GROTHENDIECK POLYNOMIALS AND QUIVER FORMULAS

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Abstract. Fulton's universal Schubert polynomials give cohomology formulas for a class of degeneracy loci, which generalize Schubert varieties. The K-theoretic quiver formula of Buch expresses the structure sheaves of these loci as integral linear combinations of products of stable Grothendieck polynomials. We prove an explicit combinatorial formula for the coefficients, which shows that they have alternating signs. Our result is applied to obtain new expansions for the Grothendieck polynomials of Lascoux and Schützenberger.

1. Introduction and main results

Let $\mathfrak{X}$ be a smooth complex algebraic variety and let

$$E_1 \to \cdots \to E_{n-1} \to E_n \to F_n \to F_{n-1} \to \cdots \to F_1 \tag{1}$$

be a sequence of vector bundles and morphisms over $\mathfrak{X}$, such that $\operatorname{rank}(F_i) = \operatorname{rank}(E_i) = i$ for $1 \leq i \leq n$. For any permutation $w \in S_{n+1}$, there is a degeneracy locus

$$\Omega_w(E_\bullet \to F_\bullet) = \{x \in \mathfrak{X} \mid \operatorname{rank}(E_q(x) \to F_p(x)) \leq r_w(p,q),\ \forall\ 1 \leq p,q \leq n\}\,, \tag{2}$$

where $r_w(p,q)$ is the number of $i \leq p$ such that $w(i) \leq q$. We will assume that the bundle maps are sufficiently general so that this degeneracy locus has the expected codimension, equal to the length $\ell(w)$. In this situation, Fulton [10] gave a formula for the cohomology class of $\Omega_w = \Omega_w(E_\bullet \to F_\bullet)$ in $H^*(\mathfrak{X},\mathbb{Z})$ as a *universal Schubert polynomial* in the Chern classes of the vector bundles involved.

While the cohomology class of Ω_w gives useful global information, there is even more data hidden in its structure sheaf $\mathcal{O}_{\Omega_w}$. The main result of this paper gives an explicit combinatorial formula for the class $[\mathcal{O}_{\Omega_w}]$ of this structure sheaf in the Grothendieck ring $K(\mathfrak{X})$ of algebraic vector bundles on $\mathfrak{X}$. To state it, we need the *degenerate Hecke algebra*, which is the associative $\mathbb{Z}$-algebra generated by s_i for $i = 1, 2, \ldots$ with relations

$$\begin{aligned} s_i^2 &= s_i \\ s_i s_j &= s_j s_i \qquad \text{for } |i-j| > 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}\,. \end{aligned}$$

We also require the stable Grothendieck polynomials $G_u(E - E')$, where u is a permutation and E, E' are vector bundles over $\mathfrak{X}$ (see Section 2 for the definition).

Theorem 1. *For $w \in S_{n+1}$ we have*

$$[\mathcal{O}_{\Omega_w}] = \sum (-1)^{\ell(u_1 \cdots u_{2n-1} w)}\, G_{u_1}(E_2 - E_1) \cdots G_{u_n}(F_n - E_n) \cdots G_{u_{2n-1}}(F_1 - F_2)$$

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in $K(\mathfrak{X})$, where the sum is over all factorizations $w = u_1 \cdots u_{2n-1}$ in the degenerate Hecke algebra such that $u_i \in S_{\min(i,2n-i)+1}$ for each i.

The above formula corresponds to computing the alternating sum of a locally free resolution of $\mathcal{O}_{\Omega_w}$ in $K(\mathfrak{X})$, and thus includes a formula for the cohomology class of Ω_w as its leading term. Theorem 1 is therefore a generalization of [7, Thm. 3].

The locus $\Omega_w(E_\bullet \to F_\bullet)$ is a special case of a *quiver variety*. In [3] a formula for the class of the structure sheaf of a general quiver variety is proved, which expresses this class as a linear combination of products of stable Grothendieck polynomials for Grassmannian permutations. Furthermore, it is conjectured that the *quiver coefficients* occurring in this formula have signs which alternate with the codimension.

The quiver formula specializes to *universal Grothendieck polynomials* $\mathcal{G}_w(F_\bullet; E_\bullet)$ in the exterior powers of the bundles (and the inverse of the top powers), which are K-theoretic analogues of universal Schubert polynomials. Given any partition α, we let $G_\alpha = G_{w_\alpha}$ denote the stable Grothendieck polynomial for the Grassmannian permutation w_α corresponding to α. Then the quiver formula has the form

$$[\mathcal{O}_{\Omega_w}] = \mathcal{G}_w(F_\bullet; E_\bullet) = \sum_\lambda c^{(n)}_{w,\lambda}\, G_{\lambda^1}(E_2 - E_1) \cdots G_{\lambda_n}(F_n - E_n) \cdots G_{\lambda^{2n-1}}(F_1 - F_2)$$

where the sum is over finitely many sequences of partitions $\lambda = (\lambda^1, \ldots, \lambda^{2n-1})$ and the $c^{(n)}_{w,\lambda}$ are quiver coefficients.

Theorem 1 combined with a result of Lascoux [15] proves that these coefficients do in fact have alternating signs. Define integers $a_{w,\beta}$ such that $G_w = \sum a_{w,\beta} G_\beta$, the sum over all partitions β. Lascoux has shown that $a_{w,\beta}$ is equal to $(-1)^{|\beta|-\ell(w)}$ times the number of paths from w to w_β in a graph of permutations. Given this result, Theorem 1 is equivalent to the following explicit combinatorial formula for quiver coefficients:

$$c^{(n)}_{w,\lambda} = (-1)^{|\lambda|-\ell(w)} \sum_{u_1 \cdots u_{2n-1} = w} \left| a_{u_1,\lambda^1}\, a_{u_2,\lambda^2} \cdots a_{u_{2n-1},\lambda^{2n-1}} \right| .$$

Our proof of Theorem 1 is based on a special case of this formula, proved in [3], together with the following Cauchy identity, which provides a K-theoretic generalization of [12, Cor. 2] (see also [10, Thm. 3.7]).

Theorem 2 (Cauchy formula)**.** *Let E_i, F_i, and H_i for $i = 1, \ldots, n$ be three collections of vector bundles on $\mathfrak{X}$. Then for any $w \in S_{n+1}$, we have*

$$\mathcal{G}_w(F_\bullet; E_\bullet) = \sum_{u \cdot v = w} (-1)^{\ell(uvw)}\, \mathcal{G}_u(H_\bullet; E_\bullet)\, \mathcal{G}_v(F_\bullet; H_\bullet)$$

where the sum is over all permutations u, v such that the product of u and v is equal to w in the degenerate Hecke algebra.

As a further consequence of our results, we obtain new formulas for the double Grothendieck polynomials of Lascoux and Schützenberger [17], which express these polynomials as linear combinations of stable Grothendieck polynomials in disjoint intervals of variables. The coefficients in these expansions are all quiver coefficients; in particular, this is true for the monomial coefficients of Grothendieck polynomials.

Very recently, the first author has shown [5] that general quiver coefficients also have alternating signs, with an approach based on Knutson, Miller, and Shimozono's

work [13]. In fact, the results of [5] show that general quiver coefficients can be realized as special cases of the coefficients studied in the present paper.

This paper is organized as follows. We review the facts about Grothendieck polynomials that we require in Section 2. The quiver varieties and universal Grothendieck polynomials are introduced in Section 3. We prove the Cauchy formula in Section 4, while our main theorem is proved in the following section. Finally in Section 6 we apply our results to obtain splitting formulas for double Grothendieck polynomials.

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2. Grothendieck polynomials

We begin by recalling the definition of Lascoux and Schützenberger's double Grothendieck polynomials [17]. Let $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ be two sequences of commuting independent variables and $w \in S_n$. If $w = w_0$ is the longest permutation in S_n, then we set

$$\mathfrak{G}_{w_0}(X;Y) = \prod_{i+j\leq n} (x_i + y_j - x_i y_j)\,.$$

If $w \neq w_0$, we can find a simple transposition $s_i = (i, i+1) \in S_n$ such that $\ell(ws_i) = \ell(w) + 1$. We then define

$$\mathfrak{G}_w = \pi_i(\mathfrak{G}_{ws_i})$$

where π_i is the isobaric divided difference operator given by

$$\pi_i(f) = \frac{(1-x_{i+1})f(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) - (1-x_i)f(x_1,\ldots,x_{i+1},x_i,\ldots,x_n)}{x_i - x_{i+1}}\,.$$

Given permutations $u_1, \ldots, u_m$, and w, we will write $u_1 \cdots u_m = w$ if the product of the u_i is equal to w in the degenerate Hecke algebra. With this notation, the Grothendieck polynomials satisfy the following Cauchy identity, which is due to Fomin and Kirillov (see [9, Thm. 8.1] and [8]):

$$\mathfrak{G}_w(X;Y) = \sum_{u\cdot v = w} (-1)^{\ell(uvw)}\, \mathfrak{G}_u(0;Y)\, \mathfrak{G}_v(X;0)\,. \tag{3}$$

Next we recall the definition of stable Grothendieck polynomials. Given a permutation $w \in S_n$, and a nonnegative integer m, let $1^m \times w \in S_{m+n}$ denote the shifted permutation which is the identity on $\{1, 2, \ldots, m\}$ and which maps j to $w(j-m)+m$ for $j > m$. It is known [8, 9] that when m grows to infinity, the coefficient of each fixed monomial in $\mathfrak{G}_{1^m\times w}$ eventually becomes stable. The double stable Grothendieck polynomial $G_w \in \mathbb{Z}[[X;Y]]$ is the resulting power series:

$$G_w = G_w(X;Y) = \lim_{m\to\infty} \mathfrak{G}_{1^m\times w}(X;Y)\,.$$

The power series $G_w(X;Y)$ is symmetric in the X and Y variables separately, and

$$G_w(1-e^{-X};1-e^{Y}) = G_w(1-e^{-x_1},1-e^{-x_2},\ldots;1-e^{y_1},1-e^{y_2},\ldots)$$

is super-symmetric, that is, if one sets $x_1 = y_1$ in this expression, then the result is independent of x_1 and y_1.

In particular, we will need stable Grothendieck polynomials for Grassmannian permutations. If $\alpha = (\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots)$ is a partition and $p \geq \ell(\alpha)$, i.e.,

$\alpha_{p+1} = 0$, the Grassmannian permutation for α with descent in position p is the permutation w_α such that $w_\alpha(i) = i + \alpha_{p+1-i}$ for $1 \leq i \leq p$ and $w_\alpha(i) < w_\alpha(i+1)$ for $i \neq p$. Now let $G_\alpha = G_{w_\alpha}$; this is independent of the choice of p. According to [4, Thm. 6.6] there are integers $d^\alpha_{\beta\gamma}$ (with alternating signs) such that

$$G_\alpha(X;Y) = \sum d^\alpha_{\beta\gamma}\, G_\beta(X;0)\, G_\gamma(0;Y)\,. \tag{4}$$

Let $\Gamma \subseteq \mathbb{Z}[[X;Y]]$ be the linear span of all stable Grothendieck polynomials. It is shown in [4] that Γ is closed under multiplication and that the elements G_α form a $\mathbb{Z}$-basis of Γ. In fact Γ is a commutative and cocommutative bialgebra with the coproduct $\Delta : \Gamma \to \Gamma \otimes \Gamma$ given by $\Delta G_\alpha = \sum_{\beta,\gamma} d^\alpha_{\beta\gamma}\, G_\beta \otimes G_\gamma$.

We will describe a formula of Lascoux for the expansion of a stable Grothendieck polynomial G_w as a linear combination of these elements. Let r be the last descent position of w, i.e. r is maximal such that $w(r) > w(r+1)$. Set $w' = w\tau_{rk}$ where $k > r$ is maximal such that $w(r) > w(k)$. We also set $I(w) = \{i < r \mid \ell(w'\tau_{ir}) = \ell(w)\}$.

Define a relation $\rhd$ on the set of all permutations as follows. If $I(w) = \emptyset$ we write $w \rhd v$ if and only if $v = 1 \times w$. Otherwise we write $w \rhd v$ if and only if there exist elements $i_1 < \cdots < i_p$ of $I(w)$, $p \geq 1$, such that $v = w'\tau_{i_1 r} \ldots \tau_{i_p r}$. The following is an immediate consequence of [15, Thm. 4].

Theorem 3 (Lascoux). *For any permutation w we have*

$$G_w = \sum_\beta a_{w,\beta}\, G_\beta$$

where the sum is over all partitions β, and $a_{w,\beta}$ is equal to $(-1)^{|\beta|-\ell(w)}$ times the number of sequences $w = w_1 \rhd w_2 \rhd \cdots \rhd w_m$ such that $w_m = w_\beta$ is a Grassmannian permutation for β and w_i is not Grassmannian for $i < m$.

Let w be any permutation and let $F = L_1 \oplus \cdots \oplus L_f$ and $E = M_1 \oplus \cdots \oplus M_e$ be vector bundles on $\mathfrak{X}$ which are both direct sums of line bundles. Buch [4] defines

$$G_w(F - E) = G_w(1 - L_1^{-1}, \ldots, 1 - L_f^{-1}; 1 - M_1, \ldots, 1 - M_e) \in K(\mathfrak{X})\,.$$

Since G_w is symmetric, this definition extends to the case where E and F do not split as direct sums. Alternatively, using the identity $\wedge^i(F^\vee) = (\wedge^{f-i}F)/(\wedge^f F)$, we may write $G_w(F-E)$ as a Laurent polynomial in the exterior powers of E and F, where only the top power of F is inverted. As $G_w(1 - e^{-X}; 1 - e^Y)$ is super-symmetric we have $G_w(F \oplus H - E \oplus H) = G_w(F - E)$ for any third vector bundle H on $\mathfrak{X}$. Finally, notice that

$$G_\alpha(F - E) = \sum_{\beta,\gamma} d^\alpha_{\beta\gamma}\, G_\beta(F - H)\, G_\gamma(H - E)\,, \tag{5}$$

which follows from (4) together with the super-symmetry property.

3. Universal Grothendieck polynomials

Consider a sequence

$$B_\bullet \;:\; B_1 \to B_2 \to \cdots \to B_n$$

of vector bundles and bundle maps over a non-singular variety $\mathfrak{X}$. Given rank conditions $r = \{r_{ij}\}$ for $1 \leq i < j \leq n$ there is a *quiver variety* given by

$$\Omega_r(B_\bullet) = \{x \in \mathfrak{X} \mid \operatorname{rank}(B_i(x) \to B_j(x)) \leq r_{ij}\ \forall i < j\}\,.$$

For convenience, we set $r_{ii} = \mathrm{rank}(B_i)$ for all i, and we demand that the rank conditions satisfy $r_{ij} \geq \max(r_{i-1,j}, r_{i,j+1})$ and $r_{ij} + r_{i-1,j+1} \geq r_{i-1,j} + r_{i,j+1}$ for all $i \leq j$. In this case, the expected codimension of $\Omega_r(B_\bullet)$ is the number $d(r) = \sum_{i<j}(r_{i,j-1} - r_{ij})(r_{i+1,j} - r_{ij})$. The main result of [3] states that when the quiver variety $\Omega_r(B_\bullet)$ has this codimension, the class of its structure sheaf is given by the formula

$$[\mathcal{O}_{\Omega_r(B_\bullet)}] = \sum_\lambda c_\lambda(r)\, G_{\lambda^1}(B_2 - B_1) \cdots G_{\lambda^{n-1}}(B_n - B_{n-1})\,. \tag{6}$$

Here the sum is over finitely many sequences of partitions $\lambda = (\lambda^1, \ldots, \lambda^{n-1})$ such that $|\lambda| = \sum |\lambda^i|$ is greater than or equal to $d(r)$. The coefficients $c_\lambda(r)$ are integers called *quiver coefficients*; they can be computed by a combinatorial algorithm which we will not reproduce here. These coefficients are uniquely determined by the condition that (6) is true for all varieties $\mathfrak{X}$ and sequences $B_\bullet$, as well as the condition that $c_\lambda(r) = c_\lambda(r')$, where $r' = \{r'_{ij}\}$ is the set of rank conditions given by $r'_{ij} = r_{ij} + 1$ for all $i \leq j$. Buch has conjectured that the signs of these coefficients alternate with codimension, that is, $(-1)^{|\lambda|-d(r)} c_\lambda(r) \geq 0$.

Suppose the index p is such that all rank conditions $\mathrm{rank}(B_i(x) \to B_p(x)) \leq r_{ip}$ and $\mathrm{rank}(B_p(x) \to B_j(x)) \leq r_{pj}$ may be deduced from other rank conditions. As in [6, §4], we will then say that the bundle B_p is *inessential*. Omitting an inessential bundle B_p from $B_\bullet$ produces a sequence

$$B'_\bullet \;:\; B_1 \to \cdots \to B_{p-1} \to B_{p+1} \to \cdots \to B_n\,,$$

where the map from B_{p-1} to B_{p+1} is the composition $B_{p-1} \to B_p \to B_{p+1}$. If r' denotes the restriction of the rank conditions to $B'_\bullet$, we have that $\Omega_{r'}(B'_\bullet) = \Omega_r(B_\bullet)$. We can use (5) to expand any factor $G_\alpha(B_{p+1} - B_{p-1})$ occurring in the quiver formula for $\Omega_{r'}(B'_\bullet)$ into a linear combination of products of the form $G_\beta(B_p - B_{p-1})G_\gamma(B_{p+1} - B_p)$, and thus arrive at the quiver formula (6) for $\Omega_r(B_\bullet)$.

The loci $\Omega_w(E_\bullet \to F_\bullet)$ of (2) are special cases of these quiver varieties. Given $w \in S_{n+1}$ we define rank conditions $r^{(n)} = \{r^{(n)}_{ij}\}$ for $1 \leq i \leq j \leq 2n$ by

$$r^{(n)}_{ij} = \begin{cases} r_w(2n+1-j, i) & \text{if } i \leq n < j \\ i & \text{if } j \leq n \\ 2n+1-j & \text{if } i \geq n+1. \end{cases}$$

Then $\Omega_w(E_\bullet \to F_\bullet)$ is identical to the quiver variety $\Omega_{r^{(n)}}(E_\bullet \to F_\bullet)$, and moreover we have $d(r) = \ell(w)$. We let $c^{(n)}_{w,\lambda} = c_\lambda(r^{(n)})$ denote the quiver coefficients corresponding to this locus.

Given vector bundles $E_1, \ldots, E_n$ and $F_1, \ldots, F_n$ on $\mathfrak{X}$ we define

$$\mathcal{G}^{(n)}_w(F_\bullet; E_\bullet) = \sum_\lambda c^{(n)}_{w,\lambda}\, G_{\lambda^1}(E_2 - E_1) \cdots G_{\lambda^n}(F_n - E_n) \cdots G_{\lambda^{2n-1}}(F_1 - F_2)\,. \tag{7}$$

It follows that $[\mathcal{O}_{\Omega_w}] = \mathcal{G}^{(n)}_w(F_\bullet; E_\bullet)$ when the bundles are part of a sequence (1) and the codimension of Ω_w is equal to $\ell(w)$.

By definition, $\mathcal{G}^{(n)}_w(F_\bullet; E_\bullet)$ is a Laurent polynomial in the exterior powers of the bundles E_i and F_i, where only the top powers are inverted. We will call these polynomials *universal Grothendieck polynomials*, in analogy with the term

'universal Schubert polynomials' which Fulton [10] used for his cohomology formula for Ω_w. The next lemma shows that the polynomial $\mathcal{G}_w(F_\bullet; E_\bullet)$ is independent of n. We will therefore drop this letter from the notation and write simply $\mathcal{G}_w(F_\bullet; E_\bullet) = \mathcal{G}_w^{(n)}(F_\bullet; E_\bullet)$ when $w \in S_{n+1}$.

Lemma 1. *Let $w \in S_{n+1}$. The polynomial $\mathcal{G}_w^{(n+1)}(F_\bullet; E_\bullet)$ is independent of E_{n+1} and F_{n+1} and agrees with $\mathcal{G}_w^{(n)}(F_\bullet; E_\bullet)$.*

Proof. Let $\mathfrak{X}$ be a non-singular variety with a bundle sequence

$$E_1 \to \cdots \to E_{n+1} \to F_{n+1} \to \cdots \to F_1$$

such that the degeneracy locus Ω_w determined by this sequence has the expected codimension. Since the same degeneracy locus is obtained by using the subsequence which skips the two middle bundles E_{n+1} and F_{n+1}, it follows that $\mathcal{G}_w^{(n+1)}(F_\bullet; E_\bullet) = [\Omega_w] = \mathcal{G}_w^{(n)}(F_\bullet; E_\bullet)$, so the polynomials agree when evaluated in the Grothendieck ring $K(\mathfrak{X})$.

To obtain the identity of polynomials, we need to construct a variety $\mathfrak{X}$ such that all monomials in exterior powers which occur in either polynomials are linearly independent in $K(\mathfrak{X})$. Here we can use that on a Grassmannian $\mathrm{Gr}(m, N)$, all monomials of total degree at most $N/m - 1$ in the exterior powers of the tautological subbundle are linearly independent. Therefore we can take a product of Grassmannians

$$\mathcal{Z} = \mathrm{Gr}(1, N) \times \cdots \times \mathrm{Gr}(n+1, N) \times \mathrm{Gr}(n+1, N) \times \cdots \times \mathrm{Gr}(1, N)\,,$$

and let $\mathfrak{X}$ be the bundle $\mathrm{Hom}(E_1, E_2) \oplus \cdots \oplus \mathrm{Hom}(E_{n+1}, F_{n+1}) \oplus \cdots \oplus \mathrm{Hom}(F_2, F_1)$, where the bundles E_i and F_i are the tautological subbundles on $\mathcal{Z}$. When N is sufficiently large, this variety $\mathfrak{X}$ fits our purpose. □

In the remainder of this paper we will use without comment that the universal Grothendieck polynomial $\mathcal{G}_w(F_\bullet; E_\bullet)$ is determined by its values, as in the above proof.

4. Proof of the Cauchy identity

In this section we prove the Cauchy identity for universal Grothendieck polynomials (Theorem 2). We will assume that $\mathfrak{X}$ is a non-singular variety equipped with vector bundles E_i and F_i for $i \geq 1$, with $\mathrm{rank}\, E_i = \mathrm{rank}\, F_i = i$.

Proposition 1. *Let $\pi : \mathcal{Y} = \mathrm{F}\ell(E_n) \to \mathfrak{X}$ be the bundle of flags in E_n, with tautological flag $0 \subset U_1 \subset U_2 \subset \cdots \subset U_n = \pi^*(E_n)$, and set*

$$\mathcal{O}_\mathcal{Z} = \prod_{1 \leq i \leq n-1} G_{(1^i)}(U_{i+1}/U_i - E_i) \;\in\; K(\mathcal{Y})\,.$$

Then we have

$$\pi_*(\mathcal{G}_w(F_\bullet; U_\bullet) \cdot \mathcal{O}_Z) = \mathcal{G}_w(F_\bullet; E_\bullet) \;\in\; K(\mathfrak{X})\,.$$

Proof. Set $\widetilde{\mathfrak{X}} = \mathrm{Hom}(E_1, E_2) \oplus \cdots \oplus \mathrm{Hom}(E_n, F_n) \oplus \cdots \oplus \mathrm{Hom}(F_2, F_1)$ and $\widetilde{\mathcal{Y}} = \mathcal{Y} \times_\mathfrak{X} \widetilde{\mathfrak{X}}$. It is enough to prove the proposition for the projection $\rho : \widetilde{\mathcal{Y}} \to \widetilde{\mathfrak{X}}$. Notice that on $\widetilde{\mathcal{Y}}$ we have a universal bundle sequence $E_\bullet \to F_\bullet$, as well as the tautological flag $U_\bullet \subset E_n$.

Let $\mathcal{Z}_{n-1} = Z(E_{n-1} \to U_n/U_{n-1}) \subset \widetilde{\mathcal{Y}}$. On this locus the map $E_{n-1} \to U_n$ factors through U_{n-1}. We then set $\mathcal{Z}_{n-2} = Z(E_{n-2} \to U_{n-1}/U_{n-2}) \subset \mathcal{Z}_{n-1}$ and

inductively $\mathcal{Z}_i = Z(E_i \to U_{i+1}/U_i)$ for $i = n-1, \ldots, 2, 1$. Notice that the structure sheaf of $\mathcal{Z} = \mathcal{Z}_1$ is given by the expression of the proposition (see e.g. [3, Thm. 2.3]). Now ρ maps the locus $\Omega_w(U_\bullet \to F_\bullet) \cap \mathcal{Z} \subset \widetilde{\mathcal{Y}}$ birationally onto $\Omega_w(E_\bullet \to F_\bullet) \subset \widetilde{\mathfrak{X}}$. In fact, the open subset of $\mathcal{Z}$ where each map $E_i \to U_i$ is an isomorphism maps isomorphically to the open subset of $\widetilde{\mathfrak{X}}$ where all maps $E_{i-1} \to E_i$ are bundle inclusions, and furthermore these subsets meet the given (irreducible) degeneracy loci in $\mathcal{Z}$ and $\widetilde{\mathfrak{X}}$. This implies the desired result, because all involved degeneracy loci are Cohen-Macaulay with rational singularities [14] and have their expected codimensions. □

This proposition allows us to prove a special case of the Cauchy formula, arguing as in [10, §3]. We let $\mathbb{C}^\bullet$ denote a sequence of trivial bundles. When used in a polynomial, the exterior power $\wedge^i \mathbb{C}^m$ equals the binomial coefficient $\binom{m}{i}$.

Corollary 1. *We have*

$$\mathcal{G}_w(F_\bullet; E_\bullet) = \sum_{u \cdot v = w} (-1)^{\ell(uvw)}\, \mathcal{G}_u(\mathbb{C}^\bullet; E_\bullet)\, \mathcal{G}_v(F_\bullet; \mathbb{C}^\bullet)\,.$$

Proof. Let $\pi : \mathcal{Y} = \mathrm{F}\ell(E_n) \to \mathfrak{X}$ be the bundle of flags in E_n, with tautological flag $U_\bullet$ as in Proposition 1. Assume at first that the bundles $F_\bullet$ form a (descending) complete flag. Then by [3, Thm. 2.1] (which generalizes [11, Thm. 3]), we have

$$\mathcal{G}_w(F_\bullet; U_\bullet) = \mathfrak{G}_w(1 - L_1^{-1}, \ldots, 1 - L_n^{-1}\,;\, 1 - M_1, \ldots, 1 - M_n)$$

in $K(\mathcal{Y})$, where $L_i = \ker(F_i \to F_{i-1})$ and $M_i = U_i/U_{i-1}$. The Cauchy identity for double Grothendieck polynomials (3) therefore implies that

$$\mathcal{G}_w(F_\bullet; U_\bullet) = \sum_{u \cdot v = w} (-1)^{\ell(uvw)}\, \mathcal{G}_u(\mathbb{C}^\bullet; U_\bullet) \cdot \mathcal{G}_v(F_\bullet; \mathbb{C}^\bullet)\,.$$

By multiplying this identity by the class $\mathcal{O}_{\mathcal{Z}}$ of Proposition 1, and pushing the result down to $\mathfrak{X}$, we get the identity of the theorem.

Now assume that the bundles F_i are arbitrary. By the case just proved we have

$$\begin{aligned}
\mathcal{G}_w(F_\bullet; U_\bullet) &= \mathcal{G}_{w^{-1}}(U_\bullet^\vee; F_\bullet^\vee) \\
&= \sum_{u \cdot v = w} (-1)^{\ell(uvw)}\, \mathcal{G}_{v^{-1}}(\mathbb{C}^\bullet; F_\bullet^\vee) \cdot \mathcal{G}_{u^{-1}}(U_\bullet^\vee; \mathbb{C}^\bullet) \\
&= \sum_{u \cdot v = w} (-1)^{\ell(uvw)}\, \mathcal{G}_u(\mathbb{C}^\bullet; U_\bullet) \cdot \mathcal{G}_v(F_\bullet; \mathbb{C}^\bullet)
\end{aligned}$$

in $K(\mathcal{Y})$. After multiplying with $\mathcal{O}_{\mathcal{Z}}$, this identity pushes forward to give the corollary in full generality. □

For the general case of the Cauchy formula we need the following vanishing theorem. We let $H_\bullet$ denote a third collection of vector bundles on $\mathfrak{X}$, $\operatorname{rank} H_i = i$.

Proposition 2. *Choose $m \geq 0$ and substitute H_j for F_j and E_j in $\mathcal{G}_w(F_\bullet; E_\bullet)$ for all $j \geq m+1$. We then have*

$$\mathcal{G}_w(F_1, \ldots, F_m, H_{m+1}, \ldots; E_1, \ldots, E_m, H_{m+1}, \ldots) = \begin{cases} \mathcal{G}_w(F_\bullet; E_\bullet) & \text{if } w \in S_{m+1} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. If $w \in S_{m+1}$, then $\mathcal{G}_w(F_\bullet, E_\bullet)$ is independent of the bundles F_j and E_j for $j \geq m+1$ by Lemma 1.

Assume that $w \in S_{n+1} \smallsetminus S_n$ where $n > m$. We claim $\mathcal{G}_w(F_\bullet, E_\bullet)$ vanishes as soon as we set $F_n = E_n$. To see this, let $\mathfrak{X}$ be a variety with bundles F_j for $1 \leq j \leq n$ and E_j for $1 \leq j \leq n-1$ such that all monomials in the polynomial $\mathcal{G}_w(F_\bullet; E_1, \ldots, E_{n-1}, F_n)$ are linearly independent.

If we set $E_n = F_n$ then we have a sequence of bundles $E_\bullet \to F_\bullet$ for which the map $E_n \to F_n$ is the identity and all other maps are zero. Since $r_w(n,n) = n-1$ it follows that the locus $\Omega_w(E_\bullet \to F_\bullet)$ is empty. Since $\mathcal{G}_w(F_\bullet; E_\bullet)$ represents the class of the structure sheaf of this locus, it must be equal to zero. □

For any commutative ring R, let $R(S_\infty)$ denote the R-module of all functions on $S_\infty = \bigcup_n S_n$ with values in R. For $f, g \in R(S_\infty)$ we define the product

$$(fg)(w) = \sum_{u \cdot v = w} (-1)^{\ell(uvw)} f(u) g(v) \tag{8}$$

where (as always) the sum is over factorizations of w in the degenerate Hecke algebra. It is straightforward to check that this multiplication is associative and that the identity element is the characteristic function $\underline{1}$ of the identity permutation. We will need the following variation of [18, (6.6)].

Lemma 2. *Let $f, g, h \in R(S_\infty)$. Assume that for any permutation $w \in S_\infty$, the sum $\sum_{u \cdot w = w} (-1)^{\ell(u)} f(u)$ is not a zero divisor in R.*

(i) *If $fg = f$ then $g = \underline{1}$.*
(ii) *If $fh = \underline{1}$ then $hf = \underline{1}$.*

Proof. Since $f(1)g(1) = fg(1) = f(1)$ and $f(1) = \sum_{u \cdot 1 = 1} (-1)^{\ell(u)} f(u)$ is a non-zero divisor, it follows that $g(1) = 1$. Let $w \neq 1 \in S_\infty$ be given and assume inductively that $g(v) = 0$ for $0 < \ell(v) < \ell(w)$. Notice that if $u \cdot v = w$ in the degenerate Hecke algebra then $\ell(v) \leq \ell(w)$, and this inequality is sharp if $v \neq w$. We therefore have $f(w) = fg(w) = f(w) + \left(\sum_{u \cdot w = w} (-1)^{\ell(u)} f(u)\right) g(w)$, which implies that $g(w) = 0$. This proves (i), and (ii) follows by setting $g = hf$. □

Theorem 2 (Cauchy formula)**.** *Let E_i, F_i, and H_i for $i = 1, \ldots, n$ be three collections of vector bundles on $\mathfrak{X}$. Then for any $w \in S_{n+1}$ we have*

$$\mathcal{G}_w(F_\bullet; E_\bullet) = \sum_{u \cdot v = w} (-1)^{\ell(uvw)}\, \mathcal{G}_u(H_\bullet; E_\bullet)\, \mathcal{G}_v(F_\bullet; H_\bullet)\,.$$

Proof. Let $\mathcal{G}(F_\bullet; E_\bullet)$ denote the function from permutations to $K(\mathfrak{X})$ which maps w to $\mathcal{G}_w(F_\bullet; E_\bullet)$. Using the product (8) we have by Corollary 1 that

$$\mathcal{G}(F_\bullet; E_\bullet) = \mathcal{G}(\mathbb{C}^\bullet; E_\bullet)\mathcal{G}(F_\bullet; \mathbb{C}^\bullet)\,.$$

Proposition 2 implies that $\mathcal{G}(\mathbb{C}^\bullet; H_\bullet)\mathcal{G}(H_\bullet, \mathbb{C}^\bullet) = \underline{1}$, and since $\mathcal{G}_w(\mathbb{C}^\bullet; H_\bullet)$ lies in the augmentation ideal of $K(\mathfrak{X})$ for $w \neq 1$, the function $f = \mathcal{G}(\mathbb{C}^\bullet; H_\bullet)$ satisfies the requirement in Lemma 2. It follows that $\mathcal{G}(H_\bullet, \mathbb{C}^\bullet)\mathcal{G}(\mathbb{C}^\bullet, H_\bullet) = \underline{1}$. We conclude that

$$\mathcal{G}(F_\bullet; E_\bullet) = \mathcal{G}(\mathbb{C}^\bullet; E_\bullet)\mathcal{G}(F_\bullet; \mathbb{C}^\bullet) = \mathcal{G}(\mathbb{C}^\bullet; E_\bullet)\mathcal{G}(H_\bullet; \mathbb{C}^\bullet)\mathcal{G}(\mathbb{C}^\bullet; H_\bullet)\mathcal{G}(F_\bullet; \mathbb{C}^\bullet)\,,$$

and therefore $\mathcal{G}(F_\bullet; E_\bullet) = \mathcal{G}(H_\bullet; E_\bullet)\mathcal{G}(F_\bullet; H_\bullet)$, as required. □

For later use, we notice that Theorem 2 and Proposition 2 together imply that

$$\begin{aligned}
&\mathcal{G}_w(F_\bullet; E_\bullet) \\
&= \sum_{u\cdot v=w} (-1)^{\ell(uvw)} \mathcal{G}_u(\mathbb{C}^1, \dots, \mathbb{C}^r, F_{r+1}, F_{r+2}, \dots ; E_\bullet) \mathcal{G}_v(F_1, \dots, F_r ; \mathbb{C}^\bullet) \\
&= \sum_{u\cdot v=w} (-1)^{\ell(uvw)} \mathcal{G}_u(\mathbb{C}^\bullet ; E_1, \dots, E_r) \mathcal{G}_v(F_\bullet ; \mathbb{C}^1, \dots, \mathbb{C}^r, E_{r+1}, E_{r+2}, \dots) .
\end{aligned} \tag{9}$$

5. Proof of Theorem 1

In this section we derive Theorem 1 from the Cauchy identity by using a K-theoretic version of the arguments found in [7]. In what follows, it will be convenient to work with the element $P_w^{(n)} \in \Gamma^{\otimes 2n-1}$ defined by

$$P_w^{(n)} = \sum_\lambda c_{w,\lambda}^{(n)} G_{\lambda^1} \otimes \cdots \otimes G_{\lambda^{2n-1}} .$$

With this notation, we can restate Theorem 1 as follows:

Theorem 1′. *For any permutation $w \in S_{n+1}$ we have*

$$P_w^{(n)} = \sum_{u_1 \cdots u_{2n-1} = w} (-1)^{\ell(u_1 \cdots u_{2n-1} w)} G_{u_1} \otimes \cdots \otimes G_{u_{2n-1}}$$

in $\Gamma^{\otimes 2n-1}$, where the sum is over all factorizations $w = u_1 \cdots u_{2n-1}$ in the degenerate Hecke algebra such that $u_i \in S_{\min(i,2n-i)+1}$ for each i.

Proof. Since $r_w(p,q) + m = r_{1^m \times w}(p+m, q+m)$ for $m \geq 0$, it follows that the coefficients $c_{w,\lambda}^{(n)}$ are uniquely defined by the condition that

$$\begin{aligned}
\mathcal{G}_{1^m \times w}(F_\bullet; E_\bullet) = \sum_\lambda c_{w,\lambda}^{(n)} G_{\lambda^1}(E_{2+m} - E_{1+m}) \cdots \\
G_{\lambda^n}(F_{n+m} - E_{n+m}) \cdots G_{\lambda^{2n-1}}(F_{1+m} - F_{2+m})
\end{aligned} \tag{10}$$

for all $m \geq 0$ (see [2] and also the discussion after the proof of Theorem 4.1 in [3]).

Given any two integers $p \leq q$ we let $P_w^{(n)}[p,q]$ denote the sum of the terms of $P_w^{(n)}$ for which λ^i is empty when $i < p$ or $i > q$:

$$P_w^{(n)}[p,q] = \sum_{\lambda : \lambda^i = \emptyset \text{ for } i \notin [p,q]} c_{w,\lambda}^{(n)} G_{\lambda^1} \otimes \cdots \otimes G_{\lambda^{2n-1}} .$$

Lemma 3. *For any $1 < i \leq 2n-1$ we have*

$$P_w^{(n)} = \sum_{u\cdot v=w} (-1)^{\ell(uvw)} P_u^{(n)}[1, i-1] \cdot P_v^{(n)}[i, 2n-1] .$$

Proof. We will do the case $i \leq n$, the other one is similar. For any element $f = \sum c_\lambda G_{\lambda^1} \otimes \cdots \otimes G_{\lambda^{2N-1}} \in \Gamma^{\otimes 2N-1}$, we set

$$f(F_\bullet; E_\bullet) = \sum c_\lambda G_{\lambda^1}(E_2 - E_1) \cdots G_{\lambda^N}(F_N - E_N) \cdots G_{\lambda^{2N-1}}(F_1 - F_2) .$$

Equation (10) implies that $P_w^{(n)} \in \Gamma^{\otimes 2n-1}$ is the unique element satisfying that $(1^{\otimes m} \otimes P_w^{(n)} \otimes 1^{\otimes m})(F_\bullet; E_\bullet) = \mathcal{G}_{1^m \times w}(F_\bullet; E_\bullet)$ in $K(\mathfrak{X})$ for all m. This uniqueness is preserved even if we make E_{i+m} trivial. The right hand side of the identity of the lemma satisfies this by equation (9) applied to $1^m \times w$. □

Lemma 4. *For $1 \leq i \leq 2n-1$ we have*

$$P_w^{(n)}[i,i] = \begin{cases} 1^{\otimes i-1} \otimes G_w \otimes 1^{\otimes 2n-1-i} & \text{if } w \in S_{m+1},\ m = \min(i, 2n-i), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For simplicity we will assume that $m = i$. If $w \notin S_{m+1}$ then it follows from Proposition 2 or the algorithm for quiver coefficients of [3, §4] that $P_w^{(n)}[1,m] = 0$, which proves the lemma. Assume now that $w \in S_{m+1}$. It is proved in [3, (5.2)] that $P_w^{(m)}[m,m] = 1^{\otimes m-1} \otimes G_w \otimes 1^{\otimes m-1}$. Let $\Phi : \Gamma^{\otimes 2m-1} \to \Gamma^{\otimes 2n-1}$ be the linear map given by

$$\Phi(G_{\lambda^1} \otimes \cdots \otimes G_{\lambda^{2m-1}}) = G_{\lambda^1} \otimes \cdots \otimes G_{\lambda^{m-1}} \otimes \Delta^{2n-2m}(G_{\lambda^m}) \otimes G_{\lambda^{m+1}} \otimes \cdots \otimes G_{\lambda^{2m-1}}$$

where $\Delta^{2n-2m} : \Gamma \to \Gamma^{\otimes 2n-2m+1}$ denotes the $(2n-2m)$-fold coproduct, that is,

$$\Delta^{2n-2m}(G_{\lambda^m}) = \sum_{\tau_1,\ldots,\tau_{2n-2m+1}} d^{\lambda^m}_{\tau_1,\ldots,\tau_{2n-2m+1}}\, G_{\tau_1} \otimes \cdots \otimes G_{\tau_{2n-2m+1}}$$

(see [4, Corollary 6.10]). In the definition of the locus $\Omega_w(E_\bullet \to F_\bullet)$, the bundles F_i and E_i for $i \geq m+1$ are inessential in the sense of Section 3, which implies that $\Phi(P_w^{(m)}) = P_w^{(n)}$. Now the result follows from the identity $P_w^{(n)}[m,2n-m] = \Phi(P_w^{(m)}[m,m]) = 1^{\otimes m-1} \otimes \Delta^{2n-2m}(G_w) \otimes 1^{\otimes m-1}$. □

Theorem 1′ follows immediately from Lemma 3 and Lemma 4. □

Corollary 2. *Let $w \in S_{n+1}$ and let $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^{2n-1})$ be a sequence of partitions. Then we have*

$$c_{w,\lambda}^{(n)} = (-1)^{|\lambda|-\ell(w)} \sum_{u_1 \cdots u_{2n-1} = w} |\, a_{u_1,\lambda^1}\, a_{u_2,\lambda^2} \cdots a_{u_{2n-1},\lambda^{2n-1}}|$$

where $|\lambda| = \sum |\lambda^i|$, a_{u_i,λ^i} is the coefficient of G_{λ^i} in $G_{u_i} \in \Gamma$, and the sum is over all factorizations of w in the degenerate Hecke algebra such that $u_i \in S_{\min(i,2n-i)+1}$ for each i.

Since Lascoux's formula (Theorem 3) implies that $a_{u_i,\lambda^i} = (-1)^{|\lambda^i|-\ell(u_i)}\, |a_{u_i,\lambda^i}|$, Corollary 2 follows immediately from Theorem 1. This verifies the alternation of signs for the quiver coefficients $c_{w,\lambda}^{(n)}$, which was conjectured in [3]. In addition, by combining the above corollary with Lascoux's formula we obtain an explicit combinatorial formula for these coefficients.

6. Splitting Grothendieck polynomials

In this section we specialize universal Grothendieck polynomials to the double Grothendieck polynomials of Lascoux and Schützenberger. This leads to new expressions for double Grothendieck polynomials in terms of quiver coefficients. Recall that a permutation w has a descent at position i if $w(i) > w(i+1)$. We say that a sequence $\{a_k\} : a_1 < \ldots < a_p$ of integers is *compatible with* w if all descent positions of w are contained in $\{a_k\}$.

Theorem 4. *Let $w \in S_{n+1}$ and let $1 \leq a_1 < \cdots < a_p \leq n$ and $1 \leq b_1 < \cdots < b_q \leq n$ be two sequences compatible with w and w^{-1}, respectively, and set $X_i = \{x_{a_{i-1}+1}, \ldots, x_{a_i}\}$ and $Y_i = \{y_{b_{i-1}+1}, \ldots, y_{b_i}\}$. Then we have*

$$\mathfrak{G}_w(X;Y) = \sum_{\mu} \widetilde{c}_{w,\mu}\, G_{\mu^1}(X_p;0) \cdots G_{\mu^p}(X_1;Y_1) \cdots G_{\mu^{p+q-1}}(0;Y_q)\,, \tag{11}$$

where the sum is over sequences of partitions $\mu = (\mu^1, \ldots, \mu^{p+q-1})$, *and* $\widetilde{c}_{w,\mu}$ *is the quiver coefficient* $c^{(n)}_{w_0 w^{-1} w_0, \lambda}$, *where* $w_0 \in S_{n+1}$ *is the longest permutation and* $\lambda = (\lambda^1, \ldots, \lambda^{2n-1})$ *is given by*

$$\lambda^i = \begin{cases} \mu^k & \text{if } i = a_{k+1} - 1 \\ \mu^p & \text{if } i = n \\ \mu^{p+q-k} & \text{if } i = 2n - b_{k+1} + 1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Proof. Let V be a vector bundle of rank $n+1$ and let

$$E_1 \subset E_2 \subset \cdots \subset E_n \subset V \twoheadrightarrow F_n \twoheadrightarrow \cdots \twoheadrightarrow F_2 \twoheadrightarrow F_1$$

be a complete flag followed by a dual complete flag of V. By [3, Thm. 2.1], the class of the structure sheaf of $\Omega_w(E_\bullet \to F_\bullet)$ is given by $\mathfrak{G}_w(X;Y)$, where we set $x_i = 1 - [\ker(F_i \to F_{i-1})]^{-1}$ and $y_i = 1 - [E_i/E_{i-1}]$ in $K(\mathfrak{X})$.

Set $E'_i = V/E_i$ and $F'_i = \ker(V \twoheadrightarrow F_i)$. This yields the sequence

$$F'_n \subset \cdots \subset F'_1 \subset V \twoheadrightarrow E'_1 \twoheadrightarrow \cdots \twoheadrightarrow E'_n$$

and it is easy to check that $\Omega_w(E_\bullet \to F_\bullet) = \Omega_{w_0 w^{-1} w_0}(F'_\bullet \to E'_\bullet)$ as subschemes of $\mathfrak{X}$, where w_0 is the longest permutation in S_{n+1}.

Define a third bundle sequence $\widetilde{E}'_\bullet \to \widetilde{F}'_\bullet$ as follows. For $a_{k-1} < i \le a_k$ we set $\widetilde{F}'_i = F'_{a_k} \oplus \mathbb{C}^{a_k - i}$ and for $b_{k-1} < i \le b_k$ we set $\widetilde{E}'_i = E'_{b_k} \oplus \mathbb{C}^{b_k - i}$. The maps of the sequence $\widetilde{E}'_\bullet \to \widetilde{F}'_\bullet$ can be chosen arbitrarily so that the subsequence

$$\widetilde{F}'_{a_p} \to \cdots \to \widetilde{F}'_{a_1} \to \widetilde{E}'_{b_1} \to \cdots \to \widetilde{E}'_{b_q}$$

agrees with the corresponding subsequence of $F'_\bullet \to E'_\bullet$, the map $\widetilde{F}'_{i+1} \to \widetilde{F}'_i$ is an inclusion of vector bundles for $i \notin \{a_k\}$, and $\widetilde{E}'_i \to \widetilde{E}'_{i+1}$ is surjective for $i \notin \{b_k\}$. Now [7, Lemma 3] implies that $\Omega_{w_0 w^{-1} w_0}(F'_\bullet \to E'_\bullet) = \Omega_{w_0 w^{-1} w_0}(\widetilde{F}'_\bullet \to \widetilde{E}'_\bullet)$. These identities of schemes show that

$$\mathfrak{G}_w(X;Y) = \mathcal{G}_{w_0 w^{-1} w_0}(\widetilde{E}'_\bullet; \widetilde{F}'_\bullet)\,.$$

Equation (11) now follows from equation (7). In fact, $G_\alpha(\widetilde{F}'_i - \widetilde{F}'_{i+1})$ is non-zero only if α is empty or $i = a_k$ for some k, and when $i = a_k$ we have $G_\alpha(\widetilde{F}'_i - \widetilde{F}'_{i+1}) = G_\alpha(X_{k+1})$. Similarly, $G_\alpha(\widetilde{E}'_{i+1} - \widetilde{E}'_i)$ is zero unless α is empty or $i = b_k$ for some k, and for $i = b_k$ we have $G_\alpha(\widetilde{E}'_{i+1} - \widetilde{E}'_i) = G_\alpha(0; Y_{k+1})$. Finally $G_\alpha(\widetilde{E}'_n - \widetilde{F}'_n) = G_\alpha(X_1; Y_1)$.

This proves (11) in the Grothendieck ring $K(\mathfrak{X})$, in which there are relations between the variables x_i and y_i (including e.g. the relations $e_j(x_1, \ldots, x_{n+1}) = e_j(y_1, \ldots, y_{n+1})$ for $1 \le j \le n+1$). We claim, however, that (11) holds as an identity of polynomials in independent variables. For this, one checks that the definition of $\widetilde{c}_{w,\mu}$ is independent of n, i.e. the coefficient $c^{(n)}_{w_0 w^{-1} w_0, \lambda}$ does not change when n is replaced with $n+1$ and w_0 with the longest element in S_{n+2}. If we choose n sufficiently large, we can construct a variety $\mathfrak{X}$ on which (11) is true, and where all relevant monomials in the variables x_i and y_i are linearly independent. This establishes the claim. □

It follows from Theorem 4 that the monomial coefficients of Grothendieck polynomials are special cases of the K-theoretic quiver coefficients $c^{(n)}_{w,\lambda}$. Explicit formulas

for the monomial coefficients in terms of 'non-reduced RC-graphs' are one of the many consequences of Fomin and Kirillov's work [8, 9]. We will finish this paper by proving a different formula which generalizes [1, Thm. 1.1] and [7, Cor. 4].

Lemma 5. *Let w be a permutation and $p \geq 0$ an integer. Then the coefficient $a_{w,(p)}$ of Theorem 3 is given by*

$$a_{w,(p)} = \begin{cases} 1 & \text{if } w = s_{i_1} \cdots s_{i_p} \text{ for integers } i_1 > \cdots > i_p, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let x be a variable and consider the degenerate Hecke algebra tensored with $\mathbb{Z}[x]$. It follows from [8, Thm. 2.3] that the Grothendieck polynomial $\mathfrak{G}_w(x) = \mathfrak{G}_w(x, 0, \ldots ; 0, 0, \ldots)$ is equal to the coefficient of w in the expansion of the product

$$(1 + xs_n)(1 + xs_{n-1}) \cdots (1 + xs_1)$$

in this algebra. In other words, $\mathfrak{G}_w(x)$ is non-zero exactly when w has a decreasing reduced word, in which case we have $\mathfrak{G}_w(x) = x^{\ell(w)}$. The same is therefore true for the stable polynomial $G_w(x)$. The lemma follows from this because $G_\beta(x) = 0$ for any partition β of length at least two, while $G_{(p)}(x) = x^p$. □

Using Fomin's identity $G_w(X;Y) = G_{w_0 w w_0}(Y;X)$ (see [4, Lemma 3.4]) we similarly obtain that $a_{w,(1^p)} = a_{w_0 w w_0,(p)}$ is equal to one if w has an increasing reduced word of length p, and $a_{w,(1^p)} = 0$ otherwise.

Corollary 3. *Let $w \in S_n$, let $x^u y^v = x_1^{u_1} \cdots x_{n-1}^{u_{n-1}} y_1^{v_1} \cdots y_{n-1}^{v_{n-1}}$ be a monomial, and set $g_i = \sum_{k=n-i}^{n-1} v_k$, $f_i = g_{n-1} + \sum_{k=1}^{i} u_k$, and $r = f_{n-1} = |u| + |v|$. Then the coefficient of $x^u y^v$ in the double Grothendieck polynomial $\mathfrak{G}_w(X;Y)$ is equal to $(-1)^{r-\ell(w)}$ times the number of factorizations $w = s_{e_1} \cdots s_{e_r}$ in the degenerate Hecke algebra such that $n - i \leq e_{g_{i-1}+1} < \cdots < e_{g_i}$ and $e_{f_{i-1}+1} > \cdots > e_{f_i} \geq i$ for all $1 \leq i \leq n-1$.*

Proof. We apply Theorem 4 to $\sigma = 1 \times w$ with $p = q = n$ and $a_i = b_i = i$, and use that $\mathfrak{G}_w(X;Y) = \mathfrak{G}_\sigma(0, x_1, \ldots, x_{n-1}; 0, y_1, \ldots, y_{n-1})$. The coefficient of $x_2^{u_1} \cdots x_n^{u_{n-1}} y_2^{v_1} \cdots y_n^{v_{n-1}}$ in $\mathfrak{G}_\sigma(X;Y)$ is equal to $\tilde{c}_{\sigma,\lambda} = c_{w_0 \sigma^{-1} w_0, \lambda}$ where $\lambda = ((u_{n-1}), \ldots, (u_1), \emptyset, (1^{v_1}), \ldots, (1^{v_{n-1}}))$. By Corollary 2 and Lemma 5 this coefficient is equal to $\pm$ the number of factorizations $w_0 \sigma^{-1} w_0 = \tau_1 \cdots \tau_{n-1} \tau_{n+1} \cdots \tau_{2n-1}$ such that each τ_i is in $S_{\min(i,2n-i)+1}$ and has a decreasing reduced word of length u_i for $i < n$ and an increasing reduced word of length v_{2n-i} for $i > n$. The sequences $(e_1, \ldots, e_r)$ of the corollary are the corresponding factorizations of w. □

Example 1. The double Grothendieck polynomials for the elements s_i of length one in S_n are given by the formula

$$\mathfrak{G}_{s_i}(X;Y) = \sum_\delta (-1)^{|\delta|-1} (xy)^\delta = \sum_\delta (-1)^{|\delta|-1} x_1^{\delta_1} \cdots x_{n-1}^{\delta_{n-1}} y_1^{\delta_n} \cdots y_{n-1}^{\delta_{2n-2}}$$

where the sum is over the $4^{n-1} - 1$ strings $\delta = (\delta_1, \ldots, \delta_{2n-2})$ with $\delta_i \in \{0, 1\}$ for each i and $|\delta| = \sum \delta_i > 0$. For instance, $\mathfrak{G}_{s_1}(X;Y) = x_1 + y_1 - x_1 y_1$ and

$$\begin{aligned} \mathfrak{G}_{s_2}(X;Y) \;=\; & x_1 + x_2 + y_1 + y_2 - x_1 x_2 - x_1 y_1 - x_1 y_2 - x_2 y_1 - x_2 y_2 - y_1 y_2 \\ & + x_1 x_2 y_1 + x_1 x_2 y_2 + x_1 y_1 y_2 + x_2 y_1 y_2 - x_1 x_2 y_1 y_2. \end{aligned}$$

This follows from Corollary 3 since the factorizations of s_i in the degenerate Hecke algebra are exactly the non-zero powers of s_i.

References

[1] S. Billey, W. Jockusch and R. P. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), no. 4, 345–374.

[2] A. S. Buch, *Stanley symmetric functions and quiver varieties,* J. of Algebra **235** (2001), no. 1, 243–260.

[3] ———, *Grothendieck classes of quiver varieties*, Duke Math. J. **115** (2002), no. 1, 75–103.

[4] ———, *A Littlewood-Richardson rule for the K-theory of Grassmannians*, Acta Math. **189** (2002), 37–78.

[5] ———, *Alternating signs of quiver coefficients*, preprint (2003).

[6] A. S. Buch and W. Fulton, *Chern class formulas for quiver varieties*, Invent. Math. **135** (1999), 665-697.

[7] A. S. Buch, A. Kresch, H. Tamvakis and A. Yong, *Schubert polynomials and quiver formulas*, Duke Math. J., to appear.

[8] S. Fomin and A. N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, Proceedings of the 6th Intern. Conf. on Formal Power Series and Algebraic Combinatorics, DIMACS (1994), 183–190.

[9] ———, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993), Discrete Math. **153** (1996), no. 1-3, 123–143.

[10] W. Fulton, *Universal Schubert polynomials*, Duke Math. J. **96** (1999), no. 3, 575–594.

[11] W. Fulton and A. Lascoux, *A Pieri formula in the Grothendieck ring of a flag bundle*, Duke Math. J. **76** (1994), 711–729.

[12] A. N. Kirillov, *Cauchy identities for universal Schubert polynomials*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **283** (2001), 123–139.

[13] A. Knutson, E. Miller, and M. Shimozono, *Four positive formulas for type A quiver polynomials*, preprint (2003).

[14] V. Lakshmibai and P. Magyar, *Degeneracy schemes, quiver schemes, and Schubert varieties*, Internat. Math. Res. Notices **1998**, no. 12, 627–640.

[15] A. Lascoux, *Transition on Grothendieck polynomials*, Physics and combinatorics, 2000 (Nagoya), 164–179, World Sci. Publishing, River Edge, NJ, 2001.

[16] A. Lascoux and M.-P. Schützenberger, *Polynômes de Schubert*, C.R. Acad. Sci. Paris Sér. I Math. **294** (1982), 447–450.

[17] ———, *Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux*, C.R. Acad. Sci. Paris Sér. I Math. **295** (1982), 629–633.

[18] I. G. Macdonald, *Notes on Schubert polynomials*, Publ. LACIM **6**, Univ. de Québec á Montréal, Montréal, 1991.

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