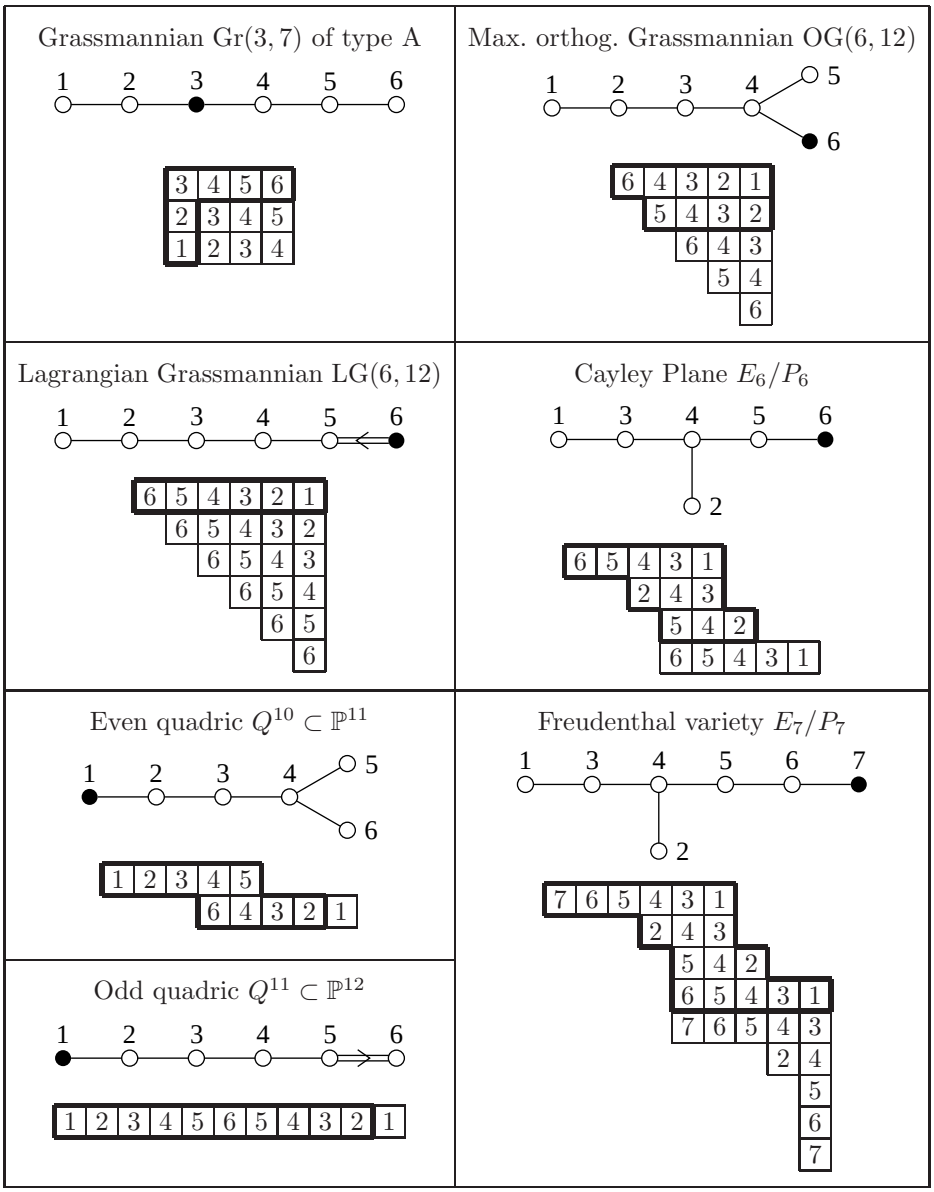
3.3. **Quantum K-theory.** Since X is cominuscule, the equivariant quantum K-theory ring $\mathrm{QK}_T(X)$ is an algebra over the power series ring $\Gamma[\![q]\!]$ in a single variable q_γ; we drop the subscript and denote this variable by q. We proceed to give a simplified construction of the product in $\mathrm{QK}_T(X)$ in the cominuscule case. Let $\psi : K^T(X) \to K^T(X)$ be the homomorphism of Γ-modules defined by $\psi(\mathcal{O}^u) = \mathcal{O}^{u(-1)}$. This map can also be defined geometrically by $\psi = (\mathrm{ev}_2)_*(\mathrm{ev}_1)^*$, where ev_1 and ev_2 are the evaluation maps from the moduli space $\overline{\mathcal{M}}_{0,2}(X, 1)$. We extend ψ by linearity to a homomorphism $\psi : \mathrm{QK}_T(X) \to \mathrm{QK}_T(X)$ of $\Gamma[\![q]\!]$-modules.

Proposition 3.2. *For $\sigma_1, \sigma_2 \in K^T(X)$ we have $\sigma_1 \star \sigma_2 = (1 - q\psi)(\sigma_1 \odot \sigma_2)$ in $\mathrm{QK}_T(X)$.*

Proof. The identity $w(-e-1) = w(-e)(-1)$ among Weyl group elements implies that the operator Ψ from Section 2.5 is given by $\Psi = 1 + q\psi + q^2\psi^2 + \cdots$. It therefore follows from Proposition 2.3 that $\sigma_1 \star \sigma_2 = \Psi^{-1}(\sigma_1 \odot \sigma_2) = (1 - q\psi)(\sigma_1 \odot \sigma_2)$. □

Let $\Gamma_d(X_u, X^v) \subset X$ denote the union of all stable curves of degree d in X that pass through X_u and X^v. Equivalently we have $\Gamma_d(X_u, X^v) = \mathrm{ev}_3(\mathrm{ev}_1^{-1}(X_u) \cap$

TABLE 1. Partially ordered sets of cominuscule varieties with $I(z_1)$ highlighted.



$\mathrm{ev}_2^{-1}(X^v)$), where $\mathrm{ev}_1, \mathrm{ev}_2, \mathrm{ev}_3 : \overline{\mathcal{M}}_{0,3}(X,d) \to X$ are the evaluation maps. The following version of the *quantum equals classical* theorem was proved in [9, Cor. 4.2].

Theorem 3.3. *For $u, v \in W^P$ we have in $K^T(X)$ that*

$$[\mathcal{O}_{\Gamma_d(X_u, X^v)}] \;=\; \sum_{w \in W^P} I_d(\mathcal{O}_u, \mathcal{O}^v, \mathcal{O}_w^\vee)\, \mathcal{O}^w \,.$$

As a consequence we have $\mathcal{O}_u \odot \mathcal{O}^v = \sum_{d \geq 0} [\mathcal{O}_{\Gamma_d(X_u, X^v)}]\, q^d$ for all $u, v \in W^P$.

3.4. **Chevalley formula for $K^T(X)$.** Let $J = 1 - \mathcal{O}^{s_\gamma} \in K^T(X)$ denote the class of the ideal sheaf of the (opposite) Schubert divisor X^{s_γ}. Our main result is an explicit combinatorial formula for any product of the form $J \star \mathcal{O}^u$ in the ring $\mathrm{QK}_T(X)$. For $u \in W^P$ we let $J_u = J|_{u.P} \in \Gamma$ denote the restriction of the class J to the T-fixed point $u.P \in X$. For $\alpha \in \Phi$ we let $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ denote the coroot of α, and we let ω_γ be the fundamental weight corresponding to γ. The following lemma is proved in Section 4.1.

Lemma 3.4. *We have $J_u = [\mathbb{C}_{u.\omega_\gamma - \omega_\gamma}] \in \Gamma$, and the weight satisfies the identity*

$$\omega_\gamma - u.\omega_\gamma = \sum_{\alpha \in I(u)} (\omega_\gamma, \alpha^\vee)\, \delta(\alpha)\,.$$

Define a homomorphism $\theta_0 : K^T(X) \to K^T(X)$ of Γ-modules by

$$\theta_0(\mathcal{O}^u) \;=\; \sum_{w/u \text{ rook strip}} (-1)^{\ell(w/u)}\, \mathcal{O}^w$$

where the sum is over all $w \in W^P$ for which $u \leq w$ and w/u is a rook strip. Equivalently, $\theta_0(\mathcal{O}^u)$ is the class of the ideal sheaf of the boundary $\partial X^u \subset X^u$ by Brion's identity (1) together with the following lemma.

Lemma 3.5. *The class $\theta_0(\mathcal{O}^u)$ is dual to $\mathcal{O}_u$ in $K^T(X)$. More precisely we have $\chi_X(\theta_0(\mathcal{O}^u) \cdot \mathcal{O}_v) = \delta_{u,v}$ for $u, v \in W^P$.*

Proof. We have $\mathcal{O}^u \cdot \mathcal{O}_v = [\mathcal{O}_{X^u \cap X_v}]$ in $K^T(X)$. Since $X^u \cap X_v$ is empty for $u \not\leq v$, and is rational with rational singularities for $u \leq v$ [59, 6], we obtain

$$\chi_X(\mathcal{O}^u \cdot \mathcal{O}_v) = \begin{cases} 1 & \text{if } u \leq v; \\ 0 & \text{if } u \not\leq v. \end{cases}$$

It follows that $\chi_X(\theta_0(\mathcal{O}^u), \mathcal{O}_v)$ is the sum of the signs $(-1)^{\ell(w/u)}$ over the set S of all $w \in W^P$ for which $u \leq w \leq v$ and w/u is a rook strip. This sum is equal to 1 if $u = v$, and it is empty if $u \not\leq v$. Assume that $u < v$ and let α be any minimal box of the skew shape $I(v) \smallsetminus I(u)$. Then the map $w \mapsto ws_\alpha$ is an involution of S. Since $\ell(ws_\alpha) = \ell(w) \pm 1$ for all $w \in S$, we have $(-1)^{\ell(w/u)} + (-1)^{\ell(ws_\alpha/u)} = 0$. This implies that $\chi_X(\theta_0(\mathcal{O}^u), \mathcal{O}_v) = 0$, as required. □

For elements $u, w \in W^P$ such that $u \leq w$ and w/u is a short rook strip, we define an element $\delta(w/u)$ of the root lattice by

$$\delta(w/u) = \sum_{\alpha \in I(w) \smallsetminus I(u)} \delta(\alpha)\,.$$

Equivalently, $\delta(w/u)$ is the sum of the (short) simple roots $\beta \in \Delta$ for which the reflection s_β appears in a reduced expression for w/u. Define an additional homomorphism of Γ-modules $\phi : K^T(X) \to K^T(X)$ by

$$\phi(\mathcal{O}^u) = \sum_{w/u \text{ short rook strip}} (-1)^{\ell(w/u)}\, [\mathbb{C}_{-\delta(w/u)}]\, \mathcal{O}^w$$

where the sum is over all $w \in W^P$ for which $u \leq w$ and w/u is a short rook strip. If the root system Φ is simply laced, then ϕ is the identity map. The following result is a special case of Lenart and Postnikov's Chevalley formula for the equivariant K-theory of arbitrary flag varieties [49].

Theorem 3.6. *Let $X = G/P$ be a cominuscule space. For any $u \in W^P$ we have*

$$J \cdot \mathcal{O}^u \;=\; J_u\, \theta_0(\phi(\mathcal{O}^u))$$

in $K^T(X)$.

Theorem 3.6 can be derived from Lenart and Postnikov's more general result as follows. Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be any ordering of the roots in $\mathcal{P}_X$ that is compatible with the partial order $\leq$, and let $\beta_1, \beta_2, \ldots, \beta_t$ be an ordering of the short roots of $\mathcal{P}_X$ that is compatible with $\leq$. Then one can show that the sequence $(\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_t)$ is what is called a λ-chain for the weight $\lambda = \omega_\gamma$ in [49], after which Theorem 3.6 is equivalent to [49, Thm. 13.1]. In Section 4 we give an alternative proof of Theorem 3.6 based on geometric considerations.

Notice that since the cominuscule root γ is long and the root system Φ does not have type G_2, we have $(\omega_\gamma, \alpha^\vee) = 2$ for any short root $\alpha \in \mathcal{P}_X$. Lemma 3.4 therefore implies that $J_w = J_u[\mathbb{C}_{-2\delta(w/u)}]$ whenever w/u is a short rook strip. For this reason we will use the notation $\sqrt{J_u J_w} = J_u[\mathbb{C}_{-\delta(w/u)}] = [\mathbb{C}_{u.\omega_\gamma - \omega_\gamma - \delta(w/u)}]$. The identity of Theorem 3.6 can be rewritten as

$$J \cdot \mathcal{O}^u \;=\; \sum_{w/u \text{ short rook strip}} (-1)^{\ell(w/u)} \sqrt{J_u J_w}\; \theta_0(\mathcal{O}^w)\,.$$

It therefore follows from Lemma 3.5 that Theorem 3.6 is equivalent to the following geometric identity, which is interesting by itself.

Theorem 3.7. *For $u, w \in W^P$ we have*

$$\chi_X(J \cdot \mathcal{O}^u \cdot \mathcal{O}_w) \;=\; \begin{cases} (-1)^{\ell(w/u)} \sqrt{J_u J_w} & \text{if } w/u \text{ is a short rook strip,} \\ 0 & \text{otherwise.} \end{cases}$$

3.5. **Chevalley formula for $\mathrm{QK}_T(X)$.** To state our Chevalley formula for the quantum K-theory ring, we define the homomorphism $\theta_1 : K^T(X) \to K^T(X)$ of Γ-modules as follows. Let $u \in W^P$. If $u \geq z_1$ then we set

$$\theta_1(\mathcal{O}^u) \;=\; \sum_w (-1)^{\ell(w/u(-1))}\, \mathcal{O}^w$$

where the sum is over all $w \in W^P$ for which $u(-1) \leq w \leq 1^\vee(-1) = z_1^\vee$ and $w/u(-1)$ is a rook strip. If $u \not\geq z_1$ then set $\theta_1(\mathcal{O}^u) = 0$.

Lemma 3.8. *We have $\theta_1 = \psi\, \theta_0$ as Γ-linear endomorphisms of $K^T(X)$.*

Proof. For any $u \in W^P$ we have

$$\psi(\theta_0(\mathcal{O}^u)) = \sum_{w/u \text{ rook strip}} (-1)^{\ell(w/u)}\, \mathcal{O}^{w(-1)}\,. \tag{4}$$

If $u \geq z_1$ then we have $w(-1) = w/z_1$ and $w(-1)/u(-1) = w/u$ for each element w in the sum. Since the map $w \mapsto w(-1)$ gives a bijection between the terms of (4) and the sum defining $\theta_1(\mathcal{O}^u)$, the lemma follows in this case. If $u \not\geq z_1$, then let α be a minimal box in the set $I(z_1) \smallsetminus I(u)$. Then the map $w \mapsto ws_\alpha$ defines an involution of the set of elements w appearing in (4). Since we have $\ell(ws_\alpha) = \ell(w) \pm 1$ and the identity $w(-1) = (w \cup z_1)/z_1$ implies that $w(-1) = (ws_\alpha)(-1)$, we deduce that $\psi(\theta_0(\mathcal{O}^u)) = 0$, as required. $\square$

Theorem 3.9. *Let $X = G/P$ be cominuscule. For $u \in W^P$ we have*

$$J \star \mathcal{O}^u \;=\; J_u\, \theta_0(\phi(\mathcal{O}^u)) \;-\; J_u\, q\, \theta_1(\phi(\mathcal{O}^u))$$

in $\mathrm{QK}_T(X)$.

Proof. Since all curves of positive degree in X meet the Schubert divisor X^{s_γ}, we have $\Gamma_d(X_u, X^{s_\gamma}) = \Gamma_d(X_u) = \Gamma_d(X_u, X)$ for all $d \geq 1$. Theorem 3.3 therefore implies that

$$I_d(\mathcal{O}_u, J, \mathcal{O}_w^\vee) \;=\; I_d(\mathcal{O}_u, 1, \mathcal{O}_w^\vee) - I_d(\mathcal{O}_u, \mathcal{O}^{s_\gamma}, \mathcal{O}_w^\vee) \;=\; 0$$

for all $u, w \in W^P$ and $d \geq 1$. By linearity we obtain $I_d(\sigma, J, \mathcal{O}_w^\vee) = 0$ for all $\sigma \in K^T(X)$ and $d \geq 1$, which is equivalent to the identity $\sigma \odot J = \sigma \cdot J$. We therefore obtain

$$\begin{aligned} J \star \mathcal{O}^u \;&=\; (1 - q\psi)(J \odot \mathcal{O}^u) \;=\; (1 - q\psi)(J \cdot \mathcal{O}^u) \\ &=\; (1 - q\psi)(J_u\, \theta_0(\phi(\mathcal{O}^u))) \;=\; J_u\, \theta_0(\phi(\mathcal{O}^u)) - J_u\, q\, \psi(\theta_0(\phi(\mathcal{O}^u))) \end{aligned}$$

so the result follows from Lemma 3.8. □

3.6. **Structure constants of the Chevalley formula.** Recall that a box α of a skew shape $I(w) \smallsetminus I(u)$ is *incomparable* if it is both a minimal and maximal box within this skew shape. Theorem 3.9 can be restated as follows.

Corollary 3.10. *Let $X = G/P$ be cominuscule. For $u \in W^P$ we have*

$$\mathcal{O}^{s_\gamma} \star \mathcal{O}^u = \sum_{w \in W^P} N_{s_\gamma, u}^{w,0}\, \mathcal{O}^w + \sum_{w \in W^P} N_{s_\gamma, u}^{w,1}\, q\, \mathcal{O}^w$$

where the constants are determined by $N_{s_\gamma, u}^{u,0} = 1 - J_u$ and the following rules.

For $u \neq w$ the constant $N_{s_\gamma, u}^{w,0}$ is non-zero if and only if $u < w$ and all non-maximal boxes of the skew shape $I(w) \smallsetminus I(u)$ are short minimal boxes. In this case

$$N_{s_\gamma, u}^{w,0} \;=\; (-1)^{\ell(w/u)-1}\, J_u \left(\prod_{\alpha} [\mathbb{C}_{-\delta(\alpha)}] \right) \left(\prod_{\alpha'} (1 + [\mathbb{C}_{-\delta(\alpha')}]) \right) \tag{5}$$

where the first product is over all non-maximal boxes α of $I(w) \smallsetminus I(u)$, and the second product is over all short incomparable boxes α' of $I(w) \smallsetminus I(u)$.

The constant $N_{s_\gamma, u}^{w,1}$ is non-zero if and only if $u(-1) \leq w \leq z_1^\vee$, all non-maximal boxes of the skew shape $I(w(1)) \smallsetminus I(u)$ are short minimal boxes, and all roots of $I(z_1) \smallsetminus I(u)$ are short minimal boxes of $I(w(1)) \smallsetminus I(u)$. In this case we have

$$N_{s_\gamma, u}^{w,1} \;=\; (-1)^{\ell(w(1)/u)}\, J_u \left(\prod_{\alpha} [\mathbb{C}_{-\delta(\alpha)}] \right) \left(\prod_{\alpha'} (1 + [\mathbb{C}_{-\delta(\alpha')}]) \right) \tag{6}$$

where the first product is over all boxes α of $I(w(1)) \smallsetminus I(u)$ that are either non-maximal or belong to $I(z_1)$, and the second product is over all short incomparable boxes α' of $I(w(1)) \smallsetminus I(u)$ for which $\alpha' \notin I(z_1)$.

Proof. Theorem 3.9 implies that the coefficient of $q\, \mathcal{O}^w$ in $J \star \mathcal{O}^u$ is given by

$$-\, N_{s_\gamma, u}^{w,1} \;=\; (-1)^{\ell(w)+\ell(z_1)+1-\ell(u)}\, J_u \sum_{v} [\mathbb{C}_{-\delta(v/u)}] \tag{7}$$

where the sum is over all $v \in W^P$ for which $\mathcal{O}^v$ appears in $\phi(\mathcal{O}^u)$ and $\mathcal{O}^w$ appears in $\theta_1(\mathcal{O}^v)$. The first condition says that $u \leq v$ and v/u is a short rook strip, and by Lemma 3.1 the second condition holds if and only if $w \leq z_1^\vee$, $z_1 \leq v \leq w(1)$, and

$w(1)/v$ is an arbitrary rook strip. Such elements v exist if and only if the conditions for $N^{w,1}_{s_\gamma,u} \neq 0$ in the corollary are satisfied, and in this case v appears in the sum if and only if $I(v)$ is the union of $I(u)$, the set of all roots α in the first product of (6), and an arbitrary subset of the roots α' in the second product. The identity (6) now follows because the last two products in this identity expand to the sum in equation (7). The proof of (5) is similar and left to the reader. □

Remark 3.11. Assume that the root system R is simply laced. Then there are no short boxes, and the statement of Corollary 3.10 simplifies considerably. Any coefficient $N^{w,0}_{s_\gamma,u}$ with $w \neq u$ is non-zero if and only if $u < w$ and w/u is a rook strip, in which case

$$N^{w,0}_{s_\gamma,u} \;=\; (-1)^{\ell(w/u)-1}\, J_u\,.$$

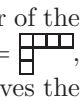
Similarly, $N^{w,1}_{s_\gamma,u}$ is non-zero if and only if $u(-1) \leq w \leq z_1^\vee$, $z_1 \leq u$, and $w(1)/u$ is a rook strip. In this case we have

$$N^{w,1}_{s_\gamma,u} \;=\; (-1)^{\ell(w(1)/u)}\, J_u\,.$$

We proceed to give two examples of the Chevalley formula in types A and C. More examples can be obtained using the *Equivariant Schubert Calculator* available at http://sites.math.rutgers.edu/~asbuch/equivcalc/.

Example 3.12. Let $X = \mathrm{Gr}(3,7) = \{V \subset \mathbb{C}^7 \mid \dim(V) = 3\}$ be the Grassmann variety of 3-planes in $\mathbb{C}^7$. This is a homogeneous space for the group $G = \mathrm{SL}(\mathbb{C}^7)$ of Lie type A_6. Let $T \subset G$ be the maximal torus of diagonal matrices and let $B \subset G$ be the Borel subgroup of upper triangular matrices. For $1 \leq i \leq 7$ we let $\varepsilon_i : T \to \mathbb{C}^*$ be the character that sends a diagonal matrix to its i-th entry. Then the set of simple roots of G is $\Delta = \{\varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq 6\}$, and all these simple roots are cominuscule. We can identify the homogeneous space G/P corresponding to $\gamma = \varepsilon_3 - \varepsilon_4$ with the Grassmannian X by the map $g.P \mapsto g.E_3$, where $E_3 = \mathrm{Span}\{e_1, e_2, e_3\}$ is the span of the first three standard basis vectors in $\mathbb{C}^7$. The partially ordered set $\mathcal{P}_X$ consists of the boxes of the following 3×4 rectangle.

γ	$\varepsilon_3-\varepsilon_5$	$\varepsilon_3-\varepsilon_6$	$\varepsilon_3-\varepsilon_7$
$\varepsilon_2-\varepsilon_4$	$\varepsilon_2-\varepsilon_5$	$\varepsilon_2-\varepsilon_6$	$\varepsilon_2-\varepsilon_7$
$\varepsilon_1-\varepsilon_4$	$\varepsilon_1-\varepsilon_5$	$\varepsilon_1-\varepsilon_6$	$\varepsilon_1-\varepsilon_7$

The partial order is given by $\varepsilon_i - \varepsilon_j \leq \varepsilon_k - \varepsilon_l$ if and only if $k \leq i$ and $j \leq l$, or equivalently, the box $\varepsilon_i - \varepsilon_j$ is north-west of $\varepsilon_k - \varepsilon_l$. The order ideals of $\mathcal{P}_X$ therefore correspond to Young diagrams λ contained in the upper-left corner of the rectangle. Notice that $\delta(\varepsilon_i - \varepsilon_j) = \varepsilon_{i+j-4} - \varepsilon_{i+j-3}$. Since we have $I(z_1) =$  , the map $w \mapsto w(-1)$ corresponds to the map of Young diagrams that removes the first row and the first column, and moves the remaining boxes one step north-west. Similarly, the map $w \mapsto w(1)$ corresponds to the map of Young diagrams that moves all boxes one step south-east, discards any boxes that leave the 3×4 rectangle, and adds the boxes of $I(z_1)$ to the result. If $w \leq z_1^\vee$ then no boxes need to be discarded; equivalently we have $w(1)(-1) = w$.

Consider the diagram $\mu =$, and let $w_\mu \in W^P$ be the corresponding Weyl group element. Using Lemma 3.4 we obtain $J_{w_\mu} = [\mathbb{C}_{\varepsilon_7+\varepsilon_5-\varepsilon_3-\varepsilon_1}]$. We will abuse notation and denote the Schubert class $\mathcal{O}^{w_\lambda}$ simply by λ. Since $\phi(\mathcal{O}^{w_\mu}) = \mathcal{O}^{w_\mu}$ and $I(w_\mu(-1)) =$, it follows from Theorem 3.9 that

$J \star$ $=$ $J_{w_\mu}\Big($ $-$ $-$ $+$ $- q$ $+ q$ $+ q$ $- q$ $\Big)$.

Equivalently, the expansion of the product $\mathcal{O}^{s_\gamma} \star \mathcal{O}^{w_\mu} = \mathcal{O}^{w_\mu} - J \star \mathcal{O}^{w_\mu}$ can be obtained using Remark 3.11.

Example 3.13 (Lagrangian Grassmannian)**.** Set $E = \mathbb{C}^{2n}$, let $\{e_1, e_2, \ldots, e_{2n}\}$ be the standard basis, and define a symplectic form on E by $(e_i, e_j) = -(e_j, e_i) = \delta_{i+j,2n+1}$ for $1 \le i \le j \le 2n$. The Lagrangian Grassmannian of maximal isotropic subspaces of E is the variety

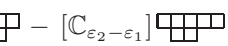
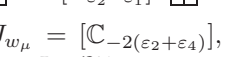
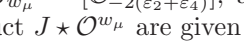
$$X = \mathrm{LG}(n, E) = \{V \subset E \mid \dim(V) = n \text{ and } (V, V) = 0\}\,.$$

This is a homogeneous space for the symplectic group $G = \mathrm{Sp}(E) = \{g \in \mathrm{GL}(E) \mid (g.x, g.y) = (x, y)\ \forall x, y \in E\}$ of Lie type C_n. Let $T \subset G$ be the maximal torus of diagonal matrices, and let $B \subset G$ be the Borel subgroup of upper triangular matrices. For $1 \le i \le 2n$ we let $\varepsilon_i : T \to \mathbb{C}^*$ be the character that sends a diagonal matrix to its i-th entry. Since all elements of T have the form $\mathrm{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1})$, we have $\varepsilon_{2n+1-i} = -\varepsilon_i$ for each i. The set of simple roots of G is $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$. This set contains a single cominuscule simple root $\gamma = 2\varepsilon_n$. The corresponding homogeneous space G/P can be identified with X by the map $g.P \mapsto g.E_n$, where $E_n = \mathrm{Span}\{e_1, \ldots, e_n\}$. The partially ordered set $\mathcal{P}_X$ consists of the boxes in the staircase diagram

γ	$\cdots$	$\varepsilon_3+\varepsilon_n$	$\varepsilon_2+\varepsilon_n$	$\varepsilon_1+\varepsilon_n$
	$\ddots$	$\vdots$	$\vdots$	$\vdots$
		$2\varepsilon_3$	$\varepsilon_2+\varepsilon_3$	$\varepsilon_1+\varepsilon_3$
			$2\varepsilon_2$	$\varepsilon_1+\varepsilon_2$
				$2\varepsilon_1$

where boxes increase from north-west to south-east as in Example 3.12. The long roots of $\mathcal{P}_X$ are the n diagonal boxes. Each element $u \in W^P$ corresponds to a shifted Young diagram $I(u)$ of boxes in the upper-left corner of the staircase diagram. Since $I(z_1)$ is the top row of boxes in $\mathcal{P}_X$, the map $w \mapsto w(-1)$ corresponds to the map of shifted Young diagrams that removes the top row and moves the remaining boxes one step north-west. The map $w \mapsto w(1)$ corresponds to the map of shifted Young diagrams that moves all boxes one step south-east, discards the boxes that leave the staircase diagram $\mathcal{P}_X$, and adds the boxes of $I(z_1)$ to the result. If $I(w)$ does not contain boxes in the right-most column of $\mathcal{P}_X$, then no boxes are discarded.

To illustrate the Chevalley formula, we consider the case $n = 5$ and $\mu =$. If we denote each Schubert class $\mathcal{O}^{w_\lambda}$ by λ as above, then we have

$\phi($ $) =$ $- [\mathbb{C}_{\varepsilon_2-\varepsilon_1}]$  $- [\mathbb{C}_{\varepsilon_4-\varepsilon_3}]$  $+ [\mathbb{C}_{\varepsilon_4-\varepsilon_3+\varepsilon_2-\varepsilon_1}]$  .

Lemma 3.4 gives $J_{w_\mu} = [\mathbb{C}_{-2(\varepsilon_2+\varepsilon_4)}]$, and Theorem 3.9 shows that the quantum terms of the product $J \star \mathcal{O}^{w_\mu}$ are given by $-J_{w_\mu} q\, \theta_1(\phi(\mathcal{O}^{w_\mu}))$, which expands to

$q[\mathbb{C}_{-\varepsilon_1-\varepsilon_2-2\varepsilon_4}]\,($ $)\, - q[\mathbb{C}_{-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4}]($ $)$.

Alternatively we can use Corollary 3.10 to compute the coefficients $N^{w_\nu,1}_{s_\gamma,w_\mu}$ in the expansion of $\mathcal{O}^{s_\gamma} \star \mathcal{O}^{w_\mu}$. For example, for $\nu =$ we obtain $I(w_\nu(1)) =$, and noting that $I(z_1) \smallsetminus \mu$ consists of the single short box $\varepsilon_1 + \varepsilon_5$, we obtain

$$N^{w_\nu,1}_{s_\gamma,w_\mu} \;=\; (-1)^3\, J_u\, [\mathbb{C}_{\varepsilon_2-\varepsilon_1}]\,(1 + [\mathbb{C}_{\varepsilon_4-\varepsilon_3}])\,.$$

4. Geometric proof of the Chevalley formula

4.1. **Minuscule varieties.** This section contains a geometric proof of Theorem 3.6. As in the previous section we assume that $X = G/P$ is a cominuscule variety, and $\gamma \in \Delta$ is the corresponding cominuscule simple root. While our proof is very simple when the root system Φ is simply laced, we need to resort to specialized arguments in the non-simply laced cases.

We may assume without loss of generality that G is simply connected, so that any integral weight of the root system Φ is represented by a character of T. The fundamental weight ω_γ extends to a character $\omega_\gamma : P \to \mathbb{C}^*$. Let $L = G \times^P \mathbb{C}_{-\omega_\gamma}$ be the associated line bundle on X. The total space of this bundle consists of all pairs $[g, z]$ with $g \in G$ and $z \in \mathbb{C}$, subject to the relation $[gp, \omega_\gamma(p)z] = [g, z]$ for all $p \in P$. We regard L as a G-equivariant line bundle with action defined by $g'.[g, z] = [g'g, z]$. In particular G acts on the vector space of global sections $H^0(X, L)$. According to the Borel-Weil theorem, $H^0(X, L)^*$ is an irreducible representation of G with highest weight ω_γ. It follows that there exists a B^--stable section $\sigma \in H^0(X, L)$ of weight $-\omega_\gamma$. Given any element $u \in W$, we will misuse notation and write $u.\sigma = \dot{u}.\sigma$, where $\dot{u} \in N_G(T)$ is a representative of u. The section $u.\sigma \in H^0(X, L)$ is well defined only up to scalar.

Proposition 4.1. *Let $u \in W^P$. The support of $Z(u.\sigma) \cap X^u$ is the boundary ∂X^u. If all minimal boxes of $\mathcal{P}_X \smallsetminus I(u)$ are long roots, then $Z(u.\sigma) \cap X^u = \partial X^u$ as (reduced) subschemes of X^u. In particular, if X is a minuscule variety, then the boundary ∂X^u is a Cartier divisor in X^u.*

Proof. Since σ is a B^--stable section of L, it follows that the zero section $Z(\sigma)$ is a B^--stable divisor, so we have $Z(\sigma) = X^{s_\gamma}$ as a set. It follows that $1.P \notin Z(\sigma)$, and hence $u.P \notin Z(u.\sigma)$, so Lemma 2.1 implies that $Z(u.\sigma) \cap X^u \subset \partial X^u$. Let $\tau \in H^0(X^u, L|_{X^u})$ denote the section obtained by restricting $u.\sigma$ to X^u. The cycle $[Z(\tau)]$ is a sum of prime divisors of X^u contained in the boundary ∂X^u. These prime divisors are exactly the Schubert varieties X^w for which w covers u in the Bruhat order on W^P, i.e. $u \leq w$ and $\ell(w) = \ell(u) + 1$. Furthermore, the classes $[X^w]$ of these prime divisors are linearly independent in $H_*(X^u; \mathbb{Z})$ as they map to linearly independent classes in $H_*(X; \mathbb{Z})$. For each $w \in W^P$ that covers u we have $I(w) \smallsetminus I(u) = \{\alpha\}$ for some root $\alpha \in \mathcal{P}_X$ such that $w = us_\alpha$, and it follows from the classical Chevalley formula [20] that the coefficient of $[X^w]$ in $[Z(\tau)]$ is equal to $(\omega_\gamma, \alpha^\vee)$. Since the cominuscule root γ is long, and since the coefficient of γ in

α is one, it follows that $(\omega_\gamma, \alpha^\vee)$ is equal to 1 if α is long and equal to 2 if α is short. In particular, τ vanishes along each Schubert divisor contained in ∂X^u, so we have $Z(u.\sigma) \cap X^u = Z(\tau) = \partial X^u$ as sets. If all minimal boxes of $\mathcal{P}_X \smallsetminus I(u)$ are long roots, then τ vanishes to the first order along each prime divisor $X^w \subset X^u$. In other words the ideal of $Z(\tau)$ agrees with the maximal ideal in each discrete valuation ring $\mathcal{O}_{X^w, X^u}$. By using that X^u is normal and the fact that any normal ring is the intersection of its localizations at prime ideals of codimension one, we deduce that $Z(\tau)$ is a reduced subscheme of X^u. This completes the proof. □

It follows from Proposition 4.1 that $X^{s_\gamma} = Z(\sigma)$ as a subscheme of X. Since σ has weight $-\omega_\gamma$ in $H^0(X, L)$, we may consider σ as a morphism $\sigma : X \times \mathbb{C}_{-\omega_\gamma} \to L$ of T-equivariant vector bundles on X. We obtain a T-equivariant exact sequence

$$0 \to L^\vee \otimes \mathbb{C}_{-\omega_\gamma} \to \mathcal{O}_X \to \mathcal{O}_{X^{s_\gamma}} \to 0$$

which reveals the identity $J = 1 - \mathcal{O}^{s_\gamma} = [L^\vee \otimes \mathbb{C}_{-\omega_\gamma}]$ in $K^T(X)$.

Proof of Lemma 3.4. The first identity of the lemma follows since $L|_{u.P} \cong \mathbb{C}_{-u.\omega_\gamma}$ as a representation of T. The second identity is clear if $u = 1$. If $u \neq 1$ then choose $u' \in W^P$ such that $u' \leq u$ and $\ell(u') = \ell(u) - 1$. Then $I(u) = I(u') \cup \{\alpha\}$ for some root $\alpha \in \mathcal{P}_X$ such that $u = u' s_\alpha$, and we have $\delta(\alpha) = u'.\alpha$. We obtain

$$\omega_\gamma - u.\omega_\gamma = \omega_\gamma - u'.(\omega_\gamma - (\omega_\gamma, \alpha^\vee)\, \alpha) = (\omega_\gamma - u'.\omega_\gamma) + (\omega_\gamma, \alpha^\vee)\, \delta(\alpha)\,,$$

so the required identity follows by induction on $\ell(u)$. □

The following result implies Theorem 3.6 for all minuscule varieties.

Proposition 4.2. *Theorem 3.7 is true whenever all minimal boxes of $\mathcal{P}_X \smallsetminus I(u)$ are long roots.*

Proof. In the situation of the proposition, Theorem 3.7 states that $J_u^{-1} J\, \mathcal{O}^u$ is dual to $\mathcal{O}_u$. Proposition 4.1 implies that $I_{\partial X^u} \cong (L^\vee \otimes \mathbb{C}_{-u.\omega_\gamma})|_{X^u}$, so $J_u^{-1} J\, \mathcal{O}^u = [L^\vee \otimes \mathbb{C}_{-u.\omega_\gamma}]\, \mathcal{O}^u = [I_{\partial X^u}]$ is dual to $\mathcal{O}_u$ by Brion's identity (1). □

4.2. **Lagrangian Grassmannians.** We next prove Theorem 3.7 for Lagrangian Grassmannians $X = \mathrm{LG}(n, E)$, using the notation introduced in Example 3.13.

Let $\mathbb{P}E$ be the projective space of lines through the origin in E, set $\mathrm{SF}(1, n; E) = \{(L, V) \in \mathbb{P}E \times X \mid L \subset V\}$, and let $\pi : \mathrm{SF}(1, n; E) \to X$ and $\varphi : \mathrm{SF}(1, n; E) \to \mathbb{P}E$ be the projections. Both of these projections are T-equivariant maps. Set $E^n = \mathrm{Span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$ and notice that the opposite Schubert divisor in X is defined by $X^{s_\gamma} = \{V \in X \mid V \cap E^n \neq 0\} = \pi(\varphi^{-1}(\mathbb{P}E^n))$. Since π maps $\varphi^{-1}(\mathbb{P}E^n)$ birationally onto X^{s_γ} we obtain $\mathcal{O}^{s_\gamma} = \pi_* \varphi^* [\mathcal{O}_{\mathbb{P}E^n}]$ in $K^T(X)$.

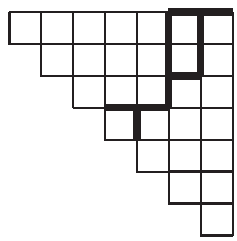
Let $u, w \in W^P$ be elements satisfying $u \leq w$. Then we have $\mathcal{O}^u \cdot \mathcal{O}_w = [\mathcal{O}_{X^u \cap X_w}]$ and $\chi_X(\mathcal{O}^u \cdot \mathcal{O}_w) = 1$. Set $Z_w^u = \varphi(\pi^{-1}(X^u \cap X_w)) \subset \mathbb{P}E$. It follows from [15, Lemma 5.1] or [40, Thm. 4.5] that $\varphi_* \pi^*(\mathcal{O}^u \cdot \mathcal{O}_w) = [\mathcal{O}_{Z_w^u}]$ in $K^T(\mathbb{P}E)$. By the projection formula we therefore obtain

$$\chi_X(J \cdot \mathcal{O}^u \cdot \mathcal{O}_w) = \chi_X(\mathcal{O}^u \cdot \mathcal{O}_w) - \chi_X(\mathcal{O}^{s_\gamma} \cdot \mathcal{O}^u \cdot \mathcal{O}_w) = 1 - \chi_{\mathbb{P}E}([\mathcal{O}_{\mathbb{P}E^n}] \cdot [\mathcal{O}_{Z_w^u}])\,.$$

It is known from [15, Lemma 5.2] that Z_w^u is a complete intersection in $\mathbb{P}E$ defined by linear and quadratic equations. We need the following explicit description of these equations, which is a consequence of the proof given in [15]. Notice that the south-east boundary of the shape $I(u)$ is a path of n line segments from the upper-right corner of the staircase diagram $\mathcal{P}_X$ to its diagonal (see Example 4.3).

The k-th step of this path will be called the k-th *boundary segment* of u. Let $\mathbb{C}[x_1, \ldots, x_{2n}]$ be the projective coordinate ring of $\mathbb{P}E$, with x_i dual to the basis element $e_i \in E$. The linear equations of Z_w^u correspond to the boundary segments that u and w have in common. Assume that the k-th boundary segment of u is also a boundary segment of w. If this boundary segment is horizontal, then it contributes the linear equation $x_{2n+1-k} = 0$, and if it is vertical, then it contributes the equation $x_k = 0$. Two boxes in the staircase diagram are *connected* if they share a side. The quadratic equations of Z_w^u correspond to connected components of the skew shape $I(w) \smallsetminus I(u)$ which do not contain any (long) diagonal boxes of $\mathcal{P}_X$. These components will be called *short components* of w/u. Assume that the north-west side of a short component consists of the boundary segments of u numbered from i to j. Then the component contributes the quadratic equation $f_{ij} = x_i x_{2n+1-i} + x_{i+1} x_{2n-i} + \cdots + x_j x_{2n+1-j} = 0$.

Example 4.3. Let $X = \mathrm{LG}(7, 14)$ be the Lagrangian Grassmannian of type C_7 and let $u \leq w$ be the elements of W^P corresponding to the marked boundaries.



Then $Z_w^u \subset \mathbb{P}^{13}$ is defined by $x_5 = x_9 = x_{14} = x_2 x_{13} + x_3 x_{12} + x_4 x_{11} = 0$.

Let $\mathcal{O}_{\mathbb{P}E}(-1) \subset \mathbb{P}E \times E$ denote the tautological subbundle, with the equivariant structure defined by the action of G on E, and set $h = [\mathcal{O}_{\mathbb{P}E}(-1)] \in K^T(\mathbb{P}E)$. Since T acts on x_i with weight $-\varepsilon_i$ and on f_{ij} with weight $\varepsilon_i + \varepsilon_{2n+1-i} = 0$, we have $[\mathcal{O}_{Z(x_i)}] = 1 - [\mathbb{C}_{-\varepsilon_i}]h$ and $[\mathcal{O}_{Z(f_{ij})}] = 1 - h^2$ in $K^T(\mathbb{P}E)$. Let l be the number of boundary segments shared by u and w, let $\mu_1, \mu_2, \ldots, \mu_l$ be the weights of the corresponding linear equations, and let q be the number of short components of w/u. We then have

$$[\mathcal{O}_{\mathbb{P}E^n}] \cdot [\mathcal{O}_{Z_w^u}] = \left(\prod_{j=1}^{n} (1 - [\mathbb{C}_{-\varepsilon_j}]h) \right) \left(\prod_{i=1}^{l} (1 - [\mathbb{C}_{\mu_i}]h) \right) (1 - h^2)^q \;=\; 1 + \sum_{k=1}^{n+l+2q} a_k \, h^k$$

for some classes $a_k \in \Gamma$. The leading coefficient is $a_{n+l+2q} = (-1)^{n+l+q} [\mathbb{C}_{\nu(u,w)}]$, where $\nu(u, w) = \sum_{i=1}^{l} \mu_i - \sum_{j=1}^{n} \varepsilon_j$.

Lemma 4.4. *Let F be any representation of T of dimension d, and consider the sheaf Euler characteristic map $\chi_{\mathbb{P}F} : K^T(\mathbb{P}F) \to \Gamma$. Then we have*

$$\chi_{\mathbb{P}F}([\mathcal{O}_{\mathbb{P}F}(-k)]) \;=\; \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } 0 < k < d, \text{ and} \\ (-1)^{d-1} \left[\bigwedge^d F\right] & \text{if } k = d. \end{cases}$$

Proof. This follows from Serre duality [33, III.7.7] because the canonical sheaf on $\mathbb{P}F$ is given by $\omega_{\mathbb{P}F} = (\bigwedge^d F^*) \otimes \mathcal{O}_{\mathbb{P}F}(-d)$ as a T-equivariant vector bundle. $\square$

Since each short component of w/u occupies at least two of the boundary segments of u, we have $l + 2q \leq n$, with equality if and only if w/u is a short rook strip, and in this case we have $\ell(w/u) \equiv n + l + q \pmod 2$. Since $[\bigwedge^{2n} E] = 1 \in \Gamma$ we obtain

$$\chi_X(J \cdot \mathcal{O}^u \cdot \mathcal{O}_w) \;=\; \begin{cases} (-1)^{\ell(w/u)}\,[\mathbb{C}_{\nu(u,w)}] & \text{if } w/u \text{ is a short rook strip;} \\ 0 & \text{otherwise.} \end{cases}$$

As special cases we have $J_u = \chi_X(J \cdot \mathcal{O}^u \cdot \mathcal{O}_u) = [\mathbb{C}_{\nu(u,u)}]$ and $J_w = [\mathbb{C}_{\nu(w,w)}]$. To finish the proof of Theorem 3.7 it is therefore enough to show that $2\,\nu(u,w) = \nu(u,u) + \nu(w,w)$ whenever w/u is a short rook strip. To see this, let $\varepsilon_i + \varepsilon_j \in I(w) \smallsetminus I(u)$, and set $k = n+i-j$. Then the k-th boundary segment of u is horizontal while the k-th boundary segment of w is vertical, so Z_u^u satisfies the equation $x_{2n+1-k} = 0$ while Z_w^w satisfies the equation $x_k = 0$. It follows that the weights $-\varepsilon_{2n+1-k}$ and $-\varepsilon_k$ of these equations cancel out in the sum $\nu(u,u) + \nu(w,w)$, and the same happens for the weights corresponding to the $k+1$-st boundary segments. This shows that $2\nu(u,w) = \nu(u,u) + \nu(w,w)$ and completes the proof.

4.3. **Odd quadric hypersurfaces.** We finally prove Theorem 3.6 for odd quadric hypersurfaces, which is the last case needed to complete the proof of the Chevalley formula. Set $E = \mathbb{C}^{2n+1}$, with basis $\{e_1, e_2, \ldots, e_{2n+1}\}$, and define an orthogonal form on E by $(e_i, e_j) = \delta_{i+j,2n+2}$. The corresponding symmetry group $G = \mathrm{SO}(E)$ has Lie type B_n. Let $T \subset G$ be the maximal torus of of diagonal matrices, and let $B \subset G$ be the Borel subgroup of upper triangular matrices. For $1 \leq i \leq 2n+1$ we let $\varepsilon_i : T \to \mathbb{C}^*$ be the character that maps a diagonal matrix to its i-th entry. We then have $\varepsilon_{n+1} = 0$ and $\varepsilon_i + \varepsilon_{2n+2-i} = 0$ for $1 \leq i \leq 2n+1$. The set of simple roots is $\Delta = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$, the unique cominuscule simple root is $\gamma = \varepsilon_1 - \varepsilon_2$, and $\omega_\gamma = \varepsilon_1$. Let $\mathbb{C}[x_1, \ldots, x_{2n+1}]$ be the coordinate ring of E and set $q = x_1x_{2n+1} + x_2x_{2n} + \cdots + x_nx_{n+2} + \frac{1}{2}x_{n+1}^2$. The cominuscule variety G/P corresponding to γ can be identified with the quadric hypersurface $X = Z(q) \subset \mathbb{P}E$ of dimension $2n-1$. The set $\mathcal{P}_X = \{\varepsilon_1 - \varepsilon_i \mid 2 \leq i \leq 2n\}$ is represented by the following diagram where the boxes increase from left to right.

γ	$\varepsilon_1{-}\varepsilon_3$	$\cdots$	$\varepsilon_1{-}\varepsilon_n$	ε_1	$\varepsilon_1{+}\varepsilon_n$	$\cdots$	$\varepsilon_1{+}\varepsilon_3$	$\varepsilon_1{+}\varepsilon_2$

The T-fixed points of X are the points $\langle e_k \rangle = \mathbb{C}e_k$ for $1 \leq k \leq 2n+1$ and $k \neq n+1$. For $k \in [0, 2n-1]$ we let X^k denote the B^--stable Schubert variety of codimension k in X. More precisely we have $X^k = \overline{B^-.\langle e_{k+1} \rangle} = Z(x_1, \ldots, x_k, q)$ for $0 \leq k \leq n-1$, and $X^k = \overline{B^-.\langle e_{k+2} \rangle} = Z(x_1, \ldots, x_{k+1})$ for $n \leq k \leq 2n-1$. Set $\mathcal{O}^k = [\mathcal{O}_{X^k}]$ and let $J_k \in K^T(\text{point})$ be the restriction of J to the T-fixed point defining X^k. Since ε_1 is the only short box in $\mathcal{P}_X$, it follows from Proposition 4.2 that the Chevalley formula $J \cdot \mathcal{O}^k = J_k\, \theta_0(\phi(\mathcal{O}^k))$ holds whenever $k \neq n-1$.

Let $\iota : X \to \mathbb{P}E$ be the inclusion and set $\widehat{J} = [\mathbb{C}_{-\varepsilon_1} \otimes \mathcal{O}_{\mathbb{P}E}(-1)] \in K^T(\mathbb{P}E)$. Then $\iota^*(\widehat{J}) = J$ in $K^T(X)$. Since the pushforward map $\iota_* : K^T(X) \to K^T(\mathbb{P}E)$ is injective, it is enough to show that the identity $\widehat{J} \cdot \mathcal{O}^{n-1} = J_{n-1}\; \theta_0(\phi(\mathcal{O}^{n-1}))$ holds in $K^T(\mathbb{P}E)$. Since ε_n is the only short simple root, we deduce that $\delta(\varepsilon_1) = \varepsilon_n$ and $\phi(\mathcal{O}^{n-1}) = \mathcal{O}^{n-1} - [\mathbb{C}_{-\varepsilon_n}]\,\mathcal{O}^n$. Using that $[\mathcal{O}_{Z(x_i)}] = 1 - [\mathbb{C}_{-\varepsilon_i}]\,[\mathcal{O}_{\mathbb{P}E}(-1)]$ in

$K^T(\mathbb{P}E)$ for each i we also have

$$\begin{aligned}[\mathcal{O}_{Z(q)}] &= 1 - [\mathcal{O}_{\mathbb{P}E}(-2)] = 1 - [\mathcal{O}_{\mathbb{P}E}(-1)] + [\mathcal{O}_{\mathbb{P}E}(-1)] \cdot [\mathcal{O}_{Z(x_{n+1})}] \\ &= [\mathcal{O}_{Z(x_{n+1})}] + [\mathbb{C}_{-\varepsilon_n}]\,(1 - [\mathcal{O}_{Z(x_{n+2})}]) \cdot [\mathcal{O}_{Z(x_{n+1})}] \\ &= [\mathcal{O}_{Z(x_{n+1})}] + [\mathbb{C}_{-\varepsilon_n}]\,[\mathcal{O}_{Z(x_{n+1})}] - [\mathbb{C}_{-\varepsilon_n}]\,[\mathcal{O}_{Z(x_{n+1},x_{n+2})}]\,.\end{aligned}$$

Since $X^{n-1} = Z(x_1, \ldots, x_{n-1}) \cap Z(q)$ and $J_{n-1} = \widehat{J}|_{\langle e_n \rangle} = [\mathbb{C}_{\varepsilon_n - \varepsilon_1}]$, we obtain

$$\begin{aligned}\widehat{J} \cdot \mathcal{O}^{n-1} &= J_{n-1}(1 - [\mathcal{O}_{Z(x_n)}]) \cdot \mathcal{O}^{n-1} = J_{n-1}\mathcal{O}^{n-1} - J_{n-1}[\mathcal{O}_{Z(x_1,\ldots,x_n)}] \cdot [\mathcal{O}_{Z(q)}] \\ &= J_{n-1}\mathcal{O}^{n-1} - J_{n-1}(\mathcal{O}^n + [\mathbb{C}_{-\varepsilon_n}]\mathcal{O}^n - [\mathbb{C}_{-\varepsilon_n}]\mathcal{O}^{n+1}) = J_{n-1}\,\theta_0(\phi(\mathcal{O}^{n-1}))\,.\end{aligned}$$

This completes the proof of Theorem 3.6.

5. Solutions to Molev-Sagan Equations

5.1. **Recursive identities.** In this section we let $X = G/P$ be an arbitrary flag manifold and show that the T-equivariant quantum K-theory ring $\mathrm{QK}_T(X)$, together with its Schubert structure constants and the underlying (three point, genus zero) Gromov-Witten invariants of X, are uniquely determined by the products that involve divisor classes. Here T denotes a maximal torus in G.

It was observed by Molev and Sagan [53] that the multiplicative structure constants of factorial Schur polynomials satisfy recursive identities that we will call *Molev-Sagan equations*. Related properties of factorial Schur polynomials had previously been studied by Okounkov and Olshanski [54, 55]. Knutson and Tao used the Molev-Sagan equations to study the equivariant Schubert structure constants of Grassmannians [41]. In this context the identities follow from the Chevalley formula together with the fact that the equivariant cohomology ring is associative. It was shown in [52] that the same method can be applied to arbitrary flag varieties. Molev-Sagan type equations have by now been used to prove Littlewood-Richardson rules in several papers [53, 41, 8, 56].

The Molev-Sagan equations provide a triangular system of identities that reduce arbitrary equivariant Schubert structure constants $C^w_{u,v}$ to the special constants of the form $C^w_{w,w}$, which are given by a simple formula of Kostant and Kumar [43]. The Chevalley formula of Mihalcea [52] for the equivariant quantum cohomology ring $\mathrm{QH}_T(X)$ can be used in a similar way to produce equations satisfied by the equivariant Gromov-Witten invariants of X, but in this case the equations are no longer triangular, and no analogue of the formula for $C^w_{w,w}$ is known. This added complexity was handled in [50, 52] with more refined recursive identities that reduce all (3 point, genus zero) equivariant Gromov-Witten invariants to those associated to multiplication with the identity class. This leads to the surprising realization that, even though the ring $\mathrm{QH}_T(X)$ may not be generated by divisor classes, the structure of this ring is completely determined by its Chevalley formula, and the same conclusion holds for the equivariant cohomology ring $H^*_T(X;\mathbb{Z})$.[1]

In this section we show that the equivariant quantum K-theory ring $\mathrm{QK}_T(X)$ is similarly determined by multiplication with divisor classes, despite the fact that a formula for multiplication with divisors is known only when X is cominuscule. We will derive this result as a formal consequence of the same property of the equivariant cohomology ring $H^*_T(X;\mathbb{Z})$. For this reason we start by considering

[1]After this paper was finished we learned from Ciocan-Fontanine that this can be explained using a localization argument. See Remark 5.10 for details.

this case, where we reprove and slightly extend some results from [52] in a form that will be useful later. This part of our analysis is also valid for the equivariant cohomology of any Kac-Moody flag variety, see Remark 5.9.

5.2. **Equivariant cohomology.** Let $\Lambda = H_T^*(\text{point};\mathbb{Z})$ denote the equivariant cohomology of a point and recall the notation from Section 2. The ring $H_T^*(X;\mathbb{Z})$ is a free Λ-module with a basis of the Schubert classes $[X^u]$ for $u \in W^P$. The *equivariant Schubert structure constants* of X are the classes $C_{u,v}^w \in \Lambda$, defined for $u, v, w \in W^P$ by the identity

$$[X^u] \cdot [X^v] \;=\; \sum_w C_{u,v}^w \, [X^w]$$

in $H_T^*(X;\mathbb{Z})$. The constant $C_{u,v}^w$ is non-zero only if $u \leq w$ and $v \leq w$ in the Bruhat order of W^P.

For any character $\lambda : T \to \mathbb{C}^*$ we set $c_T(\lambda) = c_1(\mathbb{C}_\lambda) \in \Lambda$. The ring Λ is generated by these classes, and the classes $c_T(\beta)$ for $\beta \in \Delta$ are algebraically independent. A class in Λ is *non-negative* if it can be written as a polynomial with non-negative integer coefficients in the classes $c_T(\beta)$ for $\beta \in \Delta$. Graham has proved that all equivariant structure constants $C_{u,v}^w$ are non-negative classes [30]. For $\beta \in \Delta$ we let ω_β denote the corresponding fundamental weight. Then the Chevalley formula for $H_T^*(X;\mathbb{Z})$ [20, 43] states that for $\beta \in \Delta \smallsetminus \Delta_P$ and $u, w \in W^P$ we have

$$C_{s_\beta,u}^w = \begin{cases} c_T(\omega_\beta - u.\omega_\beta) & \text{if } w = u, \\ (\omega_\beta, \alpha^\vee) & \text{if } \ell(w) = \ell(u) + 1 \text{ and } \exists\, \alpha \in \Phi^+ \text{ with } w = us_\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

The following was proved in [52]. We summarize the proof for completeness.

Lemma 5.1. *Let $u, w \in W^P$ be minimal representatives.*

(a) *If $u \leq w$, then $C_{s_\beta,w}^w - C_{s_\beta,u}^u$ is a non-negative class in Λ for every $\beta \in \Delta \smallsetminus \Delta_P$.*

(b) *If $u \neq w$, then there exists $\beta \in \Delta \smallsetminus \Delta_P$ such that $C_{s_\beta,w}^w \neq C_{s_\beta,u}^u$.*

(c) *If $u < w$ and $C_{s_\beta,w}^w \neq C_{s_\beta,u}^u$, then there exists $u' \in W^P$ such that $u < u' \leq w$, $\ell(u') = \ell(u) + 1$, and $C_{s_\beta,u}^{u'} \neq 0$.*

Proof. If $u \neq w$, then choose $\beta \in \Delta \smallsetminus \Delta_P$ such that s_β occurs in a reduced expression for $u^{-1}w$. Then it follows from [35, Thm. 1.12] that $u.\omega_\beta \neq w.\omega_\beta$. This proves part (b). Assume now that $u < w$. Choose $v \in W^P$ such that $u < v \leq w$ and $\ell(v) = \ell(u) + 1$, and choose $\alpha \in \Phi^+$ such that $v = us_\alpha$. Then $u.\alpha \in \Phi^+$. For any $\beta \in \Delta \smallsetminus \Delta_P$ we have $us_\alpha.\omega_\beta = u.(\omega_\beta - (\omega_\beta, \alpha^\vee)\alpha) = u.\omega_\beta - (\omega_\beta, \alpha^\vee)u.\alpha$, so $C_{s_\beta,v}^v - C_{s_\beta,u}^u = c_T(u.\omega_\beta - v.\omega_\beta) = (\omega_\beta, \alpha^\vee)c_T(u.\alpha)$ is a non-negative class in Λ. Part (a) follows from this by induction on $\ell(w) - \ell(u)$. Notice also that, when $u < v$ and $\ell(v) = \ell(u) + 1$, we have $C_{s_\beta,u}^v \neq 0$ if and only if $u.\omega_\beta \neq v.\omega_\beta$. To prove part (c), assume that $u.\omega_\beta = v.\omega_\beta \neq w.\omega_\beta$. By induction on $\ell(w) - \ell(u)$ we can find $v' \in W^P$ such that $v < v' \leq w$ and $C_{s_\beta,v}^{v'} \neq 0$. By part (b) we may choose $\gamma \in \Delta \smallsetminus \Delta_P$ with $C_{s_\gamma,u}^v \neq 0$. This implies that $[X^{v'}]$ occurs with non-zero coefficient in the product $[X^{s_\beta}] \cdot [X^{s_\gamma}] \cdot [X^u]$ in the ordinary cohomology ring $H^*(X;\mathbb{Z})$. Since this ring is associative, we deduce that $C_{s_\beta,u}^{u'} \neq 0$ for some $u' \in W^P$ with $u < u' < v'$. $\square$

Let Λ_0 denote the field of fractions of $\Lambda = H_T^*(\text{point};\mathbb{Z})$. We will say that a class in Λ_0 is *rationally positive* if it can be written as a quotient of non-zero non-negative classes in Λ. All rationally positive classes are non-zero, and all sums, products, and quotients of rationally positive classes are again rationally positive. However, notice that some rationally positive classes in Λ are not non-negative, for example we have $x^2 - x + 1 = (x^3+1)/(x+1)$.

Definition 5.2. Let $\{D^w_{u,v}\}$ be any vector of classes $D^w_{u,v} \in \Lambda_0$ indexed by triples $(u,v,w) \in (W^P)^3$. We will say that this vector satisfies the (generalized) *Molev-Sagan equations* for $H_T^*(X;\mathbb{Z})$ if for all $u,v,w \in W^P$ and $\beta \in \Delta \smallsetminus \Delta_P$ we have

$$(C^w_{s_\beta,w} - C^u_{s_\beta,u})\, D^w_{u,v} \;=\; \sum_{u'>u} C^{u'}_{s_\beta,u} D^w_{u',v} - \sum_{w'<w} C^w_{s_\beta,w'} D^{w'}_{u,v}$$

and

$$(C^w_{s_\beta,w} - C^v_{s_\beta,v})\, D^w_{u,v} \;=\; \sum_{v'>v} C^{v'}_{s_\beta,v} D^w_{u,v'} - \sum_{w'<w} C^w_{s_\beta,w'} D^{w'}_{u,v}\,,$$

where the sums are over all elements u', v', or w' in W^P with the indicated bounds.

The associativity relations $([X^{s_\beta}]\cdot[X^u])\cdot[X^v] = [X^{s_\beta}]\cdot([X^u]\cdot[X^v])$ and $[X^u]\cdot([X^v]\cdot[X^{s_\beta}]) = ([X^u]\cdot[X^v])\cdot[X^{s_\beta}]$ of the ring $H_T^*(X;\mathbb{Z})$ imply that the equivariant Schubert structure constants $\{C^w_{u,v}\}$ of X form a solution to the Molev-Sagan equations. It is known that the Molev-Sagan equations together with known expressions [43] for the special coefficients $C^w_{w,w}$ uniquely determine these structure constants [53, 41, 52]. It is interesting to note that every solution $\{D^w_{u,v}\}$ to the Molev-Sagan equations must satisfy the commutativity relation $D^w_{u,v} = D^w_{v,u}$, but we will not use this fact.

Example 5.3. Let $X = \mathbb{P}^1$ and $W^P = \{1,s\}$, and set $a = C^s_{s,s} \in \Lambda$. Then all solutions $\{D^w_{u,v}\}$ to the Molev-Sagan equations are given by

$$D^s_{1,s} = D^s_{s,1} = \frac{1}{a} D^s_{s,s} \quad\text{and}\quad D^s_{1,1} = \frac{1}{a^2} D^s_{s,s} - \frac{1}{a} D^1_{1,1} \quad\text{and}\quad D^1_{1,s} = D^1_{s,1} = D^1_{s,s} = 0\,,$$

where $D^1_{1,1}$ and $D^s_{s,s}$ can be chosen freely in Λ_0. Every solution to the Molev-Sagan equations defines an associative algebra $H = \mathrm{Span}_{\Lambda_0}\{\sigma_1,\sigma_s\}$ with multiplication $\sigma_u\cdot\sigma_v = \sum_w D^w_{u,v}\sigma_w$. The Chevalley formula holds in H if and only if $D^s_{s,s} = a$.

Proposition 5.4. *There exist classes $A^w_{u,v}(\tau) \in \Lambda_0$, indexed by all quadruples (u,v,w,τ) in $(W^P)^4$, with the following properties:*

(a) *Every solution $\{D^w_{u,v}\}$ to the Molev-Sagan equations for $H_T^*(X;\mathbb{Z})$ satisfies*

$$D^w_{u,v} \;=\; \sum_{\tau\in W^P} A^w_{u,v}(\tau) D^\tau_{\tau,\tau}\,.$$

(b) *The class $A^w_{u,v}(\tau)$ is non-zero only if $u \le w$, $v \le w$, and $\tau \le w$.*

(c) *If $u \le w$ and $v \le w$, then $A^w_{u,v}(w)$ is rationally positive.*

Proof. Let $(u,v,w) \in (W^P)^3$ be any triple. Assume by induction that $A^{w'}_{u',v'}(\tau)$ has been defined for all triples (u',v',w') for which $w' < w$, or $w' = w$ and $v' > v$, or $w' = w$ and $v' = v$ and $u' > u$, and for all $\tau \in W^P$. Assume also that these classes satisfy properties (a), (b), and (c). We define the classes $A^w_{u,v}(\tau)$ for $\tau \in W^P$ as follows.

Assume first that $v \neq w$. According to Lemma 5.1(b) we may choose $\beta \in \Delta \smallsetminus \Delta_P$ such that $C^w_{s_\beta,w} \neq C^v_{s_\beta,v}$. Using this choice we define

$$A^w_{u,v}(\tau) \;=\; \frac{1}{C^w_{s_\beta,w} - C^v_{s_\beta,v}} \left(\sum_{v'>v} C^{v'}_{s_\beta,v} A^w_{u,v'}(\tau) - \sum_{w'<w} C^w_{s_\beta,w'} A^{w'}_{u,v}(\tau) \right).$$

If $u \neq v = w$, then choose $\beta \in \Delta \smallsetminus \Delta_P$ such that $C^w_{s_\beta,w} \neq C^u_{s_\beta,u}$ and define

$$A^w_{u,w}(\tau) \;=\; \frac{1}{C^w_{s_\beta,w} - C^u_{s_\beta,u}} \sum_{u'>u} C^{u'}_{s_\beta,u} A^w_{u',w}(\tau)\,.$$

Finally, if $u = v = w$ then define

$$A^w_{w,w}(\tau) = \delta_{w,\tau}\,.$$

We must check that the classes $A^w_{u,v}(\tau)$ satisfy properties (a), (b), and (c). These properties are clear if $u = v = w$. In addition, property (b) follows immediately from the definition of $A^w_{u,v}(\tau)$ together with the induction hypothesis. Assume that $v \neq w$. Then property (a) holds because

$$\begin{aligned}
(C^w_{s_\beta,w} - C^v_{s_\beta,v})\, D^w_{u,v} \;&=\; \sum_{v'>v} C^{v'}_{s_\beta,v} D^w_{u,v'} \;-\; \sum_{w'<w} C^w_{s_\beta,w'} D^{w'}_{u,v} \\
&=\; \sum_{v'>v} C^{v'}_{s_\beta,v} \sum_{\tau} A^w_{u,v'}(\tau) D^\tau_{\tau,\tau} - \sum_{w'<w} C^w_{s_\beta,w'} \sum_{\tau} A^{w'}_{u,v}(\tau) D^\tau_{\tau,\tau} \\
&=\; \sum_{\tau} \left(\sum_{v'>v} C^{v'}_{s_\beta,v} A^w_{u,v'}(\tau) - \sum_{w'<w} C^w_{s_\beta,w'} A^{w'}_{u,v}(\tau) \right) D^\tau_{\tau,\tau} \\
&=\; (C^w_{s_\beta,w} - C^v_{s_\beta,v}) \sum_{\tau} A^w_{u,v}(\tau) D^\tau_{\tau,\tau}\,.
\end{aligned}$$

In addition, if $u \leq w$ and $v < w$, then it follows from property (b) that

$$A^w_{u,v}(w) \;=\; \frac{1}{C^w_{s_\beta,w} - C^v_{s_\beta,v}} \sum_{v'>v} C^{v'}_{s_\beta,v} A^w_{u,v'}(w)\,,$$

and parts (a) and (c) of Lemma 5.1 together with property (c) of the induction hypothesis imply that this sum is rationally positive. A similar argument shows that properties (a) and (c) hold when $u \neq v = w$. □

Corollary 5.5. *Let $\{D^w_{u,v}\}$ be any solution to the Molev-Sagan equations for the ring $H^*_T(X;\mathbb{Z})$ that also satisfies the additional condition*

$$\forall\, w \in W^P \;\exists\, u, v \in W^P : \; u \leq w \;\; \textit{and} \;\; v \leq w \;\; \textit{and} \;\; D^w_{u,v} = C^w_{u,v}\,.$$

Then we have $D^w_{u,v} = C^w_{u,v}$ for all $u, v, w \in W^P$.

Proof. In view of Proposition 5.4 it is enough to show that $D^\tau_{\tau,\tau} = C^\tau_{\tau,\tau}$ for all $\tau \in W^P$. Let $w \in W^P$ and assume by induction that $D^\tau_{\tau,\tau} = C^\tau_{\tau,\tau}$ for all $\tau < w$. Choose $u, v \in W^P$ such that $u \leq w$, $v \leq w$, and $D^w_{u,v} = C^w_{u,v}$. Then we have

$$A^w_{u,v}(w) D^w_{w,w} = D^w_{u,v} - \sum_{\tau<w} A^w_{u,v}(\tau) D^\tau_{\tau,\tau} = C^w_{u,v} - \sum_{\tau<w} A^w_{u,v}(\tau) C^\tau_{\tau,\tau} = A^w_{u,v}(w) C^w_{w,w}\,.$$

Since $A^w_{u,v}(w) \neq 0$ by Proposition 5.4(c), this implies that $D^w_{w,w} = C^w_{w,w}$. □

Remark 5.6. The following instances of the additional condition of Corollary 5.5 have appeared: $D^w_{w,w} = C^w_{w,w}$ in [53, 41]; $D^w_{w,1} = 1$ in [52]; and $D^w_{1,1} = \delta_{1,w}$ in [52].

Corollary 5.7. *Let H be any associative Λ-algebra that is also a free Λ-module with a basis $\{\sigma_u : u \in W^P\}$ indexed by W^P. Assume that H satisfies the (two-sided) Chevalley formula*

$$\sigma_{s_\beta} \cdot \sigma_u \;=\; \sigma_u \cdot \sigma_{s_\beta} \;=\; \sum_{w \in W^P} C^w_{s_\beta, u}\, \sigma_w$$

for all $u \in W^P$ and $\beta \in \Delta \smallsetminus \Delta_P$, as well as the identity $\sigma_1 \cdot \sigma_1 = \sigma_1$. Then the homomorphism of Λ-modules $H \to H^*_T(X;\mathbb{Z})$ defined by $\sigma_u \mapsto [X^u]$ is an isomorphism of rings.

Proof. Let $\{D^w_{u,v}\}$ be the structure constants of H with respect to the basis $\{\sigma_u\}$. Then the relations $(\sigma_{s_\beta} \cdot \sigma_u) \cdot \sigma_v = \sigma_{s_\beta} \cdot (\sigma_u \cdot \sigma_v)$ and $\sigma_u \cdot (\sigma_v \cdot \sigma_{s_\beta}) = (\sigma_u \cdot \sigma_v) \cdot \sigma_{s_\beta}$ imply that $\{D^w_{u,v}\}$ is a solution to the Molev-Sagan equations. The identity $\sigma_1 \cdot \sigma_1 = \sigma_1$ implies that $D^w_{1,1} = \delta_{w,1} = C^w_{1,1}$ for all $w \in W^P$. It therefore follows from Corollary 5.5 that $D^w_{u,v} = C^w_{u,v}$ for all $u, v, w \in W^P$, as required. □

The following conjecture implies that the vector space of all solutions to the Molev-Sagan equations has dimension equal to the cardinality of W^P.

Conjecture 5.8. *The classes $A^w_{u,v}(\tau)$ of Proposition 5.4 are uniquely determined by property* (a)*. Equivalently, for each fixed $\tau \in W^P$ the vector $\{A^w_{u,v}(\tau)\}$ is the unique solution to the Molev-Sagan equations that also satisfies $A^w_{w,w}(\tau) = \delta_{w,\tau}$ for all $w \in W^P$.*

Remark 5.9. The results proved in Section 5.2 are true also when $X = G/P$ is a homogeneous space defined by a Kac-Moody group G and a parabolic subgroup P, with the same proofs. The required Chevalley formula for the equivariant cohomology of such spaces is proved in Kumar's book [45, Thm. 11.1.7].

Remark 5.10. Ciocan-Fontanine brought to our attention that, under certain hypotheses, [22, Lemma 4.1.3] implies that the *localized* equivariant cohomology ring $H^*_T(X;\mathbb{Z}) \otimes_\Lambda \Lambda_0$ is generated by divisors. We next give an alternative proof of this conclusion. Let $D = \sum_{\beta \in \Delta \smallsetminus \Delta_P} [X^{s_\beta}]$ be the sum of the Schubert divisors in X.

Lemma 5.11. *The ring $H^*_T(X;\mathbb{Z}) \otimes_\Lambda \Lambda_0$ is generated by D as a Λ_0-algebra.*

Proof. Consider the Λ-linear endomorphism $\mu_D : H^*_T(X;\mathbb{Z}) \to H^*_T(X;\mathbb{Z})$ defined as multiplication by D. Since multiplication by D can be written in the form $D \cdot [X^u] = D_u [X^u] + \sum_{w > u} c(u;w)\, [X^w]$, where $c(u;w) \in \Lambda$ and D_u is the restriction of D to the T-fixed point $u.P$, it follows that μ_D is represented by an upper-triangular matrix. The diagonal entries are given by $D_u = c_T(\omega - u.\omega)$, where $\omega = \sum_{\beta \in \Delta \smallsetminus \Delta_P} \omega_\beta$ (see [44] or e.g. [14, Thm. 8.1]). Since we have $v.\omega = \omega$ if and only if $v \in W_P$ [5, V, §4.6], we deduce that μ_D has distinct diagonal entries. This implies that the minimal polynomial of μ_D is equal to the characteristic polynomial, so the elements $1, D, D^2, \ldots, D^{\#W_P - 1}$ are linearly independent over Λ_0 and span $H^*_T(X;\mathbb{Z}) \otimes_\Lambda \Lambda_0$. □

In relation to [22], consider the T-equivariant embedding $X \subset \mathbb{P}(V)$, where $V = H^0(X, \mathcal{O}_X(D))$. In order to apply [22, Lemma 4.1.3] to this situation, one must show that the restriction map $H^*(\mathbb{P}(V)^T;\mathbb{C}) \to H^*(X^T;\mathbb{C}) = \mathbb{C}^{\#W^P}$ is surjective. In fact, this follows from Lemma 5.11 together with the commutative diagram used

in the proof of [22, Lemma 4.1.3]. However, it is not in general true that $\mathbb{P}(V)$ has isolated T-fixed points. For example, this fails when X is an *adjoint flag variety* in the sense that $V = \mathrm{Lie}(G)$ is the adjoint representation of G, see e.g. [18].

5.3. **Equivariant quantum K-theory.** We finally show that the ring $\mathrm{QK}_T(X)$ is uniquely determined by multiplication with the divisor classes and the identity element. Our argument is based on linear algebra and could also be applied to the equivariant K-theory ring $K^T(X)$ or the equivariant quantum cohomology ring $\mathrm{QH}_T(X)$. For $u, v, w \in W^P$ we define the power series

$$N_{u,v}^w \;=\; \sum_{d\geq 0} N_{u,v}^{w,d}\, q^d$$

in the ring $\Gamma[\![q]\!] = \Gamma[\![q_\beta : \beta \in \Delta \smallsetminus \Delta_P]\!]$. The product in $\mathrm{QK}_T(X)$ is given by

$$\mathcal{O}^u \star \mathcal{O}^v \;=\; \sum_{w\in W^P} N_{u,v}^w\, \mathcal{O}^w\,.$$

Let $\Gamma[\![q]\!]_0$ be the field of fractions of $\Gamma[\![q]\!]$.

Definition 5.12. Let $\{D_{u,v}^w\}$ be a vector of classes $D_{u,v}^w \in \Gamma[\![q]\!]_0$ indexed by triples $(u, v, w) \in (W^P)^3$. We say that this vector satisfies the generalized *Molev-Sagan equations* for $\mathrm{QK}_T(X)$ if for all $u, v, w \in W^P$ and $\beta \in \Delta \smallsetminus \Delta_P$ we have

$$\sum_{u'\in W^P} N_{s_\beta,u}^{u'} D_{u',v}^w \;=\; \sum_{w'\in W^P} N_{s_\beta,w'}^{w} D_{u,v}^{w'} \;=\; \sum_{v'\in W^P} N_{s_\beta,v}^{v'} D_{u,v'}^{w}\,.$$

The equalities $(\mathcal{O}^{s_\beta} \star \mathcal{O}^u) \star \mathcal{O}^v = \mathcal{O}^{s_\beta} \star (\mathcal{O}^u \star \mathcal{O}^v) = \mathcal{O}^u \star (\mathcal{O}^{s_\beta} \star \mathcal{O}^v)$ imply that the structure constants $\{N_{u,v}^w\}$ form a solution to the Molev-Sagan equations for $\mathrm{QK}_T(X)$. Notice that if each structure constant $N_{s_\beta,u}^w$ in Definition 5.12 is replaced with $C_{s_\beta,u}^w$, then the Molev-Sagan equations for $\mathrm{QK}_T(X)$ turn into the Molev-Sagan equations for $H_T^*(X;\mathbb{Z})$. One noteworthy difference between the two cases is that the structure constants $N_{u,v}^w$ of $\mathrm{QK}_T(X)$ may be non-zero when $u \not\leq w$ and $v \not\leq w$. For this reason the Molev-Sagan equations for $\mathrm{QK}_T(X)$ are not triangular, which makes it time-consuming to solve them. Another obstruction to solving the equations is that the constants $N_{s_\beta,u}^w$ are known only when X is a cominuscule variety. Here we will only discuss how the ring $\mathrm{QK}_T(X)$ is determined by the Molev-Sagan equations. A systematic method for solving the Molev-Sagan equations for the equivariant quantum cohomology ring $\mathrm{QH}_T(X)$ can be found in [52].

Proposition 5.13. *Let $\{D_{u,v}^w\}$ be any solution to the Molev-Sagan equations for $\mathrm{QK}_T(X)$ that also satisfies the additional condition*

$$\forall\, w \in W^P\ \exists\, u, v \in W^P :\ u \leq w \ \text{ and } \ v \leq w \ \text{ and } \ D_{u,v}^w = N_{u,v}^w\,.$$

Then we have $D_{u,v}^w = N_{u,v}^w$ for all $u, v, w \in W^P$.

Before we prove Proposition 5.13, we first recall some facts about the equivariant K-theory ring $K^T(X)$. Let $\mathrm{ch} : \Gamma \to \widehat{\Lambda} := \prod_{k=0}^\infty H_T^{2k}(X;\mathbb{Q})$ denote the *equivariant Chern character* from [23, §3.1]. This injective ring homomorphism is defined by $\mathrm{ch}([\mathbb{C}_\lambda]) = \exp(c_T(\lambda))$. For $\sigma \in K^T(X)$ we may write $\mathrm{ch}(\sigma) = \sum \mathrm{ch}_m(\sigma)$, where $\mathrm{ch}_m(\sigma) \in H_T^{2m}(X;\mathbb{Q})$ is the term of degree m. Set $\mathrm{ch}_m(\sigma) = 0$ for $m < 0$. We will say that σ has *degree at least k* if $\mathrm{ch}_m(\sigma) = 0$ for all $m < k$. If $\sigma \neq 0$, then the *leading term* of σ is the first non-zero term $\mathrm{ch}_m(\sigma)$ (with m minimal). For $u, v, w \in W^P$ we set $d(u, v, w) = \ell(u) + \ell(v) - \ell(w)$. The structure constant $N_{u,v}^{w,0}$ of $K^T(X)$ has

degree at least $d(u,v,w)$, and we have $\mathrm{ch}_{d(u,v,w)}(N^{w,0}_{u,v}) = C^w_{u,v}$. This can be proved geometrically using the equivariant Riemann-Roch formula [23, Cor. 3.1] (see [13, §4.1]), or combinatorially using the Chevalley formula for $K^T(G/B)$ [49].

Proof of Proposition 5.13. According to Corollary 5.5, the vector $\{C^w_{u,v}\}$ of structure constants of $H^*_T(X;\mathbb{Z})$ is the unique solution to the equations of Definition 5.2 together with a choice of $k = \#W^P$ additional equations of the form $D^w_{u,v} = C^w_{u,v}$. Since this is a system of linear equations, we can choose a subset of k^3 of the equations that are sufficient to determine the vector $\{C^w_{u,v}\}$. Consider the corresponding set of equations for $\mathrm{QK}_T(X)$. More precisely, if the chosen equations for $\{C^w_{u,v}\}$ include the first (resp. second) equation of Definition 5.2 for some $u,v,w \in W^P$ and $\beta \in \Delta \smallsetminus \Delta_P$, then we use the first (resp. second) equality of Definition 5.12 given by the same elements u,v,w,β. For each equation of the form $D^w_{u,v} = C^w_{u,v}$ we similarly use the corresponding equation $D^w_{u,v} = N^w_{u,v}$. We claim that the vector $\{N^w_{u,v}\}$ is the unique solution to these equations for $\mathrm{QK}_T(X)$. These equations can be stated in matrix form as $AD = b$, where A is a $k^3 \times k^3$ matrix and b is a vector of length k^3, both with entries from $\Gamma[\![q]\!]$, and $D = \{D^w_{u,v}\}$ is the vector of indeterminates. Since $\{N^w_{u,v}\}$ satisfies the Molev-Sagan equations for $\mathrm{QK}_T(X)$, it suffices to show that the determinant of A is not zero. For this it is enough to show that $\det(\overline{A}) \neq 0$, where $\overline{A}$ is obtained from A by substituting zero for all quantum parameters q_β. We will show that the leading term of $\det(\overline{A})$ is the determinant of the corresponding system of linear equations for $H_T(X;\mathbb{Z})$.

Number the rows and columns of A from 1 to k^3, and define integers d_i and d'_j for $1 \le i,j \le k^3$ as follows. If the i-th row of A represents an equation coming from Definition 5.12 for a triple $(u,v,w) \in (W^P)^3$, then set $d_i = d(u,v,w)+1$. If the i-th row of A represents (the left hand side of) an equation $D^w_{u,v} = N^w_{u,v}$, then set $d_i = d(u,v,w)$. Finally, if the j-th column of A corresponds to the indeterminate $D^w_{u,v}$, then set $d'_j = d(u,v,w)$. It follows by inspection of the Molev-Sagan equations that each entry $\overline{A}_{ij}$ of $\overline{A}$ has degree at least $d_i - d'_j$. This implies that $\det(\overline{A})$ has degree at least $\sum_i d_i - \sum_j d'_j$. Furthermore, if we replace each entry $\overline{A}_{ij}$ of $\overline{A}$ with $\mathrm{ch}_{d_i - d'_j}(\overline{A}_{ij})$, then the result is the matrix of coefficients of the original k^3 equations for the structure constants of $H^*_T(X;\mathbb{Z})$. Since this system of linear equations is known to have a unique solution, we deduce that the term of degree $\sum d_i - \sum d'_j$ in $\mathrm{ch}(\det(\overline{A}))$ is indeed non-zero. This completes the proof. □

Corollary 5.14. *Let H be any associative algebra over $\Gamma[\![q]\!]$ that is also a free $\Gamma[\![q]\!]$-module with a basis $\{\sigma_u : u \in W^P\}$ indexed by W^P. Assume that H satisfies*

$$\sigma_{s_\beta} \star \sigma_u \;=\; \sigma_u \star \sigma_{s_\beta} \;=\; \sum_{w \in W^P} N^w_{s_\beta,u}\, \sigma_w$$

for all $u \in W^P$ and $\beta \in \Delta \smallsetminus \Delta_P$, as well as the identity $\sigma_1 \star \sigma_1 = \sigma_1$. Then the homomorphism of $\Gamma[\![q]\!]$-modules $H \to \mathrm{QK}_T(X)$ defined by $\sigma_u \mapsto \mathcal{O}^u$ is an isomorphism of rings.

Remark 5.15. Gorbounov and Korff have used ideas from integrable systems to construct an algebra called $qh^*(\mathrm{Gr}(m,n))$, together with a basis for this algebra that corresponds to the Schubert classes of the Grassmannian $\mathrm{Gr}(m,n)$ [28][2]. It

[2]The parameter β used in [28] must be replaced with -1 in order to match our notation.

follows from [28, Cor. 3.18] that this algebra satisfies a Chevalley formula that agrees with our Theorem 3.9. We therefore deduce from Corollary 5.14 that the algebra $qh^*(\mathrm{Gr}(m,n))$ has the same Schubert structure constants as the equivariant quantum K-theory ring $\mathrm{QK}_T(\mathrm{Gr}(m,n))$. This proves that the two algebras are isomorphic, which is Conjecture 1.2 in [28]. In particular, the presentation (Thm. 5.16) and Giambelli formula (Cor. 5.13) proved in [28] are valid for the equivariant quantum K-theory of Grassmannians. We note that the relation of $qh^*(\mathrm{Gr}(m,n))$ with the equivariant quantum cohomology ring $\mathrm{QH}_T(\mathrm{Gr}(m,n))$ and the (non-equivariant) quantum K-theory ring $\mathrm{QK}(\mathrm{Gr}(m,n))$ was established in [28] by using the structure theorems from [52, 13]. An alternative geometric proof of the presentation of $\mathrm{QK}(\mathrm{Gr}(m,n))$ has recently been given by Gonzalez and Woodward by writing the Grassmannian as a GIT quotient of a space of matrices [27].

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