PIERI RULES FOR THE K-THEORY OF COMINUSCULE GRASSMANNIANS

ANDERS SKOVSTED BUCH AND VIJAY RAVIKUMAR

Abstract. We prove Pieri formulas for the multiplication with special Schubert classes in the K-theory of all cominuscule Grassmannians. For Grassmannians of type A this gives a new proof of a formula of Lenart. Our formula is new for Lagrangian Grassmannians, and for orthogonal Grassmannians it proves a special case of a conjectural Littlewood-Richardson rule of Thomas and Yong. Recent work of Clifford, Thomas, and Yong has shown that the full Littlewood-Richardson rule for orthogonal Grassmannians follows from the Pieri case proved here. We describe the K-theoretic Pieri coefficients both as integers determined by positive recursive identities and as the number of certain tableaux. The proof is based on a computation of the sheaf Euler characteristic of triple intersections of Schubert varieties, where at least one Schubert variety is special.

1. Introduction

By a *cominuscule Grassmannian* we will mean a Grassmann variety $\mathrm{Gr}(m,n)$ of type A, a Lagrangian Grassmannian $\mathrm{LG}(n,2n)$, or a maximal orthogonal Grassmannian $\mathrm{OG}(n,2n+1) \cong \mathrm{OG}(n+1,2n+2)$. The goal of this paper is to prove (old and new) Pieri rules for the multiplication by special Schubert classes in the Grothendieck ring of any cominuscule Grassmannian.

Any homogeneous space X has a decomposition into Schubert varieties X^λ, which for cominuscule Grassmannians are indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell \geq 0)$ in a way so that the codimension of X^λ in X is equal to the weight $|\lambda| = \sum \lambda_i$. The Schubert classes $[X^\lambda] \in H^{2|\lambda|}(X;\mathbb{Z})$ defined by the Schubert varieties give a $\mathbb{Z}$-basis for the singular cohomology ring $H^*(X) = H^*(X;\mathbb{Z})$. This ring is furthermore generated as a $\mathbb{Z}$-algebra by the special Schubert classes $[X^p]$ given by partitions with a single part p. The Littlewood-Richardson coefficients $c^\nu_{\lambda\mu}$ are the structure constants for $H^*(X)$ with respect to the Schubert basis, i.e. they are determined by the identity

$$[X^\lambda] \cdot [X^\mu] = \sum_{|\nu|=|\lambda|+|\mu|} c^\nu_{\lambda\mu}\,[X^\nu] \tag{1}$$

in $H^*(X)$. Each coefficient $c^\nu_{\lambda\mu}$ in this sum depends on the type of the Grassmannians X as well as the partitions. It is equal to the number of points in the intersection of general translates of the Schubert varieties X^λ, X^μ, and $X^{\nu^\vee}$, where

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$\nu^\vee$ is the Poincare dual partition of ν. This gives the identity

$$c^{\nu}_{\lambda\mu} = \int_X [X^\lambda] \cdot [X^\mu] \cdot [X^{\nu^\vee}] \tag{2}$$

when $|\nu| = |\lambda| + |\mu|$. The Littlewood-Richardson coefficients of cominuscule Grassmannians are well understood and are described by variants of the celebrated Littlewood-Richardson rule, see e.g. [21, 11, 25, 6] and the references given there.

The cohomology ring $H^*(X)$ is generalized by various other rings, including the equivariant cohomology ring $H^*_T(X)$, the quantum cohomology ring $\mathrm{QH}(X)$, and the Grothendieck ring $K(X)$ of algebraic vector bundles on X, also called the K-theory of X. Every Schubert variety X^λ has a Grothendieck class $\mathcal{O}^\lambda = [\mathcal{O}_{X^\lambda}]$ in $K(X)$ (see §2) and these classes form a basis for the Grothendieck ring. We can therefore define *K-theoretic Schubert structure constants* for X by the identity

$$\mathcal{O}^\lambda \cdot \mathcal{O}^\mu = \sum_\nu c^\nu_{\lambda\mu}\, \mathcal{O}^\nu\,. \tag{3}$$

The coefficient $c^\nu_{\lambda\mu}$ is non-zero only if $|\nu| \geq |\lambda| + |\mu|$, and for $|\nu| = |\lambda| + |\mu|$ it agrees with the cohomological structure constant of the same name. The K-theoretic structure constants have signs that alternate with codimension, in the sense that $(-1)^{|\nu|-|\lambda|-|\mu|}\, c^\nu_{\lambda\mu} \geq 0$; this was proved by the first author for Grassmannians of type A [3] and by Brion for arbitrary homogeneous spaces [2]. If X is a Grassmann variety of type A, then several Littlewood-Richardson rules are available that express the absolute value $|c^\nu_{\lambda\mu}|$ as the number of elements in a combinatorially defined set [3, 16, 5, 24]. An earlier Pieri formula of Lenart [18] shows that the coefficients $c^\nu_{\lambda,p}$, that describe multiplication with the special classes $\mathcal{O}^p$, are equal to signed binomial coefficients. For Lagrangian and orthogonal Grassmannians it is known how to multiply with the Schubert divisor $\mathcal{O}^1$, by using Lenart and Postnikov's K-theoretic Chevalley formula which works for arbitrary homogeneous spaces [19]. More recently Thomas and Yong [24] have conjectured a full Littlewood-Richardson rule for the K-theory of maximal orthogonal Grassmannians based on K-theoretic jeu-de-taquin slides. No such conjecture is presently available for Lagrangian Grassmannians.

The main results of this paper are K-theoretic Pieri rules for maximal orthogonal Grassmannians and Lagrangian Grassmannians. These rules are stated in two equivalent versions. The first is a set of recursive identities that make it possible to compute any coefficient $c^\nu_{\lambda,p}$ from simpler ones in a way that makes the alternation of signs transparent. The second is a Littlewood-Richardson rule that expresses $|c^\nu_{\lambda,p}|$ as the number of tableaux satisfying certain properties. For maximal orthogonal Grassmannians, Itai Feigenbaum and Emily Sergel have shown us a proof that these tableaux are identical to the increasing tableaux appearing in the conjecture of Thomas and Yong [24], which confirms that this conjecture computes all Pieri coefficients correctly. The tableaux being counted for Lagrangian Grassmannians are new and contain both primed and unprimed integers; it would be very interesting to extend this rule to a full Littlewood-Richardson rule for all the K-theoretic structure constants. While the cohomological Schubert calculus of Lagrangian and maximal orthogonal Grassmannians is essentially the same, our results show that these spaces have quite different K-theoretic Schubert calculus. For example, we prove that the K-theoretic structure constants $c^{\nu^\vee}_{\lambda\mu}$ for orthogonal Grassmannians are invariant under permutations of λ, μ, and ν, but show by example that this fails for Lagrangian Grassmannians.

Thomas and Yong have conjectured a K-theoretic Littlewood-Richardson rule for all minuscule homogeneous spaces, and proved that their rule is a consequence of a well-definedness property of a jeu-de-taquin algorithm, together with a proof that their conjecture gives the correct answer for a generating set of Schubert classes [24]. For maximal orthogonal Grassmannians, the well-definedness property has later been proved by Clifford, Thomas, and Yong [9]. Since the special classes $\mathcal{O}^p$ generate the K-theory ring, the Pieri rule proved here provides the second ingredient required for the proof of Thomas and Yong's conjecture.

The standard way to prove cohomological Pieri rules is to use that any cohomological Pieri coefficient $c^{\nu}_{\lambda,p}$ counts the number of points in an intersection of three Schubert varieties, one of which is special [14, 23, 1, 7]. We prove our formulas by using a K-theoretic adaption of this method. For Grassmannians of type A, this gives a new and geometric proof of Lenart's Pieri rule [18].

The K-theoretic analogue of the triple intersections (2) for Pieri coefficients are the numbers $\chi_X(\mathcal{O}^\lambda \cdot \mathcal{O}^p \cdot \mathcal{O}^{\nu^\vee})$, where $\chi_X : K(X) \to \mathbb{Z}$ denotes the sheaf Euler characteristic map (see §2). However, the Pieri coefficients of interest are given by $c^{\nu}_{\lambda,p} = \chi_X(\mathcal{O}^\lambda \cdot \mathcal{O}^p \cdot \mathcal{O}^*_\nu)$, where $\mathcal{O}^*_\nu \in K(X)$ is the K-theoretic dual Schubert class of X^ν, a class different from $\mathcal{O}^{\nu^\vee}$. Our proof therefore requires a combinatorial translation from triple intersection numbers to structure constants. Furthermore, the K-theoretic triple intersection numbers are harder to compute because they can be non-zero when the intersection of general translates of X^λ, X^p, and $X^{\nu^\vee}$ has positive dimension. Here we use a construction from [7] to translate the computation of intersection numbers to the K-theory ring of a projective space. To make this construction work in K-theory, we also need a Gysin formula that was proved in [8] as an application of a vanishing theorem of Kollár [15]. To satisfy the conditions of the Gysin formula, we have to prove that a map from a Richardson variety to projective space has rational general fibers and that its image has rational singularities. For Grassmannians of type A we obtain that the intersection number $\chi_X(\mathcal{O}^\lambda \cdot \mathcal{O}^p \cdot \mathcal{O}^{\nu^\vee})$ is equal to one whenever the intersection of arbitrary translates of X^λ, X^p, and $X^{\nu^\vee}$ is not empty, and otherwise it is zero. For maximal orthogonal Grassmannians and Lagrangian Grassmannians, the integer $\chi_X(\mathcal{O}^\lambda \cdot \mathcal{O}^p \cdot \mathcal{O}^{\nu^\vee})$ is equal to the sheaf Euler characteristic of a complete intersection of linear and quadric hypersurfaces in projective space. We note that the earlier published proofs of K-theoretic Pieri rules in [18, 20, 19] are combinatorial and do not rely on triple intersections. On the other hand, the method used here is likely to work for other homogeneous spaces G/P.

Our paper is organized as follows. In section 2 we recall some definitions and results related to the K-theory of varieties. Grassmannians of type A are then handled in section 3, maximal orthogonal Grassmannians in section 4, and Lagrangian Grassmannians in section 5.

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2. The Grothendieck ring

In this section we recall some facts about the K-theory of algebraic varieties; more details and references can be found in [12]. The K-homology group $K_\circ(X)$ of an algebraic variety X is the Grothendieck group of coherent $\mathcal{O}_X$-modules, i.e. the free abelian group generated by isomorphism classes $[\mathcal{F}]$ of coherent sheaves on X, modulo the relations $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}'']$ whenever there exists a short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$. This group is a module over the K-cohomology ring $K^\circ(X)$, defined as the Grothendieck group of algebraic vector bundles on X. Both the multiplicative structure of $K^\circ(X)$ and the module structure of $K_\circ(X)$ is defined by tensor products. Any closed subvariety $Z \subset X$ has a *Grothendieck class* $[\mathcal{O}_Z] \in K_\circ(X)$ defined by its structure sheaf. If X is non-singular, then the implicit map $K^\circ(X) \to K_\circ(X)$ that sends a vector bundle to its sheaf of sections is an isomorphism; the inverse map is given by $[\mathcal{F}] \mapsto \sum_{i\geq 0}(-1)^i[\mathcal{E}_i]$ where $0 \to \mathcal{E}_r \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0$ is any locally free resolution of the coherent sheaf $\mathcal{F}$. In this case we will write simply $K(X)$ for both K-theory groups.

Any morphism of varieties $f : X \to Y$ gives a ring homomorphism $f^* : K^\circ(Y) \to K^\circ(X)$ defined by pullback of vector bundles. If f is proper, there is also a pushforward map $f : K_\circ(X) \to K_\circ(Y)$ defined by $f_*[\mathcal{F}] = \sum_{i\geq 0}(-1)^i[R^i f_*\mathcal{F}]$. Both pullback and pushforward are functorial with respect to composition of morphisms. The projection formula states that $f_*(f^*(\alpha)\cdot\beta) = \alpha \cdot f_*(\beta)$ for all $\alpha \in K^\circ(Y)$ and $\beta \in K_\circ(X)$. If X is a complete variety, then the sheaf Euler characteristic map $\chi_X : K_\circ(X) \to K_\circ(\text{point}) = \mathbb{Z}$ is defined as the pushforward along the structure morphism $X \to \{\text{point}\}$, i.e. $\chi_X([\mathcal{F}]) = \sum_{i\geq 0}(-1)^i \dim H^i(X,\mathcal{F})$. If X is irreducible and rational with rational singularities then $\chi_X([\mathcal{O}_X]) = 1$ [13, p. 494].

We need the following pushforward formula, which was proved in [8, Thm. 3.1] as an application of a vanishing theorem of Kollár [15, Thm. 7.1].

Lemma 2.1. *Let $f : X \to Y$ be a surjective map of projective varieties with rational singularities. Assume that $f^{-1}(y)$ is an irreducible rational variety for all closed points in a dense open subset of Y. Then $f_*[\mathcal{O}_X] = [\mathcal{O}_Y] \in K_\circ(Y)$.*

We also need the following well known fact.

Lemma 2.2. *Let X be a non-singular variety and let Y and Z be closed varieties of X with Cohen-Macaulay singularities. Assume that each component of $Y \cap Z$ has dimension $\dim(Y) + \dim(Z) - \dim(X)$. Then $Y \cap Z$ is Cohen-Macaulay and $[\mathcal{O}_Y] \cdot [\mathcal{O}_Z] = [\mathcal{O}_{Y\cap Z}]$ in $K(X)$.*

Proof. The diagonal embedding $\Delta : X \to X \times X$ is a regular embedding [12, B.7.3], and any local regular sequence defining the ideal of X in $X \times X$ restricts to a local regular sequence defining the ideal of $Y \cap Z$ in $Y \times Z$ by [12, A.7.1]. This implies that $\mathit{Tor}_i^{X\times X}(\mathcal{O}_X, \mathcal{O}_{Y\times Z}) = 0$ for all $i > 0$, so $[\mathcal{O}_Y]\cdot[\mathcal{O}_Z] = \Delta^*[\mathcal{O}_{Y\times Z}] = \sum_{i\geq 0}(-1)^i[\mathit{Tor}_i^{X\times X}(\mathcal{O}_X, \mathcal{O}_{Y\times Z})] = [\mathcal{O}_Y \otimes \mathcal{O}_Z] = [\mathcal{O}_{Y\cap Z}]$, as required. □

Now let $X = G/P$ be a homogeneous space defined by a complex connected semisimple linear algebraic group G and a parabolic subgroup P; all Grassmannians can be constructed in this fashion. The Schubert varieties of X relative to a Borel subgroup $B \subset G$ are the closures of the B-orbits in X, and the Grothendieck classes of these varieties form a $\mathbb{Z}$-basis for the K-theory ring $K(X)$. It is known that all Schubert varieties are Cohen-Macaulay and have rational singularities [22, 17].

As a special case this implies that an quadric hypersurfaces in $\mathbb{C}^n$ has rational singularities, since it is isomorphism to a product of a smaller affine space with an open subset of the Schubert divisor in an orthogonal Grassmannian $\mathrm{OG}(1,m)$ (see e.g. [7, §4]).

A Richardson variety is any non-empty intersection $Y \cap Z$, where Y is a Schubert variety relative to B and Z is a Schubert variety relative to the opposite Borel subgroup B^{op}. It was proved by Deodhar that any Richardson variety is irreducible and rational [10], and Brion has proved in that Richardson varieties have rational singularities [2]. In particular, the sheaf Euler characteristic of a Richardson variety is equal to one.

Finally we mention that a Schubert variety in the projective space $\mathbb{P}^n$ is the same as a linear subspace. The Grothendieck ring is given by $K(\mathbb{P}^n) = \mathbb{Z}[t]/(t^{n+1})$, where t is the class of a hyperplane. The class of a quadric hypersurface is equal to $2t - t^2$, and a linear subspace of codimension i has class t^i. In particular, the map $\chi_{_{\mathbb{P}^n}} : K(\mathbb{P}^n) \to \mathbb{Z}$ is determined by $\chi_{_{\mathbb{P}^n}}(t^i) = 1$ for $0 \leq i \leq n$.

3. Grassmannians of type A

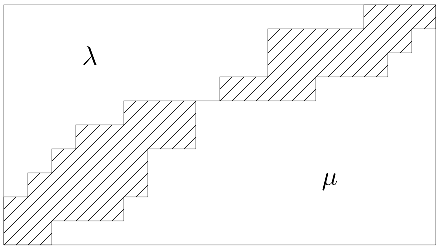
Let $X = \mathrm{Gr}(m,n) = \{V \subset \mathbb{C}^n \mid \dim(V) = m\}$ be the Grassmann variety of m-planes in $\mathbb{C}^n$. This is a non-singular variety of complex dimension mk, where $k = n - m$. The Schubert varieties in X are indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m \geq 0)$ such that $\lambda_1 \leq k$. Equivalently, the Young diagram of λ can be contained in a rectangle m rows and k columns. We will identify λ with its Young diagram. The Schubert variety for λ relative to a complete flag $0 \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_n = \mathbb{C}^n$ is defined by

$$X^\lambda(F_\bullet) = \{V \in X \mid \dim(V \cap F_{k+i-\lambda_i}) \geq i \ \forall 1 \leq i \leq m\}\,.$$

The codimension of this variety in X is equal to the weight $|\lambda| = \sum \lambda_i$. If $u_1, \ldots, u_r$ are vectors in a complex vector space, then we let $\langle u_1, \ldots, u_r \rangle$ denote their span. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$. We will mostly consider Schubert varieties relative to the standard flags in $\mathbb{C}^n$, defined by $E_i = \langle e_1, \ldots, e_i \rangle$ and $E_i^{\mathrm{op}} = \langle e_{n+1-i}, \ldots, e_n \rangle$. Let $\mathcal{O}^\lambda = [\mathcal{O}_{X^\lambda}] \in K(X)$ denote the class of $X^\lambda := X^\lambda(E_\bullet)$. The K-theoretic Schubert structure constants $c^\nu_{\lambda\mu}$ for X are defined by equation (3), where the sum includes all partitions ν contained in the $m \times k$-rectangle. The goal of this section is to give a simple geometric proof of Lenart's Pieri rule for the special coefficients $c^\nu_{\lambda,p}$, where we identify the integer $p \in \mathbb{N}$ with the one-part partition (p).

Given a partition μ contained in the $m \times k$-rectangle, let $\mu^\vee = (k - \mu_m, \ldots, k - \mu_1)$ denote the Poincare dual partition. We will also call this partition the $m \times k$–dual of μ. The intersection $X^\lambda(E_\bullet) \cap X^\mu(E_\bullet^{\mathrm{op}})$ is non-empty if and only if $\lambda \subset \mu^\vee$. Assume that $\lambda \subset \mu^\vee$ and let $\theta = \mu^\vee/\lambda$ be the skew diagram of boxes in $\mu^\vee$ that are not in λ. This is the set of boxes remaining when the boxes of λ are deleted from the upper-left corner of the $m \times k$-rectangle, and the boxes of μ (rotated by

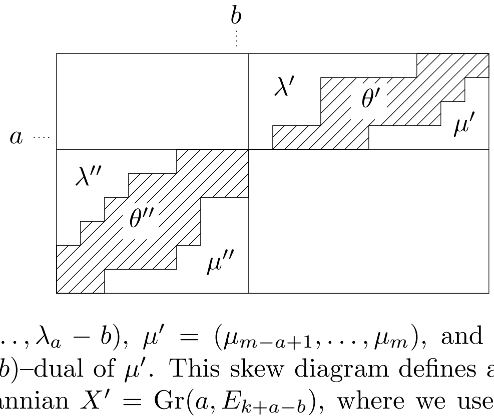
180 degrees) are deleted from the lower-right corner.



Define the Richardson variety $X_\theta = X^\lambda(E_\bullet) \cap X^\mu(E_\bullet^{\mathrm{op}}) \subset X$. This variety has dimension $|\theta| = mk - |\lambda| - |\mu|$. As a special case, notice that $X_\lambda = X^{\lambda^\vee}(E_\bullet^{\mathrm{op}})$ is the dual Schubert variety of X^λ. Let $\mathcal{O}_\theta = [\mathcal{O}_{X_\theta}] = \mathcal{O}^\lambda \cdot \mathcal{O}^\mu \in K(X)$ denote the Grothendieck class of X_θ. While X_θ depends on the partitions λ and μ, it follows from Lemma 3.2 below that its isomorphism class depends only on the skew diagram θ. For any vector $u \in \mathbb{C}^n$ we define $X_\theta(u) = \{V \in X_\theta \mid u \in V\}$. Let $\bigcup X_\theta = \bigcup_{V \in X_\theta} V \subset \mathbb{C}^n$ be the set of vectors $u \in \mathbb{C}^n$ for which $X_\theta(u) \neq \emptyset$. This set is an irreducible closed subvariety of $\mathbb{C}^n$, because $\bigcup X_\theta = \pi_2(\pi_1^{-1}(X_\theta))$ where $\pi_1 : \mathcal{S} \to X$ and $\pi_2 : \mathcal{S} \to \mathbb{C}^n$ are the natural projections from the tautological subbundle $\mathcal{S} = \{(V, u) \in X \times \mathbb{C}^n \mid u \in V\}$. We wish to show that if u is a general vector in $\bigcup X_\theta$, then $X_\theta(u)$ is a Richardson variety in the Grassmannian $\mathrm{Gr}(m-1, \mathbb{C}^n/\langle u\rangle)$, which we identify with the m-planes in X that contain u. In particular, $X_\theta(u)$ is irreducible and rational. The following example shows that this may be false without the assumption that u is general.

Example 3.1. Let $X = \mathrm{Gr}(3,5)$ and $\lambda = \mu = (1)$. Then $\bigcup X_\theta = \mathbb{C}^5$. Set $u = e_1 + e_5$. Then $X_\theta(u)$ has two components, which can be naturally identified with $\mathbb{P}(\mathbb{C}^5/\langle e_1, e_5\rangle)$ and $\mathrm{Gr}(2, \langle e_1, e_2, e_4, e_5\rangle/\langle e_1 + e_5\rangle)$. These components meet in the line $\mathbb{P}(\langle e_2, e_4\rangle)$.

Assume that $a \in [0, m]$ and $b \in [0, k]$ are integers such that $\lambda_a \geq b$ and $\mu_{m-a} \geq k - b$ (here we set $\lambda_0 = \mu_0 = k$). This implies that the diagram $\theta = \mu^\vee/\lambda$ can be split into a north-east part θ' in rows 1 through a and a south-west part θ'' in rows $a+1$ through m.



Set $\lambda' = (\lambda_1 - b, \ldots, \lambda_a - b)$, $\mu' = (\mu_{m-a+1}, \ldots, \mu_m)$, and $\theta' = {\mu'}^\vee/\lambda'$, where ${\mu'}^\vee$ is the $a \times (k-b)$–dual of μ'. This skew diagram defines a Richardson variety $X'_{\theta'}$ in the Grassmannian $X' = \mathrm{Gr}(a, E_{k+a-b})$, where we use the (ordered) basis $e_1, \ldots, e_{k+a-b}$ for E_{k+a-b}. Similarly we set $\lambda'' = (\lambda_{a+1}, \ldots, \lambda_m)$, $\mu'' = (\mu_1 - k + b, \ldots, \mu_{m-a} - k + b)$, and $\theta'' = {\mu''}^\vee/\lambda''$, which defines the Richardson variety $X''_{\theta''}$

in $X'' = \mathrm{Gr}(m-a, E^{\mathrm{op}}_{m+b-a})$, using the basis $e_{k+a-b+1}, \dots, e_n$ for E^{op}_{m+b-a}. Set $\bigcup X'_{\theta'} = \bigcup_{V' \in X'_{\theta'}} V' \subset E_{k+a-b}$ and $\bigcup X''_{\theta''} = \bigcup_{V'' \in X''_{\theta''}} V'' \subset E^{\mathrm{op}}_{m+b-a}$.

Lemma 3.2. (a) *We have* $\bigcup X_\theta = \bigcup X'_{\theta'} \times \bigcup X''_{\theta''}$ *in* $\mathbb{C}^n = E_{k+a-b} \times E^{\mathrm{op}}_{m+b-a}$.
(b) *For arbitrary vectors* $u' \in E_{k+a-b}$ *and* $u'' \in E_{m+b-a}$*, the inclusion* $X' \times X'' \subset X$ *defined by* $(V', V'') \mapsto V' \oplus V''$ *identifies* $X'_{\theta'}(u') \times X''_{\theta''}(u'')$ *with* $X_\theta(u' + u'')$.

Proof. If $V \in X_\theta$ then $\dim(V \cap E_{k+a-b}) \geq \dim(V \cap E_{k+a-\lambda_a}) \geq a$ and $\dim(V \cap E^{\mathrm{op}}_{m+b-a}) \geq \dim(V \cap E^{\mathrm{op}}_{k+(m-a)-\mu_{m-a}}) \geq m-a$. It follows that $V = V' \oplus V''$ where $V' = V \cap E_{k+a-b} \in X'$ and $V'' = V \cap E^{\mathrm{op}}_{m+b-a} \in X''$. Given arbitrary points $V' \in X'$ and $V'' \in X''$ it is immediate from the definitions that $V' \oplus V'' \in X_\theta(u' + u'')$ if and only if $V' \in X'_{\theta'}(u')$ and $V'' \in X''_{\theta''}(u'')$. The lemma follows from this. □

Let $\overline{\theta}$ be the diagram obtained from θ by removing the bottom box in each non-empty column. Let $c(\theta) = |\theta| - |\overline{\theta}|$ be the number of non-empty columns.

Lemma 3.3. (a) *The set* $\bigcup X_\theta$ *is a linear subspace of* $\mathbb{C}^n$ *of dimension* $m + c(\theta)$.
(b) *For all vectors* u *in a dense open subset of* $\bigcup X_\theta$ *we have* $X_\theta(u) \cong X_{\overline{\theta}}$.

Proof. If $\theta = \emptyset$, then X_θ is a single point and the lemma is clear. Using Lemma 3.2 it is therefore enough to assume that θ has only one component which contains the upper-right and lower-left boxes of the $m \times k$-rectangle. This implies that $\lambda_i + \mu_{m-i} \leq k$ for $0 \leq i \leq m$ (recall that we set $\lambda_0 = \mu_0 = k$). Let $u \in \mathbb{C}^n$ be any vector such that all coordinates of u are non-zero. It is enough to show that $X_\theta(u) \cong X_{\overline{\theta}}$. Set $\overline{E} = \mathbb{C}^n / \langle u \rangle$ and define flags in this vector space by $\overline{E}_i = (E_i + \langle u \rangle)/\langle u \rangle$ and $\overline{E}^{\mathrm{op}}_i = (E^{\mathrm{op}}_i + \langle u \rangle)/\langle u \rangle$, for $1 \leq i \leq n-1$. Since we have $u \notin E_i + E^{\mathrm{op}}_{n-1-i}$ for each i, it follows that $\overline{E}_i \cap \overline{E}^{\mathrm{op}}_{n-1-i} = 0$, so the flags $\overline{E}_\bullet$ and $\overline{E}^{\mathrm{op}}_\bullet$ are opposite. Identify $\overline{X} = \mathrm{Gr}(m-1, \overline{E})$ with the set of m-planes $V \in X$ for which $u \in V$. Then we have $X^\lambda(E_\bullet) \cap \overline{X} = \overline{X}^\lambda(\overline{E}_\bullet)$ and $X^\mu(E^{\mathrm{op}}_\bullet) \cap \overline{X} = \overline{X}^\mu(\overline{E}^{\mathrm{op}}_\bullet)$, so $X_\theta(u) = X_\theta \cap \overline{X} = \overline{X}^\lambda(\overline{E}_\bullet) \cap \overline{X}^\mu(\overline{E}^{\mathrm{op}}_\bullet) = \overline{X}_{\overline{\theta}}$, as required. □

We can now prove our formula for the K-theoretic triple intersection numbers $\chi_X(\mathcal{O}^\lambda \cdot \mathcal{O}^p \cdot \mathcal{O}^\mu)$. They turn out to be simpler than the corresponding Pieri coefficients $c^\nu_{\lambda,p}$ (see eqn. (6) below), despite the fact that the more general intersection numbers $\chi_X(\mathcal{O}^\lambda \cdot \mathcal{O}^\nu \cdot \mathcal{O}^\mu)$ are not well behaved [3, §8].

Proposition 3.4. *Let* θ *be a skew diagram contained in the* $m \times k$*-rectangle and let* $0 \leq p \leq k$*. Then we have*

$$\chi_X(\mathcal{O}_\theta \cdot \mathcal{O}^p) = \begin{cases} 1 & \text{if } p \leq c(\theta); \\ 0 & \text{if } p > c(\theta). \end{cases}$$

Proof. Let $Z = \mathrm{Fl}(1, m; n)$ be the variety of all two-step flags $L \subset V \subset \mathbb{C}^n$ with $\dim(L) = 1$ and $\dim(V) = m$, and let $\pi_1 : Z \to \mathbb{P}^{n-1}$ and $\pi_m : Z \to X$ be the projections. Since $\pi_m : \pi_1^{-1}(\mathbb{P}(E_{k+1-p})) \to X^p$ is a birational isomorphism of varieties with rational singularities and π_1 is flat, it follows that $\pi_{m*}\pi_1^*[\mathcal{O}_{\mathbb{P}(E_{k+1-p})}] = \pi_{m*}[\mathcal{O}_{\pi_1^{-1}(\mathbb{P}(E_{k+1-p}))}] = \mathcal{O}^p \in K(X)$. Lemma 3.3 implies that the general fibers of the map $\pi_1 : \pi_m^{-1}(X_\theta) \to \pi_1(\pi_m^{-1}(X_\theta))$ are rational. Since the Richardson variety $\pi_m^{-1}(X_\theta)$ has rational singularities, we therefore deduce from Lemma 2.1 that

$\pi_{1*}\pi_m^*[\mathcal{O}_{X_\theta}] = \pi_{1*}[\mathcal{O}_{\pi_m^{-1}(X_\theta)}] = [\mathcal{O}_{\pi_1(\pi_m^{-1}(X_\theta))}] = [\mathcal{O}_{\mathbb{P}(\bigcup X_\theta)}] \in K(\mathbb{P}^{n-1})$. Using the projection formula we obtain

$$\begin{aligned}\chi_X(\mathcal{O}_\theta \cdot \mathcal{O}^p) &= \chi_X([\mathcal{O}_{X_\theta}] \cdot \pi_{m*}\pi_1^*[\mathcal{O}_{\mathbb{P}(E_{k+1-p})}]) \\ &= \chi_{\mathbb{P}^{n-1}}(\pi_{1*}\pi_m^*[\mathcal{O}_{X_\theta}] \cdot [\mathcal{O}_{\mathbb{P}(E_{k+1-p})}]) \\ &= \chi_{\mathbb{P}^{n-1}}([\mathcal{O}_{\mathbb{P}(\bigcup X_\theta)}] \cdot [\mathcal{O}_{\mathbb{P}(E_{k+1-p})}])\,.\end{aligned}$$

Lemma 3.3 implies that this Euler characteristic equals one if $m+c(\theta)+k+1-p > n$ and is zero otherwise, as required. □

In order use the intersection numbers of Proposition 3.4 to compute the Pieri coefficients $c^\nu_{\lambda,r}$, we need the dual Schubert classes in $K(X)$. Recall that a skew diagram is a *horizontal strip* if it contains at most one box in each column, and a *vertical strip* if it has at most one box in each row. The diagram is a *rook strip* if it is both a horizontal strip and a vertical strip. For any partition ν in the $m \times k$-rectangle we define

$$\mathcal{O}^*_\nu = \sum_{\nu/\tau \text{ rook strip}} (-1)^{|\nu/\tau|}\, \mathcal{O}_\tau \tag{4}$$

where the sum is over all partitions $\tau \subset \nu$ such that ν/τ is a rook strip. The following lemma implies that $c^\nu_{\lambda\mu} = \chi_X(\mathcal{O}^\lambda \cdot \mathcal{O}^\mu \cdot \mathcal{O}^*_\nu)$. In particular, we have $c^\nu_{\lambda\mu} = 0$ whenever $\lambda \not\subset \nu$. A different proof of the lemma can be found in [3, §8].

Lemma 3.5. *Let μ and ν be partitions contained in the $m \times k$-rectangle. Then we have $\chi_X(\mathcal{O}^*_\nu \cdot \mathcal{O}^\mu) = \delta_{\nu,\mu}$ (Kronecker's delta).*

Proof. Since a non-empty Richardson variety is irreducible, rational, and has rational singularities, it follows that $\chi_X(\mathcal{O}_\tau \cdot \mathcal{O}^\mu)$ is equal to one if $\mu \subset \tau$ and is zero otherwise. This shows that $\chi_X(\mathcal{O}^*_\nu \cdot \mathcal{O}^\mu) = 0$ when $\mu \not\subset \nu$ and $\chi_X(\mathcal{O}^*_\nu \cdot \mathcal{O}^\nu) = \chi_X(\mathcal{O}_\nu \cdot \mathcal{O}^\nu) = 1$. Assume that $\mu \subsetneq \nu$ and let λ be the smallest partition such that $\mu \subset \lambda \subset \nu$ and ν/λ is a rook strip. Then $\chi_X(\mathcal{O}^*_\nu \cdot \mathcal{O}^\mu) = \sum_{\lambda \subset \tau \subset \nu} (-1)^{|\nu/\tau|}$ is a sum of $2^{|\nu/\lambda|}$ terms, half of which are negative. The lemma follows from this. □

A *south-east corner* of the skew diagram θ is any box $B \in \theta$ such that θ does not contain a box directly below or directly to the right of B. Let θ' be the diagram obtained from θ by removing its south-east corners. Notice that θ is a rook strip if and only if $\theta' = \emptyset$. Given any integer $p \in \mathbb{Z}$, we will abuse notation and write

$$\chi_X(\mathcal{O}_\theta \cdot \mathcal{O}^p) = \begin{cases} 1 & \text{if } p \leq c(\theta); \\ 0 & \text{if } p > c(\theta). \end{cases}$$

This is equivalent to setting $\mathcal{O}^p = 1$ for $p < 0$ and working on a sufficiently large Grassmannian X. Using this convention we define the constants

$$\mathcal{A}(\theta, p) = \sum_{\theta' \subset \varphi \subset \theta} (-1)^{|\theta| - |\varphi|}\, \chi_X(\mathcal{O}_\varphi \cdot \mathcal{O}^p) \tag{5}$$

where the sum is over all skew diagrams φ obtained by removing a subset of the south-east corners from θ. Proposition 3.4 and Lemma 3.5 imply the following.

Corollary 3.6. *Let $\lambda \subset \nu$ be partitions contained in the $m \times k$-rectangle and let $0 \leq p \leq k$. Then $c^\nu_{\lambda,p} = \mathcal{A}(\nu/\lambda, p)$.*

Lenart's Pieri rule [18] states that $c^{\nu}_{\lambda,p}$ is non-zero only if $\lambda \subset \nu$ and ν/λ is a horizontal strip, in which case we have

$$c^{\nu}_{\lambda,p} = (-1)^{|\nu/\lambda|-p}\binom{r(\nu/\lambda)-1}{|\nu/\lambda|-p}. \tag{6}$$

Here $r(\nu/\lambda)$ denotes the number of non-empty rows of the skew diagram ν/λ. To be consistent with our treatment of Pieri coefficients for Lagrangian and orthogonal Grassmannians, we will restate this rule as a set of recursive identities among the integers $\mathcal{A}(\theta,p)$. Given a horizontal strip θ, we let $\widehat{\theta}$ be the diagram obtained by removing the top row of θ. Notice that θ is a single row of boxes if and only if $\widehat{\theta}=\emptyset$. Lenart's formula is equivalent to the following.

Theorem 3.7. *Let θ be a skew diagram and let $p \in \mathbb{Z}$. If θ is not a horizontal strip then $\mathcal{A}(\theta,p)=0$. If $p \le 0$ then $\mathcal{A}(\theta,p)=\delta_{\theta,\emptyset}$, and $\mathcal{A}(\emptyset,p)$ is equal to one if $p \le 0$ and is zero otherwise. If $\theta \neq \emptyset$ is a horizontal strip and $p > 0$, then $\mathcal{A}(\theta,p)$ is determined by the following rules.*

(i) *If $\widehat{\theta}=\emptyset$, then $\mathcal{A}(\theta,p)=\delta_{|\theta|,p}$.*

(ii) *If $\widehat{\theta}\neq\emptyset$, then $\mathcal{A}(\theta,p)=\mathcal{A}(\widehat{\theta},p-a)-\mathcal{A}(\widehat{\theta},p-a+1)$, where $a=|\theta|-|\widehat{\theta}|$.*

Proof. We may assume that $\theta \neq \emptyset$ and $p>0$, since otherwise the result follows from Corollary 3.6. If θ is not a horizontal strip, then we can find a box $B \in \theta \smallsetminus \theta'$ such that the box directly above B is contained in θ'; if φ is any skew diagram such that $\theta' \subset \varphi \subset \theta$ and $B \notin \varphi$, then $c(\varphi)=c(\varphi \cup B)$ and $\chi_X(\mathcal{O}_\varphi \cdot \mathcal{O}^p) = \chi_X(\mathcal{O}_{\varphi\cup B}\cdot\mathcal{O}^p)$, so the terms of (5) given by φ and $\varphi \cup B$ cancel each other out. And if θ is a single row, then $\mathcal{A}(\theta,p)=\chi_X(\mathcal{O}_\theta\cdot\mathcal{O}^p)-\chi_X(\mathcal{O}_{\theta'}\cdot\mathcal{O}^p)=\delta_{|\theta|,p}$.

Assume that θ is a horizontal strip with two or more rows. Let $\psi=\theta\smallsetminus\widehat{\theta}$ be the top row and set $\psi'=\psi\cap\theta'$ and $\widehat{\theta}'=\widehat{\theta}\cap\theta'$. Each term of the sum (5) is given by a skew diagram of the form $\varphi\cup\psi$ or $\varphi\cup\psi'$, where $\widehat{\theta}'\subset\varphi\subset\widehat{\theta}$. Since $c(\varphi)=c(\varphi\cup\psi)-a=c(\varphi\cup\psi')-a+1$, it follows from Proposition 3.4 that

$$\begin{aligned}&(-1)^{|\theta|-|\varphi\cup\psi|}\left(\chi_X(\mathcal{O}_{\varphi\cup\psi}\cdot\mathcal{O}^p)-\chi_X(\mathcal{O}_{\varphi\cup\psi'}\cdot\mathcal{O}^p)\right)\\ &=(-1)^{|\widehat{\theta}|-|\varphi|}\left(\chi_X(\mathcal{O}_\varphi\cdot\mathcal{O}^{p-a})-\chi_X(\mathcal{O}_\varphi\cdot\mathcal{O}^{p-a+1})\right).\end{aligned}$$

This implies that $\mathcal{A}(\theta,p)=\mathcal{A}(\widehat{\theta},p-a)-\mathcal{A}(\widehat{\theta},p-a+1)$ by summing over φ. □

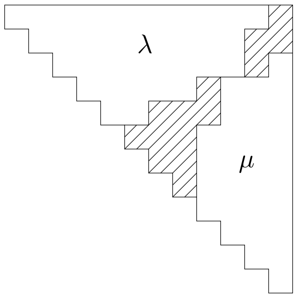
4. Maximal orthogonal Grassmannians

Let $e_1,\dots,e_{2n+1}$ be the standard basis for $\mathbb{C}^{2n+1}$. Define an orthogonal form on $\mathbb{C}^{2n}$ by $(e_i,e_j)=\delta_{i+j,2n+2}$. A vector subspace $U\subset\mathbb{C}^{2n+1}$ is called *isotropic* if $(U,U)=0$. This implies that $\dim(U)\le n$. Let $X=\mathrm{OG}(n,2n+1)=\{V\subset\mathbb{C}^{2n+1}\mid\dim(V)=n \text{ and } (V,V)=0\}$ be the orthogonal Grassmannian of maximal isotropic subspaces of $\mathbb{C}^{2n+1}$. This is a non-singular variety of dimension $\binom{n+1}{2}$. It is isomorphic to the even orthogonal Grassmannian $\mathrm{OG}(n+1,2n+2)$, so this variety is also covered in what follows. The Schubert varieties in X are indexed by strict partitions $\lambda=(\lambda_1>\lambda_2>\cdots>\lambda_\ell>0)$ for which $\lambda_1\le n$. The length of λ is the number $\ell(\lambda)=\ell$ of non-zero parts. The Schubert variety for λ relative to an isotropic flag $0\subsetneq F_1\subsetneq F_2\subsetneq\cdots\subsetneq F_n\subset\mathbb{C}^{2n+1}$ with $(F_n,F_n)=0$ is defined by

$$X^\lambda(F_\bullet)=\{V\in X\mid\dim(V\cap F_{n+1-\lambda_i})\ge i\ \forall 1\le i\le\ell(\lambda)\}.$$

The standard isotropic flags of $\mathbb{C}^{2n+1}$ are defined by $E_i = \langle e_1, \dots, e_i \rangle$ and $E_i^{\text{op}} = \langle e_{2n+2-i}, \dots, e_{2n+1} \rangle$. Set $X^\lambda = X^\lambda(E_\bullet)$. The classes $\mathcal{O}^\lambda = [\mathcal{O}_{X^\lambda}]$ form a $\mathbb{Z}$-basis for the Grothendieck ring $K(X)$, and this ring is generated as a $\mathbb{Z}$-algebra by the special classes $\mathcal{O}^1, \dots, \mathcal{O}^n$.

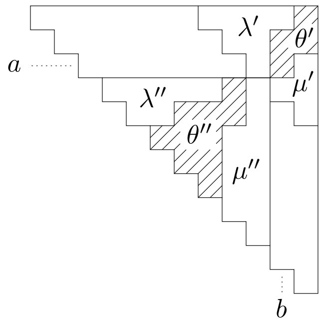
When we are working in the context of the maximal orthogonal Grassmannian $X = \text{OG}(n, 2n+1)$, we will identify a strict partition $\lambda = (\lambda_1 > \cdots > \lambda_\ell > 0)$ with its *shifted Young diagram*. The i-th row in this diagram contains λ_i boxes, which are preceded by $i-1$ unused positions. The boxes of the staircase partition $\rho_n = (n, n-1, \dots, 1)$ thus correspond to the upper-triangular positions in an $n \times n$ matrix. If μ is a strict partition with $\mu_1 \leq n$, then the n–dual partition $\mu^\vee$ consists of the parts of ρ_n which are not parts of μ. We have $X^\lambda(E_\bullet) \cap X^\mu(E_\bullet^{\text{op}}) \neq \emptyset$ if and only if $\lambda \subset \mu^\vee$. In this case the shifted skew diagram $\theta = \mu^\vee/\lambda$ is obtained by removing the boxes of λ from the upper-left corner of ρ_n and the boxes of μ from the lower-right corner, after mirroring μ in the south-west to north-east antidiagonal. For example, when $n = 12$, $\lambda = (11, 9, 8, 5, 2)$, and $\mu = (10, 8, 7, 4)$, we obtain the following diagram θ.



The set of leftmost boxes of ρ_n are called *diagonal boxes*. The above skew diagram θ contains three such boxes.

Define the Richardson variety $X_\theta = X^\lambda(E_\bullet) \cap X^\mu(E_\bullet^{\text{op}}) \subset X$. This variety has dimension $|\theta| = \binom{n+1}{2} - |\lambda| - |\mu|$. It follows from Lemma 4.1 below that the isomorphism class of X_θ depends only on the shape of the skew diagram θ, at least if it is remembered which boxes of θ are diagonal boxes. We set $\mathcal{O}_\theta = [\mathcal{O}_{X_\theta}] = \mathcal{O}^\lambda \cdot \mathcal{O}^\mu \in K(X)$. For $u \in \mathbb{C}^{2n+1}$ we define $X_\theta(u) = \{V \in X_\theta \mid u \in V\}$. Let $\bigcup X_\theta = \bigcup_{V \in X_\theta} V \subset \mathbb{C}^{2n+1}$ be the set of vectors u for which $X_\theta(u) \neq \emptyset$.

As for Grassmannians of type A, we would like to write the Richardson variety X_θ as a product, where the factors correspond to the components of θ. Assume that $a, b \geq 0$ are integers such that $0 < a + b < n$, $\lambda_a > n - a - b$, and $\mu_b > n - a - b$ (we set $\lambda_0 = \mu_0 = n + 1$). Then $\theta = \mu^\vee/\lambda$ can be split into a north-east part and a south-west part.



Set $\lambda' = (\lambda_1 + a + b - n, \dots, \lambda_a + a + b - n)$, $\mu' = (\mu_1 + a + b - n, \dots, \mu_b + a + b - n)$, and $\theta' = \mu'^\vee / \lambda'$, where $\mu'^\vee$ is the $(a+b)$–dual of μ'. Set $E' = E_{a+b} \oplus E^{\mathrm{op}}_{a+b}$ and extend the orthogonal form on E' to the vector space $E' \oplus \mathbb{C}$, with basis $e_1, \dots, e_{a+b}, e', e_{2n+2-a-b}, \dots, e_{2n+1}$, by setting $(e', e') = 1$ and $(e', e_i) = 0$ for every i. Then θ' defines a Richardson variety $X'_{\theta'}$ in $X' = \mathrm{OG}(a+b, E' \oplus \mathbb{C})$. Similarly we set $\lambda'' = (\lambda_{a+1}, \dots, \lambda_{\ell(\lambda)})$, $\mu'' = (\mu_{b+1}, \dots, \mu_{\ell(\mu)})$, and $\theta'' = \mu''^\vee / \lambda''$, using the $(n-a-b)$–dual of μ''. This defines the Richardson variety $X''_{\theta''}$ in $X'' = \mathrm{OG}(n-a-b, E'')$, where $E'' = E'^\perp = \langle e_{a+b+1}, \dots, e_{2n+1-a-b}\rangle \subset \mathbb{C}^{2n+1}$. Set $\widetilde{X}' = \{V' \in X' \mid V' \subset E'\}$. If $V' \in X'_{\theta'}$ then $\dim(V' \cap E_{a+b}) \geq a$ and $\dim(V' \cap E^{\mathrm{op}}_{a+b}) \geq b$, which implies that $V' \in \widetilde{X}'$. In particular, we have $\bigcup X'_{\theta'} \subset E'$.

Lemma 4.1. (a) *We have* $\bigcup X_\theta = \bigcup X'_{\theta'} \times \bigcup X''_{\theta''}$ *in* $\mathbb{C}^{2n+1} = E' \times E''$.

(b) *For arbitrary vectors* $u' \in E'$ *and* $u'' \in E''$, *the natural inclusion* $\widetilde{X}' \times X'' \subset X$ *defined by* $(V', V'') \mapsto V' \oplus V''$ *identifies* $X'_{\theta'}(u') \times X''_{\theta''}(u'')$ *with* $X_\theta(u' + u'')$.

Proof. If $V \in X_\theta$ then $\dim(V \cap E_{a+b}) \geq \dim(V \cap E_{n+1-\lambda_a}) \geq a$ and $\dim(V \cap E^{\mathrm{op}}_{a+b}) \geq \dim(V \cap E^{\mathrm{op}}_{n+1-\lambda_b}) \geq b$. This implies that $V' = V \cap E'$ is a maximal isotropic subspace in $E' \oplus \mathbb{C}$, so $V' \in \widetilde{X}'$. It also follows that $\dim(V \cap E'^\perp) = n - a - b$, so $V'' = V \cap E'' \in X''$. Given arbitrary points $V' \in X'$ and $V'' \in X''$, it follows from the definitions that $V' \oplus V'' \in X_\theta(u' + u'')$ if and only $V' \in X'_{\theta'}(u')$ and $V'' \in X''_{\theta''}(u'')$. The lemma follows from this. $\square$

The *south-east rim* of the shifted skew diagram θ is the set of boxes $B \in \theta$ such that no box of θ is located strictly south and strictly east of B. Let $\overline{\theta}$ denote the diagram obtained by removing the south-east rim from θ. We say that θ is a *rim* if $\overline{\theta} = \emptyset$. Let $d(\theta) = |\theta| - |\overline{\theta}|$ be number of boxes in the south-east rim. Let $N(\theta)$ denote the number of connected components of the diagram θ, where two boxes are connected if they share a side. Set $N^-(\theta) = \max(N(\theta) - 1, 0)$. For use with the Lagrangian Grassmannian, we also let $N'(\theta)$ be the number of components that do not contain any diagonal boxes. The diagram displayed above gives $d(\theta) = 10$, $N(\theta) = 2$, and $N^-(\theta) = N'(\theta) = 1$.

Lemma 4.2. (a) *The set* $\bigcup X_\theta \subset \mathbb{C}^{2n+1}$ *is a scheme theoretic complete intersection with rational singularities. It has dimension* $n + d(\theta)$ *and is defined by* $N(\theta)$ *quadratic equations and* $n + 1 - d(\theta) - N(\theta)$ *linear equations.*

(b) *For all vectors* u *in a dense open subset of* $\bigcup X_\theta$ *we have* $X_\theta(u) \cong X_{\overline{\theta}}$.

Proof. The result is clear unless $\theta \neq \emptyset$. Using Lemma 4.1 we may also assume that for all integers $a, b \geq 0$ with $0 < a + b < n$ we have $\lambda_a \leq n - a - b$ or $\mu_b \leq n - a - b$. This implies that θ has exactly one component, so $N(\theta) = 1$. Given any vector $u = (x_1, \dots, x_{2n+1}) \in \mathbb{C}^{2n+1}$ we will write $u_i = (x_1, \dots, x_i, 0, \dots, 0) \in \mathbb{C}^{2n+1}$ and $u'_i = (0, \dots, 0, x_{i+1}, \dots, x_{2n+1}) \in \mathbb{C}^{2n+1}$ for its projections to E_i and E^{op}_{2n+1-i}.

Assume first that $\ell(\lambda) + \ell(\mu) < n$. In this case θ intersects the diagonal, and since we also have $\lambda_1 < n$ and $\mu_1 < n$, it follows that $d(\theta) = n$. We will show that $\bigcup X_\theta$ is the quadric $\{u \in \mathbb{C}^{2n+1} \mid (u, u) = 0\} \subset \mathbb{C}^{2n+1}$. Let $u = (x_1, \dots, x_{2n+1}) \in \mathbb{C}^{2n+1}$ be any vector such that $(u, u) = 0$, $x_i \neq 0$ for $1 \leq i \leq 2n+1$, and $(u_i, u'_i) \neq 0$ for $1 \leq i \leq n$. It is enough to show that $X_\theta(u) \cong X_{\overline{\theta}}$. Set $\overline{E} = \langle u\rangle^\perp / \langle u\rangle$ and define isotropic flags in this vector space by $\overline{E}_i = ((E_{i+1} + \langle u\rangle) \cap \langle u\rangle^\perp)/\langle u\rangle$ and $\overline{E}^{\mathrm{op}}_i = ((E^{\mathrm{op}}_{i+1} + \langle u\rangle) \cap \langle u\rangle^\perp)/\langle u\rangle$. For each $i < n$ we have $(E_{i+1} + \langle u\rangle) \cap ((E^{\mathrm{op}}_{i+1})^\perp + \langle u\rangle) = \langle u_{i+1}, u'_{i+1}\rangle$. By the choice of u we have $(u_{i+1}, u'_{i+1}) \neq 0$, which implies that

$\overline{E}_i \cap (\overline{E}_i^{\text{op}})^\perp = 0$. Similarly we obtain $(\overline{E}_i)^\perp \cap \overline{E}_i^{\text{op}} = 0$, so the flags $\overline{E}_\bullet$ and $\overline{E}_\bullet^{\text{op}}$ are opposite. Identify $\overline{X} = \text{OG}(n-1, \overline{E})$ with the set of point $V \in X$ for which $u \in V$. Then we have $X^\lambda(E_\bullet) \cap \overline{X} = \overline{X}^\lambda(\overline{E}_\bullet)$ and $X^\mu(E_\bullet^{\text{op}}) \cap \overline{X} = \overline{X}^\mu(\overline{E}_\bullet^{\text{op}})$, so $X_\theta(u) = X_\theta \cap \overline{X} = \overline{X}^\lambda(\overline{E}_\bullet) \cap \overline{X}^\mu(\overline{E}_\bullet^{\text{op}}) = \overline{X}_{\overline{\theta}}$.

Otherwise we have $\ell(\lambda) + \ell(\mu) = n$ and $\lambda_{\ell(\lambda)} = \mu_{\ell(\mu)} = 1$. This implies that θ is disjoint from the diagonal, so $d(\theta) = n-1$. For any point $V \in X_\theta$ we must have $\dim(V \cap E_n) \geq \ell(\lambda)$ and $\dim(V \cap E_n^{\text{op}}) \geq \ell(\mu)$, so $V = (V \cap E_n) \oplus (V \cap E_n^{\text{op}})$. It follows that every vector $u = (x_1, \dots, x_{2n+1}) \in \bigcup X_\theta$ satisfies $x_{n+1} = 0$ and $(u,u) = 0$. We will show that $\bigcup X_\theta$ is the complete intersection in $\mathbb{C}^{2n+1}$ defined by these two equations. Let $u \in \mathbb{C}^{2n+1}$ be any vector such that $x_{n+1} = 0$, $(u,u) = 0$, $x_i \neq 0$ for $i \neq n+1$, and $(u_i, u_i') \neq 0$ for $1 \leq i \leq n-1$. It is enough to show that $X_\theta(u) \cong X_{\overline{\theta}}$. Set $\overline{E} = U^\perp/U$ where $U = \langle u_n, u_n' \rangle$, and define isotropic flags in $\overline{E}$ by $\overline{E}_i = ((E_{i+1} + U) \cap U^\perp)/U$ and $\overline{E}_i^{\text{op}} = ((E_{i+1}^{\text{op}} + U) \cap U^\perp)/U$ for $1 \leq i \leq n-2$. For each $i \leq n-2$ we have $(E_{i+1} + U) \cap ((E_{i+1}^{\text{op}})^\perp + U) = \langle u_n, u_n', u_{i+1}, u_{i+1}' \rangle$, so our choice of u implies that $\overline{E}_i \cap (\overline{E}_i^{\text{op}})^\perp = 0$. Identify $\overline{X} = \text{OG}(n-2, \overline{E})$ with the set of points $V \in X$ for which $U \subset V$. Then we have $X^\lambda(E_\bullet) \cap \overline{X} = \overline{X}^{\overline{\lambda}}(\overline{E}_\bullet)$ and $X^\mu(E_\bullet^{\text{op}}) \cap \overline{X} = \overline{X}^{\overline{\mu}}(\overline{E}_\bullet^{\text{op}})$, where $\overline{\lambda} = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_{\ell(\lambda)} - 1)$ and $\overline{\mu} = (\mu_1 - 1, \mu_2 - 1, \dots, \mu_{\ell(\mu)} - 1)$. We conclude that $X_\theta(u) = X_\theta \cap \overline{X} = \overline{X}^{\overline{\lambda}}(\overline{E}_\bullet) \cap \overline{X}^{\overline{\mu}}(\overline{E}_\bullet^{\text{op}}) = \overline{X}_{\overline{\theta}}$, as required. $\square$

Define a function $h : \mathbb{N} \times \mathbb{Z} \to \mathbb{Z}$ by

$$h(a,b) = \sum_{j=0}^{b} (-1)^j \, 2^{a-j} \binom{a}{j} . \tag{7}$$

Here we set $\binom{a}{j} = 0$ unless $0 \leq j \leq a$. Notice that for $b \geq a$ we have $h(a,b) = (2-1)^a = 1$. The binomial identity implies that

$$h(a+1, b) + h(a, b-1) = 2\, h(a,b) . \tag{8}$$

Proposition 4.3. *Let θ be a shifted skew diagram contained in ρ_n and $0 \leq p \leq n$. Then we have $\chi_X(\mathcal{O}_\theta \cdot \mathcal{O}^p) = h(N^-(\theta), d(\theta) - p)$.*

Proof. Notice that $\chi_X(\mathcal{O}_\theta \cdot \mathcal{O}^0) = \chi_X(\mathcal{O}_\theta) = 1$ and $\chi_X(\mathcal{O}_\emptyset \cdot \mathcal{O}^p) = \delta_{p,0}$ as claimed, so we may assume that $\theta \neq \emptyset$ and $p \geq 1$. Let $S \subset \mathbb{P}^{2n}$ be the quadric of isotropic lines in $\mathbb{C}^{2n+1}$. The subvariety $\mathbb{P}(E_{n+1-p}) \subset S$ defines the class $[\mathcal{O}_{\mathbb{P}(E_{n+1-p})}] \in K(S)$. Let $Z = \text{OF}(1, n; 2n+1)$ be the variety of two-step flags $L \subset V \subset \mathbb{C}^{2n+1}$ such $\dim(L) = 1$ and $V \in X$, and let $\pi_1 : Z \to S$ and $\pi_n : Z \to X$ be the projections. Since $\pi_n : \pi_1^{-1}(\mathbb{P}(E_{n+1-p})) \to X^p$ is a birational isomorphism and π_1 is flat, we obtain $\pi_{n*}\pi_1^*[\mathcal{O}_{\mathbb{P}(E_{n+1-p})}] = \mathcal{O}^p \in K(X)$. Since $\bigcup X_\theta$ is the affine cone over $\pi_1(\pi_n^{-1}(X_\theta))$, it follows from Lemma 4.2(a) that the later variety has rational singularities. Lemma 2.1 and Lemma 4.2(b) therefore imply that $\pi_{1*}\pi_n^*\mathcal{O}_\theta = [\mathcal{O}_{\pi_1(\pi_n^{-1}(X_\theta))}] \in K(S)$, so it follows from the projection formula that

$$\chi_X(\mathcal{O}_\theta \cdot \mathcal{O}^p) = \chi_S([\mathcal{O}_{\pi_1(\pi_n^{-1}(X_\theta))}] \cdot [\mathcal{O}_{\mathbb{P}(E_{n+1-\pi_n})}]) .$$

Let $Y \subset \mathbb{P}^{2n}$ be a complete intersection defined by the same equations as define $\bigcup X_\theta$ in $\mathbb{C}^{2n+1}$, except that one of the quadratic equations are omitted. Since $\pi_1(\pi_n^{-1}(X_\theta)) = S \cap Y$ is a proper intersection of Cohen-Macaulay varieties, we obtain $[\mathcal{O}_{\pi_1(\pi_n - 1(X_\theta))}] = \iota^*[\mathcal{O}_Y] \in K(S)$ where $\iota : S \subset \mathbb{P}^{2n}$ is the inclusion. Another application of the projection formula yields

$$\chi_X(\mathcal{O}_\theta \cdot \mathcal{O}^p) = \chi_{\mathbb{P}^{2n}}([\mathcal{O}_Y] \cdot [\mathcal{O}_{\mathbb{P}(E_{n+1-p})}]) \,.$$

Since the Grothendieck class of a quadric in $\mathbb{P}^{2n}$ is equal to $2t - t^2 \in K(\mathbb{P}^{2n}) = \mathbb{Z}[t]/(t^{2n+1})$, it follows from Lemma 4.2 that

$$[\mathcal{O}_Y] \cdot [\mathcal{O}_{\mathbb{P}(E_{n+1-p})}] = t^{2n-d(\theta)+p} \cdot (2-t)^{N^-(\theta)} \,.$$

The proposition follows from this. □

The dual Schubert classes in the K-theory of $X = \mathrm{OG}(n, 2n+1)$ are defined by

$$\mathcal{O}^*_\nu = \sum_{\nu/\tau \text{ rook strip}} (-1)^{|\nu/\tau|}\, \mathcal{O}_\tau \tag{9}$$

where the sum is over all strict partitions τ contained in ν such that the shifted skew diagram ν/τ is a rook strip. The proof of Lemma 3.5 shows that $\chi_X(\mathcal{O}^*_\nu \cdot \mathcal{O}^\mu) = \delta_{\nu,\mu}$.

A south-east corner of the shifted skew shape θ is any box B of θ such that θ does not contain a box directly below or directly to the right of B. Let θ' be the skew shape obtained by removing the south-east corners of θ. For $p \in \mathbb{Z}$ we define

$$\mathcal{B}(\theta, p) = \sum_{\theta' \subset \varphi \subset \theta} (-1)^{|\theta|-|\varphi|}\, h(N^-(\varphi), d(\varphi) - p) \tag{10}$$

where the sum is over all shifted skew diagrams φ obtained by removing a subset of the south-east corners from θ. Proposition 4.3 implies the following.

Corollary 4.4. *Let $\lambda \subset \nu$ be strict partitions with $\nu_1 \leq n$ and let $0 \leq p \leq n$. Then $c^\nu_{\lambda,p} = \mathcal{B}(\nu/\lambda, p)$.*

Lemma 4.5. *Let θ be a non-empty shifted skew diagram.*

(a) *If θ is not a rim then $\mathcal{B}(\theta, p) = 0$ for all p.*

(b) *If θ contains a row or column with a boxes, then $\mathcal{B}(\theta, p) = 0$ for $p < a$.*

(c) *If θ is a rook strip, then $\mathcal{B}(\theta, 1) = (-1)^{|\theta|-1}$.*

Proof. Assume the conditions of (a), (b), or (c) are met, and let $B \in \theta \smallsetminus \theta'$ be a south-east corner of θ; if θ is not a rim, then choose B such that θ contains a box strictly north and strictly west of θ. We claim that if φ is any shifted skew diagram such that $\theta' \subset \varphi \subset \theta$, $\varphi \neq \emptyset$, and $B \notin \varphi$, then

$$h(N^-(\varphi), d(\varphi) - p) = h(N^-(\varphi \cup B), d(\varphi \cup B) - p) \,;$$

in the situation of (c) we set $p = 1$. If θ is not a rim, then this is true because $N^-(\varphi) = N^-(\varphi \cup B)$ and $d(\varphi) = d(\varphi \cup B)$. If the conditions for (b) hold, then we must have $N^-(\varphi) \leq d(\varphi) - a + 1$ since φ contains a row or column with $a - 1$ boxes, and this implies that $h(N^-(\varphi), d(\varphi) - p) = h(N^-(\varphi \cup B), d(\varphi \cup B) - p) = 1$ for $p < a$. For (c), notice that $N^-(\varphi) = d(\varphi) - 1$ whenever φ is a non-empty rook strip.

The lemma is immediate from the claim if θ is not a rook strip, and otherwise we obtain

$$\begin{aligned} \mathcal{B}(\theta, p) &= (-1)^{|\theta|-1} \big(h(N^-(B), d(B) - p) - h(N^-(\emptyset), d(\emptyset) - p)\big) \\ &= (-1)^{|\theta|-1} \left(h(0, 1-p) - h(0, -p)\right) . \end{aligned}$$

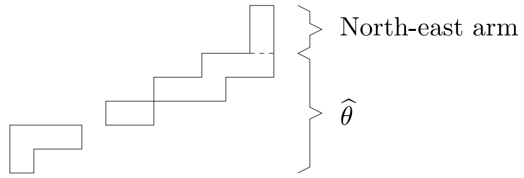
This proves the lemma when θ is a rook strip. □

Before we state our recursive Pieri rule for the coefficients $\mathcal{B}(\theta, p)$, we prove that the constants $c_{\lambda\mu}^{\nu^\vee}$ for maximal orthogonal Grassmannians are invariant under arbitrary permutations of the partitions λ, μ, ν. The same symmetry is satisfied for Grassmannians of type A (for the same reason, see [4, Cor. 1]), but Example 5.7 below shows that it fails for Lagrangian Grassmannians.

Corollary 4.6. *The K-theoretic structure constants $c_{\lambda,\mu}^{\nu}$ for $\mathrm{OG}(n, 2n+1)$ satisfy the symmetry $c_{\lambda,\mu}^{\nu} = c_{\lambda,\nu^\vee}^{\mu^\vee}$.*

Proof. Lemma 4.5 implies that $\mathcal{O}_\nu^* = (1 - \mathcal{O}^1) \cdot \mathcal{O}^{\nu^\vee}$, so we obtain the identity $c_{\lambda,\mu}^{\nu} = \chi_X(\mathcal{O}^\lambda \cdot \mathcal{O}^\mu \cdot \mathcal{O}^{\nu^\vee} \cdot (1 - \mathcal{O}^1))$. The corollary follows from this. $\square$

A shifted skew diagram is called a *row* if all its boxes belong to the same row, and a *column* if all boxes belong to the same column. If θ is a non-empty rim, then we define the *north-east arm* of θ to be the largest row or column that can be obtained by intersecting θ with a square whose upper-right box agrees with the upper-right box of θ. We let $\widehat{\theta}$ be the diagram obtained by removing the north-east arm from θ. Notice that θ is a row or a column if and only if $\widehat{\theta} = \emptyset$.



The following theorem is our recursive Pieri rule for the K-theory of maximal orthogonal Grassmannians. It implies that the Pieri coefficients $c_{\lambda,p}^{\nu} = \mathcal{B}(\nu/\lambda, p)$ have alternating signs, $(-1)^{|\nu/\lambda|-p}\, c_{\lambda,p}^{\nu} \geq 0$.

Theorem 4.7. *Let θ be a shifted skew diagram and let $p \in \mathbb{Z}$. If θ is not a rim then $\mathcal{B}(\theta, p) = 0$. If $p \leq 0$ then $\mathcal{B}(\theta, p) = \delta_{\theta,\emptyset}$, and $\mathcal{B}(\emptyset, p)$ is equal to one if $p \leq 0$ and is zero otherwise. If $\theta \neq \emptyset$ is a rim and $p > 0$, then $\mathcal{B}(\theta, p)$ is determined by the following rules, with $a = |\theta| - |\widehat{\theta}|$.*

(i) *If θ is a row or a column then $\mathcal{B}(\theta, p) = \delta_{|\theta|,p}$.*

(ii) *If $\widehat{\theta} \neq \emptyset$ and the north-east arm of θ is connected to $\widehat{\theta}$, then we have $\mathcal{B}(\theta, p) = \mathcal{B}(\widehat{\theta}, p-a) - \mathcal{B}(\widehat{\theta}, p-a+1)$.*

(iii) *Assume that $\widehat{\theta} \neq \emptyset$ and the north-east arm of θ is not connected to $\widehat{\theta}$. If $p < a$ then $\mathcal{B}(\theta, p) = 0$. Otherwise we have*

$$\mathcal{B}(\theta, p) = (2-\delta_{p,a})(\mathcal{B}(\widehat{\theta}, p-a) - \mathcal{B}(\widehat{\theta}, p-a+1)) + (1-\delta_{a,1})(\mathcal{B}(\widehat{\theta}, p-a+2) - \mathcal{B}(\widehat{\theta}, p-a+1)).$$

Proof. We may assume that θ is a non-empty rim and $p > 0$, since otherwise the theorem follows from Lemma 4.5. In particular we have $d(\theta) = |\theta|$. If θ is a row or column, then $\mathcal{B}(\theta, p) = h(0, |\theta| - p) - h(0, |\theta'| - p) = \delta_{p,|\theta|}$, as claimed in (i). Otherwise let ψ be the north-east arm of θ, let $B \in \theta \smallsetminus \theta'$ be the south-east corner farthest to the north, and set $\psi' = \psi \smallsetminus B$ and $\widehat{\theta}' = \widehat{\theta} \cap \theta'$. We have $\widehat{\theta} \neq \emptyset$ and $a = |\psi|$.

Assume that ψ is connected to $\widehat{\theta}$. Then $\widehat{\theta}$ is not a rook strip. We first consider the case where ψ is a row attached to the right side of the upper-right box of $\widehat{\theta}$.

For any rim φ with $\widehat{\theta}' \subset \varphi \subset \widehat{\theta}$ we then have

$$\begin{aligned}&(-1)^{|\theta|-|\varphi\cup\psi|}\left(h(N^-(\varphi\cup\psi),|\varphi\cup\psi|-p)-h(N^-(\varphi\cup\psi'),|\varphi\cup\psi'|-p)\right)\\&=(-1)^{|\widehat{\theta}|-|\varphi|}\left(h(N^-(\varphi),|\varphi|-p+a)-h(N^-(\varphi),|\varphi|-p+a-1)\right).\end{aligned}$$

This implies that $\mathcal{B}(\theta,p)=\mathcal{B}(\widehat{\theta},p-a)-\mathcal{B}(\widehat{\theta},p-a+1)$ by summing over φ. Next consider the case where ψ is a column attached above the upper-right box of $\widehat{\theta}$. For any diagram φ with $\widehat{\theta}' \subset \varphi \subset \widehat{\theta} \smallsetminus B$, it follows from (8) that

$$\begin{aligned}&(-1)^{|\theta|-|\varphi\cup\psi|}\left(h(N^-(\varphi\cup\psi),|\varphi\cup\psi|-p)-h(N^-(\varphi\cup B\cup\psi),|\varphi\cup B\cup\psi|-p)\right)\\&=(-1)^{|\widehat{\theta}|-|\varphi|}\left(h(N^-(\varphi)+1,|\varphi|-p+a)-h(N^-(\varphi\cup B),|\varphi\cup B|-p+a)\right)\\&=(-1)^{|\widehat{\theta}|-|\varphi|}\left(h(N^-(\varphi),|\varphi|-p+a)-h(N^-(\varphi\cup B),|\varphi\cup B|-p+a)\right)\\&\qquad\quad -h(N^-(\varphi),|\varphi|-p+a-1)+h(N^-(\varphi\cup B),|\varphi\cup B|-p+a-1)\end{aligned}$$

This again implies that $\mathcal{B}(\theta,p)=\mathcal{B}(\widehat{\theta},p-a)-\mathcal{B}(\widehat{\theta},p-a+1)$, as required by (ii).

Now assume that ψ is not connected to $\widehat{\theta}$. We may also assume that $p\geq a$, since otherwise $\mathcal{B}(\theta,p)=0$ by Lemma 4.5(b). We first consider the case $a>1$. For any non-empty rim φ with $\widehat{\theta}' \subset \varphi \subset \widehat{\theta}$, we obtain

$$\begin{aligned}&(-1)^{|\theta|-|\varphi\cup\psi|}\left(h(N^-(\varphi\cup\psi),|\varphi\cup\psi|-p)-h(N^-(\varphi\cup\psi'),|\varphi\cup\psi'|-p)\right)\\&=(-1)^{|\widehat{\theta}|-|\varphi|}\left(h(N^-(\varphi)+1,|\varphi|-p+a)-h(N^-(\varphi)+1,|\varphi|-p+a-1)\right)\\&=(-1)^{|\widehat{\theta}|-|\varphi|}(2\,h(N^-(\varphi),|\varphi|-p+a)-3\,h(N^-(\varphi),|\varphi|-p+a-1)\\&\qquad\qquad +h(N^-(\varphi),|\varphi|-p+a-2)).\end{aligned}$$

If $\widehat{\theta}$ is not a rook strip, then $\widehat{\theta}'\neq\emptyset$ and we obtain that $\mathcal{B}(\theta,p)=2\,\mathcal{B}(\widehat{\theta},p-a)-3\,\mathcal{B}(\widehat{\theta},p-a+1)+\mathcal{B}(\widehat{\theta},p-a+2)$. If $p>a$ then this is the identity of (iii), and if $p=a$ then Lemma 4.5(b) shows that $\mathcal{B}(\widehat{\theta},p-a)=\mathcal{B}(\widehat{\theta},p-a+1)=0$, so (iii) also holds in this case. If $\widehat{\theta}$ is a rook strip, then summing over $\varphi\neq\emptyset$ gives the identity

$$\begin{aligned}\mathcal{B}(\theta,p) &= 2\,\mathcal{B}(\widehat{\theta},p-a)-3\,\mathcal{B}(\widehat{\theta},p-a+1)+\mathcal{B}(\widehat{\theta},p-a+2)\\&\quad -(-1)^{|\widehat{\theta}|}\left(h(0,a-p)-2h(0,a-p-1)+h(0,a-p-2)\right).\end{aligned}$$

If $p>a$ then the three last terms vanish, and if $p=a$ then Lemma 4.5 implies that $\mathcal{B}(\theta,p)=\mathcal{B}(\widehat{\theta},2)-2\,\mathcal{B}(\widehat{\theta},1)$, as required. At last we consider the case $a=1$. Then $\psi'=\emptyset$, so for any non-empty rim φ with $\widehat{\theta}' \subset \varphi \subset \widehat{\theta}$ we get

$$\begin{aligned}&(-1)^{|\theta|-|\varphi\cup\psi|}\left(h(N^-(\varphi\cup\psi),|\varphi\cup\psi|-p)-h(N^-(\varphi\cup\psi'),|\varphi\cup\psi'|-p)\right)\\&=(-1)^{|\widehat{\theta}|-|\varphi|}\left(h(N^-(\varphi)+1,|\varphi|-p+1)-h(N^-(\varphi),|\varphi|-p)\right)\\&=(-1)^{|\widehat{\theta}|-|\varphi|}\left(2\,h(N^-(\varphi),|\varphi|-p+1)-2\,h(N^-(\varphi),|\varphi|-p)\right).\end{aligned}$$

If $\widehat{\theta}$ is not a rook strip, then this implies that $\mathcal{B}(\theta,p)=2\,\mathcal{B}(\widehat{\theta},p-1)-2\,\mathcal{B}(\widehat{\theta},p)$, and for $p=a$ we have $\mathcal{B}(\widehat{\theta},p-1)=\mathcal{B}(\widehat{\theta},p)=0$ by Lemma 4.5. Finally, if $\widehat{\theta}$ is a rook strip, then we obtain $\mathcal{B}(\theta,p)=2\,\mathcal{B}(\widehat{\theta},p-1)-2\,\mathcal{B}(\widehat{\theta},p)-(-1)^{|\widehat{\theta}|}h(0,1-p)$, which agrees with (iii) by Lemma 4.5. This establishes (iii) and completes the proof. □

Let θ be a rim. A *KOG-tableau* of shape θ is a labeling of the boxes of θ with positive integers such that (i) each row of θ is strictly increasing from left to right;

(ii) each column of θ is strictly increasing from top to bottom; and (iii) each box is either smaller than or equal to all the boxes south-west of it, or it is greater than or equal to all the boxes south-west of it. If θ is not a rim, then there are no KOG-tableaux with shape θ. The content of a KOG-tableau is the set of integers contained in its boxes.

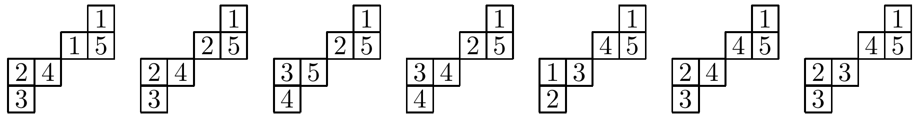
Corollary 4.8. *The constant $c^{\nu}_{\lambda,p}$ for $K(\mathrm{OG}(n, 2n + 1))$ is equal to $(-1)^{|\nu/\lambda|-p}$ times the number of KOG-tableaux of shape ν/λ with content $\{1, 2, \dots, p\}$.*

Proof. Define $\widetilde{\mathcal{B}}(\theta, p)$ to be $(-1)^{|\theta|-p}$ times the number of KOG-tableaux of shape θ with content $\{1, 2, \dots, p\}$. It is enough to show that these integers satisfy the recursions given by Theorem 4.7. We check this when the north-east arm ψ is a column that is disconnected from $\widehat{\theta}$, $\widehat{\theta} \neq \emptyset$, and $1 < a < p$ where $a = |\theta| - |\widehat{\theta}|$. The remaining cases are left to the reader.

In any KOG-tableau of shape θ, the content of (the boxes of) ψ must be either $S_1 = \{1, 2, \dots, a\}$ or $S_2 = \{1, 2, \dots, a-1, p\}$. If the content of ψ is S_1, then $\widehat{\theta}$ must be a KOG-tableau whose content is either $\{a+1, a+2, \dots, p\}$ or $\{a, a+1, \dots, p\}$. If the content of ψ is S_2, then $\widehat{\theta}$ must be a KOG-tableau whose content is one of the sets $\{a, a+1, \dots, p-1\}$, $\{a-1, a, \dots, p-1\}$, $\{a, a+1, \dots, p\}$, or $\{a-1, a, \dots, p\}$. This gives the identity $\widetilde{\mathcal{B}}(\theta, p) = 2\,\widetilde{\mathcal{B}}(\widehat{\theta}, p-a) - 3\,\widetilde{\mathcal{B}}(\widehat{\theta}, p-a+1) + \widetilde{\mathcal{B}}(\widehat{\theta}, p-a+2)$, as required. $\square$

Itai Feigenbaum and Emily Sergel have shown us a proof that KOG-tableaux are invariant under Thomas and Yong's jeu-de-taquin slides [24]. Corollary 4.8 therefore confirms that the Pieri coefficients $c^{\nu}_{\lambda,p}$ are correctly computed by Thomas and Yong's conjectured Littlewood-Richardson rule.

Example 4.9. Let $\lambda = (6, 4, 1)$ and $\nu = (7, 6, 3, 1)$. Then the constant $c^{\nu}_{\lambda,5} = -7$ for $K(\mathrm{OG}(7, 15))$ is counted by the KOG-tableaux displayed below.



5. Lagrangian Grassmannians

Let $e_1, \dots, e_{2n}$ be the standard basis of $\mathbb{C}^{2n}$ and define a symplectic form by

$$(e_i, e_j) = \begin{cases} 1 & \text{if } i + j = 2n + 1 \text{ and } i < j; \\ -1 & \text{if } i + j = 2n + 1 \text{ and } i > j; \text{ and} \\ 0 & \text{if } i + j \neq 2n + 1. \end{cases}$$

Let $X = \mathrm{LG}(n, 2n) = \{V \subset \mathbb{C}^{2n} \mid \dim(V) = n \text{ and } (V, V) = 0\}$ be the Lagrangian Grassmannian of maximal isotropic subspaces of $\mathbb{C}^{2n}$. This is a non-singular variety of dimension $\binom{n+1}{2}$. The Schubert varieties in X are indexed by strict partitions $\lambda = (\lambda_1 > \lambda_2 > \dots > \lambda_\ell > 0)$ with $\lambda_1 \leq n$, the same partitions as are used for the orthogonal Grassmannian $\mathrm{OG}(n, 2n+1)$. The notation introduced for shifted skew diagrams in section 4 will also be used in this section. The Schubert variety for the strict partition λ relative to an isotropic flag $0 \subset F_1 \subset F_2 \subset \dots \subset F_n \subset \mathbb{C}^{2n}$ with $(F_n, F_n) = 0$ is defined by

$$X^{\lambda}(F_\bullet) = \{V \in X \mid \dim(V \cap F_{n+1-\lambda_i}) \geq i \ \forall 1 \leq i \leq \ell(\lambda)\}.$$

The standard flags are defined by $E_i = \langle e_1, \dots, e_i\rangle$ and $E_i^{\mathrm{op}} = \langle e_{2n+1-i}, \dots, e_{2n}\rangle$. We have $X^\lambda(E_\bullet) \cap X^\mu(E_\bullet^{\mathrm{op}}) \neq \emptyset$ if and only if $\lambda \subset \mu^\vee$, where $\mu^\vee$ is the n–dual of μ. Set $X^\lambda = X^\lambda(E_\bullet)$. The classes $\mathcal{O}^\lambda = [\mathcal{O}_{X^\lambda}]$ form a $\mathbb{Z}$-basis for the Grothendieck ring $K(X)$, which is generated as a $\mathbb{Z}$-algebra by the special classes $\mathcal{O}^1, \dots, \mathcal{O}^n$. A shifted skew diagram $\theta = \mu^\vee/\lambda$ defines the Richardson variety $X_\theta = X^\lambda(E_\bullet) \cap X^\mu(E_\bullet^{\mathrm{op}})$ and the class $\mathcal{O}_\theta = [\mathcal{O}_{X_\theta}] = \mathcal{O}^\lambda \cdot \mathcal{O}^\mu \in K(X)$. For any vector $u \in \mathbb{C}^{2n}$ we set $X_\theta(u) = \{V \in X_\theta \mid u \in V\}$, and define $\bigcup X_\theta = \bigcup_{V \in X_\theta} V \subset \mathbb{C}^{2n}$.

Assume that $a, b \geq 0$ are integers such that $0 < a + b < n$, $\lambda_a > n - a - b$, and $\mu_b > n - a - b$. Set $\lambda' = (\lambda_1 + a + b - n, \dots, \lambda_a + a + b - n)$, $\mu' = (\mu_1 + a + b - n, \dots, \mu_b + a + b - n)$, and $\theta' = {\mu'}^\vee/\lambda'$, where ${\mu'}^\vee$ is the $(a+b)$–dual of μ'. Then θ' defines a Richardson variety $X'_{\theta'}$ in $X' = \mathrm{LG}(a+b, E')$, where $E' = E_{a+b} \oplus E^{\mathrm{op}}_{a+b}$ with basis $e_1, \dots, e_{a+b}, e_{2n+1-a-b}, \dots, e_{2n}$. Similarly we set $\lambda'' = (\lambda_{a+1}, \dots, \lambda_{\ell(\lambda)})$, $\mu'' = (\mu_{b+1}, \dots, \mu_{\ell(\mu)})$, and $\theta'' = {\mu''}^\vee/\lambda''$, using the $(n-a-b)$–dual of μ''. This defines the Richardson variety $X''_{\theta''}$ in $X'' = \mathrm{LG}(n-a-b, E'')$, where $E'' = {E'}^\perp = \langle e_{a+b+1}, \dots, e_{2n-a-b}\rangle$.

Lemma 5.1. (a) *We have $\bigcup X_\theta = \bigcup X'_{\theta'} \times \bigcup X''_{\theta''}$ in $\mathbb{C}^{2n} = E' \times E''$.*
(b) For arbitrary vectors $u' \in E'$ and $u'' \in E''$, the natural inclusion $X' \times X'' \subset X$ defined by $(V', V'') \mapsto V' \oplus V''$ identifies $X'_{\theta'}(u') \times X''_{\theta''}(u'')$ with $X_\theta(u' + u'')$.

Proof. If $V \in X_\theta$ then $\dim(V \cap E_{a+b}) \geq \dim(V \cap E_{n+1-\lambda_a}) \geq a$ and $\dim(V \cap E^{\mathrm{op}}_{a+b}) \geq \dim(V \cap E^{\mathrm{op}}_{n+1-\lambda_b}) \geq b$. This implies that $V' = V \cap E'$ is a maximal isotropic subspace in E', so $V' \in X'$. It also follows that $\dim(V \cap E'') = \dim(V \cap {E'}^\perp) = n - a - b$, so $V'' = V \cap E'' \in X''$. Given arbitrary points $V' \in X'$ and $V'' \in X''$, it follows from the definitions that $V' \oplus V'' \in X_\theta(u' + u'')$ if and only $V' \in X'_{\theta'}(u')$ and $V'' \in X''_{\theta''}(u'')$. The lemma follows from this. $\square$

Lemma 5.2. (a) *The set $\bigcup X_\theta \subset \mathbb{C}^{2n}$ is a scheme theoretic complete intersection with rational singularities. It has dimension $n + d(\theta)$ and is defined by $N'(\theta)$ quadratic equations and $n - d(\theta) - N'(\theta)$ linear equations.*
(b) *For all vectors u in a dense open subset of $\bigcup X_\theta$ we have $X_\theta(u) \cong X_{\overline{\theta}}$.*

Proof. Using Lemma 5.1 we may assume that for all integers $a, b \geq 0$ with $0 < a + b < n$ we have $\lambda_a \leq n - a - b$ or $\mu_b \leq n - a - b$. This implies that θ has exactly one component. Given any vector $u = (x_1, \dots, x_{2n}) \in \mathbb{C}^{2n}$ we will write $u_i = (x_1, \dots, x_i, 0, \dots, 0) \in \mathbb{C}^{2n}$ and $u'_i = (0, \dots, 0, x_{i+1}, \dots, x_{2n}) \in \mathbb{C}^{2n}$ for its projections to E_i and E^{op}_{2n-i}.

Assume first that $\ell(\lambda) + \ell(\mu) < n$. In this case θ intersects the diagonal, so $N'(\theta) = 0$. Since $\lambda_1 < n$ and $\mu_1 < n$ we also have $d(\theta) = n$. Let $u = (x_1, \dots, x_{2n}) \in \mathbb{C}^{2n}$ be any vector such that $x_i \neq 0$ and $(u_i, u'_i) \neq 0$ for $1 \leq i \leq 2n$. It is enough to show that $X_\theta(u) \cong X_{\overline{\theta}}$. Set $\overline{E} = u^\perp/\langle u\rangle$ and define isotropic flags in this vector space by $\overline{E}_i = ((E_{i+1} + \langle u\rangle) \cap \langle u\rangle^\perp)/\langle u\rangle$ and $\overline{E}^{\mathrm{op}}_i = ((E^{\mathrm{op}}_{i+1} + \langle u\rangle) \cap \langle u\rangle^\perp)/\langle u\rangle$. For each $i < n$ we have $(E_{i+1} + \langle u\rangle) \cap ((E^{\mathrm{op}}_{i+1})^\perp + \langle u\rangle) = \langle u_{i+1}, u'_{i+1}\rangle$. By the choice of u we have $(u_{i+1}, u'_{i+1}) \neq 0$, which implies that $\overline{E}_i \cap (\overline{E}^{\mathrm{op}}_i)^\perp = 0$. Similarly we obtain $(\overline{E}_i)^\perp \cap E^{\mathrm{op}}_i = 0$, so the flags $\overline{E}_\bullet$ and $\overline{E}^{\mathrm{op}}_\bullet$ are opposite. Identify $\overline{X} = \mathrm{LG}(n-1, \overline{E})$ with the set of point $V \in X$ for which $u \in V$. Then we have $X^\lambda(E_\bullet) \cap \overline{X} = \overline{X}^\lambda(\overline{E}_\bullet)$ and $X^\mu(E^{\mathrm{op}}_\bullet) \cap \overline{X} = \overline{X}^\mu(\overline{E}^{\mathrm{op}}_\bullet)$, so $X_\theta(u) = X_\theta \cap \overline{X} = \overline{X}^\lambda(\overline{E}_\bullet) \cap \overline{X}^\mu(\overline{E}^{\mathrm{op}}_\bullet) = \overline{X}_{\overline{\theta}}$.

Otherwise we have $\ell(\lambda) + \ell(\mu) = n$ and $\lambda_{\ell(\lambda)} = \mu_{\ell(\mu)} = 1$. This implies that θ is disjoint from the diagonal, so $N'(\theta) = 1$ and $d(\theta) = n - 1$. For any point

$V \in X_\theta$ we have $\dim(V \cap E_n) \geq \ell(\lambda)$ and $\dim(V \cap E_n^{\rm op}) \geq \ell(\mu)$, so $V = (V \cap E_n) \oplus (V \cap E_n^{\rm op})$. It follows that every vector $u \in \bigcup X_\theta$ satisfies $(u_n, u_n') = 0$. We will show that $\bigcup X_\theta \subset \mathbb{C}^{2n}$ is defined by this equation. Let $u \in \mathbb{C}^{2n}$ be any vector such that $x_i \neq 0$ for $1 \leq i \leq 2n$, $(u_n, u_n') = 0$, and $(u_i, u_i') \neq 0$ for $1 \leq i \leq n-1$. It is enough to show that $X_\theta(u) \cong X_{\overline{\theta}}$. Set $\overline{E} = U^\perp/U$ where $U = \langle u_n, u_n' \rangle$, and define isotropic flags in $\overline{E}$ by $\overline{E}_i = ((E_{i+1} + U) \cap U^\perp)/U$ and $\overline{E}_i^{\rm op} = ((E_{i+1}^{\rm op} + U) \cap U^\perp)/U$ for $1 \leq i \leq n-2$. For each $i \leq n-2$ we have $(E_{i+1} + U) \cap ((E_{i+1}^{\rm op})^\perp + U) = \langle u_n, u_n', u_{i+1}, u_{i+1}' \rangle$, so our choice of u implies that $\overline{E}_i \cap (\overline{E}_i^{\rm op})^\perp = 0$. Identify $\overline{X} = \mathrm{LG}(n-2, \overline{E})$ with the set of points $V \in X$ for which $U \subset V$. Then we have $X^\lambda(E_\bullet) \cap \overline{X} = \overline{X}^{\overline{\lambda}}(\overline{E}_\bullet)$ and $X^\mu(E_\bullet^{\rm op}) \cap \overline{X} = \overline{X}^{\overline{\mu}}(\overline{E}_\bullet^{\rm op})$, where $\overline{\lambda} = (\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_{\ell(\lambda)} - 1)$ and $\overline{\mu} = (\mu_1 - 1, \mu_2 - 1, \dots, \mu_{\ell(\mu)} - 1)$. We conclude that $X_\theta(u) = X_\theta \cap \overline{X} = \overline{X}^{\overline{\lambda}}(\overline{E}_\bullet) \cap \overline{X}^{\overline{\mu}}(\overline{E}_\bullet^{\rm op}) = \overline{X}_{\overline{\theta}}$, as required. □

Proposition 5.3. *For $0 \leq p \leq n$ we have $\chi_X(\mathcal{O}_\theta \cdot \mathcal{O}^p) = h(N'(\theta), d(\theta) - p)$.*

Proof. Let $Z = \mathrm{IF}(1, n; 2n)$ be the variety of two-step flags $L \subset V \subset \mathbb{C}^{2n}$ such that $\dim(L) = 1$ and $V \in X$, and let $\pi_1 : Z \to \mathbb{P}^{2n-1}$ and $\pi_n : Z \to X$ be the projections. Then $\pi_{n*}\pi_1^*[\mathcal{O}_{\mathbb{P}(E_{n+1-p})}] = \mathcal{O}^p \in K(X)$, and Lemmas 5.2 and 2.1 imply that $\pi_{1*}\pi_n^*\mathcal{O}_\theta = [\mathcal{O}_{\pi_1(\pi_n^{-1}(X_\theta))}] \in K(\mathbb{P}^{2n-1})$. The projection formula gives

$$\chi_X(\mathcal{O}_\theta \cdot \mathcal{O}^p) = \chi_{\mathbb{P}^{2n-1}}\left([\mathcal{O}_{\pi_1(\pi_n^{-1}(X_\theta))}] \cdot [\mathcal{O}_{\mathbb{P}(E_{n+1-p})}]\right).$$

Lemma 5.2 shows that $[\mathcal{O}_{\pi_1(\pi_n^{-1}(X_\theta))}] \cdot [\mathcal{O}_{\mathbb{P}(E_{n+1-p})}] = t^{2n-d(\theta)+p-1} \cdot (2-t)^{N'(\theta)} \in K(\mathbb{P}^{2n-1}) = \mathbb{Z}[t]/(t^{2n})$ and the proposition follows from this. □

Given a shifted skew diagram θ and $p \in \mathbb{Z}$ we define

$$\mathcal{C}(\theta, p) = \sum_{\theta' \subset \varphi \subset \theta} (-1)^{|\theta| - |\varphi|}\, h(N'(\varphi), d(\varphi) - p) \tag{11}$$

where the sum is over all shifted skew diagrams φ obtained by removing a subset of the south-east corners from θ. The dual Schubert classes $\mathcal{O}_\nu^*$ in the K-theory of $X = \mathrm{LG}(n, 2n)$ is defined by the same expression (9) as for $\mathrm{OG}(n, 2n+1)$. The identity $\chi_X(\mathcal{O}_\nu^* \cdot \mathcal{O}^\mu) = \delta_{\nu,\mu}$ and Proposition 5.3 imply the following.

Corollary 5.4. *Let $\lambda \subset \nu$ be strict partitions with $\nu_1 \leq n$ and let $0 \leq p \leq n$. Then $c^\nu_{\lambda,p} = \mathcal{C}(\nu/\lambda, p)$.*

The following result is our recursive Pieri formula for Lagrangian Grassmannians.

Theorem 5.5. *Let θ be a shifted skew diagram and let $p \in \mathbb{Z}$. If θ is not a rim then $\mathcal{C}(\theta, p) = 0$. If $p \leq 0$ then $\mathcal{C}(\theta, p) = \delta_{\theta,\emptyset}$, and $\mathcal{C}(\emptyset, p)$ is equal to one if $p \leq 0$ and is zero otherwise. If $\theta \neq \emptyset$ is a rim and $p > 0$, then $\mathcal{C}(\theta, p)$ is determined by the following rules, with $a = |\theta| - |\widehat{\theta}|$.*

(i) *If $\widehat{\theta} = \emptyset$ and θ meets the diagonal, then we have $\mathcal{C}(\theta, p) = \delta_{p,|\theta|} - \delta_{p,|\theta|-1}$ if θ is a column, and $\mathcal{C}(\theta, p) = \delta_{p,|\theta|}$ otherwise.*

(ii) *If $\widehat{\theta} = \emptyset$ and θ is disjoint from the diagonal, then $\mathcal{C}(\theta, p) = 2\,\delta_{p,|\theta|} - \delta_{p,|\theta|-1}$.*

(iii) *If $\widehat{\theta} \neq \emptyset$ and the north-east arm of θ is connected to $\widehat{\theta}$, then we have $\mathcal{C}(\theta, p) = \mathcal{C}(\widehat{\theta}, p-a) - \mathcal{C}(\widehat{\theta}, p-a+1)$.*

(iv) *Finally assume that $\widehat{\theta} \neq \emptyset$ and the north-east arm of θ is not connected to $\widehat{\theta}$. If $a = 1$ then $\mathcal{C}(\theta, p) = 2\,\mathcal{C}(\widehat{\theta}, p-a) - 2\,\mathcal{C}(\widehat{\theta}, p-a+1)$, while if $a > 1$ we have $\mathcal{C}(\theta, p) = 2\,\mathcal{C}(\widehat{\theta}, p-a) - 3\,\mathcal{C}(\widehat{\theta}, p-a+1) + \mathcal{C}(\widehat{\theta}, p-a+2)$.*

Proof. If θ is not a rim, then let $B \in \theta$ be a south-east corner such that θ contains a box strictly north and strictly west of B. For any shifted skew diagram φ with $\theta' \subset \varphi \subset \theta \smallsetminus \{B\}$ we obtain $h(N'(\varphi), d(\varphi) - p) = h(N'(\varphi \cup B), d(\varphi \cup B) - p)$ and $\mathcal{C}(\theta, p) = 0$. We also have $\mathcal{C}(\emptyset, p) = h(0, -p)$ which is equal to one when $p \leq 0$ and zero otherwise. If $\theta \neq \emptyset$ and $p \leq 0$, then $\mathcal{C}(\theta, p) = \sum_{\theta' \subset \varphi \subset \theta} (-1)^{|\theta| - |\varphi|} = 0$. If θ is a row or a column and $p > 0$, then $\mathcal{C}(\theta, p) = h(N'(\theta), |\theta| - p) - h(N'(\theta'), |\theta'| - p)$, from which (i) and (ii) are easily checked. We can therefore assume that θ is a rim, $\widehat{\theta} \neq \emptyset$, and $p > 0$. Let ψ be the north-east arm of θ, let $B \in \theta \smallsetminus \theta'$ be the south-east corner farthest to the north, and set $\psi' = \psi \smallsetminus B$ and $\widehat{\theta}' = \widehat{\theta} \cap \theta'$. We have $a = |\psi|$. Let φ be any rim such that $\widehat{\theta}' \subset \varphi \subset \widehat{\theta} \smallsetminus B$.

Assume that ψ is connected to $\widehat{\theta}$. If ψ is a is a row attached to the right side of the upper-right box of $\widehat{\theta}$, then we have

$$\begin{aligned}
&(-1)^{|\theta| - |\varphi \cup \psi|} \big(h(N^-(\varphi \cup \psi), |\varphi \cup \psi| - p) - h(N^-(\varphi \cup \psi'), |\varphi \cup \psi'| - p) \big) \\
&= (-1)^{|\widehat{\theta}| - |\varphi|} \big(h(N^-(\varphi), |\varphi| - p + a) - h(N^-(\varphi), |\varphi| - p + a - 1) \big)
\end{aligned}$$

which implies that $\mathcal{C}(\theta, p) = \mathcal{C}(\widehat{\theta}, p-a) - \mathcal{C}(\widehat{\theta}, p-a+1)$ by summing over φ. If ψ is a column attached above the upper-right box of $\widehat{\theta}$, then (8) implies that

$$\begin{aligned}
&(-1)^{|\theta| - |\varphi \cup \psi|} \big(h(N^-(\varphi \cup \psi), |\varphi \cup \psi| - p) - h(N^-(\varphi \cup B \cup \psi), |\varphi \cup B \cup \psi| - p) \big) \\
&= (-1)^{|\widehat{\theta}| - |\varphi|} \big(h(N^-(\varphi), |\varphi| - p + a) - h(N^-(\varphi \cup B), |\varphi \cup B| - p + a) \big) \\
&\qquad - h(N^-(\varphi), |\varphi| - p + a - 1) + h(N^-(\varphi \cup B), |\varphi \cup B| - p + a - 1)
\end{aligned}$$

which again implies that $\mathcal{C}(\theta, p) = \mathcal{C}(\widehat{\theta}, p-a) - \mathcal{C}(\widehat{\theta}, p-a+1)$, as required by (iii).

Now assume that ψ is not connected to $\widehat{\theta}$. If $a > 1$ then

$$\begin{aligned}
&(-1)^{|\theta| - |\varphi \cup \psi|} \big(h(N^-(\varphi \cup \psi), |\varphi \cup \psi| - p) - h(N^-(\varphi \cup \psi'), |\varphi \cup \psi'| - p) \big) \\
&= (-1)^{|\widehat{\theta}| - |\varphi|} (2\,h(N^-(\varphi), |\varphi| - p + a) - 3\,h(N^-(\varphi), |\varphi| - p + a - 1) \\
&\qquad + h(N^-(\varphi), |\varphi| - p + a - 2))
\end{aligned}$$

while implies that $\mathcal{C}(\theta, p) = 2\,\mathcal{C}(\widehat{\theta}, p-a) - 3\,\mathcal{C}(\widehat{\theta}, p-a+1) + \mathcal{C}(\widehat{\theta}, p-a+2)$. And if $a = 1$ then

$$\begin{aligned}
&(-1)^{|\theta| - |\varphi \cup \psi|} \big(h(N^-(\varphi \cup \psi), |\varphi \cup \psi| - p) - h(N^-(\varphi \cup \psi'), |\varphi \cup \psi'| - p) \big) \\
&= (-1)^{|\widehat{\theta}| - |\varphi|} \big(2\,h(N^-(\varphi), |\varphi| - p + 1) - 2\,h(N^-(\varphi), |\varphi| - p) \big)
\end{aligned}$$

implies that $\mathcal{C}(\theta, p) = 2\,\mathcal{C}(\widehat{\theta}, p-a) - 2\,\mathcal{C}(\widehat{\theta}, p-a+1)$. This establishes (iv) and completes the proof. ∎

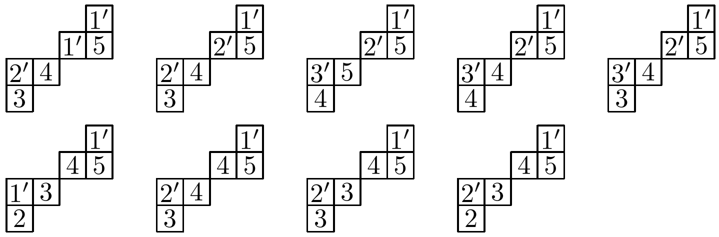
Let θ be a rim. A *KLG-tableau* of shape θ is a labeling of the boxes of θ with elements from the ordered set $\{1' < 1 < 2' < 2 < \cdots\}$ such that (i) each row of θ is strictly increasing from left to right; (ii) each column of θ is strictly increasing from top to bottom; (iii) each box containing an unprimed integer must be larger than or equal to all boxes southwest of it, (iv) each box containing a primed integer must be smaller than or equal to all boxes southwest of it, and (v) no diagonal box contains a primed integer. If θ is not a rim, then there are no KLG-tableaux of shape θ. The content of a KLG-tableau is the set of integers i such that some box contains i or i'. The proof of the following corollary is similar to Corollary 4.8 and left to the reader.

Corollary 5.6. *The constant $c^{\nu}_{\lambda,p}$ for $K(\mathrm{LG}(n,2n))$ is equal to $(-1)^{|\nu/\lambda|-p}$ times the number of KLG-tableaux of shape ν/λ with content $\{1,2,\dots,p\}$.*

It would be very interesting to extend this corollary to a full Littlewood-Richardson rule for all the structure constants $c^{\nu}_{\lambda\mu}$ of $K(\mathrm{LG}(n,2n))$.

Example 5.7. Let $X = \mathrm{LG}(2,4)$, $\lambda = \mu = (1)$, and $\nu = (2,1)$. Then $c^{\nu}_{\lambda\mu} = -1 \neq 0 = c^{\mu^\vee}_{\lambda,\nu^\vee}$.

Example 5.8. Let $\lambda = (6,4,1)$ and $\nu = (7,6,3,1)$. Then the constant $c^{\nu}_{\lambda,5} = -9$ for $K(\mathrm{LG}(7,14))$ is counted by the KLG-tableaux displayed below. Notice that the lower-left box of ν/λ is a diagonal box.



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Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

E-mail address: `asbuch@math.rutgers.edu`

E-mail address: `vravikum@math.rutgers.edu`