EIGENVALUES OF HERMITIAN MATRICES WITH POSITIVE SUM OF BOUNDED RANK

ANDERS SKOVSTED BUCH

Abstract. We give a minimal list of inequalities characterizing the possible eigenvalues of a set of Hermitian matrices with positive semidefinite sum of bounded rank. This answers a question of A. Barvinok.

1. Introduction

The combined work of A. Klyachko [8], A. Knutson, T. Tao [9] and C. Woodward [10], and P. Belkale [1] produced a minimal list of inequalities determining when three (weakly) decreasing n-tuples of real numbers can be the eigenvalues of Hermitian $n \times n$ matrices which add up to zero. The necessity of these inequalities had also been proved by S. Johnson [7] and U. Helmke and J. Rosenthal [6] (see also B. Totaro's paper [11]). We refer to [4] for a description of this work, as well as references to earlier work and applications to a surprising number of other mathematical disciplines.

S. Friedland applied these results to determine when three decreasing n-tuples of real numbers can be the eigenvalues of three Hermitian matrices with positive semidefinite sum, that is, the sum should have non-negative eigenvalues [2]. Friedland's answer included the inequalities of the above named authors, except that a trace equality was changed to an inequality. Friedland's result also needed some extra inequalities. W. Fulton has proved [5] that the extra inequalities are superfluous, and that the remaining ones form a minimal list, i.e. they correspond to the facets of the cone of permissible eigenvalues. All of these results have natural generalizations that work for any number of matrices [6, 4, 10].

In this paper we address the following more general question, which was formulated by A. Barvinok and passed along to us by Fulton. Given weakly decreasing n-tuples of real numbers $\alpha(1), \ldots, \alpha(m)$ and an integer $r \leq n$, when can one find Hermitian $n \times n$ matrices $A(1), \ldots, A(m)$ such that $\alpha(s)$ is the eigenvalues of $A(s)$ for each s and the sum $A(1) + \cdots + A(m)$ is positive semidefinite of rank at most r? The above described problems correspond to the extreme cases $r = 0$ and $r = n$.

Let $\alpha(1), \alpha(2), \ldots, \alpha(m)$ be n-tuples of reals, with $\alpha(s) = (\alpha_1(s), \ldots, \alpha_n(s))$. The requirement that these n-tuples should be decreasing is equivalent to the inequalities

$$(\dagger) \qquad \alpha_1(s) \geq \alpha_2(s) \geq \cdots \geq \alpha_n(s)$$

for all $1 \leq s \leq m$.

Date: February 15, 2006.

2000 *Mathematics Subject Classification.* Primary 15A42; Secondary 14M15, 05E15.

Key words and phrases. Hermitian; Eigenvalues; Littlewood-Richardson; Schubert calculus.

Given a set $I = \{a_1 < a_2 < \cdots < a_t\}$ of positive integers, we let $s_I = \det(h_{a_i-j})_{t\times t}$ be the Schur function for the partition $\lambda(I) = (a_t-t, \ldots, a_2-2, a_1-1)$. Here h_i denotes the complete symmetric function of degree i. Fulton's result [5] states that the n-tuples $\alpha(1), \ldots, \alpha(m)$ can be the eigenvalues of Hermitian matrices with positive semidefinite sum if and only if

$$(\triangleright_n) \qquad \sum_{s=1}^{m} \sum_{i\in I(s)} \alpha_i(s) \geq 0$$

for all sequences $(I(1), \ldots, I(m))$ of subsets of $[n] = \{1, 2, \ldots, n\}$ of the same cardinality t $(1 \leq t \leq n)$, such that the coefficient of $s_{\{n-t+1,n-t+2,\ldots,n\}}$ in the Schur expansion of the product $s_{I(1)}s_{I(2)}\cdots s_{I(m)}$ is equal to one. Notice that this coefficient is one if and only if the corresponding product of Schubert classes on the Grassmannian $\mathrm{Gr}(t, \mathbb{C}^n)$ equals a point class.

The added condition that the rank of the sum of matrices is at most r results in the additional inequalities

$$(\triangleleft_{n,r}) \qquad \sum_{s=1}^{m} \sum_{p\in P(s)} \alpha_{n+1-p}(s) \leq 0$$

for all sequences $(P(1), \ldots, P(m))$ of subsets of $[n-r]$ of the same cardinality t $(1 \leq t \leq n-r)$, such that $s_{\{n-r-t+1,\ldots,n-r\}}$ has coefficient one in the product $s_{P(1)}s_{P(2)}\cdots s_{P(m)}$. Equivalently, a product of Schubert classes on $\mathrm{Gr}(t, \mathbb{C}^{n-r})$ should be a point class. The necessity of the inequalities $(\triangleleft_{n,r})$ follows from $(\triangleright_n)$ applied to the identity $-A(1) - \cdots - A(m) + B = 0$, by noting that the $n-r$ smallest eigenvalues of the matrix $B = \sum A(i)$ are zero. We remark that without the requirement that a Hermitian matrix is positive semidefinite, rank conditions on the matrix do not correspond to linear inequalities in the eigenvalues. The following theorem is our main result.

Theorem 1. *Let $\alpha(1), \ldots, \alpha(m)$ be n-tuples of real numbers satisfying $(\dagger)$, and let $r \leq n$ be an integer. There exist Hermitian $n \times n$ matrices $A(1), \ldots, A(m)$ with eigenvalues $\alpha(1), \ldots, \alpha(m)$ such that the sum $A(1) + \cdots + A(m)$ is positive semidefinite of rank at most r, if and only if the inequalities $(\triangleright_n)$ and $(\triangleleft_{n,r})$ are satisfied. Furthermore, for $r \geq 1$ and $m \geq 3$ the inequalities $(\dagger)$, $(\triangleright_n)$, and $(\triangleleft_{n,r})$ are independent in the sense that they correspond to facets of the cone of admissible eigenvalues.*

As proved in [10], the minimal set of inequalities in the case $r = 0, m \geq 3$ consists of the inequalities $(\triangleright_n)$ for $t < n$, along with the trace equality $\sum_{s=1}^{m}\sum_{i=1}^{n} \alpha_i(s) = 0$ and, for $n > 2$, also the inequalities $(\dagger)$. The cases $r = 0, m \leq 2$, or $m = 1$ are not interesting. The situation for $m = 2$ and $r > 0$ is described by the following special cases of Weyl's inequalities [12] (see also [4, p. 3]).

Corollary 1. *Let $\alpha(1), \alpha(2)$ be n-tuples satisfying $(\dagger)$, and let $r \leq n$ be an integer. There exist Hermitian $n \times n$ matrices $A(1), A(2)$ with eigenvalues $\alpha(1), \alpha(2)$ such that the sum $A(1) + A(2)$ is positive semidefinite of rank at most r, if and only if $\alpha_i(1) + \alpha_j(2) \geq 0$ for $i + j = n + 1$ and $\alpha_i(1) + \alpha_j(2) \leq 0$ for $i + j = n + r + 1$. These inequalities are independent when $r \geq 1$; they imply $(\dagger)$ for $r = 1$, and are independent of $(\dagger)$ for $r \geq 2$.*

Proof. Given subsets $I, J \subset [n]$ of cardinality t, the coefficient of $s_{\{n-t+1,\ldots,n\}}$ in $s_I \cdot s_J$ is equal to one if and only if $J = \{n+1-i \mid i \in I\}$. This implies that the inequalities $(\triangleright_n)$ and $(\triangleleft_{n,r})$ are consequences of the inequalities of the corollary. The claims about independence of inequalities are left as an easy exercise. □

In the special case $r = 1$ of Corollary 1, the sum $A(1) + A(2)$ may be written as $\mathbf{x}\mathbf{x}^*$ for some (column) vector $\mathbf{x} \in \mathbb{C}^n$. Inspired by a question from the referee, we give an explicit description of the set of all vectors $\mathbf{x}$ that can appear in this way for fixed $\alpha(1)$ and $\alpha(2)$ satisfying the inequalities (see Proposition 1). It shows that this set is always a product of odd dimensional spheres.

Theorem 1 also has the following consequence. Although the statement does not use any inequalities, it appears to be non-trivial to prove without the use of inequalities.

Corollary 2. *Let $\alpha(1), \ldots, \alpha(m)$ be n-tuples of real numbers and let $r \leq n$. There exist Hermitian $n \times n$ matrices $A(1), \ldots, A(m)$ with these eigenvalues such that $A(1) + \cdots + A(m)$ is positive semidefinite of rank at most r, if and only if there are Hermitian $n \times n$ matrices with the same eigenvalues and positive semidefinite sum, as well as Hermitian $(n-r) \times (n-r)$ matrices $C(1), \ldots, C(m)$ with negative semidefinite sum, such that the eigenvalues of $C(s)$ are the $n-r$ smallest numbers from $\alpha(s)$.*

Proof. The inequalities $(\triangleleft_{n,r})$ for n-tuples $\alpha(1), \ldots, \alpha(m)$ are identical to the inequalities $(\triangleright_{n-r})$ for $\tilde{\alpha}(1), \ldots, \tilde{\alpha}(m)$, where $\tilde{\alpha}(s) = (-\alpha_n(s) \geq \cdots \geq -\alpha_{r+1}(s))$. □

Our proof of Theorem 1 is by induction on r, where we rely on the above mentioned results of Klyachko, Knutson, Tao, Woodward, and Belkale to cover the base case $r = 0$. To carry out the induction, we use an enhancement of Fulton's methods from [5]. We remark that Theorem 1 remains true if the Hermitian matrices are replaced with real symmetric matrices or even quaternionic Hermitian matrices. This follows because the results for zero-sum matrices hold in this generality [4, Thm. 20].

We thank Barvinok and Fulton for the communication of Barvinok's question, and Fulton for many helpful comments to our paper. We also thank the referee for inspiring comments and questions.

2. The inequalities are necessary and sufficient

In this section we prove that the inequalities of Theorem 1 are necessary and sufficient. For a subset $I = \{a_1 < a_2 < \cdots < a_t\}$ of $[n]$ of cardinality t, we let $\sigma_I \in H^* \operatorname{Gr}(t, \mathbb{C}^n)$ denote the Schubert class for the partition $\lambda(I) = (a_t - t, \ldots, a_1 - 1)$. The corresponding Schubert variety is the closure of the subset of points $V \in \operatorname{Gr}(t, \mathbb{C}^n)$ for which $V \cap \mathbb{C}^{n-a_i} \subsetneq V \cap \mathbb{C}^{n-a_i+1}$ for all $1 \leq i \leq t$. Let $S_t^n(m)$ denote the set of sequences $(I(1), \ldots, I(m))$ of subsets of $[n]$ of cardinality t, such that the product $\prod_{s=1}^n \sigma_{I(s)}$ is non-zero in $H^* \operatorname{Gr}(t, \mathbb{C}^n)$, and we let $R_t^n(m) \subset S_t^n(m)$ be the subset of sequences such that $\prod_{s=1}^n \sigma_{I(s)}$ equals the point class $\sigma_{\{n-t+1,\ldots,n-1,n\}}$.

The inequalities $(\triangleright_n)$ are indexed by all sequences $(I(1), \ldots, I(m))$ which belong to the set $R^n(m) = \bigcup_{1 \leq t \leq n} R_t^n(m)$. Furthermore, it is known [1, 10] that if $\alpha(1), \ldots, \alpha(m)$ are decreasing n-tuples of reals satisfying $(\triangleright_n)$, then they also satisfy the larger set of inequalities indexed by sequences from $S^n(m) = \bigcup_{1 \leq t \leq n} S_t^n(m)$,

that is $\sum_{s=1}^{m}\sum_{i\in I(s)}\alpha_i(s)\geq 0$ for all $(I(1),\ldots,I(m))\in S^n(m)$. Similarly, the inequalities of $(\lhd_{n,r})$ are indexed by $R^{n-r}(m)$, and if $\alpha(1),\ldots,\alpha(m)$ satisfy these inequalities, then we also have $\sum_{s=1}^{m}\sum_{p\in P(s)}\alpha_{n+1-p}(s)\leq 0$ for all sequences $(P(1),\ldots,P(m))\in S^{n-r}(m)$.

We first show that the inequalities $(\rhd_n)$ and $(\lhd_{n,r})$ are necessary. Suppose $A(1),\ldots,A(m)$ are Hermitian $n\times n$ matrices with eigenvalues $\alpha(1),\ldots,\alpha(m)$, such that the sum $B=A(1)+\cdots+A(m)$ is positive semidefinite with rank at most r. Let $\beta=(\beta_1\geq\cdots\geq\beta_r,0,\ldots,0)$ be the eigenvalues of B. For any sequence $(I(1),\ldots,I(m))\in R^n_t(m)$ we have that $(J,I(1),\ldots,I(m))$ is in $R^n_t(m+1)$ where $J=\{1,2,\ldots,t\}$. This is true because $\sigma_J\in H^*\operatorname{Gr}(t,\mathbb{C}^n)$ is the unit. Since $-B+A(1)+\cdots+A(m)=0$, it follows from [4, Thm. 11] that

$$-\sum_{j\in J}\beta_{n+1-j}+\sum_{s=1}^{m}\sum_{i\in I(s)}\alpha_i(s)\geq 0\,,$$

which implies $(\rhd_n)$ because each β_j is non-negative.

On the other hand, if $(P(1),\ldots,P(m))\in R^{n-r}_t(m)$, then $(Q,P(1),\ldots,P(m))\in R^n_t(m)$ where $Q=\{r+1,r+2,\ldots,r+t\}$. This follows from the Littlewood-Richardson rule, since $\lambda(Q)=(r)^t$ is a rectangular partition with t rows and r columns. Since $B-A(1)-\cdots-A(m)=0$, [4, Thm. 11] implies that

$$\sum_{q\in Q}\beta_q-\sum_{s=1}^{m}\sum_{p\in P(s)}\alpha_{n+1-p}(s)\geq 0\,.$$

Since $\beta_q=0$ for every $q\in Q$, this shows that $(\lhd_{n,r})$ is true.

If $I=\{i_1<i_2<\cdots<i_t\}$ is a subset of $[n]$ and P is a subset of $[t]$, we set $I_P=\{i_p\mid p\in P\}$. To prove that the inequalities are sufficient, we need the following generalization of [5, Prop. 1 (i)].

Lemma 1. *Let $(I(1),\ldots,I(m))\in S^n_t(m)$ and let $(P(1),\ldots,P(m))\in S^{t-r}_x(m)$, where $0\leq r\leq t$. Then $(I(1)_{P(1)},\ldots,I(m)_{P(m)})$ belongs to $S^{n-r}_x(m)$.*

Proof. The case $r=0$ of this Lemma is equivalent to part (i) of [5, Prop. 1]. We deduce the lemma from this case using straightforward consequences of the Littlewood-Richardson rule.

Set $Q=\{p+r\mid p\in P(1)\}$. Since $\lambda(Q)=(r)^x+\lambda(P(1))$, it follows that $\sigma_Q\cdot\prod_{s=2}^{m}\sigma_{P(s)}\neq 0$ on $\operatorname{Gr}(x,t)$. By the $r=0$ case, this implies that $\sigma_{I(1)_Q}\cdot\prod_{s=2}^{m}\sigma_{I(s)_{P(s)}}\neq 0$ on $\operatorname{Gr}(x,n)$. Now notice that if $P(1)=\{p_1<\cdots<p_x\}$ and $I(1)=\{i_1<\cdots<i_t\}$ then the jth element of $I(1)_Q$ is $i_{p_j+r}\geq i_{p_j}+r$, i.e. $\lambda(I(1)_Q)\supset(r)^x+\lambda(I(1)_{P(1)})$. This means that $\sigma_{(r)^x+\lambda(I(1)_{P(1)})}\cdot\prod_{s=2}^{m}\sigma_{I(s)_{P(s)}}$ is also non-zero on $\operatorname{Gr}(x,n)$, which implies that $\prod_{s=1}^{m}\sigma_{I(s)_{P(s)}}\neq 0$ on $\operatorname{Gr}(x,n-r)$. □

We also need the following special case of Corollary 1, which comes from reformulating the Pieri rule in terms of eigenvalues.

Lemma 2. *Let $\alpha=(\alpha_1\geq\cdots\geq\alpha_n)$ and $\gamma=(\gamma_1\geq\cdots\geq\gamma_n)$ be weakly decreasing sequences of real numbers. There exist Hermitian $n\times n$ matrices A and C with these eigenvalues such that $C-A$ is positive semidefinite of rank at most one, if and only if $\gamma_1\geq\alpha_1\geq\gamma_2\geq\alpha_2\geq\cdots\geq\gamma_n\geq\alpha_n$.*

Proof. Set $\beta = (\beta_1, 0, \ldots, 0)$ where $\beta_1 = \sum \gamma_i - \sum \alpha_i$, and assume that $\beta_1 \geq 0$. We must show that there are Hermitian matrices A, B, and C with eigenvalues α, β, and γ such that $A + B = C$ if and only if $\gamma_1 \geq \alpha_1 \geq \cdots \geq \gamma_n \geq \alpha_n$.

By approximating the eigenvalues with rational numbers and clearing denominators, we may assume that α, β, and γ are partitions. In this case it follows from the work of Klyachko [8] and Knutson and Tao [9] that the matrices A, B, C exist precisely when the Littlewood-Richardson coefficient $c_{\alpha\beta}^{\gamma}$ is non-zero (see [4, Thm. 11]). This is equivalent to the specified inequalities by the Pieri rule. □

The necessity of the inequalities of Lemma 2 also follows from Weyl's inequalities $\alpha_i(A) + \alpha_n(B) \leq \alpha_i(A + B)$ and $\alpha_i(A + B) \leq \alpha_{i-1}(A) + \alpha_2(B)$ with $B = C - A$, where $\alpha_i(A)$ denotes the ith eigenvalue of a Hermitian $n \times n$ matrix A [12]. The existence of the matrices A and C is equivalent to the existence of a (column) vector $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$ such that the matrix $D + \mathbf{x}\mathbf{x}^*$ has eigenvalues γ, where $D = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$. We will give an alternative proof that the inequalities are sufficient by explicitly solving this equation in $\mathbf{x}$ when $\gamma_1 \geq \alpha_1 \geq \cdots \geq \gamma_n \geq \alpha_n$.

Let $\widehat{\alpha} = (\widehat{\alpha}_1 \geq \cdots \geq \widehat{\alpha}_k)$ and $\widehat{\gamma} = (\widehat{\gamma}_1 \geq \cdots \geq \widehat{\gamma}_k)$ be the subsequences of α and γ obtained by removing as many equal pairs $\alpha_i = \gamma_j$ as possible. This implies that $\widehat{\gamma}_1 > \widehat{\alpha}_1 > \cdots > \widehat{\gamma}_k > \widehat{\alpha}_k$. For example, if $\alpha = (6, 5, 4, 4, 4, 3, 2, 2, 1)$ and $\gamma = (6, 6, 5, 4, 4, 3, 3, 2, 2)$, then $\widehat{\alpha} = (4, 1)$ and $\widehat{\gamma} = (6, 3)$. Now define real numbers $c_1, \ldots, c_k$ by

$$\begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} = \begin{bmatrix} \frac{1}{\widehat{\gamma}_1 - \widehat{\alpha}_1} & \cdots & \frac{1}{\widehat{\gamma}_1 - \widehat{\alpha}_k} \\ \vdots & & \vdots \\ \frac{1}{\widehat{\gamma}_k - \widehat{\alpha}_1} & \cdots & \frac{1}{\widehat{\gamma}_k - \widehat{\alpha}_k} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Notice that the matrix $\left[\frac{1}{\widehat{\gamma}_i - \widehat{\alpha}_j}\right]$ is invertible because its determinant is equal to $(\prod_{i,j}(\widehat{\gamma}_i - \widehat{\alpha}_j))^{-1}(\prod_{i<j}(\widehat{\alpha}_i - \widehat{\alpha}_j)(\widehat{\gamma}_j - \widehat{\gamma}_i))$. The following proposition is inspired by and answers a question from the referee, who suggested that exactly 2^n real solutions $\mathbf{x} \in \mathbb{R}^n$ exist when $\gamma_1 > \alpha_1 > \cdots > \gamma_n > \alpha_n$.

Proposition 1. *Assume that $\gamma_1 \geq \alpha_1 \geq \cdots \geq \gamma_n \geq \alpha_n$. Then each real number c_p is strictly positive. The matrix $D + \mathbf{x}\mathbf{x}^*$ has eigenvalues γ if and only if*

$$\sum_{j:\alpha_j = \widehat{\alpha}_p} |x_j|^2 = c_p$$

for each $1 \leq p \leq k$, and $x_j = 0$ whenever $\alpha_j \notin \{\widehat{\alpha}_1, \ldots, \widehat{\alpha}_k\}$.

Proof. The characteristic polynomial of the matrix $D + \mathbf{x}\mathbf{x}^*$ is given by $P(T) = \left(\prod_j (\alpha_j - T)\right)\left(1 + \sum_j \frac{|x_j|^2}{\alpha_j - T}\right)$. Suppose $\alpha_j \notin \{\widehat{\alpha}_p\}$ and let m be the number of occurrences of α_j in α. Since α_j occurs at least m times in γ, it must be a root of $P(T)$ of multiplicity at least m, which is possible only if $x_i = 0$ whenever $\alpha_i = \alpha_j$. It is enough to prove the proposition after removing all occurrences of α_j from α and equally many occurrences of α_j from γ. We may therefore assume that if an eigenvalue γ_i is also found in α, then α contains more copies of γ_i than γ.

It follows from the expression for $P(T)$ that the requirement that γ is the list of roots of $P(T)$ is equivalent to a system of linear equations in $|x_1|^2, \ldots, |x_n|^2$. If $\alpha_{p-1} > \alpha_p = \cdots = \alpha_q > \alpha_{q+1}$, then each of these equations has the same coefficient in front of $|x_p|^2, \ldots, |x_q|^2$, so this group of unknowns can be replaced with its sum. We do this explicitly by discarding $\alpha_{p+1}, \ldots, \alpha_q$ from α and $\gamma_{p+1}, \ldots, \gamma_q$ from γ,

which replaces $|x_p|^2 + \cdots + |x_q|^2$ with $|x_p|^2$ in the equations. This reduces to the situation where $\alpha = \widehat{\alpha}$ and $\gamma = \widehat{\gamma}$, in which case $D + \mathbf{x}\mathbf{x}^*$ has eigenvalues γ if and only if $|x_i|^2 = c_i$ for each i. It remains to show that $c_i > 0$.

We first note that this is true for at least one choice of eigenvalues γ. In fact, if $\mathbf{x} \in \mathbb{C}^n$ is any vector with non-zero coordinates and $\alpha_1 > \cdots > \alpha_n$, then the list γ of eigenvalues of the matrix $D + \mathbf{x}\mathbf{x}^*$ contains none of the numbers α_j. By Weyl's inequalities, we must therefore have $\gamma_1 > \alpha_1 > \cdots > \gamma_n > \alpha_n$, and the numbers c_j defined by γ are strictly positive because $c_j = |x_j|^2$. If some choice of eigenvalues γ with $\gamma_1 > \alpha_1 > \cdots > \gamma_n > \alpha_n$ results in non-positive real numbers c_j, then by continuity one may also choose γ such that $c_1, \ldots, c_n \geq 0$ and $c_j = 0$ for some j. But then for any vector $\mathbf{x}$ with $|x_i|^2 = c_i$ for each i, α_j is in the list of eigenvalues γ of the matrix $D + \mathbf{x}\mathbf{x}^*$, a contradiction. This shows that $c_j > 0$ for each j and finishes the proof. □

Finally, we need the following statement, which is equivalent to the Claim proved in [5, p. 30].

Lemma 3 (Fulton)**.** *Let $\alpha(1), \ldots, \alpha(m)$ be weakly decreasing n-tuples of real numbers which satisfy $(\triangleright_n)$. Suppose that for some sequence $(I(1), \ldots, I(m)) \in S_t^n(m)$ we have $\sum_{s=1}^m \sum_{i \in I(s)} \alpha_i(s) = 0$. For $1 \leq s \leq m$ we let $\alpha'(s)$ be the sequence of $\alpha_i(s)$ for $i \in I(s)$ and let $\alpha''(s)$ be the sequence of $\alpha_i(s)$ for $i \notin I(s)$, both in weakly decreasing order. Then $\{\alpha'(s)\}$ satisfy $(\triangleright_t)$ and $\{\alpha''(s)\}$ satisfy $(\triangleright_{n-t})$.*

We prove that the inequalities $(\triangleright_n)$ and $(\triangleleft_{n,r})$ are sufficient by a 'lexicographic' induction on (n, r). As the starting point we take the cases where $r = 0$, which are already known [8, 1, 10], [4, Thm. 17]. For the induction step we let $1 \leq r \leq n$ be given and assume that the inequalities are sufficient in all cases where n is smaller, as well as the cases with the same n and smaller r. Using this hypothesis, we start by proving the following fact. Given two decreasing n-tuples α and β, we write $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for all i.

Lemma 4. *Let β, γ, and $\alpha(2), \ldots, \alpha(m)$ be weakly decreasing n-tuples with $\beta \geq \gamma$, such that $\beta, \alpha(2), \ldots, \alpha(m)$ satisfy $(\triangleright_n)$ and $\gamma, \alpha(2), \ldots, \alpha(m)$ satisfy $(\triangleleft_{n,r})$. There exists a decreasing n-tuple $\alpha(1)$ such that $\beta \geq \alpha(1) \geq \gamma$ and $\alpha(1), \ldots, \alpha(m)$ satisfy both $(\triangleright_n)$ and $(\triangleleft_{n,r})$.*

Proof. We start by decreasing some entries of β in the following way. First decrease β_n until an inequality $(\triangleright_n)$ becomes an equality, or until $\beta_n = \gamma_n$. If the latter happens, then we continue by decreasing β_{n-1} until an inequality $(\triangleright_n)$ becomes an equality, or until $\beta_{n-1} = \gamma_{n-1}$. If the latter happens we continue by decreasing β_{n-2}, etc. If we are able to decrease all entries in β so that $\beta = \gamma$, then we can use $\alpha(1) = \gamma$.

Otherwise we may assume that for some sequence $(I(1), \ldots, I(m)) \in R_t^n(m)$ we have an equality $\sum_{i \in I(1)} \beta_i + \sum_{s=2}^m \sum_{i \in I(s)} \alpha_i(s) = 0$. For each $s \geq 2$ we let $\alpha'(s)$ be the decreasing t-tuple of numbers $\alpha_i(s)$ for $i \in I(s)$, and we let $\alpha''(s)$ be the decreasing $(n-t)$-tuple of numbers $\alpha_i(s)$ for $i \notin I(s)$. Similarly we define decreasing tuples $\beta' = (\beta_i)_{i \in I(1)}$, $\beta'' = (\beta_i)_{i \notin I(1)}$, and $\gamma'' = (\gamma_i)_{i \notin I(1)}$. By Lemma 3 we know that $\beta', \alpha'(2), \ldots, \alpha'(m)$ satisfy $(\triangleright_t)$ and that $\beta'', \alpha''(2), \ldots, \alpha''(m)$ satisfy $(\triangleright_{n-t})$. In particular, since the entries of the t-tuples add up to zero, we can find Hermitian $t \times t$ matrices $A'(1), \ldots, A'(m)$ with eigenvalues $\gamma', \alpha'(2), \ldots, \alpha'(m)$ such that $\sum A'(s) = 0$.

We claim that the $(n-t)$-tuples $\gamma'', \alpha''(2), \ldots, \alpha''(m)$ satisfy $(\triangleleft_{n-t,r})$. This is clear if $n-t \leq r$. Otherwise set $J(s) = \{n+1-i \mid i \notin I(s)\}$. Since $\lambda(J(s))$ is the conjugate partition of $\lambda(I(s))$, it follows that $(J(1), \ldots, J(m)) \in R^n_{n-t}(m)$. For any sequence $(P(1), \ldots, P(m)) \in R^{n-t-r}_x(m)$, we obtain from Lemma 1 that the sequence $(J(1)_{P(1)}, \ldots, J(m)_{P(m)})$ belongs to $S^{n-r}_x(m)$. Notice that if $J(s) = \{j_1 < j_2 < \cdots < j_{n-t}\}$, then $\alpha''_{n-t+1-p}(s) = \alpha_{n+1-j_p}(s)$. The claim therefore follows because

$$\sum_{p\in P(1)} \gamma''_{n-t+1-p} + \sum_{s=2}^{m} \sum_{p\in P(s)} \alpha''_{n-t+1-p}(s) \;=\; \sum_{j\in J(1)_{P(1)}} \gamma_{n+1-j} + \sum_{s=2}^{m} \sum_{j\in J(s)_{P(s)}} \alpha_{n+1-j}(s) \;\leq\; 0\,.$$

By induction on n there exists a decreasing $(n-t)$-tuple $\alpha''(1)$ such that $\beta'' \geq \alpha''(1) \geq \gamma''$ and $\alpha''(1), \ldots, \alpha''(m)$ satisfy both of $(\triangleright_{n-t})$ and $(\triangleleft_{n-t,r})$. By the cases of Theorem 1 that we assume are true by induction, we can find Hermitian $(n-t)\times(n-t)$ matrices $A''(1), \ldots, A''(m)$ with eigenvalues $\alpha''(1), \ldots, \alpha''(m)$ and with positive semidefinite sum of rank at most r. We can finally take $\alpha(1)$ to be the eigenvalues of $A'(1) \oplus A''(1)$. □

We can now finish the proof that the inequalities of Theorem 1 are sufficient. Let $\gamma = (\alpha_2(1), \alpha_3(1), \ldots, \alpha_n(1), M)$ for some large negative number $M \ll 0$. We claim that when M is sufficiently small, the n-tuples $\gamma, \alpha(2), \ldots, \alpha(m)$ satisfy $(\triangleleft_{n,r-1})$. In fact, let $(P(1), \ldots, P(m)) \in R^{n-r+1}_t(m)$. If $1 \in P(1)$ then the inequality for this sequence holds by choice of M. Otherwise we have that $(Q, P(2), \ldots, P(m)) \in R^{n-r}_t(m)$ where $Q = \{p-1 \mid p \in P(1)\}$, and the required inequality follows because

$$\sum_{q\in Q} \alpha_{n+1-q}(1) + \sum_{s=2}^{m} \sum_{p\in P(s)} \alpha_{n+1-p}(s) \;\leq\; 0\,.$$

By Lemma 4 we may now find a decreasing n-tuple $\tilde{\alpha}(1)$ with $\alpha(1) \geq \tilde{\alpha}(1) \geq \gamma$, such that $\tilde{\alpha}(1), \alpha(2), \ldots, \alpha(m)$ satisfy $(\triangleright_n)$ and $(\triangleleft_{n,r-1})$. By induction on r there exist Hermitian $n\times n$ matrices $\tilde{A}(1), A(2), \ldots, A(m)$ with eigenvalues $\tilde{\alpha}(1)$, $\alpha(2), \ldots, \alpha(m)$, such that $\tilde{A}(1) + A(2) + \cdots + A(m)$ is positive semidefinite of rank at most $r-1$. Finally, using Lemma 2 and the choice of γ we may find a Hermitian matrix $A(1)$ with eigenvalues $\alpha(1)$ such that $A(1) - \tilde{A}(1)$ is positive semidefinite of rank at most 1. The matrices $A(1), A(2), \ldots, A(m)$ now satisfy the requirements.

3. Minimality of the inequalities

In this section we prove that when $r \geq 1$ and $m \geq 3$, the inequalities (†), $(\triangleright_n)$, and $(\triangleleft_{n,r})$ are independent, thereby proving the last statement of Theorem 1. It is enough to show that for each inequality among $(\triangleright_n)$ or $(\triangleleft_{n,r})$, there exist strictly decreasing n-tuples $\alpha(1), \ldots, \alpha(m)$ such that the given inequality is an equality and all other inequalities $(\triangleright_n)$ and $(\triangleleft_{n,r})$ are strict. In addition we must show that for each $1 \leq i \leq n-1$ there exist $\alpha(1) = (\alpha_1(1) > \cdots > \alpha_i(1) = \alpha_{i+1}(1) > \cdots > \alpha_n(1))$ and strictly decreasing n-tuples $\alpha(2), \ldots, \alpha(m)$, such that all inequalities $(\triangleright_n)$ and $(\triangleleft_{n,r})$ are strict.

We start with the latter case. If $n = 2$ we can take $\alpha(1) = (0, 0)$ and $\alpha(s) = (2, -1)$ for $2 \leq s \leq m$. For $n \geq 3$, it was shown in [3, Lemma 1] that the n-tuples $\beta(1) = \beta(2) = \cdots = \beta(m) = (n-1, n-3, \ldots, 3-n, 1-n)$ satisfy that $\sum_{s=1}^{m} \sum_{i \in I(s)} \beta_i(s) \geq 2$ for all sequences $(I(1), \ldots, I(m)) \in R_t^n(m)$ of subsets of cardinality $t < n$. In fact, this follows because $\sum_{s=1}^{m} \sum_{i \in I(s)} i = t(n-t) + m\binom{t+1}{2}$. Using this fact, one easily checks that both $(\triangleright_n)$ and $(\triangleleft_{n,r})$ are strict for $\alpha(1) = (n-1, n-3, \ldots, n-2i, n-2i, \ldots, 3-n, 1-n)$, with $n-2i$ as the ith and $i+1$st entries, and $\alpha(2) = \cdots = \alpha(m) = (n, n-3, n-5, \ldots, 3-n, 1-n)$.

Now consider an inequality from $(\triangleright_n)$, given by a sequence $(I(1), \ldots, I(m)) \in R_t^n(m)$. By [10, Thm. 9] we can choose strictly decreasing n-tuples $\alpha(1), \ldots, \alpha(m)$ such that $\sum_{s=1}^{m} \sum_{i=1}^{n} \alpha_i(s) = \sum_{s=1}^{m} \sum_{i \in I(s)} \alpha_i(s) = 0$ and all other inequalities $(\triangleright_n)$ are strict. If $(P(1), \ldots, P(m)) \in R_x^{n-r}(m)$ then we have $(Q, P(2), \ldots, P(m)) \in R_x^n(m)$ where $Q = \{p + r \mid p \in P(1)\}$. Since the negated n-tuples $\tilde{\alpha}(1), \ldots, \tilde{\alpha}(m)$ given by $\tilde{\alpha}(s) = (-\alpha_n(s) > \cdots > -\alpha_1(s))$ must satisfy $(\triangleright_n)$, we obtain that $\sum_{s=1}^{m} \sum_{p \in P(s)} \alpha_{n+1-p}(s) < \sum_{q \in Q} \alpha_{n+1-q}(1) + \sum_{s=2}^{m} \sum_{p \in P(s)} \alpha_{n+1-p}(s) \leq 0$. This shows that the inequalities $(\triangleleft_{n,r})$ are strict. If $t < n$ we may finally replace $\alpha_{i_0}(1)$ with $\alpha_{i_0}(1) + \epsilon$, where $i_0 \notin I(1)$, to obtain that $\sum_{s=1}^{m} \sum_{i=1}^{n} \alpha_i(s) > 0$.

At last we consider an inequality of $(\triangleleft_{n,r})$ given by a sequence $(P(1), \ldots, P(m)) \in R_x^{n-r}(m)$. We once more apply [10, Thm. 9] to obtain strictly decreasing $(n-r)$-tuples $\beta(1), \ldots, \beta(m)$ such that $\sum_{s=1}^{m} \sum_{p=1}^{n-r} \beta_p(s) = \sum_{s=1}^{m} \sum_{p \in P(s)} \beta_p(s) = 0$, and all other inequalities of $(\triangleright_{n-r})$ are strict. Set $\alpha(s) = (N + r, N + r - 1, \ldots, N + 1, -\beta_{n-r}(s), \ldots, -\beta_1(s))$ for $1 \leq s \leq m$, where $N \gg 0$ is a large number. Then the n-tuples $\alpha(1), \ldots, \alpha(m)$ strictly satisfy all inequalities from $(\triangleleft_{n,r})$, except for the equalities $\sum_{s=1}^{m} \sum_{p=1}^{n-r} \alpha_{n+1-p}(s) = \sum_{s=1}^{m} \sum_{p \in P(s)} \alpha_{n+1-p}(s) = 0$. We must show that $\sum_{s=1}^{m} \sum_{i \in I(s)} \alpha_i(s) > 0$ for every sequence $(I(1), \ldots, I(m)) \in R_t^n(m)$. If $I(1) \cap [r] \neq \emptyset$ then this follows from our choice of N. Otherwise we have $(J, I(2), \ldots, I(m)) \in R_t^{n-r}(m)$ where $J = \{i - r \mid i \in I(1)\}$. Since $\alpha_i(s) > -\beta_{n-r+1-i}(s)$ for $i \in [n-r]$, we obtain that $\sum_{s=1}^{m} \sum_{i \in I(s)} \alpha_i(s) > \sum_{i \in J} (-\beta_{n-r+1-i}(1)) + \sum_{s=2}^{m} \sum_{i \in I(s)} (-\beta_{n-r+1-i}(s)) \geq 0$. Finally, if $x \neq n-r$ we replace $\alpha_{n+1-p_0}(1)$ with $\alpha_{n+1-p_0}(1) - \epsilon$, $p_0 \notin P(1)$, to obtain a strict inequality $\sum_{s=1}^{m} \sum_{p=1}^{n-r} \alpha_{n+1-p}(s) < 0$. This completes the proof that the inequalities are independent.

References

[1] P. Belkale, *Local systems on $\mathbb{P}^1 \smallsetminus s$ for s a finite set*, Ph.D. thesis, University of Chicago, 1999.

[2] S. Friedland, *Finite and infinite dimensional generalizations of Klyachko's theorem*, Linear Algebra Appl. **319** (2000), no. 1-3, 3–22. MR 2002a:15023

[3] W. Fulton, *Eigenvalues of sums of Hermitian matrices (after A. Klyachko)*, Astérisque (1998), no. 252, Exp. No. 845, 5, 255–269, Séminaire Bourbaki. Vol. 1997/98. MR 2000e:14092

[4] ———, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. (N.S.) **37** (2000), no. 3, 209–249 (electronic). MR 2001g:15023

[5] ———, *Eigenvalues of majorized Hermitian matrices and Littlewood-Richardson coefficients*, Linear Algebra Appl. **319** (2000), no. 1-3, 23–36. MR 2002a:15024

[6] U. Helmke and J. Rosenthal, *Eigenvalue inequalities and Schubert calculus*, Math. Nachr. **171** (1995), 207–225. MR MR1316359 (96b:15039)

[7] S. Johnson, *The schubert calculus and eigenvalue inequalities for sums of hermitian matrices*, Ph.D. thesis, University of California, Santa Barbara, 1979.

[8] A. A. Klyachko, *Stable bundles, representation theory and Hermitian operators*, Selecta Math. (N.S.) **4** (1998), no. 3, 419–445. MR 2000b:14054

[9] A. Knutson and T. Tao, *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products. I. Proof of the saturation conjecture*, J. Amer. Math. Soc. **12** (1999), no. 4, 1055–1090. MR 2000c:20066

[10] A. Knutson, T. Tao, and C. Woodward, *The honeycomb model of* $\mathrm{GL}_n(\mathbb{C})$ *tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone*, J. Amer. Math. Soc. **17** (2004), no. 1, 19–48 (electronic). MR 2 015 329

[11] B. Totaro, *Tensor products in p-adic Hodge theory*, Duke Math. J. **83** (1996), no. 1, 79–104. MR MR1388844 (97d:14032)

[12] H. Weyl, *Das asymptotische Verteilungsgesetz der Eigenwerte lineare partieller Differentialgleichungen*, Math. Ann. **71** (1912), 441–479.

Department of Mathematics, Rutgers University, 110 Frelinghuysen Road, Piscataway, NJ 08854, USA

E-mail address: asbuch@math.rutgers.edu