MUTATIONS OF PUZZLES AND EQUIVARIANT COHOMOLOGY OF TWO-STEP FLAG VARIETIES

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Abstract. We introduce a mutation algorithm for puzzles that is a three-direction analogue of the classical jeu de taquin algorithm for semistandard tableaux. We apply this algorithm to prove our conjectured puzzle formula for the equivariant Schubert structure constants of two-step flag varieties. This formula gives an expression for the structure constants that is positive in the sense of Graham. Thanks to the equivariant version of the 'quantum equals classical' result, our formula specializes to a Littlewood-Richardson rule for the equivariant quantum cohomology of Grassmannians.

1. Introduction

In 1999 Allen Knutson circulated a conjecture stating that any Schubert structure constant of the cohomology ring of a partial flag variety $X = \mathrm{GL}(n)/P$ can be expressed as the number of puzzles that can be created using a list of triangular puzzle pieces with matching side labels [21]. While this conjecture was proved in the special case where X is a Grassmann variety [24, 23], Knutson discovered counter examples to his general conjecture. In later work by Kresch, Tamvakis, and the author [9] it was proved that the (3 point, genus zero) Gromov-Witten invariants of Grassmannians are special cases of the Schubert structure constants of two-step flag varieties $\mathrm{Fl}(a, b; n)$. In fact, the map that sends a rational curve to its kernel-span pair [7] provides a bijection between the curves counted by a Gromov-Witten invariant and the points of intersection of three general Schubert varieties in a two-step flag variety. Supported by computer verification, it was suggested in [9] that Knutson's conjecture might correctly predict the Schubert structure constants of two-step flag varieties. This case of the conjecture has recently been proved [8]. A different positive combinatorial formula for the cohomological structure constants of two-step flag varieties had earlier been proved by Coşkun [13]. See also [22] for a relation between puzzles and Belkale-Kumar coefficients.

The cohomology ring of a homogeneous space $X = G/P$ generalizes to the equivariant cohomology ring $H_T^*(X;\mathbb{Z})$, whose structure incorporates the action of a torus T. The Schubert structure constants of this ring are elements of $H_T^*(\mathrm{pt},\mathbb{Z})$, which can be identified with the polynomial ring $\mathbb{Z}[y_1,\dots,y_n]$. Graham has proved that the equivariant Schubert structure constants are polynomials with positive coefficients in the differences $y_{i+1} - y_i$ [18]. Knutson and Tao's paper [23] proves an equivariant generalization of the puzzle rule for Grassmannians that makes Graham's positivity result explicit. The equivariant puzzles in this rule are composed

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by triangular puzzle pieces as well as rhombus shaped *equivariant puzzle pieces*. The equivariant pieces are required to be vertical, and each equivariant piece contributes a *weight* $y_j - y_i$ with $i < j$, where the values of i and j depend on the location of the equivariant piece in the puzzle. Knutson and Tao define the weight of an equivariant puzzle to be the product of the weights of its equivariant pieces, and prove that any equivariant Schubert structure constant of a Grassmann variety is equal to the sum of the weights of a collection of equivariant puzzles. A different Graham-positive formula for the equivariant structure constants of Grassmannians was later obtained independently by Molev [31] and Kreiman [27]. In addition, Knutson has recently generalized the puzzle rule for Grassmannians to equivariant K-theory [20].

Efforts to prove Knutson's conjecture more than a decade ago resulted in a conjectured Graham-positive formula for the equivariant Schubert structure constants of any two-step flag variety $\mathrm{Fl}(a, b; n)$, which generalizes both the equivariant puzzle rule for Grassmannians and the cohomological puzzle rule for two-step flag varieties. This conjecture was published in Coşkun and Vakil's survey [14] together with a suggested correction of Knutson's cohomological conjecture for three-step flag varieties. The main result in the present paper is a proof of the conjectured equivariant puzzle formula for two-step flag varieties (Theorem 2.1).

Our paper [10] with Mihalcea proves that the equivariant Gromov-Witten invariants of Grassmannians are special cases of the equivariant Schubert structure constants of two-step flag varieties, thus generalizing the 'quantum equals classical' result from [9]. Our puzzle formula therefore specializes to a Littlewood-Richardson rule for the equivariant quantum cohomology of Grassmannians that accords with Mihalcea's result [29] that the equivariant Gromov-Witten invariants satisfy Graham positivity. While different formulas for equivariant Gromov-Witten invariants are known [28, 30, 3], positive formulas have not been available earlier for either the equivariant cohomology of two-step flag varieties or the equivariant quantum cohomology of Grassmannians.

The main combinatorial construction in our paper is an algorithm called *mutation of puzzles*, which is analogous to Schützenberger's jeu de taquin algorithm for semistandard Young tableaux. Recall that the jeu de taquin algorithm operates on Young tableaux that contain a *flaw* in the form of an empty box, and works by making natural changes to move the empty box to a different location in the tableau. Our mutation algorithm similarly operates on *flawed puzzles*. A flaw in a puzzle can be a *pair of gashes* on the boundary, a *marked scab*, or a *temporary puzzle piece*. Gashes and scabs are also present in Knutson and Tao's work [23], whereas temporary puzzle pieces are new in this paper. Flawed puzzles that contain a gash pair or a marked scab can be mutated in exactly one way. On the other hand, a puzzle containing a temporary puzzle piece has exactly three mutations, which correspond to moving the temporary piece in three different directions. The mutation algorithm therefore organizes the set of all flawed puzzles into a trivalent graph with leaves (see Figure 4.9). In contrast, the jeu de taquin algorithm offers only two choices for moving an empty box in a tableau. Our definition of flawed puzzles allows equivariant puzzle pieces to appear in arbitrary orientations and also allows the shape of a puzzle to be a hexagon. This ensures that rotations of puzzles are again puzzles, which in turn simplifies the definition of the mutation algorithm.

The changes made to a puzzle during a mutation are based on the following observation. Suppose that a puzzle contains a *gash*, i.e. the labels of two puzzle pieces next to each other do not match. Then there is *at most one way* to replace either of these pieces with a different puzzle piece such that the gash disappears and only one new gash is created by the replacement. This provides a natural method for moving a gash around in a puzzle, which we call *propagation* of the gash. Given a flawed puzzle, the mutation algorithm first *resolves* the flaw by replacing it with two gashes. Both of these gashes are then propagated as far as possible. Our main technical result states that the two gashes will propagate to the same location in the puzzle, where they create a new flaw. While the mutation algorithm itself can be formulated in terms of general principles, the proof that it works requires some case by case analysis. For example, the proof of the above-mentioned technical result is a winding number argument that is justified with case checking. Even so, our construction of the mutation algorithm applies without change to the puzzles appearing in the conjectured formula for the cohomology of three-step flag varieties; this will be explained in [5] together with the consequences for this conjecture. It is natural to speculate that a correction of Knutson's general conjecture for $\mathrm{GL}(n)/P$, if one exists, should be formulated in terms of puzzles that can be mutated.

Thanks to an idea that originates in Molev and Sagan's work on products of factorial Schur functions [32], any formula for the equivariant Schubert structure constants of a homogeneous space $X = G/P$ can be proved by verifying certain recursive identities associated to multiplication with divisor classes, together with showing that the formula is compatible with restriction of equivariant Schubert classes to torus-fixed points [26, 1, 4]. This method was used in [23]. Molev and Sagan's method requires the verification of $2r$ families of recursive identities, where r is the rank of the Picard group of X. By working with equivariant cohomology with coefficients in the polynomial ring $R = \mathbb{C}[\delta_0, \delta_1, \delta_2]$, we combine the 4 families of identities required for a two-step flag variety into a single recursive identity that involves powers of a 12-th root of unity $\zeta \in \mathbb{C}$. The proof that this identity is satisfied by the constants defined by our puzzle formula involves assigning an *aura* in the ring R to various objects related to puzzles. Here the powers of ζ are used as unit vectors whose directions correspond to puzzle angles, and the variables δ_0, δ_1, and δ_2 correspond to simple puzzle labels. The recursive identity then follows from the mutation algorithm together with simple identities among auras. Our proof of the puzzle formula is logically self-contained starting from basic properties of equivariant cohomology [26, 2] and the Monk/Chevalley formula [12, 33].

The proofs of the puzzle formulas in [23, 8] rely on bijections of puzzles to establish certain basic identities. These bijections are formulated in terms of *propagation rules* stating that a small region of a puzzle with a particular look must be changed in a specified way. Knutson and Tao's bijection requires around 10 rules, while the bijection in [8] uses a list of 80 propagation rules. In contrast the mutation algorithm is defined without lists of rules. In Section 4.11 we explain how mutations of puzzles can be used to give a new construction of the bijections from [23, 8]. This construction involves that some areas of a puzzle can be changed by more than one mutation, which, at least for two-step puzzles, is simpler than giving a direct description of the end result of the bijection. We also sketch how the *breathing* construction of Knutson, Tao, and Woodward [24] can be carried out using mutations; this application was pointed out by the referee.

This paper is organized as follows. In Section 2 we state our puzzle formula for the equivariant cohomology of two-step flag varieties and specialize it to the equivariant Gromov-Witten invariants of Grassmannians. Section 3 explains the recursive identity required to prove the formula. Section 4 starts with an informal introduction of the mutation algorithm, after which we give the precise definitions and prove that the mutation algorithm works as required. Finally, Section 5 defines auras associated to various objects in puzzles and uses this concept to prove our main result.

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2. The equivariant puzzle formula

2.1. **Two-step flag varieties.** Fix integers $0 \leq a \leq b \leq n$ and let $X = \mathrm{Fl}(a,b;n)$ be the variety of two-step flags $A \subset B \subset \mathbb{C}^n$ such that $\dim(A) = a$ and $\dim(B) = b$. A *012-string* for X is a sequence $u = (u_1, u_2, \ldots, u_n) \in \mathbb{Z}^n$ consisting of a zeros, $b-a$ ones, and $n-b$ twos. The Schubert varieties in X are indexed by these 012-strings. Let $e_1, e_2, \ldots, e_n$ be the standard basis for $\mathbb{C}^n$, let $\mathbf{B} \subset \mathrm{GL}_n(\mathbb{C})$ be the Borel subgroup of upper triangular matrices, and let $\mathbf{B}^- \subset \mathrm{GL}_n(\mathbb{C})$ be the opposite Borel subgroup of lower triangular matrices. We also let $T = \mathbf{B} \cap \mathbf{B}^-$ be the maximal torus of diagonal matrices. Given any 012-string u for X, we define the subspaces $A_u = \mathrm{Span}_{\mathbb{C}}\{e_i \mid u_i = 0\}$ and $B_u = \mathrm{Span}_{\mathbb{C}}\{e_i \mid u_i \leq 1\}$ of $\mathbb{C}^n$. Then (A_u, B_u) is a point in X, and the T-fixed points in X are exactly the points of this form. Let $X_u = \overline{\mathbf{B}.(A_u, B_u)}$ the *Schubert variety* defined by u, and let $X^u = \overline{\mathbf{B}^-.(A_u, B_u)}$ the *opposite Schubert variety* defined by u. We have $\dim(X_u) = \mathrm{codim}(X^u, X) = \ell(u) = \#\{i < j \mid u_i > u_j\}$.

Let $H_T^*(X;\mathbb{Z})$ denote the T-equivariant cohomology ring of X. An introduction to this ring can be found in [2]. Each T-stable closed subvariety $Z \subset X$ defines an equivariant class $[Z] \in H_T^{2d}(X;\mathbb{Z})$, where $d = \mathrm{codim}(Z, X)$. Pullback along the structure morphism $X \to \{\mathrm{pt}\}$ gives $H_T^*(X;\mathbb{Z})$ the structure of an algebra over the ring $\Lambda := H_T^*(\mathrm{pt};\mathbb{Z})$, and $H_T^*(X;\mathbb{Z})$ is a free Λ-module with a basis consisting of the Schubert classes $[X^u]$ indexed by all 012-strings for X. The *equivariant Schubert structure constants* of X are the unique classes $C_{u,v}^w \in \Lambda$ defined by the equation

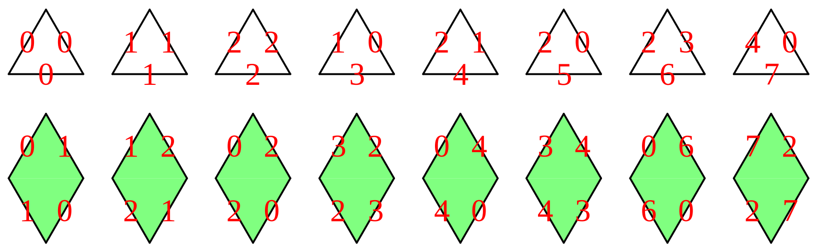
$$[X^u] \cdot [X^v] = \sum_w C_{u,v}^w [X^w] \ \in H_T^*(X;\mathbb{Z})\,, \tag{1}$$

where the sum is over all 012-strings w for X. Let $\int_X : H_T^*(X;\mathbb{Z}) \to \Lambda$ denote the pushforward along the map $X \to \{\mathrm{pt}\}$. For arbitrary 012-strings u and v for X we then have $\int_X [X^u] \cdot [X_v] = \delta_{u,v}$. It follows that the equivariant structure constants of X are given by

$$C_{u,v}^w \ = \ \int_X [X^u] \cdot [X^v] \cdot [X_w]\,.$$

Each basis element e_i for $\mathbb{C}^n$ defines a one-dimensional T-representation $\mathbb{C}e_i$, where the action is given by $(t_1, \ldots, t_n).e_i = t_i e_i$. This representation can be regarded as a T-equivariant line bundle over a point. We let $y_i = -c_1(\mathbb{C}e_i) \in \Lambda$ be the corresponding equivariant Chern class with negated sign.[1] We then have $\Lambda = \mathbb{Z}[y_1, \dots, y_n]$. Since $H_T^*(X;\mathbb{Z})$ is a graded ring, it follows that each structure constant $C_{u,v}^w \in \Lambda$ is a homogeneous polynomial of total degree $\ell(u) + \ell(v) - \ell(w)$. The constants $C_{u,v}^w \in \mathbb{Z}$ for which $\ell(u) + \ell(v) = \ell(w)$ are the structure constants of the ordinary cohomology ring $H^*(X;\mathbb{Z})$. These constants are given by the cohomological puzzle rule proved in [8]. A result of Graham [18] asserts that every equivariant structure constant $C_{u,v}^w \in \Lambda$ is a polynomial with positive coefficients in the differences $y_{i+1} - y_i$, i.e. we have $C_{u,v}^w \in \mathbb{Z}_{\geq 0}[y_2 - y_1, \dots, y_n - y_{n-1}]$. We proceed to state our manifestly positive formula for these constants.

2.2. **Equivariant puzzles.** A *puzzle piece* is a figure from the following list.



The triangular puzzle pieces come from the cohomological puzzle rule [8], which was originally conjectured by Knutson [21]. In Knutson's notation the side labels were parenthesized strings of the integers 0, 1, and 2. The labels that are greater than two can be translated to such strings as follows:

$$3 = 10,\ 4 = 21,\ 5 = 20,\ 6 = 2(10),\ 7 = (21)0\,.$$

The labels 0, 1, 2 are called *simple* and the other labels 3, 4, 5, 6, 7 are called *composed*. Notice that a triangular puzzle piece is uniquely determined if the labels on two of its sides are known.

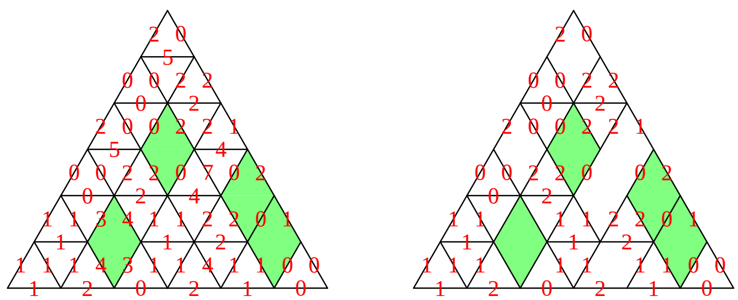
The rhombus-shaped puzzle pieces are called *equivariant puzzle pieces*. The first equivariant piece comes from Knutson and Tao's puzzle formula for the equivariant structure constants of Grassmannians [23]. In fact, the first five equivariant pieces are very natural from the statement of this formula together with the cohomological puzzle rule for two-step flag varieties. The last three equivariant pieces are more surprising, as each of them embeds the same simple label on all sides, which appears to violate the philosophy of Knutson's original conjecture [21]. Puzzle pieces may be rotated arbitrarily, but they may not be reflected. An equivariant puzzle piece is called *vertical* if it is oriented as in the above list.

Define a *triangular puzzle* to be any equilateral triangle made from puzzle pieces with matching labels, i.e. any two puzzle pieces next to each other assign the same label to the side that they share. We also demand that all labels on the boundary of the triangle are simple, and that the triangle is 'right side up', i.e. its bottom border is a horizontal line segment. The sides of the puzzle pieces in a puzzle are called *puzzle edges*, and the three sides of the boundary of the puzzle are called *border segments*. We will say that a triangular puzzle P has *boundary* $\triangle_w^{u,v}$, also written as $\partial P = \triangle_w^{u,v}$, if u is the string of labels on the left border segment, v is

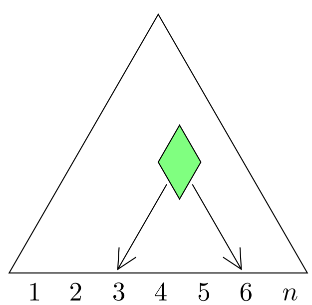
[1]The sign ensures consistency with standard notation for double Schubert polynomials.

the string of labels on the right border segment, and w is the string of labels on the bottom border segment, all read in left to right order.

The triangular puzzle P is called an *equivariant puzzle* for X if all its equivariant pieces are vertical and the boundary of P is $\triangle_w^{u,v}$ where u, v, and w are 012-strings for X. The composed labels in any puzzle are uniquely determined by the simple labels, so we will often omit them in pictures of puzzles. The following are two pictures of the same equivariant puzzle for the variety $\mathrm{Fl}(2,4;6)$, with and without the composed labels. This puzzle has boundary $\triangle_w^{u,v}$ where $u = (1,1,0,2,0,2)$, $v = (0,2,1,2,1,0)$, and $w = (1,2,0,2,1,0)$.



Given an equivariant puzzle P for X, we number the edges of the bottom border segment from 1 to n, starting from the left. Each equivariant puzzle piece q in P has a *weight* defined by $\mathrm{wt}(q) = y_j - y_i$, where i is the number of the bottom edge obtained by following a south-west line from q, and j is the bottom edge number obtained by following a south-east line from q. For example, the following equivariant puzzle piece has weight $y_6 - y_3$.



The weight of the equivariant puzzle P is the product of the weights of all equivariant puzzle pieces in P.

Our main result is the following Graham-positive combinatorial formula for the equivariant Schubert structure constants of X, which generalizes both Knutson and Tao's equivariant rule for Grassmannians [23] and the cohomological puzzle rule for two-step flag varieties [8]. We conjectured this formula more than 10 years ago, and our conjecture was printed in Coşkun and Vakil's survey [14].[2]

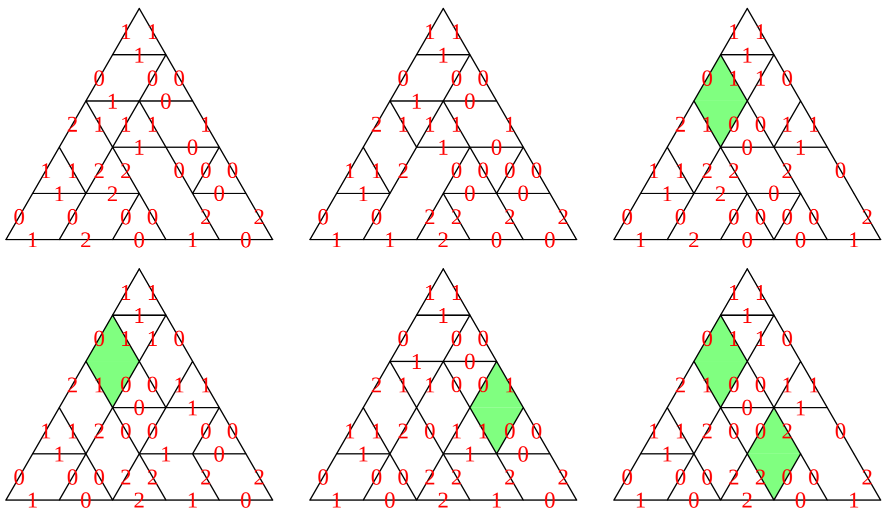
[2]The statement in [14] lacks two of the equivariant puzzle pieces; the author of the present paper is entirely responsible for this.

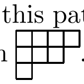
Theorem 2.1. *Let u, v, and w be 012-strings for the two-step flag variety $X = \mathrm{Fl}(a, b; n)$. Then the equivariant Schubert structure constant $C^w_{u,v} \in \Lambda$ is given by*

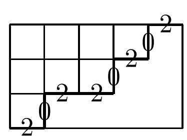
$$C^w_{u,v} \;=\; \sum_{\partial P = \triangle^{u,v}_w} \mathrm{wt}(P)$$

where the sum is over all equivariant puzzles P for X with boundary $\triangle^{u,v}_w$.

Example 2.2. For $X = \mathrm{Fl}(2, 4; 5)$ we have $[X^{01201}]\cdot[X^{10102}] = [X^{12010}]+[X^{11200}]+(y_4 - y_1)[X^{12001}] + (y_5 + y_4 - y_3 - y_1)[X^{10210}] + (y_4 - y_3)(y_4 - y_1)[X^{10201}]$. The puzzles required to compute this product are:



2.3. **Equivariant quantum cohomology.** Let $X = \mathrm{Gr}(m, n)$ be the Grassmann variety of m-dimensional subspaces of $\mathbb{C}^n$. By identifying X with the variety $\mathrm{Fl}(m, m; n)$, we may index the Schubert varieties in X by strings $\lambda = (\lambda_1, \ldots, \lambda_n)$ containing m zeros and $n - m$ twos. Each such string λ can be identified with a *Young diagram* contained in the rectangle with m rows and $n - m$ columns. More precisely, λ defines a path from the lower-left corner to the upper-right corner of this rectangle, where the i-th step is vertical if $\lambda_i = 0$ and horizontal if $\lambda_i = 2$, and λ is identified with the portion of the rectangle that is north-west of this path. For example, we identify $\lambda = (2, 0, 2, 2, 0, 2, 0, 2)$ with the Young diagram .



Given a degree $d \in \mathbb{N}$, let $M_d = \overline{\mathcal{M}}_{0,3}(X, d)$ denote the Kontsevich moduli space of 3-pointed stable maps to X of genus zero and degree d. This variety parametrizes morphisms of varieties $f : C \to X$ defined on a tree of projective lines C with three ordered marked non-singular points, such that $f_*[C] = d\,[X_{\square}] \in H_2(X; \mathbb{Z})$, and any component of C that is mapped to a single point in X contains at least three special points, where a special point is either marked or singular. The variety M_d is equipped with *evaluation maps* $\mathrm{ev}_i : M_d \to X$ for $1 \le i \le 3$, where ev_i sends a stable map to the image of the i-th marked point in its domain. We refer to [17] for a careful construction of this space.

The *equivariant quantum cohomology ring* $\mathrm{QH}_T(X)$ is an algebra over the ring $\Lambda[q]$, which as a module is defined by $\mathrm{QH}_T(X) = H_T^*(X;\mathbb{Z}) \otimes_\Lambda \Lambda[q]$. The multiplicative structure of $\mathrm{QH}_T(X)$ is determined by

$$[X^\lambda] \star [X^\mu] = \sum_{\nu, d \geq 0} N_{\lambda,\mu}^{\nu,d}\, q^d\, [X^\nu]\,,$$

where the structure constants $N_{\lambda,\mu}^{\nu,d} \in \Lambda$ are the *equivariant Gromov-Witten invariants* defined by

$$N_{\lambda,\mu}^{\nu,d} \; = \; \int_{M_d} \mathrm{ev}_1^*[X^\lambda] \cdot \mathrm{ev}_2^*[X^\mu] \cdot \mathrm{ev}_3^*[X_\nu]\,.$$

It is a non-trivial fact that this construction defines an associative ring [34, 25, 19]. The structure constant $N_{\lambda,\mu}^{\nu,d}$ is a homogeneous polynomial in $\Lambda = \mathbb{Z}[y_1, \dots, y_n]$ of total degree $|\lambda| + |\mu| - |\nu| - nd$. If this degree is zero, then $N_{\lambda,\mu}^{\nu,d}$ is the number of stable maps $f \in M_d$ for which $\mathrm{ev}_1(f)$, $\mathrm{ev}_2(f)$, and $\mathrm{ev}_3(f)$ belong to (fixed) general translates of the Schubert varieties X^λ, X^μ, and X_ν.

In [7] we introduced the *kernel* and *span* of a rational curve in a Grassmann variety as a tool to study its Gromov-Witten invariants. The kernel of a stable map $f : C \to X$ is defined as the intersection of the m-planes in its image, and the span of f is the linear span of these m-planes:

$$\mathrm{Ker}(f) = \bigcap_{V \in f(C)} V \qquad \text{and} \qquad \mathrm{Span}(f) = \sum_{V \in f(C)} V\,.$$

Define the two-step flag variety $Y_d = \mathrm{Fl}(m-d, m+d; n)$ and the three-step flag variety $Z_d = \mathrm{Fl}(m-d, m, m+d; n)$; these varieties can be regarded as empty if $d > \min(m, n-m)$. Let $p : Z_d \to X$ and $h : Z_d \to Y_d$ be the projections. It was proved in [9] that, when $N_{\lambda,\mu}^{\nu,d}$ has degree zero, the map $f \mapsto (\mathrm{Ker}(f), \mathrm{Span}(f))$ defines an explicit bijection between the set of stable maps counted by $N_{\lambda,\mu}^{\nu,d}$ and the set of points in the intersection of general translates of the Schubert varieties $h(p^{-1}(X^\lambda))$, $h(p^{-1}(X^\mu))$, and $h(p^{-1}(X_\nu))$ in Y_d. It follows that $N_{\lambda,\mu}^{\nu,d}$ is equal to a classical triple intersection number of Schubert varieties in Y_d. The following equivariant generalization of this result was obtained in [10, Thm. 4.2].

Theorem 2.3 ([9, 10]). *We have* $N_{\lambda,\mu}^{\nu,d} = \int_{Y_d} h_* p^*[X^\lambda] \cdot h_* p^*[X^\mu] \cdot h_* p^*[X_\nu]$.

Let $J^d(\lambda)$ denote the 012-string obtained from λ by replacing the first d occurrences of 2 and the last d occurrences of 0 with 1. For example, we obtain $J^2((2,0,2,2,0,2,0,2)) = (1,0,1,2,1,2,1,2)$. We then have $h(p^{-1}(X^\lambda)) = Y_d^{J^d(\lambda)}$, i.e. $h(p^{-1}(X^\lambda))$ is the (opposite) Schubert variety in Y_d defined by the 012-string $J^d(\lambda)$. Furthermore, the varieties $p^{-1}(X^\lambda)$ and $h(p^{-1}(X^\lambda))$ have the same dimension if and only if the first d occurrences of 2 in λ come before the last d occurrences of 0; equivalently, the Young diagram of λ contains a $d \times d$ rectangle. Let $\lambda^\vee = (\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$ denote the string λ in reverse order. Then X_λ is a translate of $X^{\lambda^\vee}$. We obtain the following consequence of Theorem 2.1 and Theorem 2.3.

Corollary 2.4. *The Gromov-Witten invariant $N^{\nu,d}_{\lambda,\mu}$ is non-zero only if each of the Young diagrams of λ, μ, and $\nu^\vee$ contains a $d \times d$ rectangle. In this case we have*

$$N^{\nu,d}_{\lambda,\mu} \;=\; \sum_P \operatorname{wt}(P)$$

where the sum is over all equivariant puzzles P for Y_d with boundary $\triangle^{J^d(\lambda),J^d(\mu)}_{J^d(\nu^\vee)^\vee}$.

Proof. If the Young diagram of λ, μ, or $\nu^\vee$ does not contain a $d \times d$ rectangle, then one of the classes $h_*p^*[X^\lambda]$, $h_*p^*[X^\mu]$, or $h_*p^*[X_\nu]$ is equal to zero. On the other hand, if each of these Young diagrams contain a $d \times d$ rectangle, then we obtain

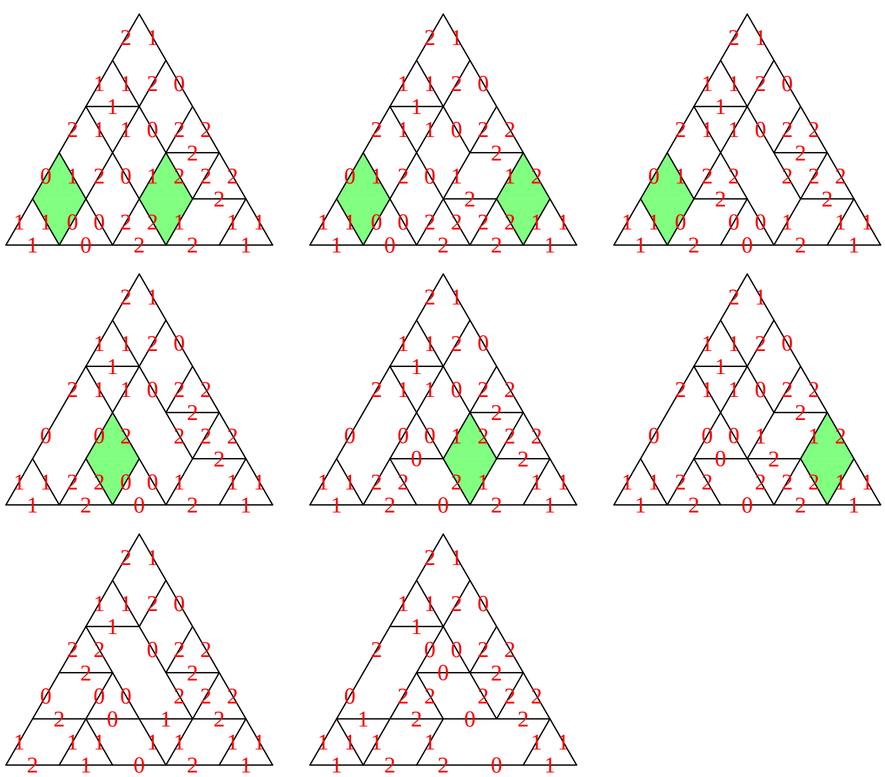
$$\begin{aligned} N^{\nu,d}_{\lambda,\mu} &= \int_{Y_d} h_*p^*[X^\lambda] \cdot h_*p^*[X^\mu] \cdot h_*p^*[X_\nu] \\ &= \int_{Y_d} [Y_d^{J^d(\lambda)}] \cdot [Y_d^{J^d(\mu)}] \cdot [(Y_d)_{J^d(\nu^\vee)^\vee}] \;=\; C^{J^d(\nu^\vee)^\vee}_{J^d(\lambda),J^d(\mu)}\,, \end{aligned}$$

and the result follows from Theorem 2.1. $\square$

Example 2.5. In the equivariant quantum cohomology ring of $X = \operatorname{Gr}(2,5)$ we have

$$\begin{aligned} &[X^{(2,1)}] \star [X^{(3,1)}] \;=\; \\ &(y_5-y_3)(y_5-y_1)(y_2-y_1)\,[X^{(3,1)}] \;+\; (y_5-y_1)^2\,[X^{(3,2)}] \;+\; (y_5-y_1)\,[X^{(3,3)}] \;+ \\ &(y_5-y_3)(y_2-y_1)\,q \;+\; (y_5-y_1)\,q\,[X^{(1)}] \;+\; q\,[X^{(1,1)}] \;+\; q\,[X^{(2)}]\,. \end{aligned}$$

The last four terms involving q are accounted for by the following puzzles.



3. Recursive equations

An observation that originates in Molev and Sagan's work [32] shows that all the equivariant structure constants $C_{u,v}^w$ are determined by the structure constants of the form $C_{w,w}^w$ by a set of recursive identities. This observation was used to prove the equivariant puzzle rule for Grassmannians [23], and it was extended to equivariant quantum cohomology in [30]. Molev and Sagan's recursions apply to the equivariant structure constants of any homogeneous space Y, and in general involves $2r$ families of identities where r is the rank of the Picard group of Y. The structure constants of the form $C_{w,w}^w$ are given by a formula of Kostant and Kumar [26]. In this section we will arrange the recursive identities into a single family, focusing on the two-step flag variety $X = \mathrm{Fl}(a, b; n)$.

For this purpose we will work with T-equivariant cohomology with coefficients in the polynomial ring $R = \mathbb{C}[\delta_0, \delta_1, \delta_2]$. The variables of this ring correspond to the simple puzzle labels, and the field of complex numbers $\mathbb{C}$ will be utilized as a two-dimensional real plane where puzzle angles can be encoded. This will later make it possible to use the triangular geometry of puzzles to prove the required recursive identities. We have $H_T^*(\mathrm{pt}; R) = R[y] := R[y_1, \dots, y_n]$ where $y_i = -c_1(\mathbb{C}e_i)$, and this ring contains Λ as a subring. Furthermore, the ring $H_T^*(X; R) = H_T^*(X; \mathbb{Z}) \otimes_{\mathbb{Z}} R$ is an $R[y]$-algebra with an $R[y]$-basis consisting of the equivariant Schubert classes $[X^u]$. The defining equation (1) for the equivariant structure constants $C_{u,v}^w \in \Lambda$ is also valid in $H_T^*(X; R)$.

The *Bruhat order* on the set of 012-strings for X is defined by $u \leq v$ if and only if $X_u \subset X_v$. We will write $u \to u'$ if u' covers u in the Bruhat order, i.e. we have $u \leq u'$ and $\ell(u') = \ell(u) + 1$. Equivalently, u' can be obtained from u by replacing a connected subsequence in one of the following three ways:

$$(0, 2^m, 1) \to (1, 2^m, 0) \quad \text{or} \quad (0, 2) \to (2, 0) \quad \text{or} \quad (1, 0^m, 2) \to (2, 0^m, 1)\,.$$

Here x^m denotes a sequence of m copies of x. Given a covering $u \to u'$ we set

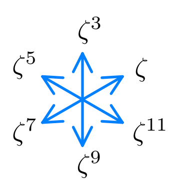
$$\delta\left(\frac{u}{u'}\right) = \delta_{u_i} - \delta_{u'_i}\,,$$

where i is the smaller index for which $u_i \neq u'_i$. For example, we have $\delta(\frac{10221}{11220}) = \delta_0 - \delta_1$ and $\delta(\frac{12021}{12201}) = \delta_0 - \delta_2$. Finally, given any 012-string u for X we define

$$C_u = \sum_{i=1}^{n} \delta_{u_i} y_i \ \in R[y]\,.$$

For example, $C_{01021} = \delta_0 y_1 + \delta_1 y_2 + \delta_0 y_3 + \delta_2 y_4 + \delta_1 y_5$.

Set $\zeta = \exp(\pi i/6) \in \mathbb{C}$. Notice that the odd powers of ζ are unit vectors perpendicular to puzzle edges.



Theorem 3.1. *The equivariant Schubert structure constants $C_{u,v}^w$ of the two-step partial flag variety $X = \mathrm{Fl}(a, b; n)$ satisfy the identities*

$$C_{w,w}^w = \prod_{i<j:\, w_i > w_j} (y_j - y_i) \tag{2}$$

in Λ *and*

$$(3)\quad (C_u\zeta^{11} + C_v\zeta^7 + C_w\zeta^3)\, C^w_{u,v} \;=\; \zeta^5 \sum_{u\to u'} \delta\left(\frac{u}{u'}\right) C^w_{u',v} + \zeta \sum_{v\to v'} \delta\left(\frac{v}{v'}\right) C^w_{u,v'} + \zeta^9 \sum_{w'\to w} \delta\left(\frac{w'}{w}\right) C^{w'}_{u,v}$$

in $R[y]$*, for all 012-strings* u*,* v*, and* w *for* X*. Furthermore, the equivariant structure constants of* X *are uniquely determined by these identities, i.e. any family of classes* $C^w_{u,v}$ *that satisfy (2) and (3) are the equivariant structure constants of* X*.*

Theorem 3.1 will be proved at the end of this section after some additional notation has been introduced. Given a class $\Omega \in H^*_T(X;R)$ and a 012-string u for X, we let $\Omega_u \in R[y]$ denote the restriction of Ω to the T-fixed point (A_u, B_u). The following identity is a special case of a formula of Kostant and Kumar [26, Prop. 4.24(a)].

Lemma 3.2. *For any 012-string* w *for* X *we have* $[X^w]_w = \prod_{i<j:\, w_i>w_j} (y_j - y_i)$.

Proof. It follows from [16, Ex. 3.3.2] that $[X^w]_w$ is the top Chern class of the fiber of the normal bundle of X^w in X over the point (A_w, B_w). For $i \neq j$ we define maps $\gamma_{i,j} : \mathbb{C} \to \mathrm{GL}_n(\mathbb{C})$ and $\varphi_{i,j} : \mathbb{C} \to X$ by $\gamma_{i,j}(s).e_k = e_k + s\,\delta_{jk}\,e_i$ for all k, and $\varphi_{i,j}(s) = \gamma_{i,j}(s).(A_w, B_w)$. Here δ_{jk} is Kronecker's delta. The tangent space T_wX of X at (A_w, B_w) has the basis $\{\varphi'_{i,j}(0) \mid w_i > w_j\}$, and T_wX^w has basis $\{\varphi'_{i,j}(0) \mid w_i > w_j \text{ and } i > j\}$. It follows that the normal space T_wX/T_wX^w has basis $\{\varphi'_{i,j}(0) \mid w_i > w_j \text{ and } i < j\}$. Since the torus T acts on T_wX by $t.\varphi'_{i,j}(0) = t_i/t_j\,\varphi'_{i,j}(0)$, we obtain $c_1(\mathbb{C}\,\varphi'_{i,j}(0)) = y_j - y_i$. This proves the lemma. ☐

Lemma 3.3. *Let* $\Omega \in H^*_T(X;R)$*, let* u *be a 012-string for* X*, and consider the expansion* $\Omega \cdot [X^u] = \sum_w d_w [X^w]$ *where* $d_w \in R[y]$*. Then* d_w *is non-zero only if* $u \le w$*, and we have* $d_u = \Omega_u$*.*

Proof. We have $d_w = \int_X \Omega \cdot [X^u] \cdot [X_w] = \int_X \Omega \cdot [X^u \cap X_w]$, and the intersection $X^u \cap X_w$ is non-empty if and only if $u \le w$. The last identity follows because $d_u = \int_X \Omega \cdot [(A_u, B_u)] = \Omega_u$. ☐

Lemma 3.3 implies that we have $C^w_{u,w} = [X^u]_w$ for arbitrary 012-strings u and w for X. In particular, the identity (2) follows from Kostant and Kumar's formula for $[X^w]_w$. A formula for the more general restrictions $[X^u]_w$ has been proved by Andersen, Jantzen, and Soergel [1, App. D] and by Billey [4].

Let $D_1 = X^{\lambda(1)}$ and $D_2 = X^{\lambda(2)}$ be the Schubert divisors on X, defined by the 012-strings $\lambda(1) = (0^{a-1}, 1, 0, 1^{b-a-1}, 2^{n-b})$ and $\lambda(2) = (0^a, 1^{b-a-1}, 2, 1, 2^{n-b-1})$. We will work with the class $\mathcal{D} = (\delta_0 - \delta_1)\,[D_1] + (\delta_1 - \delta_2)\,[D_2] \in H^*_T(X;R)$ that encodes both of these divisors. We also set $C_0 = C_{(0^a,1^{b-a},2^{n-b})} \in R[y]$.

Lemma 3.4. *For any 012-string* u *for* X *we have*

$$\mathcal{D} \cdot [X^u] = (C_u - C_0)\,[X^u] + \sum_{u\to u'} \delta\left(\frac{u}{u'}\right) [X^{u'}]\,.$$

Proof. The equivariant ring $H^*_T(X;R)$ has a natural grading by complex codimension given by $\deg\,[X^w] = \ell(w)$, $\deg(y_i) = 1$, and $\deg(\delta_j) = 0$. It therefore follows from Lemma 3.3 that, if the coefficient of $[X^w]$ is non-zero in the expansion of $\mathcal{D} \cdot [X^u]$, then we have either $u = w$ or $u \to w$. Recall that the classical Monk/Chevalley formula [12, 33] states that the product $[D_1] \cdot [X^u]$ in the ordinary cohomology ring $H^*(X; \mathbb{Z})$ is equal to the sum of all classes $[X^w]$ for which $u \to w$ and $\delta(\frac{u}{w}) \in \{\delta_0 - \delta_1, \delta_0 - \delta_2\}$, and the product $[D_2] \cdot [X^u]$ is the sum of all classes $[X^w]$ for which $u \to w$ and $\delta(\frac{u}{w}) \in \{\delta_0 - \delta_2, \delta_1 - \delta_2\}$. It follows that the coefficient of $[X^w]$ in $\mathcal{D} \cdot [X^u]$ is equal to $\delta(\frac{u}{w})$ whenever $u \to w$. It remains to show that $\mathcal{D}_u = C_u - C_0$. This has been proved in higher generality by Kostant and Kumar [26, Prop. 4.24(c)]. For completeness we give the following argument from [11, §8].

For $m \leq n$ we set $\mathbb{C}^m = \mathrm{Span}_{\mathbb{C}}\{e_1, \dots, e_m\} \subset \mathbb{C}^n$ and $\mathbb{C}^m_X = \mathbb{C}^m \times X$. There is a natural sequence of vector bundles over X given by $\mathcal{A} \subset \mathcal{B} \subset \mathbb{C}^n_X \to \mathbb{C}^b_X \to \mathbb{C}^a_X$, where $\mathcal{A}$ and $\mathcal{B}$ are the tautological subbundles on X and the last two maps are the projections to the first b and a coordinates in $\mathbb{C}^n$. The Schubert divisors on X are the zero sections $D_1 = Z(\bigwedge^a \mathcal{A} \to \bigwedge^a \mathbb{C}^a_X)$ and $D_2 = Z(\bigwedge^b \mathcal{B} \to \bigwedge^b \mathbb{C}^b_X)$. It follows that

$$\begin{aligned}\mathcal{D}_u &= (\delta_0 - \delta_1)\ (c_1(\mathbb{C}^a_X) - c_1(\mathcal{A}))_u + (\delta_1 - \delta_2)\ \big(c_1(\mathbb{C}^b_X) - c_1(\mathcal{B})\big)_u \\ &= \delta_0\big(c_1(\mathbb{C}^a) - c_1(A_u)\big) + \delta_1\big(c_1(\mathbb{C}^b/\mathbb{C}^a) - c_1(B_u/A_u)\big) \\ &\quad + \delta_2\big(c_1(\mathbb{C}^n/\mathbb{C}^b) - c_1(\mathbb{C}^n/B_u)\big) \\ &= C_u - C_0\,.\end{aligned}$$

This completes the proof. □

Proof of Theorem 3.1. The identity (2) follows from Lemma 3.2 and Lemma 3.3. To prove (3), we use Lemma 3.4 and the equivariant structure constants of X to expand both sides of the associativity relation $(\mathcal{D} \cdot [X^u]) \cdot [X^v] = \mathcal{D} \cdot ([X^u] \cdot [X^v])$ in the basis of Schubert classes. Since the coefficient of $[X^w]$ in both sides is the same, we obtain the identity

$$(C_u - C_w)\, C^w_{u,v} = \sum_{w' \to w} \delta\left(\frac{w'}{w}\right) C^{w'}_{u,v} - \sum_{u \to u'} \delta\left(\frac{u}{u'}\right) C^w_{u',v}\,. \tag{4}$$

Similarly, the relation $[X^u] \cdot ([X^v] \cdot \mathcal{D}) = ([X^u] \cdot [X^v]) \cdot \mathcal{D}$ implies the identity

$$(C_v - C_w)\, C^w_{u,v} = \sum_{w' \to w} \delta\left(\frac{w'}{w}\right) C^{w'}_{u,v} - \sum_{v \to v'} \delta\left(\frac{v}{v'}\right) C^w_{u,v'}\,. \tag{5}$$

Finally, the identity (3) is obtained by multiplying both sides of (4) with ζ^{11}, multiplying both sides of (5) with ζ^7, and adding the resulting equations.

We next observe that $C_u - C_w$ is non-zero whenever $u \neq w$, and $C_v - C_w$ is non-zero whenever $v \neq w$. Since both $C_u - C_w$ and $C_v - C_w$ are elements of the polynomial ring $\mathbb{Z}[\delta_0, \delta_1, \delta_2, y_1, \dots, y_n]$, and the powers ζ^{11} and ζ^7 are linearly independent over this ring, it follows that the factor $(C_u\zeta^{11} + C_v\zeta^7 + C_w\zeta^3) = (C_u - C_w)\zeta^{11} + (C_v - C_w)\zeta^7$ of (3) is non-zero whenever $u \neq w$ or $v \neq w$.

We finally prove that the equivariant Schubert structure constants of X are uniquely determined by (2) and (3) by descending induction on $\deg(C^w_{u,v}) = \ell(u) + \ell(v) - \ell(w)$. The basis step is vacuous because $\deg(C^w_{u,v}) \leq 2\dim(X)$. For the inductive step, let u, v, and w be given. If $u = v = w$, then the constant $C^w_{u,v}$ is determined by (2). Otherwise notice that all structure constants appearing on the right side of equation (3) have degree equal to $\deg(C^w_{u,v}) + 1$, so these constants are uniquely determined by the induction hypothesis. Since $R[y]$ is a domain and $(C_u\zeta^{11} + C_v\zeta^7 + C_w\zeta^3) \neq 0$, we deduce that $C^w_{u,v}$ is uniquely determined as well. □

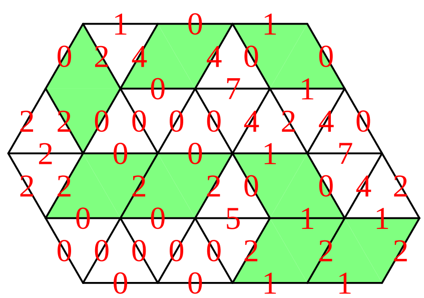
Remark 3.5. The real scalar product of two vectors $x, y \in \mathbb{C}$ is defined by $(x, y) = \mathrm{Re}(x\overline{y})$, and this scalar product has an $\mathbb{R}$-linear extension to the ring $R[y]$. By taking the scalar product of both sides of equation (3) with the vector ζ^{10}, we recover the identity (4) associated to the relation $(\mathcal{D}\cdot[X^u])\cdot[X^v] = \mathcal{D}\cdot([X^u]\cdot[X^v])$, and by taking the scalar product with ζ^8, we recover the identity (5) associated to the relation $[X^u]\cdot([X^v]\cdot\mathcal{D}) = ([X^u]\cdot[X^v])\cdot\mathcal{D}$. One may check that the scalar product of equation (3) with $1 \in \mathbb{C}$ results in an identity associated to the relation $([X^u]\cdot\mathcal{D})\cdot[X^v] = [X^u]\cdot(\mathcal{D}\cdot[X^v])$.

4. Mutations of puzzles

4.1. **Puzzles.** Define a *puzzle* to be any hexagon made from puzzle pieces with matching side labels, such that all boundary labels are simple. In contrast to the conventions used in [23] we allow all puzzle pieces to be rotated arbitrarily, including equivariant pieces. This means that rotations of puzzles are again puzzles, which will be exploited to simplify constructions and proofs. We shall work only with puzzles whose edges are parallel to the sides of a right-side-up triangle. Define the *dual* of a puzzle to be the result of reflecting it in a vertical line and applying the following substitution to its labels:

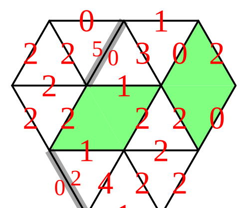
$$0 \mapsto 2\,,\ 1 \mapsto 1\,,\ 2 \mapsto 0\,,\ 3 \mapsto 4\,,\ 4 \mapsto 3\,,\ 5 \mapsto 5\,,\ 6 \mapsto 7\,,\ 7 \mapsto 6\,.$$

For example, the following two puzzles are dual to each other.



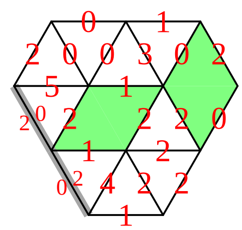
The line segments that make up the boundary of a puzzle are called *border segments*. We allow border segments to have length zero. In particular, the shape of a puzzle may be an equilateral triangle.

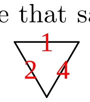
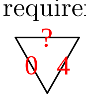
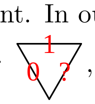
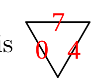
A *gashed puzzle* is a hexagon made of puzzle pieces, not necessarily with matching side labels, but still with simple boundary labels. The puzzle edges where the labels do not match are called *gashes*. We think about gashes as edges that have two labels, one on each side. We also allow gashes on the boundary of a gashed puzzle, by artificially imposing an extra label on the far side of a boundary edge. The following gashed puzzle has two gashes.

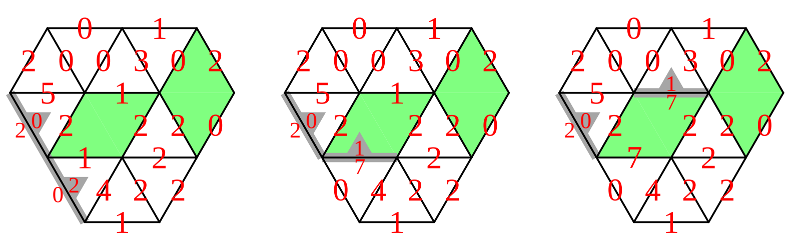


We will use the textual notation $\frac{a}{b} = \frac{a}{b}$, $b/a = {}^{b}/_{a}$, and $a\backslash b = {}_{a}\backslash^{b}$ for gashes of the three possible orientations. For example, the two gashes in the above example are denoted $5/0$ and $0\backslash 2$.

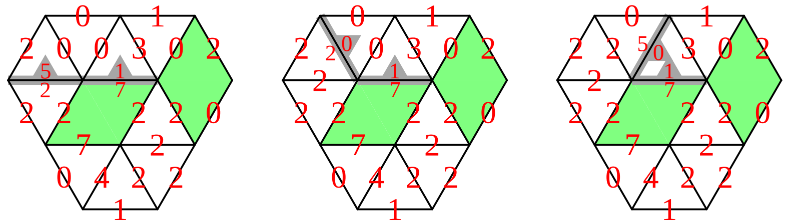
4.2. **Introduction to mutations.** The main new combinatorial construction in this paper is an algorithm called *mutation of puzzles*. Before we state the precise definition, we will give a more informal introduction by working through some examples. Consider the following gashed puzzle, where both gashes are located on the south-west border segment.



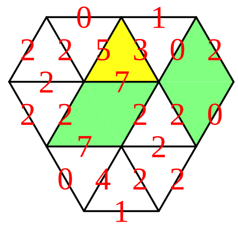
Such a pair of gashes can be introduced on an ungashed puzzle if one wishes to change the labels of a border segment. For example, the bottom gash $0\backslash 2$ indicates that the label 2 should be changed to 0. We will always make such a change by replacing the puzzle piece that contributes the unwanted label of the gash with a new piece of the same shape, and this new piece must be chosen such that only one new gash is created by the replacement. It is a fundamental observation that there is always *at most one* puzzle piece that satisfies this requirement. In our example we must replace the puzzle piece  with either  or , where the question marks can be arbitrary labels, and the only possible choice is . The replacement makes the bottom gash move to the top side of the replaced puzzle piece. We say that the gash has been *propagated*. After the gash has been moved, the process can be repeated to propagate it one more step. However, after two propagations have been carried out, no further propagations are possible. The steps are displayed in the following sequence of gashed puzzles, where we have also indicated the direction in which each gash is supposed to move.



Propagation of the second gash gives the following continuation.

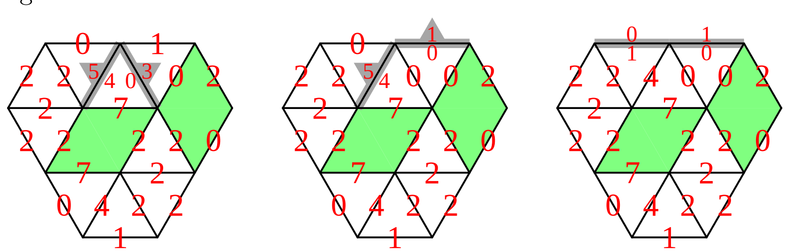


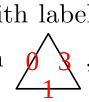
At this point both gashes are stuck at two sides of the same puzzle piece. We will show in Theorem 4.6 below that this is no coincidence. At this time we change the labels of the gashed edges to what the gashes suggest. The result is the following *flawed puzzle*, where one of the small triangles is a *temporary puzzle piece*. A temporary puzzle piece is analogous to an empty box in a Young tableau during a sequence of jeu de taquin slides; in fact, it is also possible to regard the temporary piece as a hole in the puzzle where no valid puzzle piece will fit.

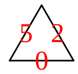


We would like to end up with a valid puzzle made from the puzzle pieces listed in Section 2, which means that we have to get rid of the temporary puzzle piece. A temporary puzzle piece can be *resolved* in three different ways, each of which preserves one of its sides and replaces the other two sides with gashes. This is done by replacing the temporary piece with a valid piece that has the same label on the side that is preserved. When we work with two-step puzzles, this valid piece is the unique puzzle piece whose largest label is the preserved label.

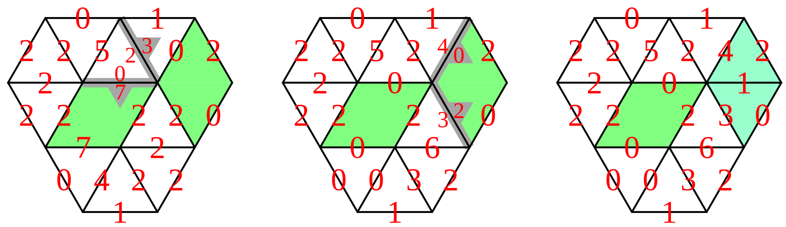
In our example, if we choose to preserve the bottom side with label 7 of the temporary piece, then we replace this piece with . The resulting gashes can be propagated as follows.



On the other hand, if we preserve the right side with label 3 of the temporary puzzle piece, then the temporary piece is replaced with , and by propagating the resulting gashes we recover the puzzle that we started with.

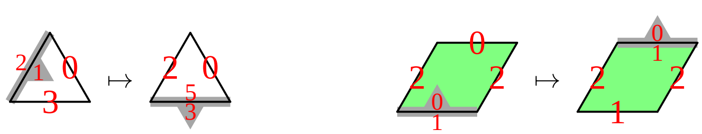
Finally, if we choose to preserve the left side with label 5 of the temporary piece, then this piece is replaced with 

, and the resulting gashes can be propagated as follows (skipping some steps).



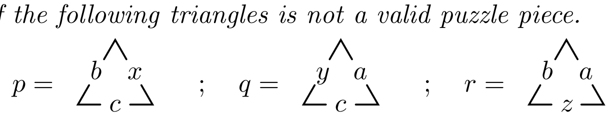
The middle picture shows the gashes at the positions where they get stuck, which is on two sides of an equivariant puzzle piece. We can change the labels of these edges to what the gashes suggest by replacing the equivariant piece with a rhombus made from two triangular puzzle pieces. Following [23], we call this rhombus a *scab*, and it has been colored light blue to *mark* its position. This allows the mutation to be inverted.

4.3. **Propagation of gashes.** We now give a detailed definition of the mutation algorithm, starting with several related concepts. Define a *directed gash* to be a gash together with a direction perpendicular to its edge. In pictures we will indicate the direction with a gray arrow. The label that the direction points to is called the *original label* and the other label is called the *new label*. Assume that a directed gash g points to a puzzle piece q that contributes the original label of g, and that no other gashes are located on the sides of q. Assume also that there exists a puzzle piece q' of the same shape as q, such that q' has the new label of g on its side corresponding to g, and another label of q' is equal to the label of q on the same side. In this case the gash g can be *propagated* by replacing q with q'. This replaces the gash g with its new label and creates a new gash on a different side of q'. The following are examples of propagations.

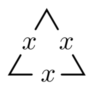


If the puzzle piece q is equivariant, then the only possible way to propagate g is to move this gash to the opposite side of q; this follows because opposite sides of any equivariant piece have the same label. On the other hand, if q is a triangular puzzle piece, then the following lemma implies that g can be propagated in at most one way.

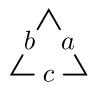
Lemma 4.1. *Let a, b, c, x, y, z be labels such that $a \neq x$, $b \neq y$, and $c \neq z$. Then at least one of the following triangles is not a valid puzzle piece.*



Proof. Since there are finitely many puzzle pieces, this lemma can be checked case by case. However, the lemma is also true with the more general definition of puzzle pieces that Knutson gave in [21]. We will prove the lemma in this generality. In this proof we will therefore use the definition of puzzle pieces from [21], which can be stated as follows. Each $x \in \mathbb{N}$ is a label and we set $\min(x) = \max(x) = x$. Whenever a and b are labels such that $\max(a) < \min(b)$, we declare that $c = (b, a)$ is also a label and set $\min(c) = \min(a)$ and $\max(c) = \max(b)$. A triangular puzzle piece is any small triangle of the form



or

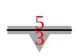


where $x \in \mathbb{N}$ and $c = (b, a)$ is a label. For labels a and b we will write $a < b$ if $\max(a) < \min(b)$. The *depth* of a label is its depth as a rooted binary tree.

Now assume that a, b, c, x, y, z are labels in this sense and the triangles p, q, r of the lemma are puzzle pieces. If $z = a = b \in \mathbb{N}$, then we have either $y = (a, c)$ and $c < a = b$, or $c = (y, a) = (y, b)$. In both cases the triangle p is not a puzzle piece. We may therefore assume that all three triangles are composed.

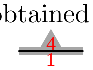
We claim that exactly one of the identities $x = (c, b)$, $y = (a, c)$, $z = (b, a)$ is true. If two of the identities are true, say $x = (c, b)$ and $y = (a, c)$, then $b < c < a$ implies that r is not a puzzle piece. On the other hand, if none of the identities are true, then we may assume without loss of generality that c is the deepest of the labels a, b, c, and we must have $c = (b, x) = (y, a)$, contradicting that $a \neq x$.

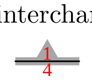
By the claim, we may assume that $x \neq (c, b)$ and $y \neq (a, c)$ and $z = (b, a)$. In particular, we have $a < b$. If $c = (b, x)$, then $c \neq (y, a)$, so we must have $a = (c, y) = ((b, x), y)$, contradicting $a < b$. It follows that $b = (x, c)$. Since we have either $c = (y, a)$ or $a = (c, y)$, we again deduce that $a < b$ is impossible. This completes the proof. $\square$

4.4. **Equivalence classes of gashes.** We will consider a *directed gash* as an object that exists independently of its appearance in a puzzle. In other words, a directed gash consists of a direction and two labels, but not a location. We will use the textual notation $\dfrac{a}{b}$, b/a, and $a\backslash b$ also for directed gashes when the direction of the gash is clear from the context. Given directed gashes g and h, we say that h is *immediately reachable* from g if h can be obtained by propagating g across a single triangular puzzle piece. For example, the first propagation displayed in Section 4.3 shows that the gash 

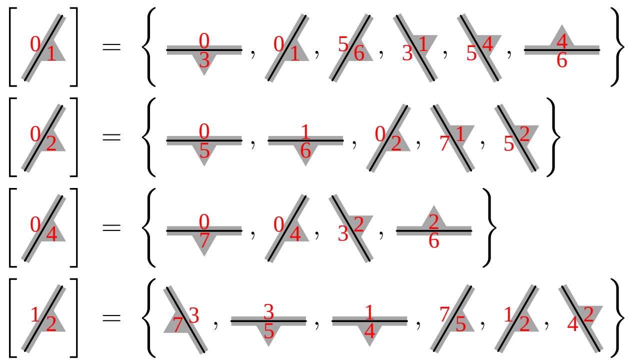
is immediately reachable from 

. Notice that, if h is obtained from g by a propagation that replaces a puzzle piece q with another piece q', then g is obtained from h by a propagation that replaces the 180 degree rotation of q with the 180 degree rotation of q'. It follows that 'immediately reachable' is a symmetric relation.

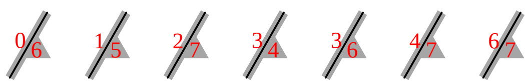
Let $[g]$ denote the set of directed gashes that can be reached from g by a series of propagations, i.e. we have $h \in [g]$ if and only if there exists a sequence $g = g_0, g_1, \ldots, g_k = h$ such that g_i is immediately reachable from g_{i-1} for each i. The set $[g]$ is called the *class* of g. Define the *opposite gash* of g to be the gash $\widehat{g}$ obtained by interchanging the labels of g and keeping the direction. For example, 

and 

are opposite gashes. Notice that $[\widehat{g}] = \{\widehat{h} \mid h \in [g]\}$. Similarly, if g' is obtained by rotating g by some angle, then $[g']$ is obtained from $[g]$ by rotating all elements by the same angle. The directed gashes g and h are said to be in *opposite* *classes* if $[\widehat{g}] = [h]$. All gash classes that contain at least two gashes are rotations of one of the following four classes or their opposites.

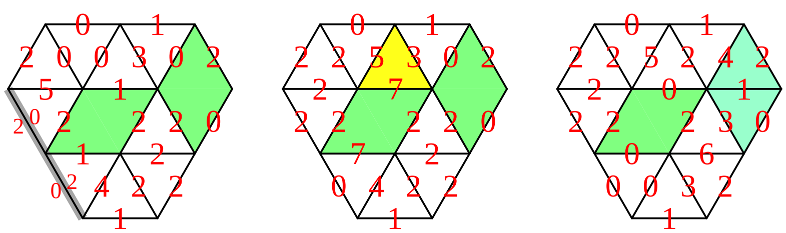


The directed gashes g for which $[g] = \{g\}$ are the gashes that can never be propagated. These gashes are rotations of the following seven gashes or their opposites.



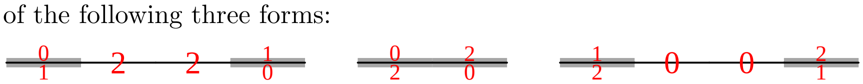
To see that the displayed gashes account for everything, notice that none of them are rotations of (opposites of) each other, there are 28 of them, and $28 \cdot 12 = 336$ is the total number of directed gashes.

4.5. **Flawed puzzles.** A *flawed puzzle* is a puzzle that contains a flaw. The flaw can be of three different types: a *gash pair* on a border segment, a *temporary puzzle piece*, or a *marked scab*. All types of flaws are represented in the following three puzzles, which have already been encountered in Section 4.2.



All boundary labels of a flawed puzzle must be simple. Any flawed puzzle has one or more *resolutions* where the flaw is replaced with two directed gashes. These resolutions are used to define the *mutations* of the flawed puzzle. As we will see, the gashes of a resolution are always in opposite classes. We proceed to discuss each type of flaw in more detail.

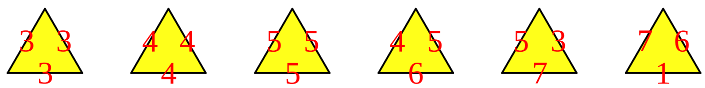
4.6. **Gash pairs.** A *gash pair* is a pair of gashes located on a single border segment of a puzzle. If the puzzle is rotated so that the gashed border segment is at the top of the puzzle, then the segment of edges between the two gashed edges should have one of the following three forms:



In the first and third forms, the middle segment may consist of any number of edges with the indicated labels, including zero. Notice that if u is the sequence of labels on or above the border segment, and u' is the sequence of labels on or below the segment, then we have $u \to u'$ with the notation of Section 3.

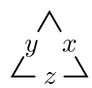
The gashes of a gash pair should be considered as directed towards the interior of the puzzle. A flawed puzzle containing a gash pair is therefore its own resolution. However, we usually omit the direction of gash pairs in pictures. Notice also that the gashes of a gash pair are opposite to each other.

4.7. **Temporary puzzle pieces.** According to Definition 4.2 below, a *temporary puzzle piece* is a small triangle from the following list. Temporary puzzle pieces are colored yellow and may be rotated.

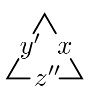


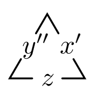
A flawed puzzle containing a temporary piece is the same as a puzzle, except that exactly one temporary puzzle piece is used together with the valid puzzle pieces from Section 2. The following formal definition of temporary puzzle pieces and their resolutions is valid also for three-step puzzles, see [5].

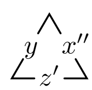
Definition 4.2. Let x, y, and z be puzzle labels. The triangle

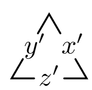
$t =$  .

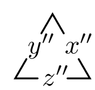
with these labels is a *temporary puzzle piece* if and only if there exist puzzle labels $x', x'', y', y'', z', z''$ such that all of the following triangles are valid puzzle pieces:

$r_1 =$ 

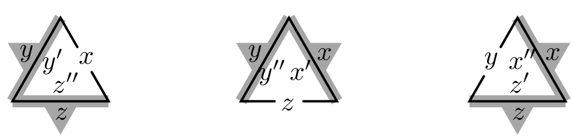
$r_2 =$ 

$r_3 =$ 

$t' =$ 

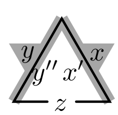
$t'' =$ 

In this case the *resolutions* of t are obtained by replacing two of the sides of t with gashes directed away from t, such that the original labels come from t and the new labels come from r_1, r_2, or r_3:



We need the following properties and classification of the resolutions of temporary puzzle pieces.

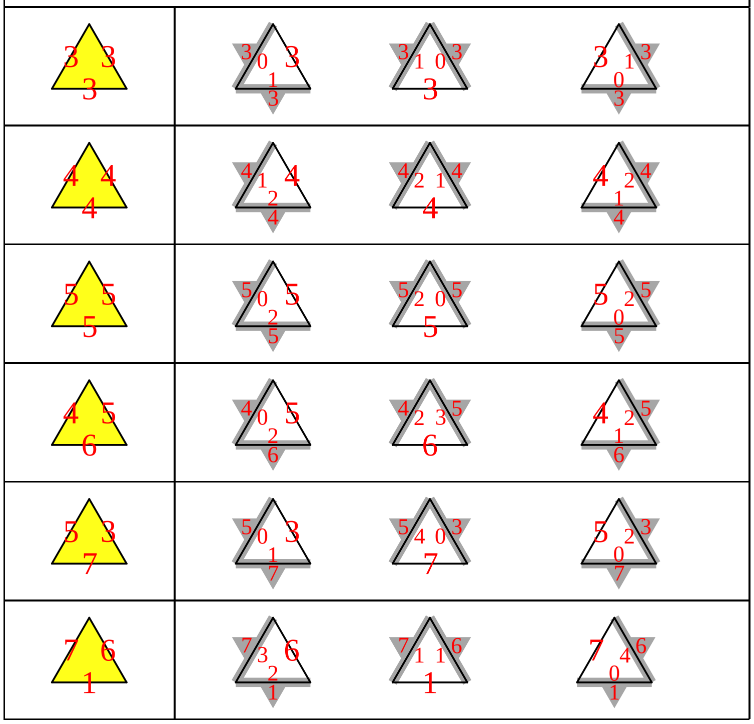
Proposition 4.3. (a) *Let x, y, z, x', y'' be puzzle labels. The gashed triangle*

$\widetilde{t} =$ 

is a resolution of a temporary puzzle piece if and only if its two gashes are in opposite classes and the triangle r_2 of Definition 4.2 is a valid puzzle piece.

(b) *Each temporary puzzle piece t has exactly three resolutions. In other words, the valid puzzle pieces r_1, r_2, r_3 of Definition 4.2 are uniquely determined by t.*

Table 4.7. Temporary puzzle pieces and their resolutions.



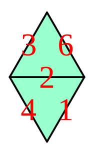
Proof. Assume first that $\widetilde{t}$ is a resolution of a temporary puzzle piece t, and let z', z'', r_1, r_3, t', and t'' be as in Definition 4.2. Let g_1 be the left directed gash of $\widetilde{t}$ and let g_2 be the right gash. The valid puzzle pieces r_3 and t'' then show that $[g_1] = [\frac{z'}{z''}]$, while r_1 and t' show that $[g_2] = [\frac{z''}{z'}]$, with both horizontal gashes directed towards the north. This shows that g_1 and g_2 are in opposite classes.

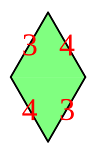
To establish the rest of the proposition, one first checks that each triangle t in the left column of Table 4.7 is a temporary puzzle piece. In fact, if x, y, z are the labels of t, and we let $x', x'', y', y'', z', z''$ be the unique labels such that $y', z'' \leq x$, $z', x'' \leq y$, $x', y'' \leq z$, and the triangles r_1, r_2, r_3 of Definition 4.2 are valid puzzle pieces, then t' and t'' are also valid puzzle pieces. This shows that t is a temporary puzzle piece, and also that the three gashed triangles next to t in Table 4.7 are resolutions of t. On the other hand, by inspection of the gash classes displayed in Section 4.4 it is easy to check that, up to rotation, all gashed triangles that satisfy the condition in part (a) are represented in the right column of Table 4.7. The proposition follows from this. $\square$

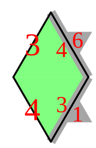
Remark 4.4. Given a temporary puzzle piece t, the valid puzzle pieces used to form the resolutions of t are obtained by keeping one label z of t and replacing the two other labels with the unique integers smaller than or equal to z such that the resulting triangle is a valid puzzle piece. This is a coincidence that holds for two-step puzzles but not for three-step puzzles [5].

4.8. **Scabs.** A *scab* means a small rhombus consisting of two triangular puzzle pieces with matching labels next to each other, so that the rhombus is not invariant under 180 degree rotation. In other words, the triangular puzzle pieces are not rotations of each other. Any puzzle containing one or more scabs can be turned into a flawed puzzle by marking one of the scabs. Marked scabs are colored light blue in pictures.

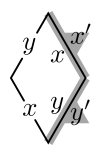
Let s be a scab and assume that q is an equivariant puzzle piece of the same shape as s, such that two sides of s and q share the same labels. In this case the two labels that s and q agree about must be on sides connected by an obtuse angle. The gashed rhombus $\widetilde{s}$ resulting from replacing s with q is then called a *resolution* of s. More precisely, $\widetilde{s}$ is obtained from q by replacing two of its sides with gashes directed away from q. These are the sides where the labels of q and s disagree, and the original labels of the gashes come from s while the new labels come from q. The following is an example.

$s =$ 

; $q =$ 

; $\widetilde{s} =$ 

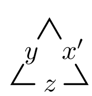
Proposition 4.5. (a) *Let x, x', y, y' be puzzle labels. The gashed rhombus*

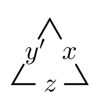


is a resolution of a scab if and only if the inner labels form an equivariant puzzle piece and the two gashes are in opposite classes.

(b) *Each scab has exactly one resolution.*

Proof. Assume that the gashed rhombus is a resolution of a scab, and let z be the label of the middle edge in this scab. Then the valid puzzle pieces

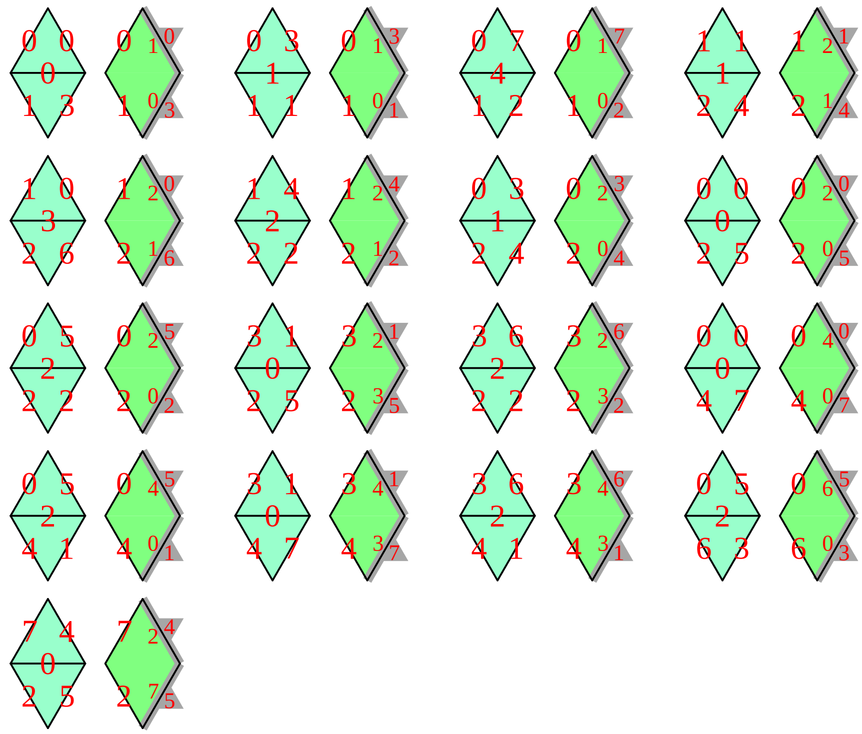


and


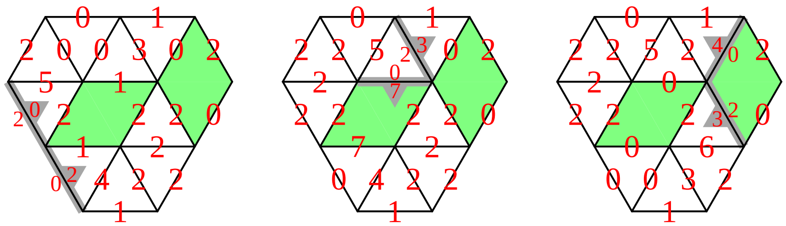
show that the two gashes of the resolution are in opposite classes. On the other hand, by inspection of the gash classes displayed in Section 4.4 it is easy to check that, up to rotation, all gashed rhombuses that satisfy the condition in part (a) are represented in Table 4.8. Since Table 4.8 also documents that every scab has a resolution, this completes the proof. □

4.9. **Mutations.** Let P be a flawed puzzle and let $\widetilde{P}$ be the result of replacing the flaw in P with one of its resolutions. Then $\widetilde{P}$ is a gashed puzzle called a *resolution* of P. The two directed gashes in $\widetilde{P}$ are in opposite classes, and these gashes are either connected or separated by a sequence of edges from the same border segment. The *right gash* of $\widetilde{P}$ is the rightmost of the two gashes for an observer standing between the gashes and facing the direction of the gashes. The other gash in $\widetilde{P}$ is called

Table 4.8. Scabs and their resolutions.



the *left gash*. The following gashed puzzles are resolutions of the flawed puzzles displayed in Section 4.5. The right gashes of these puzzles are $0\backslash 2$, $\frac{0}{7}$, and $4/0$.

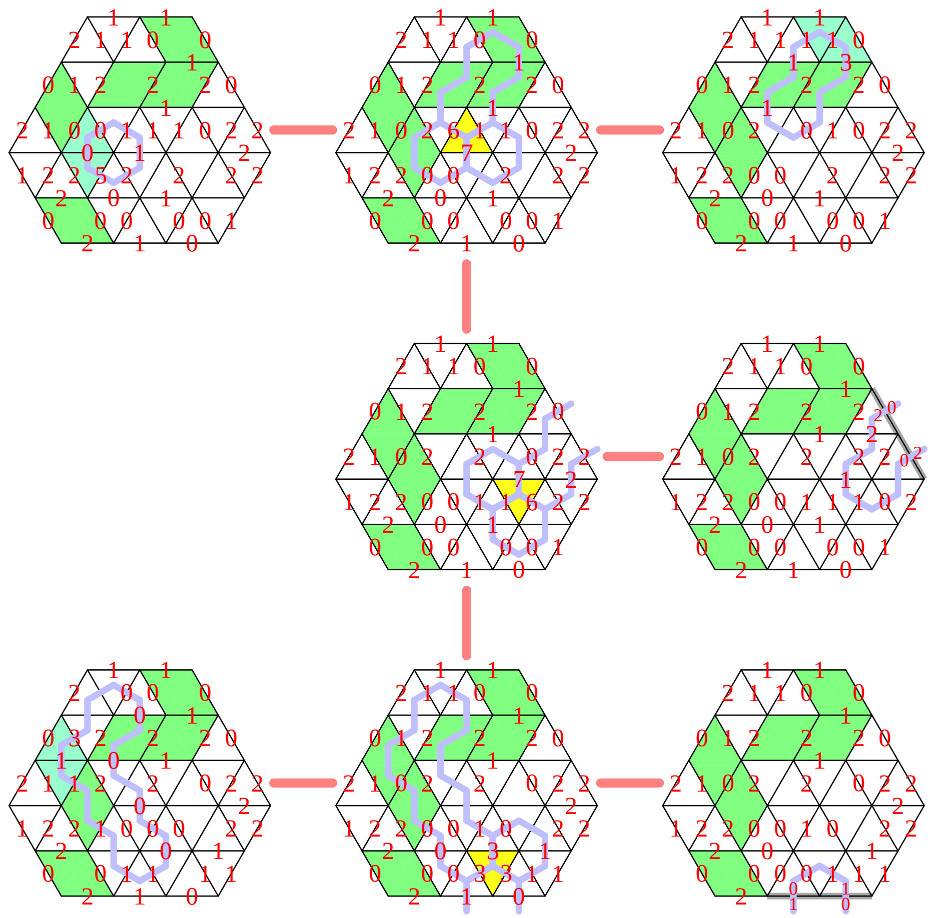


Define the *propagation path* of a gash in $\widetilde{P}$ to be the sequence of edges that change if we repeatedly propagate the gash until no more propagations are possible. We let $\Phi(\widetilde{P})$ denote the result of propagating both gashes in $\widetilde{P}$ as far as possible and then reversing the directions of the gashes. This is well defined by the first claim in the following result.

Theorem 4.6. *Let $\widetilde{P}$ be a resolution of a flawed puzzle. Then the propagation paths of the two gashes in $\widetilde{P}$ are disjoint. Furthermore, $\Phi(\widetilde{P})$ is a resolution of a unique flawed puzzle, and we have $\Phi(\Phi(\widetilde{P})) = \widetilde{P}$.*

Theorem 4.6 will be proved in Section 4.10. We will say that two flawed puzzles P and Q are *mutations* of each other if P has a resolution $\widetilde{P}$ such that $\Phi(\widetilde{P})$ is

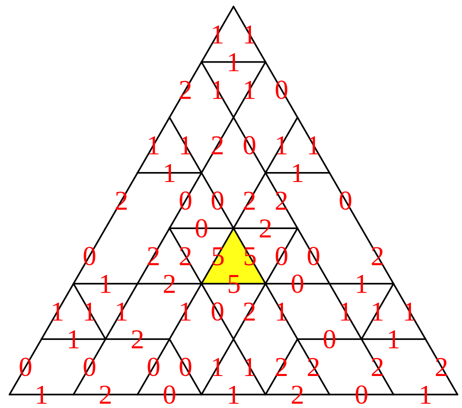
FIGURE 4.9. A connected component of the mutation graph.



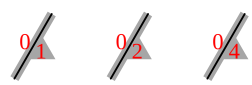
a resolution of Q. The set of all flawed puzzles can be arranged in a *mutation graph*, where each flawed puzzle is connected to its mutations. Figure 4.9 shows one component of this graph. For each flawed puzzle in the figure we have also indicated the set of edges that are changed by at least one mutation.

It should be noted that resolutions of flawed puzzles and propagation of gashes commute with rotations and dualization. This simplifies our proof of Theorem 4.6, and it implies that mutation commutes with rotations and dualization.

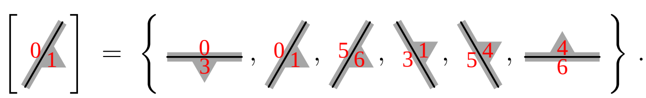
Example 4.7. In early versions of this paper we conjectured that every connected component of the mutation graph is a tree. However, the reader may check that the following puzzle belongs to a cycle of length 13 in its component. Notice also that since this puzzle is dual to itself, dualization of puzzles provides an involution of this component.



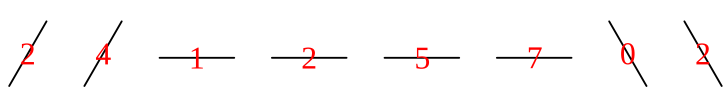
4.10. **Proof of Theorem 4.6.** Let $\widetilde{P}$ be a resolution of a flawed puzzle. After rotating and possibly dualizing this puzzle, we may assume that the right gash in $\widetilde{P}$ is equivalent to one of the following directed gashes:



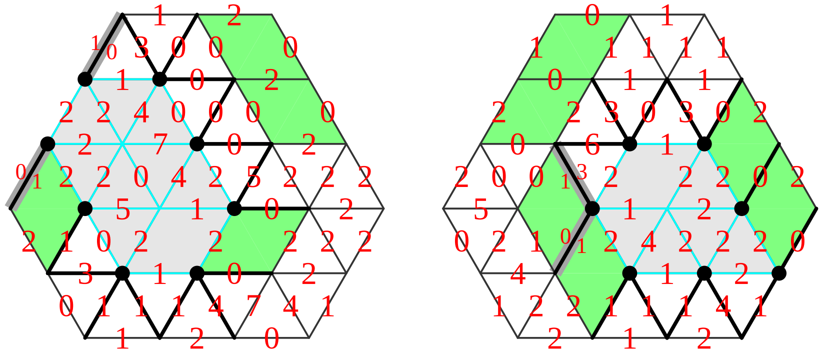
We first assume that the right gash is in the equivalence class

$[0/1] =$ 

The left and right gashes in $\widetilde{P}$ are connected by a node or a sequence of edges. In the latter case, these edges have the label 2 and are located on the north-west border segment of $\widetilde{P}$. Consider the set of all edges in $\widetilde{P}$ that come from the following list (with the indicated orientations). These edges can also be found in the center of Figure 4.10(a).

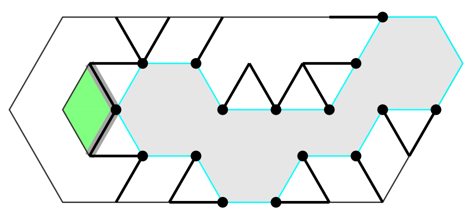


Let I be the connected component in this set of edges that includes the node or edges connecting the left and right gashes. The edges of $\widetilde{P}$ that are connected to I but not contained in I will be called the *spikes* of I. In particular, the left and right gashes of P are spikes of I. In the following two examples the edges of I have been colored light blue while the spikes have been made thick. The nodes where the spikes are connected to I are drawn as fat dots. One can show that no edge outside I can be connected to I in both ends, but we will not rely on this fact. In such a situation the edge would count as two spikes.



Let $s_0, s_1, \ldots, s_\ell$ be the sequence of spikes obtained when we start with the right gash and follow the boundary of I in counter clockwise direction. Then s_0 is the right gash of $\widetilde{P}$ and s_ℓ is the left gash. Any pair of consecutive spikes s_{k-1}, s_k in the sequence is separated either by a single puzzle piece or by the boundary of $\widetilde{P}$; the latter happens when part of the boundary of I is contained in the boundary of $\widetilde{P}$.

Let $\theta_0 \in (0, 2\pi]$ be the direction of the first spike s_0 in $\widetilde{P}$. Then choose angles $\theta_1, \ldots, \theta_\ell \in \mathbb{R}$ for the other spikes relative to θ_0. More precisely, if $\theta_0, \ldots, \theta_{k-1}$ have been chosen, then let θ_k be the result of adding or subtracting an amount to θ_{k-1} that represents the change in direction from s_{k-1} to s_k. For example, in the hypothetical situation



we have $\ell = 20$ and

$$(\theta_0, \theta_1, \ldots, \theta_{20}) =$$
$$\left(\tfrac{4}{3}\pi, \pi, \tfrac{4}{3}\pi, \tfrac{4}{3}\pi, \pi, 2\pi, \tfrac{5}{3}\pi, \tfrac{5}{3}\pi, \tfrac{5}{3}\pi, \tfrac{4}{3}\pi, 3\pi, 3\pi, \tfrac{8}{3}\pi, \tfrac{7}{3}\pi, \tfrac{8}{3}\pi, \tfrac{7}{3}\pi, \tfrac{7}{3}\pi, \tfrac{7}{3}\pi, \tfrac{8}{3}\pi, 3\pi, \tfrac{8}{3}\pi\right).$$

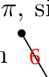
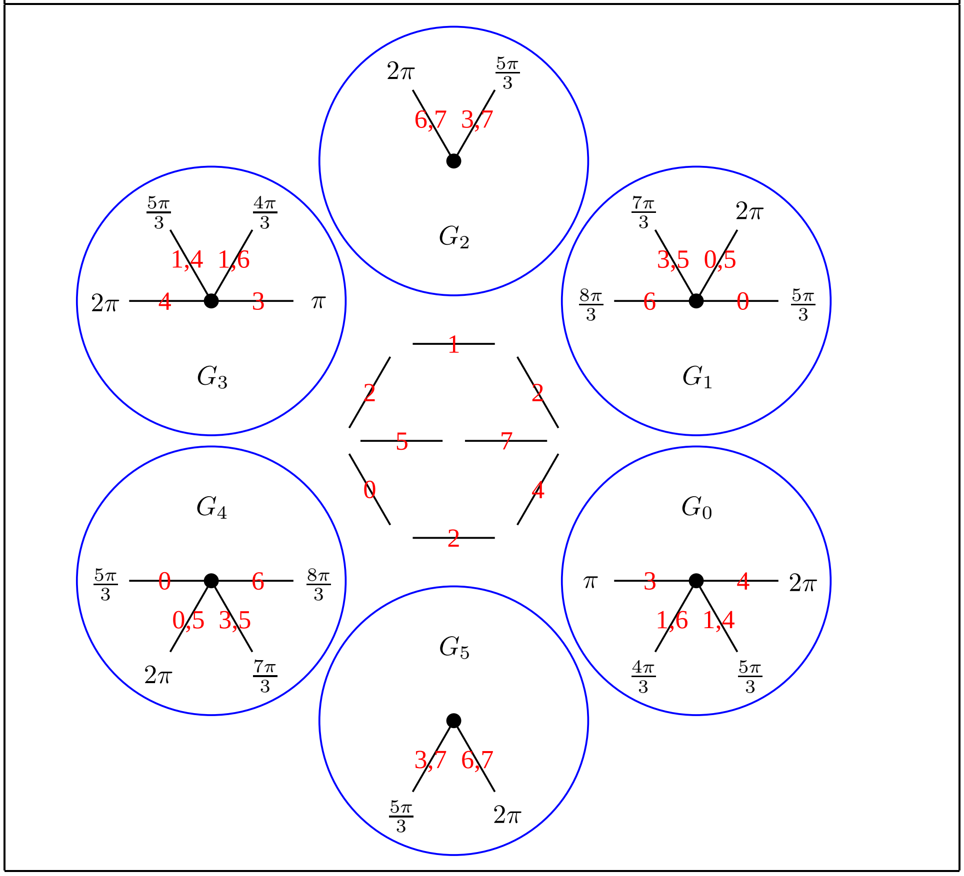
To each spike s_k we now define an *adjusted angle* $\widehat{\theta}_k$ that is obtained by subtracting an amount from θ_k that depends on both the label of s_k and (θ_k mod 2π). For s_0 and s_ℓ we use the original labels of the corresponding gashes. Figure 4.10(a) shows all possible spikes of I together with the amount that should be subtracted in each case. Notice that many edges in the figure are used to represent several spikes with different labels, which is done by listing the relevant labels. For example, if $\theta_k = \frac{11}{3}\pi$ and s_k has label 6, then we obtain $\widehat{\theta}_k = \frac{5}{3}\pi$, since the amount 2π must be subtracted from the angle of any spike of the form  .

Figure 4.10(a) separates the collection of possible spikes to I into the six groups G_0, G_1, G_2, G_3, G_4, and G_5. Notice that the right gash s_0 belongs to G_0, while the left gash s_ℓ belongs to G_1. In particular, we have $\pi \leq \theta_0 \leq 2\pi$ and $\widehat{\theta}_0 = 0$.

FIGURE 4.10(a). Spike groups and adjustment angles for the gash class [0/1].



Lemma 4.8. *We have $\widehat{\theta}_0 \leq \widehat{\theta}_1 \leq \cdots \leq \widehat{\theta}_\ell$. Furthermore, if two consecutive spikes s_{k-1} and s_k belong to different spike groups, or if s_{k-1} and s_k are separated by the boundary of $\widetilde{P}$, then $\widehat{\theta}_k - \widehat{\theta}_{k-1} \geq \frac{1}{3}\pi$.*

Proof. Assume first that s_{k-1} and s_k are separated by a puzzle piece q. Then the difference $\theta_k - \theta_{k-1}$ is determined by q. Since there are finitely many possibilities for q, the lemma can be checked case by case.

Table 4.10(a) lists all possibilities for the puzzle piece q when s_{k-1} belongs to G_0, G_1, or G_2. The spikes s_{k-1} and s_k are also identified in each case. The puzzle pieces q for which s_{k-1} is in G_3, G_4, or G_5 can be obtained by rotating the puzzle pieces in Table 4.10(a) by 180 degrees. Notice also that the puzzle pieces in the table are organized into four rows, depending on the exact spike groups that s_{k-1} and s_k belong to. This will be convenient later.

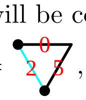
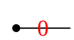
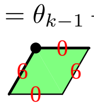
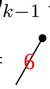
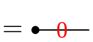
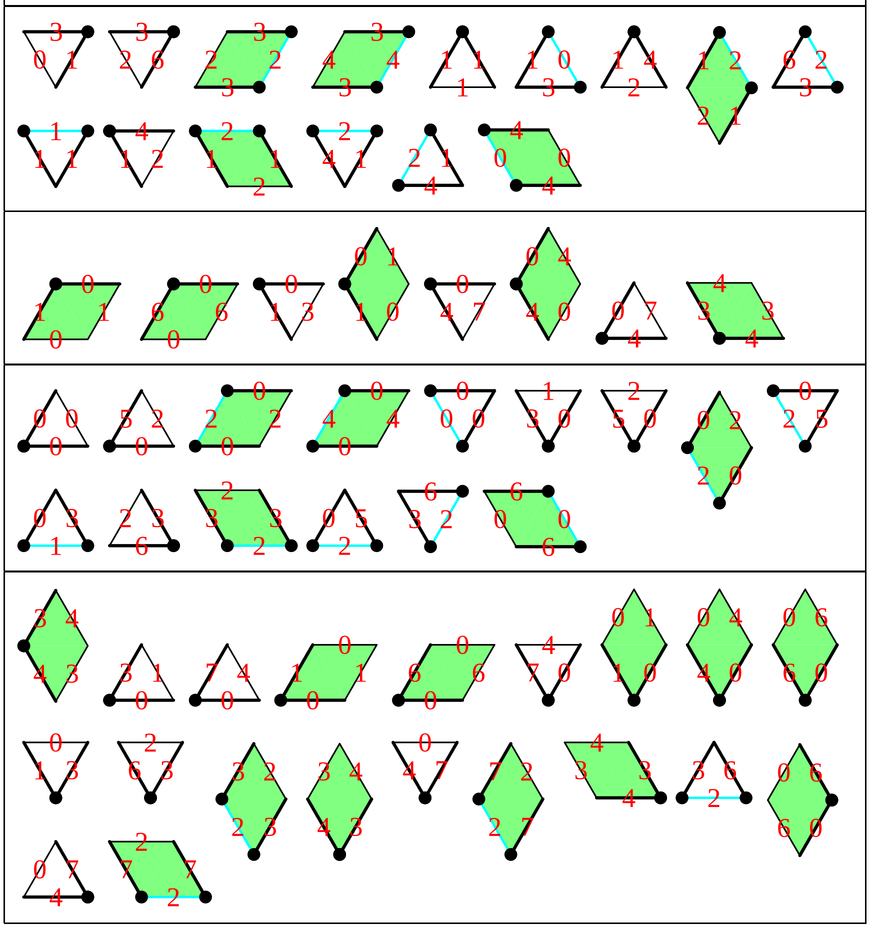
As an example, if $q =$  , then $s_{k-1} =$  and $s_k =$  are both in the group G_1, and we have $\theta_k = \theta_{k-1} - \frac{\pi}{3}$, $\widehat{\theta}_{k-1} = \theta_{k-1} - 2\pi$, and $\widehat{\theta}_k = \theta_k - \frac{5\pi}{3} = \widehat{\theta}_{k-1}$. On the other hand, if $q =$  , then $s_{k-1} =$  is in the group G_0, $s_k =$ 

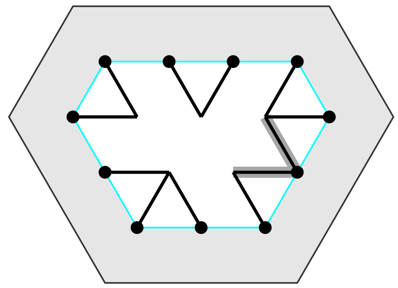
TABLE 4.10(a). Consecutive spikes for the gash class [0/1].



is in G_1, $\theta_k = \theta_{k-1} + \frac{2\pi}{3}$, $\widehat{\theta}_{k-1} = \theta_{k-1} - \frac{4\pi}{3}$, and $\widehat{\theta}_k = \theta_k - \frac{5\pi}{3} = \widehat{\theta}_{k-1} + \frac{\pi}{3}$. We leave the remaining cases to the reader.

We finally assume that s_{k-1} and s_k are separated by the boundary of $\widetilde{P}$. Then we have $\theta_k - \theta_{k-1} \geq \pi$, and since all boundary labels of $\widetilde{P}$ are simple, it follows that s_{k-1} and s_k have simple labels. Based on these observations one may check from Figure 4.10(a) that $\widehat{\theta}_k - \widehat{\theta}_{k-1} \geq \frac{\pi}{3}$, as required. $\square$

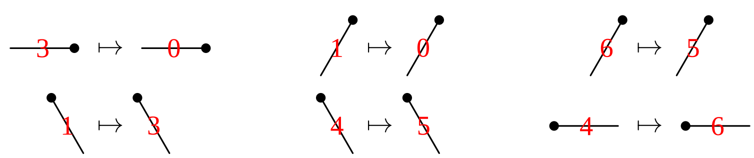
We first deduce from Lemma 4.8 that our sequence of spikes goes around the outer boundary of I in counter clockwise direction, as opposed to going around a hole in I in clockwise direction. In other words, situations like the following are impossible.



In fact, if the sequence of spikes went around a hole in I, then we would have $\theta_0 - \theta_\ell \in \{\frac{7\pi}{3}, \frac{8\pi}{3}\}$, $-\frac{5}{3}\pi \leq \theta_\ell \leq -\pi$, $\widehat{\theta}_0 = 0$, and $\widehat{\theta}_\ell = -\frac{11}{3}\pi$, which contradicts Lemma 4.8.

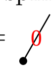
Since the sequence of spikes goes counter clockwise around the outer boundary of I, we obtain $\theta_\ell - \theta_0 \in \{\pi, \frac{4\pi}{3}, \frac{5\pi}{3}\}$, $2\pi \leq \theta_\ell \leq 3\pi$, $\widehat{\theta}_0 = 0$, and $\widehat{\theta}_\ell = \frac{\pi}{3}$. Lemma 4.8 then implies that for some $r \in [1, \ell]$ we have $\widehat{\theta}_0 = \widehat{\theta}_1 = \cdots = \widehat{\theta}_{r-1} = 0$ and $\widehat{\theta}_r = \widehat{\theta}_{r+1} = \cdots = \widehat{\theta}_\ell = \frac{\pi}{3}$. Furthermore, we have $s_k \in G_0$ for $0 \leq k \leq r-1$ and $s_k \in G_1$ for $r \leq k \leq \ell$. This implies that the first r spikes are separated by puzzle pieces from the first row of Table 4.10(a), the two middle spikes s_{r-1} and s_r are separated either by the boundary of $\widetilde{P}$ or by a puzzle piece from the second row of the table, and the last $\ell - r + 1$ spikes are separated by puzzle pieces from the third row.

When the right gash of $\widetilde{P}$ is propagated, this gash moves through the spikes s_k for $0 \leq k \leq r-1$. Each spike s_k is first replaced with the unique gash in the gash class $[0/1]$ that has the same orientation as s_k and whose original label is equal to the label of s_k. Then s_k attains the new label of the same gash, and the gash moves on. This follows by observing that the following substitution of spikes replaces all puzzle pieces in the first row of Table 4.10(a) with different valid puzzle pieces. These substitutions correspond to the gashes in the gash class $[0/1]$.



The above propagations will replace the spike s_{r-1} with the unique gash in the class $[0/1]$ whose orientation and original label agree with s_{r-1}, and this gash points to either the boundary of $\widetilde{P}$ or a puzzle piece from the second row of Table 4.10(a). An inspection of the puzzle pieces in this row then shows that the right gash cannot be propagated further.

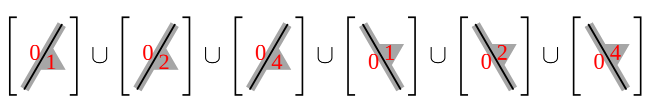
Similarly, the left gash of $\widetilde{P}$ propagates through the spikes s_k for $r \leq k \leq \ell$ in reverse order. Each spike s_k is first replaced with the unique gash in the opposite gash class $[1/0]$ that has the same orientation as s_k and whose original label is equal to the label of s_k. Then s_k attains the new label of this gash, and the gash moves on. This follows because the substitution of spikes corresponding to the opposite class $[1/0]$ replaces all puzzle pieces in the third row of Table 4.10(a) with different valid puzzle pieces. Eventually s_r is replaced with the unique gash from the opposite class $[1/0]$ with the same orientation and original label. At this point an inspection of the second row of Table 4.10(a) shows that the left gash cannot be propagated further (this is also true if the left gash is propagated before the right gash).

At this point $\Phi(\widetilde{P})$ is obtained by reversing the directions of both gashes. If the spikes s_{r-1} and s_r are both on the boundary of $\widetilde{P}$, then the (original) labels of these spikes are simple, and we have $\theta_r - \theta_{r-1} \geq \pi$. An inspection of the spike groups G_0 and G_1 of Figure 4.10(a) then shows that $s_{r-1} =$  and $s_r =$ . This implies that $\Phi(\widetilde{P})$ is a flawed puzzle with a gash-pair on the south-east border segment. Otherwise s_{r-1} and s_r are separated by a puzzle piece from the second row of Table 4.10(a), and this puzzle piece appears in $\Phi(\widetilde{P})$ with gashes on two sides that are in opposite classes. In this case it follows from Proposition 4.3(a) or Proposition 4.5(a) that $\Phi(\widetilde{P})$ is a resolution of a flawed puzzle.

Theorem 4.6 follows from this when the right gash of $\widetilde{P}$ is in the gash class [0/1]. The same argument also works if the right gash is in one of the classes [0/2] or [0/4], except that Figure 4.10(a) and Table 4.10(a) must be replaced with Figure 4.10(b) and Table 4.10(b) for the class [0/2] and with Figure 4.10(c) and Table 4.10(c) for the class [0/4]. This completes the proof of Theorem 4.6.

4.11. **Bijections of puzzles.** We finish this section by explaining how the mutation algorithm can be used to give new constructions of certain bijections of puzzles defined in the papers [24, 23, 8]. These constructions are not required for our proof of Theorem 2.1. We start by generalizing the bijections from [23, 8] which were applied to prove special cases of Theorem 2.1.

Let $\overrightarrow{\mathcal{G}}$ be the union of equivalence classes of gashes defined by

$\overrightarrow{\mathcal{G}} =$  .

Let $\overrightarrow{\mathcal{R}}$ be the set of all resolutions of flawed puzzles for which the right gash belongs to $\overrightarrow{\mathcal{G}}$, and let $\overrightarrow{\mathcal{P}}$ be the set of all flawed puzzles for which at least one resolution belongs to $\overrightarrow{\mathcal{R}}$. We also let $\overleftarrow{\mathcal{R}}$ and $\overleftarrow{\mathcal{P}}$ denote the sets obtained by rotating the objects in $\overrightarrow{\mathcal{R}}$ and $\overrightarrow{\mathcal{P}}$ by 180 degrees. Given any set of flawed puzzles S, we write S_{gash}, S_{scab}, and S_{temp} for the subsets of puzzles in S whose flaws have the indicated types. Notice that the gash pair of any puzzle in $\overrightarrow{\mathcal{P}}_{\text{gash}}$ is located on one of the left border segments, while the gash pair of a puzzle in $\overleftarrow{\mathcal{P}}_{\text{gash}}$ is located on one of the right border segments.

Lemma 4.9. *We have* $\overrightarrow{\mathcal{P}}_{\text{temp}} = \overleftarrow{\mathcal{P}}_{\text{temp}} = \overrightarrow{\mathcal{P}} \cap \overleftarrow{\mathcal{P}}$. *Furthermore, any puzzle in* $\overrightarrow{\mathcal{P}} \cap \overleftarrow{\mathcal{P}}$ *has exactly one resolution in* $\overrightarrow{\mathcal{R}}$ *and exactly one resolution in* $\overleftarrow{\mathcal{R}}$.

Proof. The (right-side-up) temporary puzzle pieces that occur in $\overrightarrow{\mathcal{P}}_{\text{temp}}$, and the resolutions of these pieces that provide elements of $\overrightarrow{\mathcal{R}}$ and $\overleftarrow{\mathcal{R}}$, are listed in Table 4.11. $\square$

FIGURE 4.10(b). Spike groups and adjustment angles for the gash class [0/2].

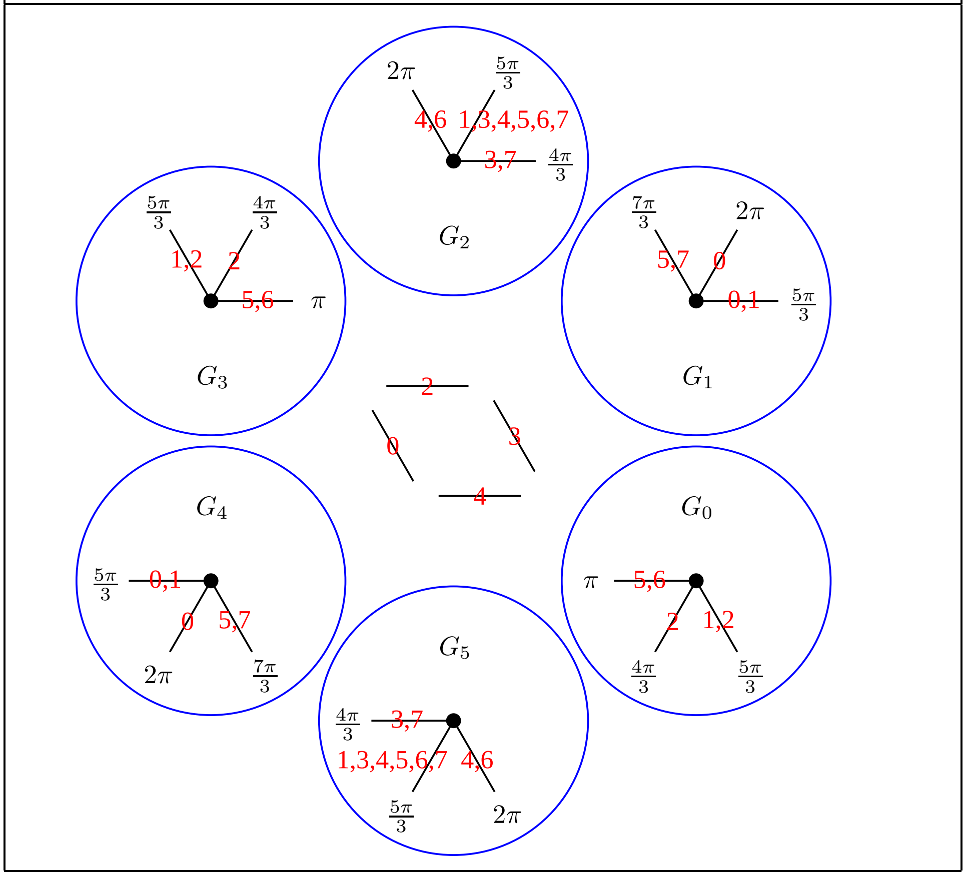
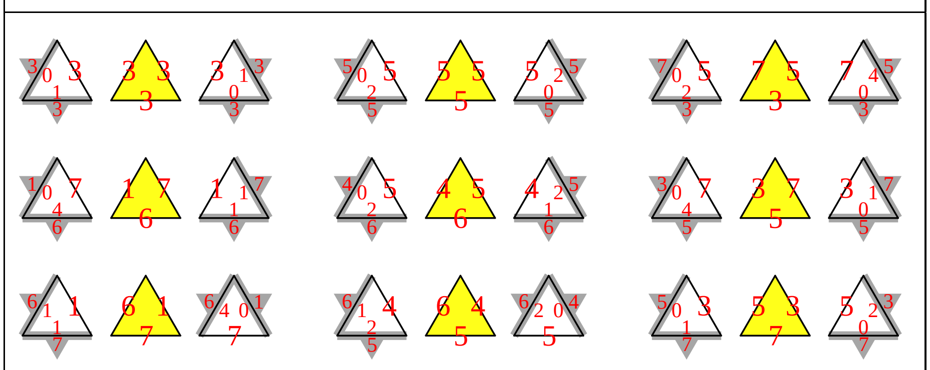
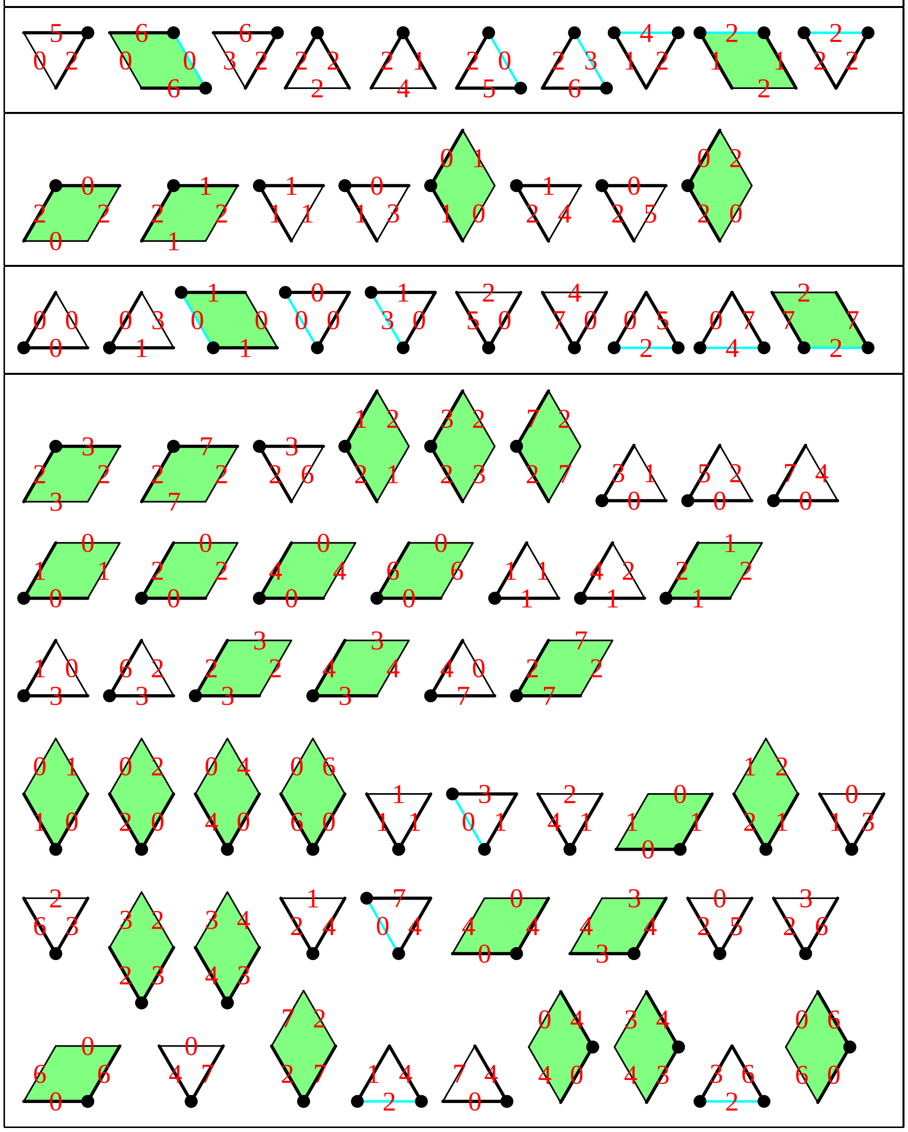


TABLE 4.11. Temporary puzzle pieces encountered in $\overrightarrow{\mathcal{P}} \cap \overleftarrow{\mathcal{P}}$.



The involution Φ defined in Section 4.9 restricts to a bijection from $\overrightarrow{\mathcal{R}}$ to $\overleftarrow{\mathcal{R}}$. We can therefore define a bijection $\Psi : \overrightarrow{\mathcal{P}} \to \overleftarrow{\mathcal{P}}$ as follows. Given $P \in \overrightarrow{\mathcal{P}}$, let $\widetilde{P}$ be the unique resolution of P that belongs to $\overrightarrow{\mathcal{R}}$, and let $\Psi(P)$ be the unique flawed

TABLE 4.10(b). Consecutive spikes for the gash class [0/2].

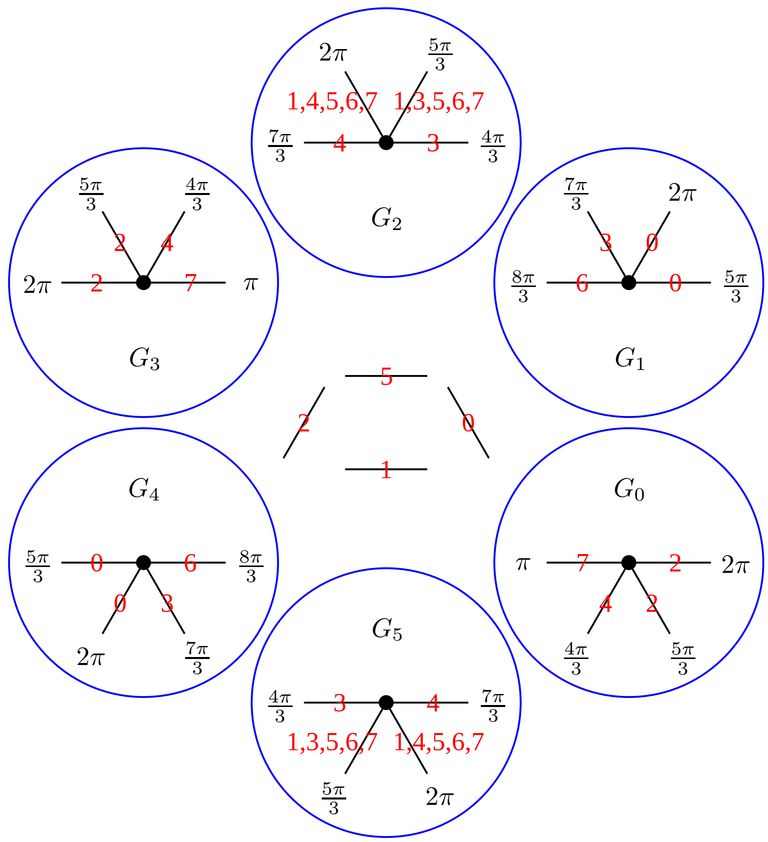


puzzle that has $\Phi(\widetilde{P})$ as a resolution. Notice that if $\Psi(P) \in \overrightarrow{\mathcal{P}} \cap \overleftarrow{\mathcal{P}}$, then we may apply Ψ an additional time. Let $\Psi^\infty(P)$ denote the result of applying Ψ to P until we obtain a flawed puzzle in the set $\overleftarrow{\mathcal{P}} \smallsetminus \overrightarrow{\mathcal{P}} = \overleftarrow{\mathcal{P}}_{\text{gash}} \cup \overleftarrow{\mathcal{P}}_{\text{scab}}$. The restriction of Ψ^∞ to $\overrightarrow{\mathcal{P}} \smallsetminus \overleftarrow{\mathcal{P}}$ is a bijection

$$\Psi^\infty : \overrightarrow{\mathcal{P}}_{\text{gash}} \cup \overrightarrow{\mathcal{P}}_{\text{scab}} \xrightarrow{\simeq} \overleftarrow{\mathcal{P}}_{\text{gash}} \cup \overleftarrow{\mathcal{P}}_{\text{scab}} .$$

For example, Ψ^∞ maps the top-left puzzle in Figure 4.9 to the top-right puzzle, and it maps the bottom-left puzzle to the middle-right puzzle. Related bijections can be obtained by conjugating Ψ^∞ by rotations and/or dualization of flawed puzzles. This corresponds to rotating and/or dualizing the gashes in $\overrightarrow{\mathcal{G}}$.

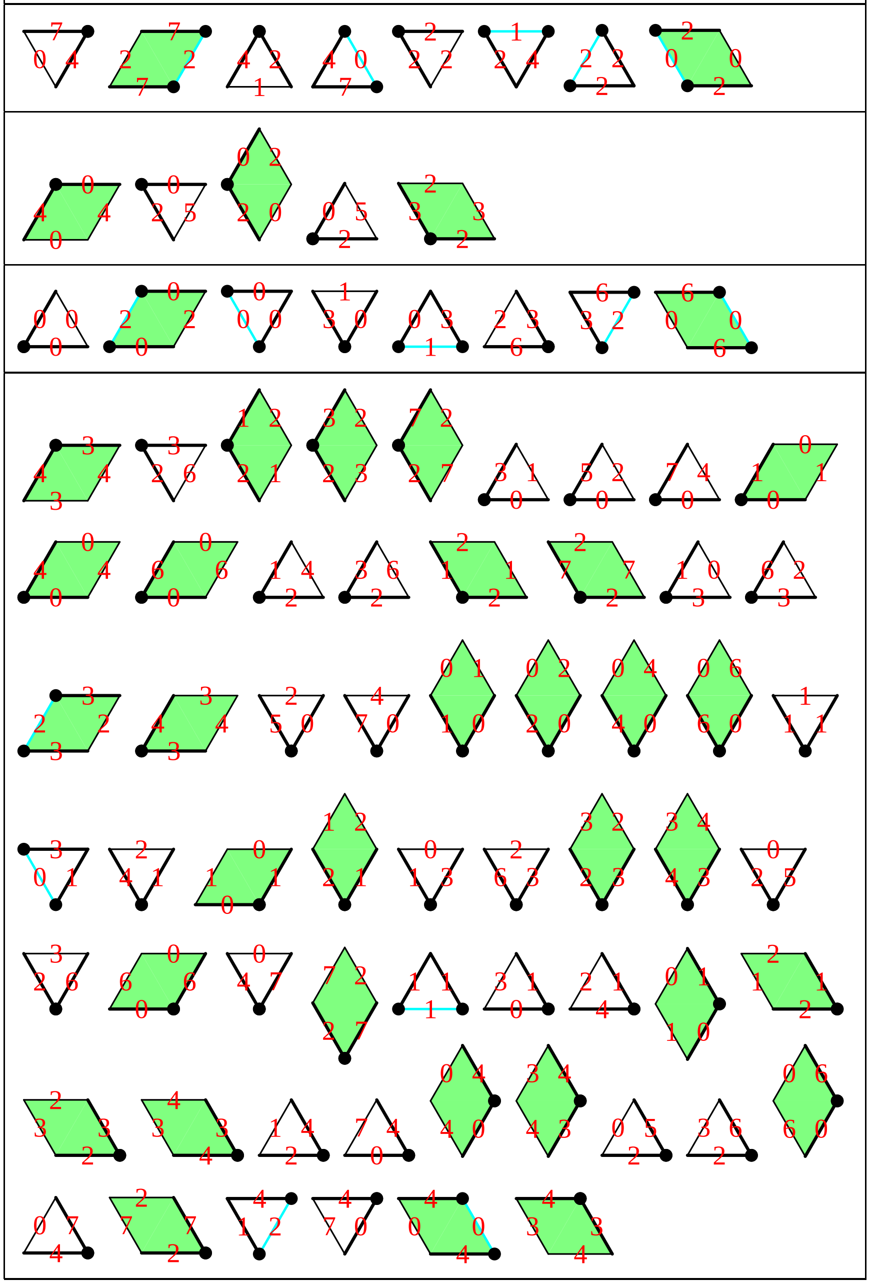
FIGURE 4.10(c). Spike groups and adjustment angles for the gash class [0/4].



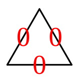
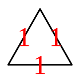
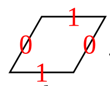
The bijections of puzzles from [23, 8] related to multiplication with divisors are special cases of Ψ^∞ and its conjugates. Notice that our definition of Ψ^∞ involves modifying some areas of a puzzle multiple times. In contrast the constructions used in [23, 8] directly describe the end results of the respective bijections. By factoring the bijection Ψ^∞ into a series of mutations, we have obtained a simpler and more conceptual description.

Remark 4.10. The classical Littlewood-Richardson rule expresses any Littlewood-Richardson coefficient $c^\nu_{\lambda,\mu}$ as the number of *LR tableaux* of shape ν/λ and weight μ. The precise definitions can be found in e.g. [15]. By composing bijections of Fulton [6] and of Knutson, Tao, and Woodward [24], one may obtain a bijection between these LR tableaux and the set of puzzles counted by the cohomological puzzle rule for Grassmannians. A more general bijection between equivariant LR tableaux and equivariant puzzles for Grassmannians has been defined by Kreiman [27]. Given a LR tableau T of shape ν/λ and a Young diagram $\lambda' \subset \lambda$ with one box less than λ, the jeu de taquin algorithm can be used to produce a LR tableau T' of some shape ν'/λ', where $\nu' \subset \nu$ has one box less than ν. The bijection Ψ^∞ is compatible with the jeu de taquin algorithm in the sense that the puzzle corresponding to T' may be obtained from the puzzle corresponding to T by applying one of the

TABLE 4.10(c). Consecutive spikes for the gash class [0/4].

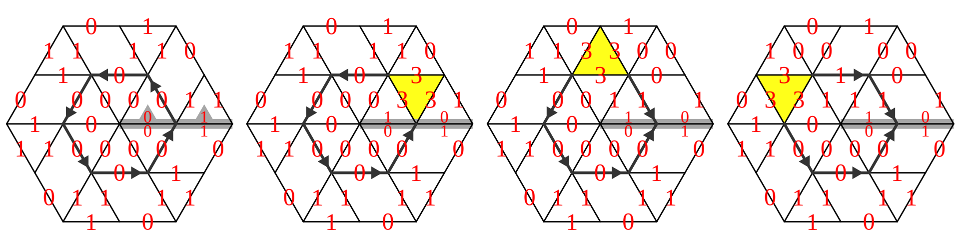


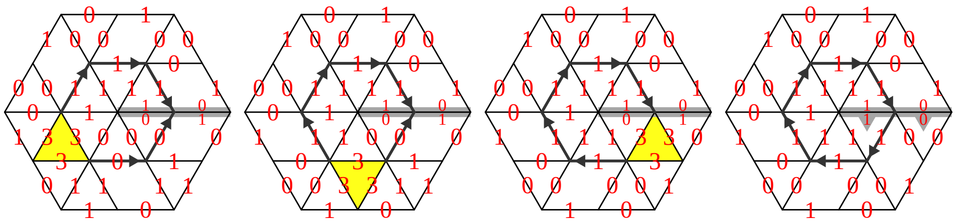
conjugates of Ψ^∞. However, it is not possible to extend the bijection between LR tableaux and puzzles to a bijection between tableaux with empty boxes and flawed puzzles in a way such that individual jeu de taquin slides correspond to individual mutations. For example, if the box of λ/λ' is an outer corner of ν, then the jeu de taquin algorithm involves zero slides, whereas an arbitrary number of mutations may be required to transform the corresponding puzzles. Similarly one can construct examples where two mutations correspond to an arbitrary number of jeu de taquin slides. Notice also that not all conjugates of Ψ^∞ correspond to the jeu de taquin algorithm.

4.12. **Breathing gentle loops.** We finally address a construction of Knutson, Tao, and Woodward that was used in [24] to characterize Littlewood-Richardson coefficients equal to one and to prove a related conjecture of Fulton. Recall from [24] that any Littlewood-Richardson coefficient $c^{\nu}_{\lambda,\mu}$ counts puzzles made from the pieces , , and . A *gentle loop* in such a puzzle is defined to be an oriented cycle of puzzle edges, with turns of $\pm 60°$, such that each edge in the cycle separates two puzzle pieces of different types. In addition, each edge must be directed so that it has either a 0-triangle on its left side or a 1-triangle on its right side. It is proved in [24, Lemma 6] that, if γ is any gentle loop of minimal length in a Grassmannian puzzle, then a new valid puzzle can be obtained by replacing all puzzle pieces in the radius-1 neighborhood of γ with different pieces. This is called *breathing* the gentle loop, and it demonstrates that the corresponding Littlewood-Richardson coefficient must be at least 2. The breathing construction in [24] is defined by specifying how to modify each local region in the radius-1 neighborhood of γ. We will sketch how a minimal gentle loop can also be breathed by applying a sequence of mutations. We thank the referee for providing this application.

Given a minimal gentle loop in a puzzle, consider a *normal line* consisting of two puzzle edges of equal slope that cuts across the loop (see [24, §4.2]). On this normal line we place two gash pairs infinitesimally close to each other, so that the outer labels agree with the original labels of the normal line. The gentle loop can then be breathed by propagating one of the gash pairs around the loop. Whenever a temporary puzzle piece is created, this piece must be resolved in the direction of the gentle loop. Eventually the moving gash pair will reach the other side of the normal line, where it cancels the stationary gash pair. It is important to perform the propagations in the direction of the gentle loop, as otherwise the process will run astray. Notice also that, while resolutions of temporary puzzle pieces in the construction of Ψ^∞ are chosen to keep the propagations moving in a constant direction, the breathing construction chooses resolutions that steer the propagations around the loop.

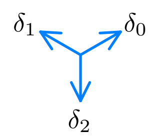
Example 4.11. The two shortest gentle loops have length 6 and are interchanged by breathing. We list the initial and terminal double-gashed puzzles as well as all intermediate puzzles that contain both a temporary puzzle piece and the stationary gash pair. Notice that the gentle loop changes orientation during the process, and that the normal line can be chosen in several ways.



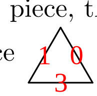


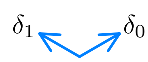
5. Auras of puzzles and the proof of the puzzle formula

5.1. **Aura.** In this section we assign an *aura* to certain objects related to puzzles and use this concept together with the mutation algorithm to prove Theorem 2.1. An aura is a linear form in the ring $R = \mathbb{C}[\delta_0, \delta_1, \delta_2]$ from Section 3. We will represent auras graphically as a collection of unit vectors labeled with linear forms from $\mathbb{Z}[\delta_0, \delta_1, \delta_2]$. The aura is then the sum of the unit vectors multiplied to their labels. For example, we have

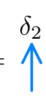
$$\delta_0\zeta + \delta_1\zeta^5 + \delta_2\zeta^9 \ =$$ 

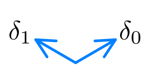
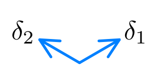
where $\zeta = \exp(\pi i/6) \in \mathbb{C}$.

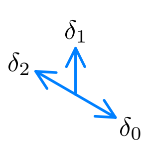
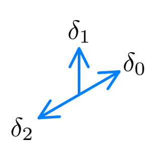
Define a *semi-labeled edge* to be a puzzle edge that has a label only on one side. We will use the textual notation $a/$, $/a$, $a\backslash$, $\backslash a$, $\overset{a}{-\!-}$, and $\underset{a}{-\!-}$ for such edges. The aura $\mathcal{A}(e)$ of a semi-labeled edge e is defined as follows. If the label a of e is simple, then we set $\mathcal{A}(e) = \delta_a v$, where $v \in \mathbb{C}$ is a unit vector perpendicular to e that points towards the side of the label. Otherwise $\mathcal{A}(e)$ is determined by the rule that, whenever the sides of a valid puzzle piece are changed to semi-labeled edges by moving their labels slightly inside the puzzle piece, the sum of the auras of the sides is zero. For example, using the puzzle piece  we obtain

$$\mathcal{A}(\overset{3}{-\!-}) = -\mathcal{A}(/1) - \mathcal{A}(0\backslash) = \delta_1\zeta^5 + \delta_0\zeta =$$  .

The auras of all semi-labeled edges can be obtained by rotating the following identities.

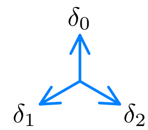
$\mathcal{A}(\overset{0}{-\!-}) =$ 
$\mathcal{A}(\overset{1}{-\!-}) =$ 
$\mathcal{A}(\overset{2}{-\!-}) =$ 

$\mathcal{A}(\overset{3}{-\!-}) =$ 
$\mathcal{A}(\overset{4}{-\!-}) =$ 
$\mathcal{A}(\overset{5}{-\!-}) =$ 

$\mathcal{A}(\overset{6}{-\!-}) =$ 
$\mathcal{A}(\overset{7}{-\!-}) =$ 

A gash can be regarded as a union of two semi-labeled edges. We define the aura of a gash to be the sum of the auras of the two semi-labeled edges. For example, we have

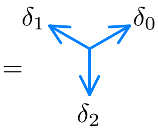
$$\mathcal{A}(\frac{0}{4}) \;=\; \mathcal{A}(\frac{0}{\;}) + \mathcal{A}(\frac{\;}{4}) \;=\; \delta_0\zeta^3 + \delta_1\zeta^7 + \delta_2\zeta^{11} \;=$$



.

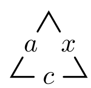
The aura of a directed gash is the aura of the underlying undirected gash. The following are additional examples of auras of gashes.

$$\mathcal{A}(\frac{0}{1}) = \overset{\delta_0-\delta_1}{\uparrow} \qquad \mathcal{A}(\frac{0}{2}) = \overset{\delta_0-\delta_2}{\uparrow} \qquad \mathcal{A}(\frac{1}{2}) = \overset{\delta_1-\delta_2}{\uparrow} \qquad \mathcal{A}(\frac{3}{2}) =$$

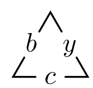


Lemma 5.1. *Any two gashes in the same gash class have the same aura.*

Proof. Let g and h be gashes that are immediately reachable from each other. We must show that $\mathcal{A}(g) = \mathcal{A}(h)$. After rotating and possibly interchanging the gashes, we may assume that $g = a/b$ and $h = x\backslash y$. Furthermore, the labels of the gashes appear on puzzle pieces of the form:



and



.

By definition of the aura of semi-labeled edges we therefore obtain

$$\mathcal{A}(a/b) = \mathcal{A}(a/) + \mathcal{A}(/b) = \mathcal{A}(\frac{c}{\;}) + \mathcal{A}(x\backslash) + \mathcal{A}(\frac{\;}{c}) + \mathcal{A}(\backslash y) = \mathcal{A}(x\backslash y)\,,$$

as required. $\square$

Let P be a flawed puzzle and let $\widetilde{P}$ be a resolution of P. We define $\mathcal{A}(\widetilde{P})$ to be the aura of the right gash of $\widetilde{P}$. If the flaw in P is a gash pair or a marked scab, so that $\widetilde{P}$ is the only resolution of P, then we also write $\mathcal{A}(P) = \mathcal{A}(\widetilde{P})$. Recall that, if S is any set of flawed puzzles, then we write S_{gash}, S_{scab}, and S_{temp} for the subsets of puzzles with flaws of the indicated types. Our main application of the mutation algorithm is the following identity, which is proved in the generality of hexagonal puzzles with equivariant puzzle pieces in arbitrary orientations. The two sums in this identity will later be related to the two sides of the recursive identity (3).

Proposition 5.2. *Let S be any finite set of flawed puzzles that is closed under mutations. Then we have*

$$\sum_{P\in S_{\text{scab}}} \mathcal{A}(P) \;+\; \sum_{P\in S_{\text{gash}}} \mathcal{A}(P) \;=\; 0\,.$$

Proof. Let $\widetilde{S}$ be the set of all resolutions of the flawed puzzles in S. Since S is closed under mutations, it follows that the involution Φ defined in Section 4.9 restricts to an involution of $\widetilde{S}$. Since Lemma 5.1 implies that $\mathcal{A}(\widetilde{P}) + \mathcal{A}(\Phi(\widetilde{P})) = 0$ for any $\widetilde{P} \in \widetilde{S}$, we deduce that

$$\sum_{\widetilde{P}\in\widetilde{S}} \mathcal{A}(\widetilde{P}) = 0\,.$$

It suffices to show that, if P is any flawed puzzle containing a temporary puzzle piece, then the sum of the auras of the three resolutions of P is equal to zero.

Assume that P contains the temporary piece t displayed in Definition 4.2, and let the labels $x, x', x'', y, y', y'', z, z', z''$ and the puzzle pieces r_1, r_2, r_3, t', t'' be as in this definition. Then the right gashes of the three resolutions of P are y/y', $x'\backslash x$, and $\frac{z'}{z}$. Thanks to the puzzle pieces r_3, r_1, and r_2 we have

$$\begin{aligned}\mathcal{A}(y/y') &= \mathcal{A}(x''\backslash) + \mathcal{A}(/y') + \mathcal{A}(\frac{z'}{})\ ,\\ \mathcal{A}(x'\backslash x) &= \mathcal{A}(x'\backslash) + \mathcal{A}(/y') + \mathcal{A}(\frac{z''}{})\ ,\ \text{and}\\ \mathcal{A}(\frac{z'}{z}) &= \mathcal{A}(x'\backslash) + \mathcal{A}(/y'') + \mathcal{A}(\frac{z'}{})\,.\end{aligned}$$

The last two puzzle pieces t' and t'' therefore imply that

$$\mathcal{A}(y/y') + \mathcal{A}(x'\backslash x) + \mathcal{A}(\frac{z'}{z}) = 0\,.$$

This completes the proof. $\square$

5.2. **The constants $C_{w,w}^{w}$.** We first apply the notion of aura to prove that the equivariant puzzle rule is compatible with restrictions of Schubert classes to torus fixed points. Let $X = \mathrm{Fl}(a, b; n)$ be a two-step flag variety.

Lemma 5.3. *Let P be any triangle made from puzzle pieces (in any orientations) with matching side labels, and let u, v, and w be strings of labels such that $\partial P = \triangle_w^{u,v}$. If u and v are 012-strings for X, then so is w. In particular, w consists of simple labels.*

Proof. Consider all pairs (q, e) where q is a puzzle piece in P and e is a side of q. For each such pair we regard e as a semi-labeled edge, where the label is slightly inside the puzzle piece q. Now consider the sum

$$\phi = \sum_{(q,e)} \mathcal{A}(e)$$

over all such pairs. Since the sum over the sides of each puzzle piece q is zero, we have $\phi = 0$. On the other hand, since each interior edge of P appears twice in the sum with its label on opposite sides, it follows that the sum of the auras of all boundary edges of P is equal to zero. Set $\gamma = a\delta_0 + (b-a)\delta_1 + (n-b)\delta_2$. The assumption that u and v are 012-strings for X implies that the sum of the auras of the left border edges is equal to $\gamma\,\zeta^{11}$, and the sum of the auras of the right border edges is equal to $\gamma\,\zeta^{7}$. We deduce that

$$\sum_{i=1}^{n} \mathcal{A}(\frac{w_i}{}) = \gamma\,\zeta^3\,. \tag{6}$$

Since the coefficient of δ_0 in this expression is a multiple of the vertical vector ζ^3, an inspection of the auras of horizontal semi-labeled edges listed in Section 5.1 shows that w does not contain any of the labels 3, 5, 6, and 7. Similarly, since the coefficient of δ_2 is a multiple of ζ^3, we deduce that w does not contain any of the labels 4, 5, 6, and 7. It follows that w consists of simple labels, after which (6) shows that w is a 012-string for X. $\square$

Let P be an equivariant puzzle for X and recall from Section 2 that we number the edges of the bottom border segment from 1 to n, starting from the left. An edge in P will be called SW-NE if it is parallel to the left border segment, NW-SE if it is parallel to the right border segment, and horizontal otherwise. Given any NW-SE edge e in P, define the *left projection* of e to be the number of the bottom edge obtained by following a line parallel to the left border segment. Similarly, the *right projection* of a SW-NE edge is the number of the bottom edge obtained by following a line parallel to the right border segment.

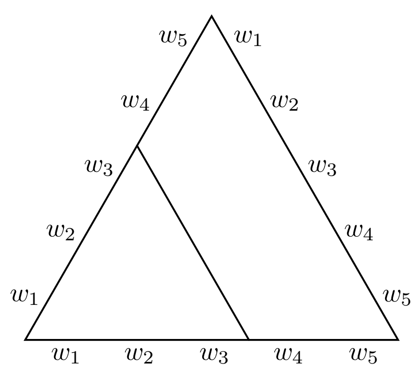
Since all equivariant puzzle pieces in P are vertical, we may dissect P into $\binom{n}{2}$ small vertical rhombuses together with n triangular puzzle pieces along the bottom border. Each small vertical rhombus s is either an equivariant puzzle piece or the union of two triangular puzzle pieces. We will say that s is in position (i, j) if i is the left projection of its NW-SE edges and j is the right projection of its SW-NE edges. In this case we define the *weight* of s to be $\mathrm{wt}(s) = y_j - y_i$. This extends the definition of the weight of an equivariant puzzle piece given in Section 2.

The following result implies that the constants $C_{u,v}^w$ defined by the equivariant puzzle rule satisfy equation (2) from Theorem 3.1.

Proposition 5.4. *Let w be any 012-string for X. Then there exists a unique equivariant puzzle P for X with boundary $\triangle_w^{w,w}$, and this puzzle satisfies*

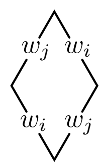
$$\mathrm{wt}(P) \;=\; \prod_{i<j:\, w_i > w_j} (y_j - y_i) \;\in \Lambda\,.$$

Proof. Let P be any equivariant puzzle for X with $\partial P = \triangle_w^{w,w}$, and consider any separation of P into two subpuzzles by any NW-SE line that goes along puzzle edges.



Notice that two of the border segments of the triangular subpuzzle are 012-strings for the same two-step flag variety. It therefore follows from Lemma 5.3 that all labels on the separating line are simple. We deduce that all NW-SE puzzle edges in P have simple labels, and a symmetric argument shows that all SW-NE edges have simple labels. In particular, each small vertical rhombus in P has simple border labels. An inspection of the puzzle pieces from Section 2 shows that, if all border labels of a small rhombus are simple, then opposite border edges have the same label. We deduce that the border labels of the small vertical rhombus in position

(i, j) are given by:



This shows that P is uniquely determined by its boundary, and also provides a recipe for constructing P. Finally, the expression for $\mathrm{wt}(P)$ is correct because the small vertical rhombus in position (i, j) is an equivariant puzzle piece if and only if $w_i > w_j$. $\square$

5.3. **Equivariant Aura.** Let P be an equivariant puzzle for $X = \mathrm{Fl}(a, b; n)$ and let e be an edge in P. If e is a NW-SE edge, then we set $\mathrm{wt}(e) = y_i$ where i is the left projection of e. If e is a SW-NE edge, then set $\mathrm{wt}(e) = y_j$ where j is the right projection of e. Finally, if e is a horizontal edge, then we set $\mathrm{wt}(e) = y_i$ if e is the i-th edge of the bottom border segment, and otherwise we set $\mathrm{wt}(e) = 0$.

An *equivariant aura* is an element of the ring $R[y] = R[y_1, \dots, y_n]$. If e is a semi-labeled edge in P, then we define the equivariant aura of e to be $\mathcal{A}_T(e) = \mathrm{wt}(e)\mathcal{A}(e)$. Given any puzzle piece q in P we let $\mathcal{A}_T(q)$ be the sum of the equivariant auras of the sides of q, where these sides are regarded as semi-labeled edges by moving their labels slightly inside q. If s is any small vertical rhombus in P consisting of two triangular puzzle pieces, then we let $\mathcal{A}_T(s)$ be the sum of the equivariant auras of these pieces.

Proposition 5.5. *Let u, v, and w be 012-strings for X, and let P be an equivariant puzzle for X with boundary $\partial P = \triangle_w^{u,v}$. Then we have*

$$\sum_{s \in \mathrm{scabs}(P)} \mathcal{A}_T(s) \;=\; C_u \zeta^{11} + C_v \zeta^7 + C_w \zeta^3$$

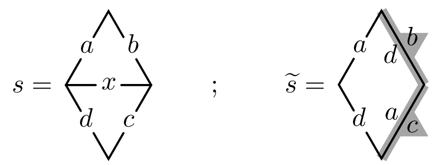
where the sum is over all vertical scabs in P.

Proof. Consider the sum $\phi = \sum_q \mathcal{A}_T(q)$ over all puzzle pieces q in P. Since the equivariant aura of all inner puzzle edges cancel, ϕ is equal to the right hand side of the claimed identity. On the other hand, if s is any vertical rhombus in P that is not a scab, then $\mathcal{A}_T(s) = 0$. In addition we have $\mathcal{A}_T(q) = 0$ whenever q is a triangular puzzle piece on the bottom border of P. This implies that ϕ is equal to the left hand side of the claimed identity. $\square$

A flawed puzzle P is called a *flawed puzzle for X* if P is a right-side-up triangle with boundary $\triangle_w^{u,v}$ where u, v, and w are 012-strings for X, and all equivariant puzzle pieces and marked scabs in P are vertical. By the first condition we mean that u, v, and w are the strings of labels on or outside the three border segments of P. If P is a flawed puzzle for X that contains a marked scab s, then we set $\mathcal{A}_T(P) = \mathcal{A}_T(s)$. Recall also that $\mathcal{A}(P)$ is the aura of the right gash in the resolution of P.

Lemma 5.6. *If P is any flawed puzzle for X containing a marked scab s, then we have $\mathcal{A}_T(P) = -\,\mathrm{wt}(s)\,\mathcal{A}(P)$.*

Proof. Let (i,j) be the position of s, and assume that the labels of s and its resolution $\widetilde{s}$ are as follows.



Since the gashes a/c and $d\backslash b$ are in opposite classes by Proposition 4.5, we obtain $\mathcal{A}_T(s) = \mathcal{A}(c/a)y_j + \mathcal{A}(b\backslash d)y_i = -\mathcal{A}(a/c)(y_j - y_i) = -\mathcal{A}(P)\,\mathrm{wt}(s)$. The same calculation holds if the gashes are on the left side of the resolution of s. $\square$

Proof of Theorem 2.1. For each triple (u,v,w) of 012-strings for X we let $\widehat{C}^w_{u,v} \in \Lambda$ denote the equivariant class defined by the right hand side of Theorem 2.1. In other words we set $\widehat{C}^w_{u,v} = \sum_P \mathrm{wt}(P)$ where the sum is over all equivariant puzzles for X with boundary $\triangle^{u,v}_w$. It follows from Proposition 5.4 that these constants satisfy equation (2). We must show that they also satisfy equation (3).

Fix u, v, and w, and let S be the set of all flawed puzzles for X with boundary $\triangle^{u,v}_w$. Since the mutation algorithm preserves the set of positions of equivariant pieces and marked scabs in a flawed puzzle, it follows from Proposition 5.2 and Lemma 5.6 that

$$\sum_{P \in S_{\mathrm{scab}}} \mathcal{A}_T(P)\,\mathrm{wt}(P) \;=\; \sum_{P \in S_{\mathrm{gash}}} \mathcal{A}(P)\,\mathrm{wt}(P)\,. \tag{7}$$

Here the weight of a flawed puzzle is defined as the product of the weights of its equivariant pieces. By rewriting the left hand side of (7) as a sum over (flawless) equivariant puzzles for X and applying Proposition 5.5 we obtain

$$\begin{aligned}
\sum_{P \in S_{\mathrm{scab}}} \mathcal{A}_T(P)\,\mathrm{wt}(P) \;&=\; \sum_{\partial P = \triangle^{u,v}_w} \mathrm{wt}(P) \sum_{s \in \mathrm{scabs}(P)} \mathcal{A}_T(s) \\
&=\; \sum_{\partial P = \triangle^{u,v}_w} \mathrm{wt}(P)\,(C_u\zeta^{11} + C_v\zeta^7 + C_w\zeta^3) \\
&=\; (C_u\zeta^{11} + C_v\zeta^7 + C_w\zeta^3)\,\widehat{C}^w_{u,v}\,.
\end{aligned}$$

Assume that P is a puzzle in the second sum of (7) with a gash-pair on the left border segment. If u' is the string of labels on or inside this border segment, then we have $u \to u'$. Furthermore, if i is the smallest index for which $u_i \neq u'_i$, then $\mathcal{A}(P) = \mathcal{A}(u_i/u'_i) = \zeta^5\,\delta(\frac{u}{u'})$. Similar identities hold for puzzles with gash pairs on the right or bottom border segments. The second sum in (7) can therefore be rewritten as:

$$\sum_{P \in S_{\mathrm{gash}}} \mathcal{A}(P)\,\mathrm{wt}(P) \;=\; \sum_{u \to u'} \zeta^5\,\delta(\frac{u}{u'})\,\widehat{C}^w_{u',v} + \sum_{v \to v'} \zeta\,\delta(\frac{v}{v'})\,\widehat{C}^w_{u,v'} + \sum_{w' \to w} \zeta^9\,\delta(\frac{w'}{w})\,\widehat{C}^{w'}_{u,v}\,.$$

We conclude that the identity (7) is equivalent to equation (3). Since the constants $\widehat{C}^w_{u,v}$ satisfy the identities (2) and (3), it follows from Theorem 3.1 that they are the equivariant Schubert structure constants of X. This completes the proof. $\square$

References

[1] H. H. Andersen, J. C. Jantzen, and W. Soergel, *Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p*, Astérisque (1994), no. 220. MR 1272539 (95j:20036)

[2] D. Anderson and W. Fulton, *Equivariant cohomology in algebraic geometry*, book in preparation, `http://people.math.osu.edu/anderson.2804/eilenberg/`.

[3] E. Beazley, A. Bertiger, and K. Taipale, *Equivariant quantum cohomology of the Grassmannian via the rim hook rule*, arXiv:1403.6218.

[4] S. Billey, *Kostant polynomials and the cohomology ring for G/B*, Proc. Nat. Acad. Sci. U.S.A. **94** (1997), no. 1, 29–32. MR 1425869 (98e:14051)

[5] A. S. Buch, *On the puzzle conjecture for three-step flag manifolds*, in preparation.

[6] ———, *The saturation conjecture (after A. Knutson and T. Tao)*, Enseign. Math. (2) **46** (2000), no. 1-2, 43–60, With an appendix by William Fulton. MR 1769536 (2001g:05105)

[7] ———, *Quantum cohomology of Grassmannians*, Compositio Math. **137** (2003), no. 2, 227–235. MR 1985005 (2004c:14105)

[8] A. S. Buch, A. Kresch, K. Purbhoo, and H. Tamvakis, *The puzzle conjecture for the cohomology of two-step flag manifolds*, arXiv:1401.1725.

[9] A. S. Buch, A. Kresch, and H. Tamvakis, *Gromov-Witten invariants on Grassmannians*, J. Amer. Math. Soc. **16** (2003), no. 4, 901–915 (electronic). MR 1992829 (2004h:14060)

[10] A. S. Buch and L. Mihalcea, *Quantum K-theory of Grassmannians*, Duke Math. J. **156** (2011), no. 3, 501–538. MR 2772069 (2011m:14092)

[11] ———, *Curve neighborhoods of Schubert varieties*, J. Differential Geom. **99** (2015), no. 2, 255–283. MR 3302040

[12] C. Chevalley, *Sur les décompositions cellulaires des espaces G/B*, Proc. Sympos. Pure Math., vol. 56, 1994, pp. 1–23. MR 1278698 (95e:14041)

[13] I. Coşkun, *A Littlewood-Richardson rule for two-step flag varieties*, Invent. Math. **176** (2009), no. 2, 325–395. MR 2495766 (2010e:14048)

[14] I. Coşkun and R. Vakil, *Geometric positivity in the cohomology of homogeneous spaces and generalized Schubert calculus*, Proc. Sympos. Pure Math., vol. 80, 2009, pp. 77–124. MR 2483933 (2010d:14074)

[15] W. Fulton, *Young tableaux*, London Mathematical Society Student Texts, vol. 35, 1997. MR 1464693 (99f:05119)

[16] ———, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 2, Springer-Verlag, Berlin, 1998. MR 1644323 (99d:14003)

[17] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Proc. Sympos. Pure Math., vol. 62, 1997, pp. 45–96. MR 1492534 (98m:14025)

[18] W. Graham, *Positivity in equivariant Schubert calculus*, Duke Math. J. **109** (2001), no. 3, 599–614. MR 1853356 (2002h:14083)

[19] B. Kim, *On equivariant quantum cohomology*, Internat. Math. Res. Notices (1996), no. 17, 841–851. MR 1420551 (98h:14013)

[20] A. Knutson, *Puzzles, Positroid varieties, and equivariant K-theory of Grassmannians*, arXiv:1008.4302.

[21] ———, *A conjectural rule for GL_n Schubert calculus*, unpublished manuscript, 1999.

[22] A. Knutson and K. Purbhoo, *Product and puzzle formulae for* GL_n *Belkale-Kumar coefficients*, Electron. J. Combin. **18** (2011), no. 1, Paper 76, 20. MR 2788693 (2012f:14095)

[23] A. Knutson and T. Tao, *Puzzles and (equivariant) cohomology of Grassmannians*, Duke Math. J. **119** (2003), no. 2, 221–260. MR 1997946 (2006a:14088)

[24] A. Knutson, T. Tao, and C. Woodward, *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), no. 1, 19–48. MR 2015329 (2005f:14105)

[25] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), no. 3, 525–562. MR 1291244 (95i:14049)

[26] B. Kostant and S. Kumar, *The nil Hecke ring and cohomology of G/P for a Kac-Moody group G*, Adv. in Math. **62** (1986), no. 3, 187–237. MR 866159 (88b:17025b)

[27] V. Kreiman, *Equivariant Littlewood-Richardson skew tableaux*, Trans. Amer. Math. Soc. **362** (2010), no. 5, 2589–2617. MR 2584612 (2011d:05385)

[28] L. Mihalcea, *Equivariant quantum Schubert calculus*, Adv. Math. **203** (2006), no. 1, 1–33. MR 2231042 (2007c:14061)

[29] ———, *Positivity in equivariant quantum Schubert calculus*, Amer. J. Math. **128** (2006), no. 3, 787–803. MR 2230925 (2007c:14062)

[30] ———, *On equivariant quantum cohomology of homogeneous spaces: Chevalley formulae and algorithms*, Duke Math. J. **140** (2007), no. 2, 321–350. MR 2359822 (2008j:14106)

[31] A. Molev, *Littlewood-Richardson polynomials*, J. Algebra **321** (2009), no. 11, 3450–3468. MR 2510056 (2010e:05316)

[32] A. Molev and B. Sagan, *A Littlewood-Richardson rule for factorial Schur functions*, Trans. Amer. Math. Soc. **351** (1999), no. 11, 4429–4443. MR 1621694 (2000a:05212)

[33] D. Monk, *The geometry of flag manifolds*, Proc. London Math. Soc. (3) **9** (1959), 253–286. MR 0106911 (21 #5641)

[34] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, Math. Res. Lett. **1** (1994), no. 2, 269–278. MR 1266766 (95b:58025)

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