CURVE NEIGHBORHOODS OF SCHUBERT VARIETIES

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Abstract. A previous result of the authors with Chaput and Perrin states that the closure of the union of all rational curves of fixed degree passing through a Schubert variety in a homogeneous space G/P is again a Schubert variety. In this paper we identify this Schubert variety explicitly in terms of the Hecke product of Weyl group elements. We apply our result to give an explicit formula for any two-point Gromov-Witten invariant as well as a new proof of the quantum Chevalley formula and its equivariant generalization. We also recover a formula for the minimal degree of a rational curve between two given points in a cominuscule variety.

1. Introduction

Let $X = G/P$ be a homogeneous space defined by a semisimple complex Lie group G and a parabolic subgroup P. The quantum cohomology ring of X is closely related to the geometry of rational curves in X and has received much attention since the mid 1990's. Given a subvariety $\Omega \subset X$ and an effective degree $d \in H_2(X)$, define the *curve neighborhood* $\Gamma_d(\Omega)$ to be the closure of the union of all rational curves of degree d in X that meet Ω. Recent developments suggest that this variety is a key object in the study of the quantum K-theory ring of X.

The study of $\Gamma_d(\Omega)$ was initiated in the recent paper [5] by Chaput, Perrin, and the authors. Using that the Kontsevich moduli space of stable maps to X is irreducible it was proved that, if Ω is an irreducible subvariety of X, then $\Gamma_d(\Omega)$ is also irreducible. In particular, if Ω is a Schubert variety in X, then so is $\Gamma_d(\Omega)$. The applications to quantum K-theory require a precise description of this locus. This was obtained in [5] when X is any cominuscule homogeneous space. For example, when X is a Grassmann variety of type A and $\Omega = \Omega_\lambda$ is a Schubert variety corresponding to a Young diagram λ, the Young diagram associated to $\Gamma_d(\Omega_\lambda)$ is obtained by removing the first d rows and columns from λ. This operation on Young diagrams has appeared in several references, possibly starting with [14]. The main result of this paper is an explicit combinatorial formula for the Weyl group element corresponding to $\Gamma_d(\Omega)$ when $\Omega \subset X$ is a Schubert variety in an arbitrary homogeneous space. A description of the curve neighborhood of a Richardson variety has been obtained in [22] when d is the degree of a line and the Fano variety of lines of degree d in X is a homogeneous space.

Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. Let W be the Weyl group of G and W_P the Weyl group of P. Each element $w \in W$ defines a Schubert variety $X(w) = \overline{Bw.P}$ in X and an opposite Schubert variety

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$Y(w) = \overline{B^{\mathrm{op}} w.P}$, where B^{op} is the opposite Borel subgroup. If w is the minimal representative for its coset in W/W_P, then we have $\dim X(w) = \operatorname{codim} Y(w) = \ell(w)$. Given a positive root α with $s_\alpha \notin W_P$, let $C_\alpha \subset X$ be the unique T-stable curve that contains the T-fixed points $1.P$ and $s_\alpha.P$. The homology group $H_2(X) = H_2(X;\mathbb{Z})$ can be identified with the quotient $\mathbb{Z}\Delta^\vee/\mathbb{Z}\Delta_P^\vee$, where $\mathbb{Z}\Delta^\vee$ is the coroot lattice of G and $\mathbb{Z}\Delta_P^\vee$ is the coroot lattice of P. The degree $[C_\alpha] \in H_2(X)$ is equal to the image of the coroot $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ under this identification.

Our description of $\Gamma_d(X(w))$ is formulated using the *Hecke product* on W, which by definition is the unique associative monoid product such that, for any simple reflection s_β and $w \in W$ we have

$$w \cdot s_\beta = \begin{cases} w s_\beta & \text{if } \ell(w s_\beta) > \ell(w); \\ w & \text{otherwise.} \end{cases}$$

Theorem 1. *Assume that $0 < d \in H_2(X)$, and let α be any maximal root with the property that $\alpha^\vee \leq d$ as elements in $H_2(X)$. Then we have $\Gamma_d(X(w)) = \Gamma_{d-\alpha^\vee}(X(w \cdot s_\alpha))$.*

A root α will be called *P-cosmall* if $s_\alpha \notin W_P$ and α satisfies the condition of Theorem 1 for any positive degree $d \in H_2(X)$. This condition makes simultaneous use of the partial orders of the root system and the dual root system of G, which gives rise to some interesting combinatorics.

Let $z_d^P \in W$ denote the minimal representative for the curve neighborhood of a point, i.e. $\Gamma_d(X(1)) = X(z_d^P)$. Theorem 1 then implies that $\Gamma_d(X(w)) = X(w \cdot z_d^P)$. Much of our paper therefore focuses on the curve neighborhood of a point.

Curve neighborhoods are related to Fulton and Woodward's work [14] on determining the smallest degree of the quantum parameter that appears in a product of Schubert classes in the (small) quantum ring $\mathrm{QH}(X)$. Let $\overline{\mathcal{M}}_{0,n}(X,d)$ denote the Kontsevich moduli space of n-pointed stable maps to X of degree d, with evaluation map $\mathrm{ev} = (\mathrm{ev}_1, \dots, \mathrm{ev}_n) : \overline{\mathcal{M}}_{0,n}(X,d) \to X^n$. Then the curve neighborhood of $X(w)$ can be defined by $\Gamma_d(X(w)) = \mathrm{ev}_1(\mathrm{ev}_2^{-1}(X(w)))$. It is proved in [14] that the quantum product $[Y(u)] \star [Y(w)]$ contains a term $q^{d'}[Y(v)]$ with $d' \leq d$ if and only if the Gromov-Witten variety $\mathrm{ev}_1^{-1}(Y(u)) \cap \mathrm{ev}_2^{-1}(X(w_0 w))$ is not empty in $\overline{\mathcal{M}}_{0,2}(X,d)$, where w_0 is the longest element in W. The later condition is equivalent to $Y(u) \cap \Gamma_d(X(w_0 w)) \neq \emptyset$, which holds if and only if $uW_P \leq w_0 w \cdot z_d^P W_P$ in the Bruhat order of W/W_P.

Define the Gromov-Witten variety $\mathrm{GW}_d(w) = \mathrm{ev}_2^{-1}(X(w)) \subset \overline{\mathcal{M}}_{0,2}(X,d)$ and consider the surjective map $\mathrm{ev}_1 : \mathrm{GW}_d(w) \to \Gamma_d(X(w))$. It was proved in [5] that the general fibers of this map are unirational. This implies that the pushforward $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] \in H_T^*(X)$ is equal to $[X(w \cdot z_d^P)]$ whenever $\dim \mathrm{GW}_d(w) = \dim X(w \cdot z_d^P)$, and otherwise $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] = 0$. It follows that any (equivariant) two-point Gromov-Witten invariant of X is given by

$$\begin{aligned} I_d([Y(u)],[X(w)]) &= \int_{\overline{\mathcal{M}}_{0,2}(X,d)} \mathrm{ev}_1^*[Y(u)] \cdot \mathrm{ev}_2^*[X(w)] \\ &= \begin{cases} 1 & \text{if } \dim \mathrm{GW}_d(w) = \dim X(w \cdot z_d^P) \text{ and } w \cdot z_d^P W_P = uW_P\,; \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

It turns out that, if this invariant is non-zero, then $d = \alpha^\vee \in H_2(X)$ for a unique P-cosmall root α, and we have $uW_P = ws_\alpha W_P$. A similar formula holds for the more general 2-point K-theoretic Gromov-Witten invariants, see Remark 7.5 below.

We apply our methods to give a new proof of the (equivariant) quantum Chevalley formula for any product of a Schubert divisor with an arbitrary Schubert class in the equivariant quantum ring $\mathrm{QH}_T(X)$. In fact, all Gromov-Witten invariants required in such a product can be obtained from (1) combined with the divisor axiom in Gromov-Witten theory [18]. The quantum Chevalley formula was first stated in a lecture given by Dale Peterson at M.I.T., and a proof was later supplied by Fulton and Woodward [14]. The equivariant generalization is due to the second author [24] and states that all terms of a product involving a Schubert divisor in $\mathrm{QH}_T(X)$ are also visible in the equivariant cohomology $H_T^*(X)$ or in $\mathrm{QH}(X)$.

We remark that if P is not a Borel subgroup of G, then the ring $\mathrm{QH}_T(X)$ is not generated by divisor classes. However, it was demonstrated in [24] that all (3 point, genus zero) equivariant Gromov-Witten invariants of X can be computed with an explicit algorithm based on the equivariant quantum Chevalley formula. This has been applied by Lam and Shimozono to prove that the equivariant Gromov-Witten invariants of X coincide with certain structure constants of the equivariant homology of the affine Grassmannian [20].

Our paper is organized as follows. In section 2 we recall some basic facts about Schubert classes on X. Section 3 defines the Hecke product and gives combinatorial proofs of its main properties. In section 4 we use the statement of Theorem 1 to give a combinatorial construction of the element $z_d^P \in W$. We then prove a key technical result stating that for all effective degrees $0 \leq d' \leq d \in H_2(X)$ we have $z_{d'}^P \cdot z_{d-d'}^P W_p \leq z_d^P W_P$. We remark that this inequality is easy to deduce from the geometric definition $X(z_d^P) := \Gamma_d(X(1))$, but we need to work combinatorially to obtain a complete proof of Theorem 1. The results in Section 4 include some basic facts concerning cosmall roots and Hecke products of reflections that are proved using the classification of root systems (Lemmas 4.4, 4.7, and 4.10). All other results in our paper are deduced from these facts in a type independent setup. Theorem 1 is established in section 5, where we also apply this result to recover a well known formula [27, 16, 9] for the minimal degree of a rational curve between two given points in a cominuscule variety. In Section 6 we prove some equivalent conditions for P-cosmall roots, one of them stating that α is P-cosmall if and only if $\dim X(s_\alpha) = \int_{C_\alpha} c_1(T_X) - 1$. In combinatorial terms this implies that α is B-cosmall if and only if $\ell(s_\alpha) = 2\,\mathrm{height}(\alpha^\vee) - 1$. Section 7 uses these results and related inequalities to prove an explicit formula for any two-point Gromov-Witten invariant of X, and Section 8 proves the equivariant quantum Chevalley formula. While this proof logically depends on many earlier results, including the combinatorial construction of z_d^P, we finish our paper by noting that the concept of curve neighborhoods can be used to give a very short geometric proof of the quantum Chevalley formula.

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2. Schubert varieties

In this section we fix our notation for Schubert varieties and state some basic facts. Proofs can be found in e.g. [15]. Let $X = G/P$ be a homogeneous space defined by a connected, simply connected, semisimple complex Lie group G and a parabolic subgroup P. Fix also a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. Let R be the associated root system, with positive roots R^+ and simple roots $\Delta \subset R^+$. Let $W = N_G(T)/T$ be the Weyl group of G and $W_P = N_P(T)/T$ the Weyl group of P. The parabolic subgroup P corresponds to the set of simple roots $\Delta_P = \{\beta \in \Delta \mid s_\beta \in W_P\}$. The group W_P is generated by the simple reflections s_β for $\beta \in \Delta_P$. Set $R_P = R \cap \mathbb{Z}\Delta_P$ and $R_P^+ = R^+ \cap \mathbb{Z}\Delta_P$, where $\mathbb{Z}\Delta_P = \mathrm{Span}_{\mathbb{Z}}(\Delta_P)$ is the group spanned by Δ_P.

For each element $w \in W$ we let $I(w) = R^+ \cap w^{-1}(-R^+) = \{\alpha \in R^+ \mid w(\alpha) < 0\}$ denote the *inversion set* of w. The second expression uses the partial order $\leq$ on $\mathbb{R}\Delta = \mathrm{Span}_{\mathbb{R}}(\Delta)$ defined by $\alpha \geq \beta$ if and only if $\alpha - \beta$ is a linear combination with non-negative coefficients of the simple roots Δ. The length of w is defined by $\ell(w) = |I(w)|$. Equivalently, $\ell(w)$ is the minimal number of simple reflections that w can be a product of. Define the length of the coset $wW_P \in W/W_P$ to be $\ell(wW_P) = |I(w) \smallsetminus R_P^+|$. The element w can be written uniquely as $w = uv$ such that $I(u) \cap R_P^+ = \emptyset$ and $v \in W_P$. We then have $uW_P = wW_P$ and $\ell(u) = \ell(wW_P)$. The element u is called the *minimal representative* for the coset wW_P. Similarly, if w_P denotes the longest element of W_P, then uw_P is the *maximal representative* for wW_P. Let $W^P \subset W$ be the set of all minimal representatives for cosets in W/W_P.

Let w_0 be the longest element in W and let $B^{\mathrm{op}} = w_0 B w_0 \subset G$ be the Borel subgroup opposite to B. For $w \in W$ we define the B-stable Schubert variety $X(w) = \overline{Bw.P} \subset X$ and the B^{op}-stable Schubert variety $Y(w) = \overline{B^{\mathrm{op}}w.P} \subset X$. These varieties depend only on the coset wW_P and we have $\dim X(w) = \mathrm{codim}\, Y(w) = \ell(wW_P)$. We also have $X(w) \cap Y(w) = \{w.P\}$. The collection of points $w.P$ for $w \in W^P$ is the set of all T-fixed points in X.

The *Bruhat order* on W/W_P is defined by $uW_P \leq wW_P$ if and only if $X(u) \subset X(w)$. This order is compatible with the Bruhat order on W in the sense that $uW_P \leq wW_P$ whenever $u \leq w$ in W. This follows because $X(u)$ is the image of $\overline{Bu.B}$ under the projection $G/B \to X$.

Let $(-,-)$ denote the W-invariant inner product on $\mathbb{R}\Delta$. Each root $\alpha \in R$ has a coroot $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$. The coroots form the dual root system $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$, with basis of simple coroots $\Delta^\vee = \{\beta^\vee \mid \beta \in \Delta\}$. For $\beta \in \Delta$ we let $\omega_\beta \in \mathbb{R}\Delta$ denote the corresponding fundamental weight, defined by $(\omega_\beta, \alpha^\vee) = \delta_{\alpha,\beta}$ for $\alpha \in \Delta$.

Lemma 2.1. *Let $\alpha \in R$ and let $S \subset R$ be any set of roots such that $s_\alpha(S) = S$. Then we have*

$$\sum_{\gamma \in S} (\alpha, \gamma^\vee) = \sum_{\gamma \in S} (\gamma, \alpha^\vee) = 0\,.$$

Proof. Since s_α is an involution of S defined by $s_\alpha(\gamma) = \gamma - (\gamma, \alpha^\vee)\alpha$, we obtain

$$\sum_{\gamma \in S} (\gamma, \alpha^\vee) = \sum_{\gamma \in S} \frac{\gamma - s_\alpha(\gamma)}{\alpha} = 0\,.$$

Since we have $s_{\alpha^\vee}(S^\vee) = S^\vee$, the first sum of the lemma is also equal to zero. $\square$

We need the following observation, which can also be found in [14, p. 648].

Lemma 2.2. *Let $\alpha \in R^+ \smallsetminus R_P^+$. Then α is uniquely determined by the coset $s_\alpha W_P \in W/W_P$.*

Proof. Set $\lambda = \sum_{\beta \in \Delta \smallsetminus \Delta_P} \omega_\beta$. Then W_P acts trivially on λ, while $\lambda - s_\alpha.\lambda = (\lambda, \alpha^\vee)\, \alpha$ is a non-zero multiple of α. The lemma follows from this because distinct positive roots are never parallel. □

All homology and cohomology groups in this paper are taken with integer coefficients. Any closed irreducible subvariety $Z \subset X$ defines a fundamental homology class $[Z] \in H_{2\dim(Z)}(X)$. We will also use the notation $[Z]$ for its Poincare dual class in $H^{2\,\mathrm{codim}(Z)}(X)$. The *Schubert classes* $[Y(w)]$ for $w \in W^P$ form a basis for the cohomology ring $H^*(X)$. It is convenient to identify $H^2(X)$ with the span $\mathbb{Z}\{\omega_\beta \mid \beta \in \Delta \smallsetminus \Delta_P\}$ and $H_2(X)$ with the quotient $\mathbb{Z}\Delta^\vee/\mathbb{Z}\Delta_P^\vee$. More precisely, for each $\beta \in \Delta \smallsetminus \Delta_P$ we identify the class $[X(s_\beta)] \in H_2(X)$ with $\beta^\vee + \mathbb{Z}\Delta_P^\vee \in \mathbb{Z}\Delta^\vee/\mathbb{Z}\Delta_P^\vee$ and we identify $[Y(s_\beta)] \in H^2(X)$ with ω_β. The Poincare pairing $H^2(X) \otimes H_2(X) \to \mathbb{Z}$ is then induced by the W-invariant inner product $(-,-)$ on $\mathbb{R}\Delta$.

For each positive root $\alpha \in R^+ \smallsetminus R_P^+$ there is a unique irreducible T-stable curve $C_\alpha \subset X$ that contains $1.P$ and $s_\alpha.P$. We can restate [14, Lemma 3.4] as the identity

$$[C_\alpha] = \alpha^\vee + \mathbb{Z}\Delta_P^\vee \in H_2(X)\,. \tag{2}$$

According to [14, Lemma 3.5] we have

$$c_1(T_X) \;=\; \sum_{\gamma \in R^+ \smallsetminus R_P^+} \gamma \;\in H^2(X)\,. \tag{3}$$

Indeed, if we let $c_1 = \sum_{\gamma \in R^+ \smallsetminus R_P^+} \gamma$ denote the right hand side of (3), then the cited lemma implies that $(c_1, \beta^\vee) = \int_{X(s_\beta)} c_1(T_X)$ for $\beta \in \Delta \smallsetminus \Delta_P$, and Lemma 2.1 shows that $(c_1, \beta^\vee) = 0$ for each $\beta \in \Delta_P$, hence $c_1 \in \mathbb{Z}\{\omega_\beta \mid \beta \in \Delta \smallsetminus \Delta_P\} = H^2(X)$.

If $\lambda \in \mathbb{Z}\{\omega_\beta \mid \beta \in \Delta \smallsetminus \Delta_P\}$ is an integral weight, then the assumption that G is simply connected implies that λ is represented by a character $\lambda : T \to \mathbb{C}^*$. It therefore defines the line bundle $L_\lambda := G \times^P \mathbb{C}_{-\lambda} = (G \times \mathbb{C})/P$ over X, where P acts on $G \times \mathbb{C}$ by $p.(g,z) = (gp^{-1}, \lambda(p)^{-1}z)$. By [4, p. 71] we then have

$$c_1(L_\lambda) = \lambda \in H^2(X)\,. \tag{4}$$

3. The Hecke product

Our description of curve neighborhoods is formulated in terms of the Hecke product, which provides a monoid structure on the Weyl group W. This product describes the multiplication of basis elements in a Hecke algebra that was first studied in the context of Tits buildings [2, Ch. 4, §2.1]. It also describes the composition of Demazure operators [11] and plays a key role in the combinatorial study of K-theory of homogeneous spaces, see e.g. [19, 12]. While the Hecke product and its properties are well known, we do not know about a reference that gives a short unified exposition, so we have taken the opportunity to provide one here. Everything in this section works more generally if W is a Coxeter group, see [1] for definitions.

For $u \in W$ and $\beta \in \Delta$, define

$$u \cdot s_\beta = \begin{cases} us_\beta & \text{if } us_\beta > u; \\ u & \text{if } us_\beta < u. \end{cases} \tag{5}$$

Let $u, v \in W$ and let $v = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_\ell}$ be any reduced expression for v. Define the *Hecke product* of u and v by

$$u \cdot v = u \cdot s_{\beta_1} \cdot s_{\beta_2} \cdot \ldots \cdot s_{\beta_\ell}\,,$$

where the simple reflections are multiplied to u in left to right order.

We claim that this product is independent of the chosen reduced expression for v. In fact, any reduced expression for v can be obtained from any other by using finitely many *braid relations*, i.e. by steps that replace a subexpression of the form $t_0 t_1 \cdots t_{m-1}$ with $t_1 t_2 \cdots t_m$, where $t_{2i} = s_\alpha$ and $t_{2i+1} = s_\beta$ for given simple roots $\alpha, \beta \in \Delta$ and all $i \in \mathbb{N}$, and where m equals the order of $s_\alpha s_\beta$ in W. It is enough to show that, for each such relation $t_0 t_1 \cdots t_{m-1} = t_1 t_2 \cdots t_m$ in W, we have

$$u \cdot t_0 \cdot t_1 \cdot \ldots \cdot t_{m-1} = u \cdot t_1 \cdot t_2 \cdot \ldots \cdot t_m\,. \tag{6}$$

Let $W_{\alpha,\beta} \subset W$ denote the parabolic subgroup generated by s_α and s_β. Then $t_0 t_1 \cdots t_{m-1} = t_1 t_2 \cdots t_m$ is the longest element of $W_{\alpha,\beta}$, and both sides of (6) are equal to the unique maximal representative for the coset $uW_{\alpha,\beta}$ in $W/W_{\alpha,\beta}$.

Given $u, v \in W$, we will say that the product uv is *reduced* if $\ell(uv) = \ell(u) + \ell(v)$. This implies that $w \cdot uv = (w \cdot u) \cdot v$ for all $w \in W$. Notice also that for $\beta \in \Delta$ and $v \in W$ we have

$$s_\beta \cdot v = \begin{cases} s_\beta v & \text{if } s_\beta v > v; \\ v & \text{if } s_\beta v < v. \end{cases} \tag{7}$$

In fact, if we set $v' = s_\beta v$, then the identity is clear if $\ell(v') > \ell(v)$, and otherwise $v = s_\beta v'$ is a reduced product, hence $s_\beta \cdot v = (s_\beta \cdot s_\beta) \cdot v' = s_\beta \cdot v' = v$.

Proposition 3.1. *Let $u, v, v', w \in W$.*

(a) *The Hecke product is associative, i.e. $(u \cdot v) \cdot w = u \cdot (v \cdot w)$.*

(b) *We have $(u \cdot v)^{-1} = v^{-1} \cdot u^{-1}$.*

(c) *If $v \leq v'$ then $u \cdot v \cdot w \leq u \cdot v' \cdot w$.*

(d) *We have $u \leq u \cdot v$, $v \leq u \cdot v$, $uv \leq u \cdot v$, and $\ell(u \cdot v) \leq \ell(u) + \ell(v)$.*

(e) *The element $u' = (u \cdot v) v^{-1}$ satisfies $u' \leq u$ and $u'v = u' \cdot v = u \cdot v$.*

(f) *$I(v) \subset I(u \cdot v)$.*

Proof. To prove (a) it is enough to show that $(u \cdot s_\beta) \cdot w = u \cdot (s_\beta \cdot w)$ for each $\beta \in \Delta$. This is clear if $s_\beta \cdot w = s_\beta w$, and it is also clear if $u \cdot s_\beta = u$ and $s_\beta \cdot w = w$. Assume that $u \cdot s_\beta = u s_\beta$ and $s_\beta \cdot w = w$, and set $w' = s_\beta w$. Since $w = s_\beta w'$ is a reduced product, we obtain $(u \cdot s_\beta) \cdot w = ((u \cdot s_\beta) \cdot s_\beta) \cdot w' = (u \cdot s_\beta) \cdot w' = u \cdot w = u \cdot (s_\beta \cdot w)$, as required. Part (b) follows from the associativity together with (5) and (7). To prove (c) it is enough to show that $v \cdot s_\beta \leq v' \cdot s_\beta$ for each $\beta \in \Delta$. This is true because $W_\beta = \{1, s_\beta\}$ is a parabolic subgroup of W, $v \cdot s_\beta$ is the maximal representative for vW_β in W/W_β, $v' \cdot s_\beta$ is the maximal representative for $v'W_\beta$, and $vW_\beta \leq v'W_\beta$. For (d), the inequality $u \leq u \cdot v$ follows from (5), and $v \leq u \cdot v$ follows from (7). If we write $v = v' s_\beta$ as a reduced product with $\beta \in \Delta$, then it follows from (c) and induction on $\ell(v)$ that $u \cdot v = (u \cdot v') \cdot s_\beta \geq uv' \cdot s_\beta \geq uv' s_\beta = uv$. The inequality $\ell(u \cdot v) \leq \ell(u) + \ell(v)$ is clear from the definition. For (e), let $u = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_\ell}$ be a reduced expression for u, and set $y_j = s_{\alpha_j} \cdot s_{\alpha_{j+1}} \cdot \ldots \cdot s_{\alpha_\ell} \cdot v$ for each j. Let $\{i_1 < i_2 < \cdots < i_p\}$ be the set of indices j for which $y_j \neq y_{j+1}$. Then $u \cdot v = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_p}} v$ is a reduced product, so $u' = s_{\alpha_{i_1}} s_{\alpha_{i_2}} \cdots s_{\alpha_{i_p}}$ satisfies the requirements. Finally, part (f) follows from (7). $\square$

Proposition 3.2. *Let $u, v \in W$. The following are equivalent.*

(a) *The product uv is reduced.*

(b) $\ell(u \cdot v) = \ell(u) + \ell(v)$.

(c) $u \cdot v = uv$.

(d) $I(v) \subset I(uv)$.

(e) $I(u) \cap I(v^{-1}) = \emptyset$.

Proof. All of the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a) follow easily from the definitions and Proposition 3.1. $\square$

The Hecke product also defines a product $W \times W/W_P \to W/W_P$ given by

$$u \cdot (wW_P) = (u \cdot w)W_P \,.$$

To see that this is well defined, write $w = w'w''$ with $w' \in W^P$ and $w'' \in W_P$, and set $v = u \cdot w'$ and $p = v^{-1}(v \cdot w'')$. Then $u \cdot w = (u \cdot w') \cdot w'' = v \cdot w'' = vp$, and since $p \leq w''$ by Proposition 3.1(e) we must have $p \in W_P$. It follows that $(u \cdot w)W_P = (u \cdot w')W_P$, as required.

Notice also that for $\beta \in \Delta$ we have

$$s_\beta \cdot (wW_P) = \begin{cases} s_\beta w W_P & \text{if } s_\beta w W_P > wW_P; \\ wW_P & \text{if } s_\beta w W_P \leq wW_P. \end{cases} \tag{8}$$

In fact, if $s_\beta w W_P > wW_P$, then we must have $s_\beta w > w$ by compatibility of the Bruhat orders, and otherwise the inequality $wW_P \leq (s_\beta \cdot w)W_P$ implies that $s_\beta w W_P = wW_P$. The identity follows from this.

Proposition 3.3. *For $u, v \in W$ we have $\ell(u \cdot vW_P) \leq \ell(u) + \ell(vW_P)$. Moreover, if $\ell(u \cdot vW_P) = \ell(u) + \ell(vW_P)$ then we must have $u \cdot vW_P = uvW_P$.*

Proof. This follows from equation (8). $\square$

4. Combinatorial construction of z_d

4.1. **Complete flag varieties G/B.** Given a degree $d \in H_2(G/B) = \mathbb{Z}\Delta^\vee$, the maximal elements of the set $\{\alpha \in R^+ \mid \alpha^\vee \leq d\}$ are called *maximal roots* of d. The root $\alpha \in R^+$ is *cosmall* if α is a maximal root of $\alpha^\vee$. For example, this holds if α is a simple root, a long root, or if R is simply laced. Notice also that if $\alpha \in R^+$ is a maximal root of any degree, then α is automatically cosmall. The term *cosmall* was chosen because it is related to what we call a *large root* in Definition 6.7 below, see also Proposition 6.8.

Example 4.1. If R has one of the classical Lie types A_{n-1}, B_n, C_n, or D_n, then we can identify R with a subset of $\mathbb{R}^n$ as follows. Let $e_1, \dots, e_n$ be the standard basis for $\mathbb{R}^n$ and set $\beta_i = e_{i+1} - e_i$ for $1 \leq i \leq n-1$. We also set $\beta_0 = e_1$, $\widehat{\beta}_0 = 2e_1$, and $\beta_{-1} = e_2 + e_1$. The following table lists the simple, long, short, and cosmall (positive) roots in each type.

A_{n-1}	Simple	$\beta_1, \ldots, \beta_{n-1}$	
	Long	$e_j - e_i = \beta_i + \beta_{i+1} + \cdots + \beta_{j-1}$	$1 \leq i < j \leq n$
	Cosmall	All positive roots.	
B_n	Simple	$\beta_0, \beta_1, \ldots, \beta_{n-1}$	
	Long	$e_j - e_i = \beta_i + \beta_{i+1} + \cdots + \beta_{j-1}$	$1 \leq i < j \leq n$
		$e_j + e_i =$ $2\beta_0 + 2\beta_1 + \cdots + 2\beta_{i-1} + \beta_i + \cdots + \beta_{j-1}$	$1 \leq i < j \leq n$
	Short	$e_i = \beta_0 + \beta_1 + \cdots + \beta_{i-1}$	$1 \leq i \leq n$
	Cosmall	e_1, and $e_j - e_i$ for $1 \leq i < j \leq n$, and $e_j + e_i$ for $1 \leq i < j \leq n$.	
C_n	Simple	$\widehat{\beta}_0, \beta_1, \ldots, \beta_{n-1}$	
	Long	$2e_i = \widehat{\beta}_0 + 2\beta_1 + \cdots + 2\beta_{i-1}$	$1 \leq i \leq n$
	Short	$e_j - e_i = \beta_i + \beta_{i+1} + \cdots + \beta_{j-1}$	$1 \leq i < j \leq n$
		$e_j + e_i =$ $\widehat{\beta}_0 + 2\beta_1 + \cdots + 2\beta_{i-1} + \beta_i + \cdots + \beta_{j-1}$	$1 \leq i < j \leq n$
	Cosmall	$2e_i$ for $1 \leq i \leq n$, and $e_j - e_i$ for $1 \leq i < j \leq n$.	
D_n	Simple	$\beta_{-1}, \beta_1, \ldots, \beta_{n-1}$	
	Long	$e_j - e_i = \beta_i + \beta_{i+1} + \cdots + \beta_{j-1}$	$1 \leq i < j \leq n$
		$e_j + e_1 = \beta_{-1} + \beta_2 + \beta_3 + \cdots + \beta_{j-1}$	$2 \leq j \leq n$
		$e_j + e_i =$ $\beta_{-1} + \beta_1 + 2\beta_2 + \cdots + 2\beta_{i-1} + \beta_i + \cdots + \beta_{j-1}$	$2 \leq i < j \leq n$
	Cosmall	All positive roots.	

Example 4.2. Assume that R has type G_2, with $\Delta = \{\beta_1, \beta_2\}$ where β_2 is the long root. Then the cosmall roots of R consist of β_1, β_2, $3\beta_1 + \beta_2$, and $3\beta_1 + 2\beta_2$.

Example 4.3. Assume that R has type F_4, with $\Delta = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ and Dynkin diagram 1 — 2 =>= 3 — 4. Then the cosmall roots of R consist of β_1, β_2, β_3, β_4, $\beta_1 + \beta_2$, $\beta_3 + \beta_4$, $\beta_2 + 2\beta_3$, $\beta_1 + \beta_2 + 2\beta_3$, $\beta_1 + 2\beta_2 + 2\beta_3$, $\beta_2 + 2\beta_3 + 2\beta_4$, $\beta_1 + \beta_2 + 2\beta_3 + 2\beta_4$, $\beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4$, $\beta_1 + 2\beta_2 + 4\beta_3 + 2\beta_4$, $\beta_1 + 3\beta_2 + 4\beta_3 + 2\beta_4$, and $2\beta_1 + 3\beta_2 + 4\beta_3 + 2\beta_4$.

For $\alpha \in R^+$ we set $\operatorname{supp}(\alpha) = \{\beta \in \Delta \mid \beta \leq \alpha\}$. Two positive roots α and β are *separated* if $\operatorname{supp}(\alpha) \cup \operatorname{supp}(\beta)$ is a disconnected subset of the Dynkin diagram. Equivalently, every root in the support of α is perpendicular to every root in the support of β. Notice that if α and β are separated roots, then $s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha = s_\alpha s_\beta$. Given $d, d' \in \mathbb{R}\Delta$ we let $d \bigvee d'$ denote the smallest element in $\mathbb{R}\Delta$ that is greater than or equal to both d and d'.

Lemma 4.4.
(a) *For each $\alpha \in R^+$ there exists exactly one maximal root of $\alpha^\vee$.*
(b) *If $\alpha, \beta \in R^+$ are non-separated roots, then $\alpha \bigvee \beta \in R^+$ is also a root.*

Proof. The lemma has been checked case by case when R has exceptional Lie type, so we will assume that R has classical type and use the notation of Example 4.1. Let $\alpha \in R^+$. If α is a long root, then α is the unique maximal root of $\alpha^\vee$, so assume that α is short. If R has type B_n, then $\alpha = e_i$ for some i. If $i = 1$, then α is the unique maximal root of $\alpha^\vee$, and otherwise the unique maximal root of $\alpha^\vee$ is $e_i + e_{i-1}$. If R has type C_n, then we have either $\alpha = e_j - e_i$ in which case α is the unique maximal root of $\alpha^\vee$, or $\alpha = e_j + e_i$ in which case the unique maximal root of $\alpha^\vee$ is $2e_j$. This proves part (a). Part (b) follows by inspection of the table in Example 4.1. □

Corollary 4.5. *If $\alpha, \beta \in R^+$ are non-separated roots and α is a maximal root of $\alpha^\vee \bigvee \beta^\vee$, then $\beta \leq \alpha$.*

Proof. It follows from Lemma 4.4(b) that $\alpha^\vee \bigvee \beta^\vee = \gamma^\vee$ for some root $\gamma \in R^+$, after which Lemma 4.4(a) implies that α is the only maximal root of $\alpha^\vee \bigvee \beta^\vee$. It follows that $\beta \leq \alpha$. □

Definition 4.6. Given an effective degree $d \in \mathbb{Z}\Delta^\vee$, we recursively define an element $z_d \in W$ as follows. If $d = 0$ then set $z_d = 1$. Otherwise we set $z_d = s_\alpha \cdot z_{d-\alpha^\vee}$ where α is any maximal root of d.

We prove that this is well defined by induction on d. If α and β are distinct maximal roots of d, then Corollary 4.5 implies that α and β are separated. It follows that α is a maximal root of $d - \beta^\vee$ and β is a maximal root of $d - \alpha^\vee$, so we obtain $s_\beta \cdot z_{d-\beta^\vee} = s_\beta \cdot s_\alpha \cdot s_{d-\beta^\vee-\alpha^\vee} = s_\alpha \cdot s_\beta \cdot s_{d-\alpha^\vee-\beta^\vee} = s_\alpha \cdot z_{d-\alpha^\vee}$, as required.

Lemma 4.7. *Let $\alpha, \beta \in R^+$ be cosmall roots.*
(a) *We have $\alpha \leq \beta$ if and only if $\alpha^\vee \leq \beta^\vee$.*
(b) *If $\alpha < \beta$, then there exists a cosmall root γ such that $\alpha < \gamma \leq \beta$ and $\gamma^\vee - \alpha^\vee$ is a simple coroot.*

Proof. The lemma has been checked case by case when R has exceptional Lie type, so we will assume that R has classical type. It follows by inspection of Example 4.1 that, if $\alpha < \beta$ is a covering relation in the partially ordered set of cosmall roots in R^+, i.e. no cosmall root is strictly in between α and β, then $\beta^\vee - \alpha^\vee$ is a simple coroot. This proves part (b), which in turn shows that $\alpha \leq \beta$ implies $\alpha^\vee \leq \beta^\vee$. On the other hand, if $\alpha^\vee \leq \beta^\vee$, then since Lemma 4.4(a) implies that β is the unique maximal root of $\beta^\vee$, we obtain $\alpha \leq \beta$. □

Proposition 4.8. *Let $\alpha, \beta \in R^+$.*
(a) *Assume that $\beta \in \Delta$. Then we have $s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha$ if and only if $(\alpha, \beta) \geq 0$.*
(b) *If α is a maximal root of $\alpha^\vee + \beta^\vee$, then $s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha$.*

Proof. Assume that β is a simple root. Then we have $s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha$ if and only $s_\alpha s_\beta = s_\beta s_\alpha$ or $\ell(s_\alpha s_\beta) < \ell(s_\alpha)$. The inequality $\ell(s_\alpha s_\beta) < \ell(s_\alpha)$ is equivalent to $(\alpha, \beta) > 0$, and the identity $s_\alpha s_\beta = s_\beta s_\alpha$ holds if and only if $\alpha = \beta$ or $(\alpha, \beta) = 0$. Part (a) follows from this.

Now let $\alpha, \beta \in R^+$ be arbitrary positive roots such that α is a maximal root of $\alpha^\vee + \beta^\vee$. We must show that $s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha$. The assumptions imply that α is a maximal root of $\alpha^\vee + \gamma^\vee$ for all $\gamma \in \operatorname{supp}(\beta)$. Since s_β is a product of simple reflections s_γ for $\gamma \in \operatorname{supp}(\beta)$, we may assume that β is a simple root.

Assume that $(\alpha, \beta) < 0$, and set $\delta = s_\beta(\alpha)$. Then $\delta = \alpha - (\alpha, \beta^\vee)\beta > \alpha$ and $\delta^\vee = \alpha^\vee - (\beta, \alpha^\vee)\beta^\vee > \alpha^\vee$. Let $\gamma' \geq \delta$ be a maximal root of $\delta^\vee$. Since $\alpha < \gamma'$, it follows from Lemma 4.7 that there exists a cosmall root γ such that $\alpha < \gamma \leq \gamma'$ and $\gamma^\vee - \alpha^\vee$ is a simple coroot. Since we also have $\gamma^\vee \leq {\gamma'}^\vee \leq \delta^\vee$, we must have $\gamma^\vee - \alpha^\vee = \beta^\vee$, which contradicts that α is a maximal root of $\alpha^\vee + \beta^\vee$. We conclude that $(\alpha, \beta) \geq 0$, so part (b) follows from part (a). □

Let $d \in \mathbb{Z}\Delta^\vee$ be an effective degree. A *greedy decomposition* of d is a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive roots satisfying the recursive condition that α_1 is a maximal root of d, and $(\alpha_2, \dots, \alpha_k)$ is a greedy decomposition of $d-\alpha_1^\vee$. The empty sequence is the only greedy decomposition of the degree $0 \in \mathbb{Z}\Delta^\vee$. If $(\alpha_1, \dots, \alpha_k)$ is a greedy decomposition of d, then it follows from the definition that $z_d = s_{\alpha_1} \cdot s_{\alpha_2} \cdot \dots \cdot s_{\alpha_k}$. Furthermore, it follows from Proposition 4.8(b) that $s_{\alpha_i} \cdot s_{\alpha_j} = s_{\alpha_j} \cdot s_{\alpha_i}$ for all $1 \leq i, j \leq k$. We record the following consequence.

Corollary 4.9. *For any effective degree $d \in \mathbb{Z}\Delta^\vee$ we have $(z_d)^{-1} = z_d$.*

Notice also that if $(\alpha_1, \dots, \alpha_k)$ and $(\beta_1, \dots, \beta_l)$ are greedy decompositions of the same degree d, then these decompositions are equal up to reordering. To see this, notice that if $\alpha_1 \neq \beta_1$, then Corollary 4.5 shows that that β_1 is a maximal root of $d - \alpha_1^\vee$ and α_1 is a maximal root of $d - \beta_1^\vee$. Let $(\gamma_1, \dots, \gamma_p)$ be a greedy decomposition of $d - \alpha_1^\vee - \beta_1^\vee$. Then $(\alpha_2, \dots, \alpha_k)$ and $(\beta_1, \gamma_1, \dots, \gamma_p)$ are both greedy decompositions of $d - \alpha_1^\vee$, and $(\beta_2, \dots, \beta_l)$ and $(\alpha_1, \gamma_1, \dots, \gamma_p)$ are both greedy decompositions of $d - \beta_1^\vee$. It therefore follows by induction on d that all of the sequences $(\alpha_1, \dots, \alpha_k)$, $(\alpha_1, \beta_1, \gamma_1, \dots, \gamma_p)$, and $(\beta_1, \dots, \beta_l)$ are reorderings of each other.

Our proof of the following lemma relies heavily on the classification of root systems. It would be very interesting to find a type independent proof.

Lemma 4.10.

(a) *If $\alpha \in R^+$ is not cosmall, then $s_\alpha < z_{\alpha^\vee}$.*

(b) *If α and β are cosmall roots, then $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee + \beta^\vee}$.*

Proof. The lemma has been checked case by case when R has exceptional Lie type, so we will assume that R has classical type. We will use the notation of Example 4.1 for the roots in R. The coroots are given by $(e_j - e_i)^\vee = e_j - e_i$, $(e_j + e_i)^\vee = e_j + e_i$, $(e_i)^\vee = 2e_i$, and $(2e_i)^\vee = e_i$.

The Weyl group W is a subgroup of $\mathrm{Aut}(\mathbb{R}^n)$. For $-1 \leq i \leq n-1$ we set $s_i = s_{\beta_i} \in \mathrm{Aut}(\mathbb{R}^n)$. Depending on the Lie type of R, these reflections may or may not be elements of W. We also set $\sigma_k = s_{e_k}$ for $1 \leq k \leq n$, and set $\nu_{i,j} = s_{e_j + e_i}$, $\tau_{i,j} = s_{e_j - e_i}$, $u_{i,j} = s_i s_{i+1} \cdots s_{j-1}$, and $d_{i,j} = s_{j-1} s_{j-2} \cdots s_i$ for $1 \leq i < j \leq n$. To reduce the number of special cases, we furthermore set $\tau_{i,j} = u_{i,j} = d_{i,j} = 1$ for $i \geq j$. Notice that the Hecke products of these elements depend on the Lie type of R. For example, if R has type D_n, then s_{-1} is a simple reflection and $s_{-1} \cdot s_{-1} = s_{-1}$, whereas $s_{-1} \cdot s_{-1} = s_0 s_1 s_0 s_1 = s_{-1} s_1$ if R has type B_n.

Assume that $\alpha \in R^+$ is not cosmall. Then the root system R is not simply laced. If R has type B_n, then $\alpha = e_i$ where $2 \leq i \leq n$, the greedy decomposition of $\alpha^\vee$ is $(e_i + e_{i-1}, e_i - e_{i-1})$, and $s_\alpha = \sigma_i = s_{i-1} \cdot \sigma_{i-1} \cdot s_{i-1} < \sigma_{i-1} \cdot s_{i-1} \cdot \sigma_{i-1} \cdot s_{i-1} = \nu_{i-1,i} \cdot s_{i-1} = z_{\alpha^\vee}$. If R has type C_n, then $\alpha = e_j + e_i$ where $1 \leq i < j \leq n$, the greedy decomposition of $\alpha^\vee$ is $(2e_j, 2e_i)$, and $s_\alpha = \nu_{i,j} = \sigma_i \cdot \tau_{i,j} \cdot \sigma_i = \sigma_i \cdot d_{i+1,j} \cdot u_{i,j} \cdot \sigma_i = d_{i+1,j} \cdot \sigma_i \cdot u_{i,j} \cdot \sigma_i < d_{i,j} \cdot \sigma_i \cdot u_{i,j} \cdot \sigma_i = \sigma_j \cdot \sigma_i = z_{\alpha^\vee}$. This proves part (a).

Now let $\alpha, \beta \in R^+$ be cosmall roots. We must prove the inequality $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee + \beta^\vee}$ in the Bruhat order of W. Notice that this implies that $s_\beta \cdot s_\alpha = (s_\alpha \cdot s_\beta)^{-1} \leq (z_{\alpha^\vee + \beta^\vee})^{-1} = z_{\alpha^\vee + \beta^\vee}$, hence we are free to interchange α and β. We may assume that α (and β) is not a maximal root of $\alpha^\vee + \beta^\vee$, since otherwise (α, β) is a greedy decomposition and there is nothing to prove. In what follows we will use the convention that zero vectors should be omitted in any specification of a greedy decomposition. We consider five main cases. All commutations of factors in the identities below follow from Proposition 4.8.

Case 1: Assume that $\alpha = e_k - e_i$ and $\beta = e_l - e_j$ for some $i < k$ and $j < l$. Up to interchanging α and β, the assumption that α and β are not maximal roots of $\alpha^\vee + \beta^\vee$ implies that $i < j < k < l$. In this case the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_l - e_i, e_k - e_j)$, and $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\begin{aligned}\tau_{j,l} \cdot \tau_{i,k} &= \tau_{j,l} \cdot d_{j,k} \cdot \tau_{i,j} \cdot u_{j,k} = \tau_{j,l} \cdot d_{j+1,k} \cdot \tau_{i,j} \cdot u_{j,k} = \tau_{j,l} \cdot \tau_{i,j} \cdot d_{j+1,k} \cdot u_{j,k} \\ &= \tau_{j,l} \cdot \tau_{i,j} \cdot \tau_{j,k} = \tau_{j,l} \cdot u_{i,j-1} \cdot d_{i,j} \cdot \tau_{j,k} = u_{i,j-1} \cdot \tau_{j,l} \cdot d_{i,j} \cdot \tau_{j,k} \\ &\leq u_{i,j} \cdot \tau_{j,l} \cdot d_{i,j} \cdot \tau_{j,k} = \tau_{i,l} \cdot \tau_{j,k}\,.\end{aligned}$$

Case 2: Assume that $\alpha = 2e_j$ for some j. Then R has type C_n, and since α and β are not maximal roots of $\alpha^\vee + \beta^\vee$ we must have $\beta = e_k - e_i$ where $i \leq j \leq k$. The greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(2e_k, e_j - e_i)$; as mentioned above, the vector $e_j - e_i$ is omitted if $i = j$. The inequality $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\begin{aligned}\sigma_j \cdot \tau_{i,k} &= d_{i,j} \cdot \sigma_i \cdot u_{i,j} \cdot \tau_{i,k} = d_{i,j} \cdot \sigma_i \cdot u_{i+1,j} \cdot \tau_{i,k} = d_{i,j} \cdot u_{i+1,j} \cdot \sigma_i \cdot \tau_{i,k} \\ &= \tau_{i,j} \cdot \sigma_i \cdot \tau_{i,k} = \tau_{i,j} \cdot \sigma_i \cdot d_{i+1,k} \cdot u_{i,k} = \tau_{i,j} \cdot d_{i+1,k} \cdot \sigma_i \cdot u_{i,k} \leq \\ &= \tau_{i,j} \cdot d_{i,k} \cdot \sigma_i \cdot u_{i,k} = \tau_{i,j} \cdot \sigma_k\,.\end{aligned}$$

Case 3: Assume that $\alpha = e_1$. Then R has type B_n, and since α and β are not maximal roots of $\alpha^\vee + \beta^\vee$ we must have $\beta = e_i - e_1$ for some $i > 1$. The greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_i + e_1)$, and $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$s_0 \cdot \tau_{1,i} \leq s_0 \cdot \tau_{1,i} \cdot s_0 = \nu_{1,i}\,.$$

Case 4: Assume that $\alpha = e_j + e_i$ and $\beta = e_l - e_k$ for some $i < j$ and $k < l$. Then R has type B_n or D_n. The assumption that α is not a maximal root of $\alpha^\vee + \beta^\vee$ holds if and only if we have either $k \leq j < l$, or $i < l \leq j$ and $k \leq i < j - 1$. If $k \leq j < l$, then the inequality $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ follows from the subcases 4a, 4b, and 4c below, and if $i < l \leq j$ and $k \leq i < j - 1$, then it follows from the subcases 4d and 4e.

Case 4a: Assume that $\alpha = e_j + e_i$ and $\beta = e_k - e_i$ where $i < j < k$. Then the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_k + e_j)$. If R has type B_n, then the inequality $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\begin{aligned}\nu_{i,j} \cdot \tau_{i,k} &= \sigma_i \cdot \tau_{i,j} \cdot \sigma_i \cdot \tau_{i,k} = \sigma_i \cdot u_{i,j-1} \cdot d_{i,j} \cdot \sigma_i \cdot \tau_{i,k} \\ &= \sigma_i \cdot u_{i,j-1} \cdot d_{i,j} \cdot \sigma_i \cdot u_{i,j} \cdot \tau_{j,k} \cdot d_{i,j} = \sigma_i \cdot u_{i,j-1} \cdot \sigma_j \cdot \tau_{j,k} \cdot d_{i,j} \\ &= \sigma_i \cdot \sigma_j \cdot \tau_{j,k} \cdot u_{i,j-1} \cdot d_{i,j} = \sigma_i \cdot \sigma_j \cdot \tau_{j,k} \cdot \tau_{i,j} \\ &= \sigma_j \cdot \tau_{j,k} \cdot \sigma_i \cdot \tau_{i,j} \leq \sigma_j \cdot \tau_{j,k} \cdot \sigma_j = \nu_{j,k} \qquad \text{(by Case 2).}\end{aligned}$$

Otherwise R has type D_n and the inequality holds because

$$\begin{aligned}
\nu_{i,j} \cdot \tau_{i,k} &= \nu_{i,j} \cdot u_{i,k} \cdot d_{i,k-1} = d_{1,i} \cdot d_{2,j} \cdot s_{-1} \cdot u_{2,j} \cdot u_{1,i} \cdot u_{i,k} \cdot d_{i,k-1} \\
&= d_{1,i} \cdot d_{2,j} \cdot s_{-1} \cdot u_{2,j} \cdot u_{1,k} \cdot d_{i,k-1} = d_{1,i} \cdot d_{2,j} \cdot s_{-1} \cdot u_{1,k} \cdot u_{1,j-1} \cdot d_{i,k-1} \\
&= d_{1,i} \cdot d_{2,j} \cdot s_{-1} \cdot s_1 \cdot u_{2,k} \cdot u_{1,j-1} \cdot d_{i,k-1} \\
&= d_{1,i} \cdot d_{2,j} \cdot s_1 \cdot s_{-1} \cdot u_{2,k} \cdot u_{1,j-1} \cdot d_{i,k-1} \\
&= d_{1,i} \cdot d_{1,j} \cdot s_{-1} \cdot u_{2,k} \cdot u_{1,j-1} \cdot d_{i,k-1} \\
&= d_{1,i} \cdot d_{1,j} \cdot s_{-1} \cdot u_{2,k} \cdot u_{1,j-1} \cdot d_{j,k-1} \cdot d_{i,j} \\
&= d_{1,i} \cdot d_{1,j} \cdot s_{-1} \cdot u_{2,k} \cdot d_{j,k-1} \cdot u_{1,j-1} \cdot d_{i,j} \\
&= d_{1,i} \cdot d_{1,j} \cdot s_{-1} \cdot u_{2,k} \cdot d_{j,k-1} \cdot u_{1,j} \cdot d_{i,j-1} \\
&= d_{1,i} \cdot d_{1,j} \cdot s_{-1} \cdot u_{2,k} \cdot d_{j,k-1} \cdot d_{i+1,j} \cdot u_{1,j} \\
&= d_{1,i} \cdot d_{1,j} \cdot s_{-1} \cdot u_{2,k} \cdot d_{i+1,k-1} \cdot u_{1,j} = d_{1,i} \cdot d_{1,j} \cdot s_{-1} \cdot d_{i+2,k} \cdot u_{2,k} \cdot u_{1,j} \\
&= d_{1,j} \cdot d_{2,i+1} \cdot s_{-1} \cdot d_{i+2,k} \cdot u_{2,k} \cdot u_{1,j} = d_{1,j} \cdot d_{i+2,k} \cdot d_{2,i+1} \cdot s_{-1} \cdot u_{2,k} \cdot u_{1,j} \\
&\leq d_{1,j} \cdot d_{2,k} \cdot s_{-1} \cdot u_{2,k} \cdot u_{1,j} = \nu_{j,k} \,.
\end{aligned}$$

Case 4b: Assume that $\alpha = e_k + e_j$ and $\beta = e_l - e_i$ where $i \leq j < k < l$. Then the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_l + e_k, e_j - e_i)$, and $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\begin{aligned}
\nu_{j,k} \cdot \tau_{i,l} = \nu_{j,k} \cdot u_{i,j} \cdot \tau_{j,l} \cdot d_{i,j} = u_{i,j} \cdot \nu_{j,k} \cdot \tau_{j,l} \cdot d_{i,j} \leq \qquad &\text{(by Case 4a)} \\
u_{i,j} \cdot \nu_{k,l} \cdot d_{i,j} = \nu_{k,l} \cdot u_{i,j} \cdot d_{i,j} = \nu_{k,l} \cdot \tau_{i,j} \,. &
\end{aligned}$$

Case 4c: Assume that $\alpha = e_k + e_i$ and $\beta = e_l - e_j$ where $i < j \leq k < l$. Then the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_l + e_i, e_k - e_j)$, and $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\begin{aligned}
\nu_{i,k} \cdot \tau_{j,l} &= \nu_{i,k} \cdot u_{j,k} \cdot \tau_{k,l} \cdot d_{j,k} = \nu_{i,k} \cdot u_{j,k} \cdot d_{k+1,l} \cdot u_{k,l} \cdot d_{j,k} \\
&= u_{j,k} \cdot d_{k+1,l} \cdot \nu_{i,k} \cdot u_{k,l} \cdot d_{j,k} \leq u_{j,k} \cdot d_{k,l} \cdot \nu_{i,k} \cdot u_{k,l} \cdot d_{j,k} \\
&= u_{j,k} \cdot \nu_{i,l} \cdot d_{j,k} = \nu_{i,l} \cdot u_{j,k} \cdot d_{j,k} = \nu_{i,l} \cdot \tau_{j,k} \,.
\end{aligned}$$

Case 4d: Assume that $\alpha = e_k + e_j$ and $\beta = e_k - e_i$ where $i \leq j < k-1$. Then the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_k + e_{k-1}, e_k - e_{k-1}, e_j - e_i)$, and $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\begin{aligned}
\nu_{j,k} \cdot \tau_{i,k} &= \nu_{j,k} \cdot u_{i,j} \cdot \tau_{j,k} \cdot d_{i,j} = \nu_{j,k} \cdot u_{i,j} \cdot d_{j+1,k} \cdot u_{j,k} \cdot d_{i,j} \\
&= u_{i,j} \cdot d_{j+1,k} \cdot \nu_{j,k} \cdot u_{j,k} \cdot d_{i,j} \leq u_{i,j} \cdot d_{j,k} \cdot \nu_{j,k} \cdot u_{j,k} \cdot d_{i,j} \\
&= u_{i,j} \cdot s_{k-1} \cdot d_{j,k-1} \cdot \nu_{j,k} \cdot u_{j,k-1} \cdot s_{k-1} \cdot d_{i,j} \\
&= u_{i,j} \cdot s_{k-1} \cdot \nu_{k-1,k} \cdot s_{k-1} \cdot d_{i,j} = u_{i,j} \cdot \nu_{k-1,k} \cdot s_{k-1} \cdot s_{k-1} \cdot d_{i,j} \\
&= u_{i,j} \cdot \nu_{k-1,k} \cdot s_{k-1} \cdot d_{i,j} = \nu_{k-1,k} \cdot s_{k-1} \cdot u_{i,j} \cdot d_{i,j} = \nu_{k-1,k} \cdot s_{k-1} \cdot \tau_{i,j} \,.
\end{aligned}$$

Case 4e: Assume that $\alpha = e_l + e_j$ and $\beta = e_k - e_i$ where $i \leq j < k < l$. Then the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_l + e_k, e_j - e_i)$, and $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\begin{aligned}
\nu_{j,l} \cdot \tau_{i,k} &= \nu_{j,l} \cdot u_{i,j} \cdot \tau_{j,k} \cdot d_{i,j} = \nu_{j,l} \cdot u_{i,j} \cdot d_{j+1,k} \cdot u_{j,k} \cdot d_{i,j} \\
&= u_{i,j} \cdot d_{j+1,k} \cdot \nu_{j,l} \cdot u_{j,k} \cdot d_{i,j} \leq u_{i,j} \cdot d_{j,k} \cdot \nu_{j,l} \cdot u_{j,k} \cdot d_{i,j} \\
&= u_{i,j} \cdot \nu_{k,l} \cdot d_{i,j} = \nu_{k,l} \cdot u_{i,j} \cdot d_{i,j} = \nu_{k,l} \cdot \tau_{i,j} \,.
\end{aligned}$$

Case 5: Assume that $\alpha = e_l + e_k$ and $\beta = e_j + e_i$ for some $i < j$ and $k < l$. Then R has type B_n or D_n. The assumption that α and β are not maximal roots of $\alpha^\vee + \beta^\vee$ implies that $i < l$ and $k < j$. If $j \neq l$, then the inequality $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$

follows from the subcases 5a, 5b, and 5c below, and if $j = l$, then it follows from the subcases 5d and 5e.

Case 5a: Assume that $\alpha = e_l + e_j$ and $\beta = e_k + e_i$ where $i < j < k < l$. Then the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_l + e_k, e_j + e_i)$. If R has type B_n, then the inequality $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\begin{aligned}\nu_{j,l} \cdot \nu_{i,k} &= \nu_{j,l} \cdot \sigma_i \cdot \tau_{i,k} \cdot \sigma_i = \sigma_i \cdot \nu_{j,l} \cdot \tau_{i,k} \cdot \sigma_i \leq \sigma_i \cdot \nu_{k,l} \cdot \tau_{i,j} \cdot \sigma_i \quad \text{(by Case 4)}\\ &= \nu_{k,l} \cdot \sigma_i \cdot \tau_{i,j} \cdot \sigma_i = \nu_{k,l} \cdot \nu_{i,j}\,.\end{aligned}$$

Otherwise R has type D_n and the inequality holds because

$$\begin{aligned}\nu_{j,l} \cdot \nu_{i,k} &= \nu_{j,l} \cdot d_{1,i} \cdot \nu_{1,k} \cdot u_{1,i} = \nu_{j,l} \cdot d_{1,i} \cdot s_{-1} \cdot \tau_{2,k} \cdot s_{-1} \cdot u_{1,i}\\ &= d_{1,i} \cdot s_{-1} \cdot \nu_{j,l} \cdot \tau_{2,k} \cdot s_{-1} \cdot u_{1,i} \leq d_{1,i} \cdot s_{-1} \cdot \nu_{k,l} \cdot \tau_{2,j} \cdot s_{-1} \cdot u_{1,i} \quad \text{(by Case 4)}\\ &= \nu_{k,l} \cdot d_{1,i} \cdot s_{-1} \cdot \tau_{2,j} \cdot s_{-1} \cdot u_{1,i} = \nu_{k,l} \cdot d_{1,i} \cdot \nu_{1,j} \cdot u_{1,i} = \nu_{k,l} \cdot \nu_{i,j}\,.\end{aligned}$$

Case 5b: Assume that $\alpha = e_l + e_i$ and $\beta = e_k + e_j$ where $i < j < k < l$. Then the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_l + e_k, e_j + e_i)$. If R has type B_n, then the inequality $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\begin{aligned}\nu_{i,l} \cdot \nu_{j,k} &= \sigma_i \cdot \tau_{i,l} \cdot \sigma_i \cdot \nu_{j,k} = \sigma_i \cdot \tau_{i,l} \cdot \nu_{j,k} \cdot \sigma_i \leq \sigma_i \cdot \nu_{k,l} \cdot \tau_{i,j} \cdot \sigma_i \quad \text{(by Case 4)}\\ &= \nu_{k,l} \cdot \sigma_i \cdot \tau_{i,j} \cdot \sigma_i = \nu_{k,l} \cdot \nu_{i,j}\,.\end{aligned}$$

Otherwise R has type D_n and the inequality holds because

$$\begin{aligned}\nu_{i,l} \cdot \nu_{j,k} &= d_{1,i} \cdot \nu_{1,l} \cdot u_{1,i} \cdot \nu_{j,k} = d_{1,i} \cdot s_{-1} \cdot \tau_{2,l} \cdot s_{-1} \cdot u_{1,i} \cdot \nu_{j,k}\\ &= d_{1,i} \cdot s_{-1} \cdot \tau_{2,l} \cdot \nu_{j,k} \cdot s_{-1} \cdot u_{1,i} \leq d_{1,i} \cdot s_{-1} \cdot \nu_{k,l} \cdot \tau_{2,j} \cdot s_{-1} \cdot u_{1,i} \quad \text{(by Case 4)}\\ &= \nu_{k,l} \cdot d_{1,i} \cdot s_{-1} \cdot \tau_{2,j} \cdot s_{-1} \cdot u_{1,i} = \nu_{k,l} \cdot d_{1,i} \cdot \nu_{1,j} \cdot u_{1,i} = \nu_{k,l} \cdot \nu_{i,j}\,.\end{aligned}$$

Case 5c: Assume that $\alpha = e_k + e_i$ and $\beta = e_j + e_i$ where $i < j < k$. If $i = 1$, then the assumption that α is not a maximal root of $\alpha^\vee + \beta^\vee$ implies that R has type B_n. In this case the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_k + e_j, e_1)$, and the inequality $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\nu_{1,k} \cdot \nu_{1,j} = \nu_{1,k} \cdot s_0 \cdot \tau_{1,j} \cdot s_0 = \nu_{1,k} \cdot \tau_{1,j} \cdot s_0 \leq \nu_{j,k} \cdot s_0 \qquad \text{(by Case 4)}.$$

Otherwise we have $1 < i < j < k$, the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_k + e_j, e_i + e_{i-1}, e_i - e_{i-1})$, and the inequality follows from Case 5a because

$$\nu_{i,k} \cdot \nu_{i,j} = \nu_{i,k} \cdot s_{i-1} \cdot \nu_{i-1,j} \cdot s_{i-1} = \nu_{i,k} \cdot \nu_{i-1,j} \cdot s_{i-1} \leq \nu_{j,k} \cdot \nu_{i-1,i} \cdot s_{i-1}\,.$$

Case 5d: Assume that $\alpha = e_k + e_j$ and $\beta = e_k + e_i$ where $i < j < k-1$. Then the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_k + e_{k-1}, e_k - e_{k-1}, e_j + e_i)$, and the inequality $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\begin{aligned}\nu_{j,k} \cdot \nu_{i,k} = \nu_{j,k} \cdot s_{k-1} \cdot \nu_{i,k-1} \cdot s_{k-1} = \nu_{j,k} \cdot \nu_{i,k-1} \cdot s_{k-1} \leq \qquad &\text{(by Case 5a)}\\ \nu_{k-1,k} \cdot \nu_{i,j} \cdot s_{k-1} = \nu_{k-1,k} \cdot s_{k-1} \cdot \nu_{i,j}\,.&\end{aligned}$$

Case 5e: Assume that $\alpha = e_j + e_i$ and $\beta = e_j + e_i$ where $i < j-1$. If $i = 1$, then the assumption that α is not a maximal root of $\alpha^\vee + \beta^\vee$ implies that R has type B_n. In this case the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_j + e_{j-1}, e_j - e_{j-1}, e_1)$, and the inequality $s_\alpha \cdot s_\beta \leq z_{\alpha^\vee+\beta^\vee}$ holds because

$$\nu_{1,i} \cdot \nu_{1,i} = \nu_{1,i} \cdot s_0 \cdot \tau_{1,i} \cdot s_0 = \nu_{1,i} \cdot \tau_{1,i} \cdot s_0 \leq \nu_{i-1,i} \cdot s_{i-1} \cdot s_0 \qquad \text{(by Case 4)}.$$

Otherwise we have $1 < i < j-1$, the greedy decomposition of $\alpha^\vee + \beta^\vee$ is $(e_j + e_{j-1}, e_j - e_{j-1}, e_i + e_{i-1}, e_i - e_{i-1})$, and the inequality follows from Case 5d because

$$\nu_{i,j} \cdot \nu_{i,j} = \nu_{i,j} \cdot s_{i-1} \cdot \nu_{i-1,j} \cdot s_{i-1} = \nu_{i,j} \cdot \nu_{i-1,j} \cdot s_{i-1} \leq \nu_{j-1,j} \cdot s_{j-1} \cdot \nu_{i-1,i} \cdot s_{i-1}\,.$$

Since the above cases cover all possibilities for α and β, the proof is complete. $\square$

Theorem 4.11. *Let $d \in \mathbb{Z}\Delta^\vee$ be an effective degree.*
(a) *If $\alpha \in R^+$ satisfies $\alpha^\vee \leq d$, then $s_\alpha \cdot z_{d-\alpha^\vee} \leq z_d$.*
(b) *If $0 \leq d' \leq d$, then $z_{d'} \cdot z_{d-d'} \leq z_d$.*

Proof. We proceed by induction on d, the case $d = 0$ being clear. Let $d > 0$ and assume that the theorem is true for all strictly smaller degrees. We first show that part (b) follows from part (a). Given an effective degree d' with $0 < d' < d$, let $\alpha \in R^+$ be a maximal root of d'. Then part (a) and the induction hypothesis imply that

$$z_{d'} \cdot z_{d-d'} = s_\alpha \cdot z_{d'-\alpha^\vee} \cdot z_{d-d'} \leq s_\alpha \cdot z_{d-\alpha^\vee} \leq z_d$$

as required.

We prove part (a) by descending induction on α. The statement is true by definition if α is a maximal root of d. Assume α is not a maximal root of d. If α is not cosmall, then let $\beta > \alpha$ be a maximal root of $\alpha^\vee$. We then obtain from Lemma 4.10(a) that

$$s_\alpha \cdot z_{d-\alpha^\vee} \leq z_{\alpha^\vee} \cdot z_{d-\alpha^\vee} = s_\beta \cdot z_{\alpha^\vee - \beta^\vee} \cdot z_{d-\alpha^\vee} \leq s_\beta \cdot z_{d-\beta^\vee} \leq z_d \,.$$

We may therefore assume that α is cosmall. Since α is not a maximal root of d, we can choose a cosmall root γ such that $\alpha < \gamma$ and $\gamma^\vee \leq d$. By Lemma 4.7 we may assume that $\gamma^\vee - \alpha^\vee$ is a simple coroot. We can therefore choose a maximal root β of $d - \alpha^\vee$ such that $\gamma^\vee - \alpha^\vee \leq \beta^\vee$. Now let $\gamma' \geq \gamma$ be a maximal root of $\alpha^\vee + \beta^\vee$. We finally obtain from Lemma 4.10(b) that

$$\begin{aligned} s_\alpha \cdot z_{d-\alpha^\vee} = s_\alpha \cdot s_\beta \cdot z_{d-\alpha^\vee-\beta^\vee} &\leq z_{\alpha^\vee+\beta^\vee} \cdot z_{d-\alpha^\vee-\beta^\vee} \\ &= s_{\gamma'} \cdot z_{\alpha^\vee+\beta^\vee-\gamma'^\vee} \cdot z_{d-\alpha^\vee-\beta^\vee} \leq s_{\gamma'} \cdot z_{d-\gamma'^\vee} \leq z_d \,. \end{aligned}$$

This completes the proof. $\square$

4.2. **General homogeneous spaces.** We finish this section by extending the construction of z_d to an arbitrary homogeneous space $X = G/P$. Given a degree $d \in H_2(X) = \mathbb{Z}\Delta^\vee/\mathbb{Z}\Delta_P^\vee$, the maximal elements of the set $\{\alpha \in R^+ \smallsetminus R_P^+ \mid \alpha^\vee + \Delta_P^\vee \leq d\}$ are called *maximal roots* of d. The root $\alpha \in R^+ \smallsetminus R_P^+$ is called *P-cosmall* if α is a maximal root of $\alpha^\vee + \Delta_P^\vee \in H_2(X)$. Notice that any P-cosmall root is cosmall, and a B-cosmall root is the same as a cosmall root. The highest root in R is P-cosmall for every parabolic subgroup P. If α is P-cosmall, then Proposition 4.8 implies that $s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha$ for each $\beta \in \Delta_P$. It follows that $s_\alpha \cdot w_P = w_P \cdot s_\alpha$.

Define a *greedy decomposition* of $d \in H_2(X)$ to be a sequence of positive roots $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that $\alpha_1 \in R^+ \smallsetminus R_P^+$ is a maximal root of d and $(\alpha_2, \ldots, \alpha_k)$ is a greedy decomposition of $d - \alpha_1^\vee \in H_2(X)$. The empty sequence is the only greedy decomposition of $0 \in H_2(X)$. If $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a greedy decomposition of $d \in H_2(X)$, then for any sufficiently large degree $e \in H_2(G/B)$ such that $e + \mathbb{Z}\Delta_P^\vee = d$ there exist positive roots $\gamma_1, \ldots, \gamma_m \in R_P^+$ such that $(\alpha_1, \ldots, \alpha_k, \gamma_1, \ldots, \gamma_m)$ is a greedy decomposition of e. It follows from this that any other greedy decomposition of d is a reordering of $(\alpha_1, \ldots, \alpha_k)$.

If $d \in H_2(X)$ is any effective degree and $(\alpha_1, \ldots, \alpha_k)$ is a greedy decomposition of d, then we let $z_d^P \in W^P$ be the unique element satisfying

$$z_d^P w_P = s_{\alpha_1} \cdot s_{\alpha_2} \cdot \ldots \cdot s_{\alpha_k} \cdot w_P \,.$$

Notice that $w_P \cdot z_d^P w_P = z_d^P w_P$.

Corollary 4.12. *Let $d \in H_2(X)$ be an effective degree.*

(a) *If $\alpha \in R^+$ satisfies $\alpha^\vee + \mathbb{Z}\Delta_P^\vee \leq d$, then $s_\alpha \cdot z_{d-\alpha^\vee}^P w_P \leq z_d^P w_P$.*

(b) *If $0 \leq d' \leq d$, then $z_{d'}^P \cdot z_{d-d'}^P w_P \leq z_d^P w_P$.*

(c) *If $\alpha \in R^+ \smallsetminus R_P^+$ is a maximal root of d, then $s_\alpha \cdot z_{d-\alpha^\vee}^P w_P = z_d^P w_P$.*

(d) *For all sufficiently large degrees $e \in H_2(G/B)$ such that $e + \mathbb{Z}\Delta_P = d \in H_2(X)$ we have $z_d^P w_P = z_e$.*

Proof. Parts (a) and (b) follow from Theorem 4.11, and parts (c) and (d) are clear from the definitions. $\square$

Remark 4.13. Let $0 \leq d \in H_2(X)$. For any root $\alpha \in R^+$ such that $\alpha^\vee \leq d \in H_2(X)$ we have (1) $z_{d'}^P w_P \cdot s_\alpha \leq z_d^P w_P$, (2) $s_\alpha \cdot z_{d'}^P w_P \leq z_d^P w_P$, (3) $z_{d'}^P \cdot s_\alpha W_P \leq z_d^P W_P$, and (4) $s_\alpha \cdot z_{d'}^P W_P \leq z_d^P W_P$, where $d' = d - \alpha^\vee \in H_2(X)$. Furthermore, if $\alpha \in R^+ \smallsetminus R_P^+$ is any maximal root of d, then (1), (2), (3), and (4) hold with equality.

5. Curve Neighborhoods

5.1. **The main theorem.** Given an effective degree $d \in H_2(X) = \mathbb{Z}\Delta^\vee / \mathbb{Z}\Delta_P^\vee$, let $\overline{\mathcal{M}}_{0,n}(X, d)$ be the Kontsevich space of n-pointed stable maps of degree d to X, with total evaluation map $\mathrm{ev} = (\mathrm{ev}_1, \ldots, \mathrm{ev}_n) : \overline{\mathcal{M}}_{0,n}(X, d) \to X^n$. We have

$$\dim \overline{\mathcal{M}}_{0,n}(X, d) = \dim(X) + (c_1(T_X), d) + n - 3$$

where $c_1(T_X)$ is given by (3). Given any subvariety $\Omega \subset X$, define the degree d *curve neighborhood* of Ω to be $\Gamma_d(\Omega) = \mathrm{ev}_1(\mathrm{ev}_2^{-1}(\Omega))$. A geometric argument in [5, Cor. 3.3(a)] shows that, if Z is a B-stable Schubert variety, then so is $\Gamma_d(\Omega)$. The following result gives a more combinatorial proof of this fact, and also identifies the Weyl group element of the curve neighborhood. This result is equivalent to Theorem 1.

Theorem 5.1. *For any $w \in W$ we have $\Gamma_d(X(w)) = X(w \cdot z_d^P)$.*

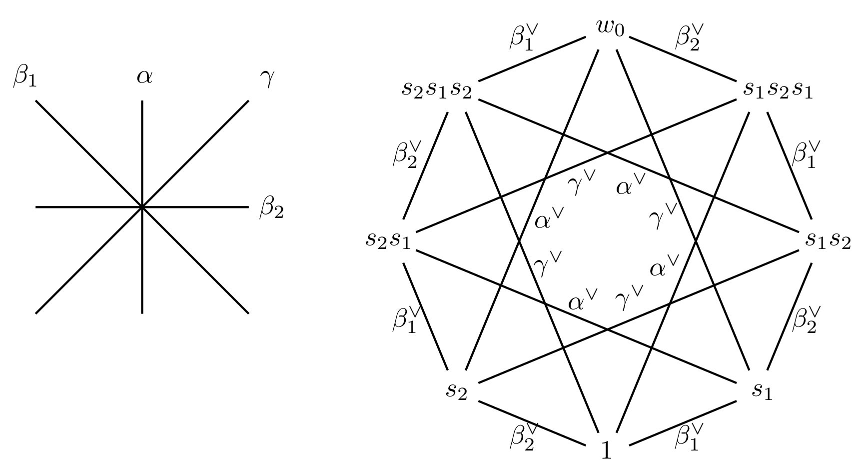
Proof. We prove the result by induction on d, the case $d = 0$ being clear. Assume that $d > 0$ and that $\Gamma_{d'}(X(u)) = X(u \cdot z_{d'}^P)$ for all $d' < d$ and all $u \in W$. Let $\alpha \in R^+ \smallsetminus R_P^+$ be a maximal root of d and set $v = (w \cdot s_\alpha) s_\alpha$. Since $v.P \in X(w)$ and $v.C_\alpha$ is a curve of degree $\alpha^\vee + \mathbb{Z}\Delta_P^\vee$ from $v.P$ to $(w \cdot s_\alpha).P$, it follows that $X(w \cdot s_\alpha) \subset \Gamma_{\alpha^\vee}(X(w))$. We therefore obtain $X(w \cdot z_d^P) = X(w \cdot s_\alpha \cdot z_{d-\alpha^\vee}^P) = \Gamma_{d-\alpha^\vee}(X(w \cdot s_\alpha)) \subset \Gamma_{d-\alpha^\vee}(\Gamma_{\alpha^\vee}(X(w))) \subset \Gamma_d(X(w))$.

On the other hand, let $u.P \in \Gamma_d(X(w))$ be any T-fixed point. Since the locus of curves of degree d from $X(w)$ to $u.P$ is a closed T-stable subvariety of $\overline{\mathcal{M}}_{0,2}(X, d)$, it follows that this locus contains a T-stable curve C connecting $u.P$ to a point $v.P \in X(w)$ where $v \in W^P$. This curve must be a connected union of irreducible T-stable components. At least one component contains $v.P$, and any such component has the form $v.C_\alpha$ with $\alpha \in R^+ \smallsetminus R_P^+$. Choose a component $v.C_\alpha$ such that $C' = \overline{C \smallsetminus v.C_\alpha}$ connects $vs_\alpha.P$ to $u.P$. Since $vs_\alpha.P \in X(w \cdot s_\alpha)$ and $[C'] \leq d - \alpha^\vee \in H_2(X)$, it follows from Corollary 4.12(a) that $u.P \in \Gamma_{d-\alpha^\vee}(X(w \cdot s_\alpha)) = X(w \cdot s_\alpha \cdot z_{d-\alpha^\vee}^P) \subset X(w \cdot z_d^P)$. Since $\Gamma_d(X(w))$ is B-stable and all its T-fixed points belong to $X(w \cdot z_d^P)$, we deduce that $\Gamma_d(X(w)) \subset X(w \cdot z_d^P)$. This completes the proof. $\square$

Remark 5.2. The curve neighborhood of the opposite Schubert variety $Y(w)$ is given by $\Gamma_d(Y(w)) = Y(w_0(w_0 w \cdot z_d^P))$, where $w_0(w_0 w \cdot z_d^P)$ may be regarded as a 'decreasing' Hecke product of w and z_d^P.

5.2. **The moment graph.** The element $z_d^P \in W^P$ can also be constructed using the *moment graph* of X. The vertices of this graph are the T-fixed points X^T and the edges are the irreducible T-stable curves in X. More precisely, there is an edge between $u.P$ and $w.P$ if and only if $wW_P = us_\alpha W_P$ for some root $\alpha \in R^+ \smallsetminus R_P^+$; the corresponding T-stable curve is $u.C_\alpha \subset X$ which has degree $[C_\alpha] = \alpha^\vee + \mathbb{Z}\Delta_P \in H_2(X)$. Define the *weight* of a path in the moment graph to be the sum of the degrees of its edges. Given $d \in H_2(X)$, let $Z_d \subset X^T$ be the subset of points $w.P$ for which there exists a path from $1.P$ to $w.P$ of weight at most d. Then Theorem 1 implies that $z_d^P.P$ is the unique maximal element of Z_d in the Bruhat order on X^T, defined by $u.P \leq w.P$ if and only if $uW_P \leq wW_P$.

Example 5.3. Let $X = \mathrm{SO}(5)/B = \mathrm{OF}(5)$ be the variety of isotropic flags in the vector space $\mathbb{C}^5$ equipped with an orthogonal form. The corresponding root system has type B_2. Let $\Delta = \{\beta_1, \beta_2\}$ be the simple roots, with β_1 long and β_2 short. The remaining positive roots are $\alpha = \beta_1 + \beta_2$ and $\gamma = \beta_1 + 2\beta_2$, with coroots $\alpha^\vee = 2\beta_1^\vee + \beta_2^\vee$ and $\gamma^\vee = \beta_1^\vee + \beta_2^\vee$. Write $s_i = s_{\beta_i}$ for $i = 1, 2$. The moment graph of X is displayed below, with each edge labeled by its degree. Since the paths of weight $\gamma^\vee$ starting at 1 are $1 \to s_1 \to s_1 s_2$, $1 \to s_2 \to s_2 s_1$, and $1 \to s_\gamma = s_2 s_1 s_2$, we have $z_{\gamma^\vee} = s_\gamma$. On the other hand, the paths of weight $\alpha^\vee$ starting at 1 include $1 \to s_1 \to w_0$, so $z_{\alpha^\vee} = w_0 \neq s_\alpha = s_1 s_2 s_1$.



5.3. **Line neighborhoods.** Let $\gamma \in \Delta \smallsetminus \Delta_P$ and consider the Dynkin diagram of the simple roots in the set $\Delta_P \cup \{\gamma\}$. We will say that γ is a *Fano root* of X if γ is at least as long as all other roots in its connected component in this diagram. It has been proved by Strickland [26] and by Landsberg and Manivel [21] that γ is a Fano root for X if and only if the Fano variety of lines in X of degree $\gamma^\vee \in H_2(X)$ is a homogeneous space. In this case we will compute the element $z_{\gamma^\vee}^P$ that describes neighborhoods defined by lines of degree $\gamma^\vee$.

Lemma 5.4. *Let $\gamma \in \Delta \smallsetminus \Delta_P$. Then $w_P(\gamma)$ is the largest root in $R \cap (\gamma + \mathbb{Z}\Delta_P)$.*

Proof. Let ρ be any maximal root in the set $R \cap (\gamma + \mathbb{Z}\Delta_P)$. Then $w_P(\rho) - \gamma$ is a non-negative linear combination of Δ_P. Since $w_P(\Delta_P) \subset R^-$ we obtain $\rho - w_P(\gamma) = w_P(w_P(\rho) - \gamma) \leq 0$. Since $w_P(\gamma) \in R \cap (\gamma + \mathbb{Z}\Delta_P)$, this implies that $\rho = w_P(\gamma)$. $\square$

Proposition 5.5. *Let $\gamma \in \Delta \smallsetminus \Delta_P$ be any Fano root for X. Then $w_P(\gamma)$ is the unique maximal root of $\gamma^\vee + \mathbb{Z}\Delta_P^\vee \in H_2(X)$.*

Proof. Let $\alpha \in R^+ \smallsetminus R_P^+$ be any maximal root of $\gamma^\vee + \mathbb{Z}\Delta_P^\vee$. Then we have $\alpha^\vee = \gamma^\vee + y$ for some $y \in \mathbb{Z}\Delta_P^\vee$, and $\alpha = \frac{|\alpha|^2}{|\gamma|^2}\gamma + \frac{|\alpha|^2}{2} y$. Since γ is a Fano root of X and the support of α is contained in the connected component of γ in the Dynkin diagram of $\Delta_P \cup \{\gamma\}$, it follows that $|\alpha| \leq |\gamma|$. We deduce that $\alpha \in R \cap (\gamma + \mathbb{Z}\Delta_P)$, so Lemma 5.4 implies that $\alpha \leq \rho := w_P(\gamma)$. Finally, since $\rho^\vee + \mathbb{Z}\Delta_P^\vee = \gamma^\vee + \mathbb{Z}\Delta_P^\vee \in H_2(X)$, we must have $\alpha = \rho$. $\square$

Corollary 5.6. *If $\gamma \in \Delta \smallsetminus \Delta_P$ is a Fano root for X, then $z^P_{\gamma^\vee} W_P = w_P s_\gamma W_P$.*

Proof. The root $\rho = w_P(\gamma)$ is the unique maximal root of $\gamma^\vee \in H_2(X)$, so we have $z^P_{\gamma^\vee} w_P = s_\rho \cdot w_P = w_P s_\gamma w_P \cdot w_P = w_P s_\gamma \cdot w_P$. $\square$

5.4. **Degree distances in cominuscule varieties.** A simple root $\gamma \in \Delta$ is *cominuscule* if, when the highest root in R^+ is written as a linear combination of simple roots, the coefficient of γ is one. The variety $X = G/P$ is cominuscule if P is a maximal parabolic subgroup and $\Delta_P = \Delta \smallsetminus \{\gamma\}$ for a cominuscule root γ. In this case $H_2(X) \cong \mathbb{Z}$ is generated by $\gamma^\vee + \mathbb{Z}\Delta_P^\vee$, so any degree can be identified with an integer. The following result is known from combining lemmas 4.2 and 4.4 in [5].

Theorem 5.7. *Assume that $X = G/P$ is cominuscule with $\Delta_P = \Delta \smallsetminus \{\gamma\}$. For any effective degree $d \in H_2(X)$ we have $z^P_d W_P = (w_P s_\gamma) \cdot (w_P s_\gamma) \cdot \ldots \cdot (w_P s_\gamma) W_P$ where the Hecke product has d factors.*

Proof. Lemma 5.4 and the cominuscule condition imply that $\rho = w_P(\gamma)$ is the highest root in R^+. Since $\rho^\vee + \mathbb{Z}\Delta_P^\vee = \gamma^\vee + \mathbb{Z}\Delta_P^\vee$ is the smallest positive degree in $H_2(X)$, it follows that ρ is the only P-cosmall root in $R^+ \smallsetminus R_P^+$. The greedy decomposition of $d \in H_2(X)$ is therefore $(\rho, \rho, \ldots, \rho)$, with ρ repeated d times. Finally, since $s_\rho \cdot w_P = w_P s_\gamma \cdot w_P$, we obtain

$$z^P_d w_P = s_\rho \cdot s_\rho \cdot \ldots \cdot s_\rho \cdot w_P = (w_P s_\gamma) \cdot (w_P s_\gamma) \cdot \ldots \cdot (w_P s_\gamma) \cdot w_P \,,$$

as required. $\square$

Given two point $x, y \in X$ in a cominuscule variety, define the *degree distance* $d(x, y)$ to be the smallest possible degree of a rational curve from x to y. The study of this integer was suggested by Zak [27] and plays a fundamental role in Chaput, Manivel, and Perrin's generalization [9] of the 'quantum equals classical' principle from [6]. The maximal value of $d(x, y)$ was computed by Hwang and Kebekus [16]. Notice that we may choose $g \in G$ such that $g.x = 1.P$ and $g.y = u.P$ for some $u \in W^P$, in which case $d(x, y) = d(1.P, u.P)$. The function $d(x, y)$ is therefore determined by the following corollary, which was obtained earlier in [9, Prop. 18].

Corollary 5.8. *Assume that $X = G/P$ is cominuscule with $\Delta_P = \Delta \smallsetminus \{\gamma\}$, and let $u \in W^P$. Then $d(1.P, u.P)$ is the number of occurrences of s_γ in any reduced expression for u.*

Proof. For any degree $d \in H_2(X)$ it follows from Theorem 5.7 that $u.P \in \Gamma_d(X(1))$ if and only if u has a reduced expression with at most d occurrences of s_γ. Set $d = d(1.P, u.P)$. Since we have $u.P \in \Gamma_d(X(1))$ and $u.P \notin \Gamma_{d-1}(X(1))$, we deduce that u has a reduced expression with exactly d occurrences of s_γ. We finally appeal to Stembridge's result [25] that all elements of W^P are *fully commutative*, i.e. any reduced expression for an element of W^P can be obtained from any other by interchanging commuting simple transpositions (see also [8, §2] for an alternative proof of this fact). This implies that all reduced expressions for u have the same number of occurrences of s_γ. □

6. Criteria for cosmall roots

In this section we prove several criteria for cosmall roots. These results will be useful for proving the quantum Chevalley formula. We start with the following two theorems which are proved together.

Theorem 6.1. *Given any root $\alpha \in R^+ \smallsetminus R_P^+$ we have $\ell(s_\alpha W_P) \leq (c_1(T_X), \alpha^\vee) - 1$. Moreover, the following are equivalent.*

(a) *The root α is P-cosmall.*

(b) *We have equality $\ell(s_\alpha W_P) = (c_1(T_X), \alpha^\vee) - 1$.*

(c) *We have $(R^+ \smallsetminus R_P^+) \cap s_\alpha(R_P^+) = \emptyset$ and $(\gamma, \alpha^\vee) = 1$ for all $\gamma \in I(s_\alpha) \smallsetminus (R_P^+ \cup \{\alpha\})$.*

Theorem 6.2. *Let $0 < d \in H_2(X)$. Then $\ell(z_d^P) \leq (c_1(T_X), d) - 1$. Furthermore, if $\ell(z_d^P) = (c_1(T_X), d) - 1$, then $d = \alpha^\vee + \mathbb{Z}\Delta_P^\vee$ for a unique P-cosmall root α.*

Proof of Theorems 6.1 and 6.2. Given $\alpha \in R^+ \smallsetminus R_P^+$ we consider the sets $A = I(s_\alpha) \smallsetminus (R_P^+ \cup \{\alpha\})$, $B = (R^+ \smallsetminus R_P^+) \cap s_\alpha(R_P^+)$, and $C = (R^+ \smallsetminus R_P^+) \cap s_\alpha(R^+ \smallsetminus R_P^+)$. Since $R^+ \smallsetminus R_P^+$ is the disjoint union of $\{\alpha\}$, A, B, and C, we obtain from (3) that

$$(c_1(T_X), \alpha^\vee) = 2 + \sum_{\gamma \in A} (\gamma, \alpha^\vee) + \sum_{\gamma \in B} (\gamma, \alpha^\vee) + \sum_{\gamma \in C} (\gamma, \alpha^\vee) \,.$$

Notice that $|A| = \ell(s_\alpha W_P) - 1$, $(\gamma, \alpha^\vee) \geq 1$ for all $\gamma \in A \cup B$, and Lemma 2.1 implies that $\sum_{\gamma \in C} (\gamma, \alpha^\vee) = 0$. The inequality $\ell(s_\alpha W_P) \leq (c_1(T_X), \alpha^\vee) - 1$ and the equivalence of (b) and (c) in Theorem 6.1 follow from this.

We next prove Theorem 6.2. Let $0 < d \in H_2(X)$ and let γ be a maximal root of d. If $d - \gamma^\vee \neq 0 \in H_2(X)$, then we obtain by induction that $\ell(z_d^P) = \ell(z_{d-\gamma^\vee}^P \cdot s_\gamma W_P) \leq \ell(z_{d-\gamma^\vee}^P) + \ell(s_\gamma W_P) \leq (c_1(T_X), d - \gamma^\vee) - 1 + (c_1(T_X), \gamma^\vee) - 1 = (c_1(T_X), d) - 2$. We deduce that, if $\ell(z_d^P) = (c_1(T_X), d) - 1$, then $d = \gamma^\vee + \mathbb{Z}\Delta_P^\vee$ in $H_2(X)$. The uniqueness of γ follows because the greedy decomposition (γ) of d is unique.

Assume that $\alpha \in R^+ \smallsetminus R_P^+$ satisfies $\ell(s_\alpha W_P) = (c_1(T_X), \alpha^\vee) - 1$. Since $s_\alpha W_P \leq z_{\alpha^\vee}^P W_P$, we deduce from Theorem 6.2 that $\alpha^\vee = \gamma^\vee \in H_2(X)$ for a P-cosmall root $\gamma \in R^+ \smallsetminus R_P^+$, and we have $s_\alpha W_P = z_{\alpha^\vee}^P W_P = s_\gamma W_P$. Lemma 2.2 therefore implies that $\alpha = \gamma$. This proves the implication (b) $\Rightarrow$ (a) in Theorem 6.1.

On the other hand, assume that condition (c) fails. If there exists a root $\gamma \in A$ with $(\gamma, \alpha^\vee) \geq 2$, then α is short, γ is long, and $(\alpha, \gamma^\vee) = 1$. Set $\beta = -s_\alpha(\gamma) \in R^+$. Then $\beta^\vee + \gamma^\vee = \gamma^\vee - s_\alpha(\gamma^\vee) = \alpha^\vee$, and $(\alpha, \beta^\vee) = (\alpha, \alpha^\vee - \gamma^\vee) = 1$. Since $|\beta| = |\gamma|$ we obtain $s_\beta(\alpha) + s_\gamma(\alpha) = 2\alpha - (\beta + \gamma) = 2\alpha - \frac{|\gamma|^2}{|\alpha|^2}\alpha \leq 0$. Up to interchanging β and γ, this implies that $s_\gamma(\alpha) = \alpha - \gamma < 0$, so $\alpha < \gamma$ and $\gamma^\vee \leq \alpha^\vee \in H_2(X)$. This shows that α is not P-cosmall. Otherwise $B \neq \emptyset$ and we can choose $\beta \in R_P^+$

such that $s_\alpha(\beta) \in R^+ \smallsetminus R^+_P$. Since the support of $s_\alpha(\beta)$ is not contained in the support of β, we must have $(\beta, \alpha^\vee) < 0$. Set $\gamma = s_\beta(\alpha) = \alpha - (\alpha, \beta^\vee)\beta > \alpha$. Then $\gamma^\vee = s_\beta(\alpha^\vee) = \alpha^\vee - (\beta, \alpha^\vee)\beta^\vee = \alpha^\vee \in H_2(X)$, hence α is not P-cosmall. This establishes the implication (a) $\Rightarrow$ (c) and completes the proof. □

Remark 6.3. Theorem 6.1 implies that $\ell(s_\alpha) = 2\min(\text{height}(\alpha), \text{height}(\alpha^\vee)) - 1$ for each $\alpha \in R^+$, where height(α) is the sum of the coefficients obtained when α is expressed in the basis of simple roots. In fact, we have $(c_1(T_{G/B}), \alpha^\vee) = 2\,\text{height}(\alpha^\vee)$, and either α or $\alpha^\vee$ is cosmall.

The following criterion has so far only been proved for B-cosmall roots, but we believe that it holds for P-cosmall roots when the root system R is simply laced.

Proposition 6.4. *Let $\alpha \in R^+$. Then α is cosmall if and only if $z_{\alpha^\vee} = s_\alpha$.*

Proof. This follows from Lemma 4.10(a) and the definition of $z_{\alpha^\vee}$. □

Conjecture 6.5. *Assume that R is simply laced and let $\alpha \in R^+ \smallsetminus R^+_P$. Then α is P-cosmall if and only if $z^P_{\alpha^\vee} W_P = s_\alpha W_P$.*

Example 6.6. Let G be a group of type B_2, let $\Delta = \{\beta_1, \beta_2\}$, $\alpha = \beta_1 + \beta_2$, and $\gamma = \beta_1 + 2\beta_2$ be as in Example 5.3, and let $P \subset G$ be the parabolic subgroup defined by $\Delta_P = \{\beta_2\}$. Then $s_\alpha W_P = s_1 s_2 s_1 W_P = w_0 W_P$ is the longest coset, hence $s_\alpha W_P = z^P_{\alpha^\vee} W_P$. However, the greedy decomposition of $\alpha^\vee + \mathbb{Z}\Delta^\vee_P$ is (γ, γ), so α is not P-cosmall. In fact, Example 5.3 shows that α is not even B-cosmall.

The following definition together with Proposition 6.8 below is our reason for choosing the name '*cosmall*' in section 4.

Definition 6.7. A positive root $\alpha \in R^+$ is *large* if it is long and can be written as the sum of two short positive roots. Otherwise α is *small*.

Proposition 6.8. *Let $\alpha \in R^+$. Then α is cosmall if and only if the coroot $\alpha^\vee$ is a small root of the dual root system $R^\vee$.*

Proof. By Theorem 6.1 it is enough to show that $\alpha^\vee$ is large if and only if there exists a root $\gamma \in I(s_\alpha) \smallsetminus \{\alpha\}$ for which $(\gamma, \alpha^\vee) \geq 2$. If $\alpha^\vee$ is large, then α is short and we can write $\alpha^\vee = \beta^\vee + \gamma^\vee$ where β and γ are long positive roots. Since we have $(\alpha, \beta^\vee) \leq 1$, $(\alpha, \gamma^\vee) \leq 1$, and $(\alpha, \beta^\vee) + (\alpha, \gamma^\vee) = (\alpha, \alpha^\vee) = 2$, we deduce that $(\alpha, \gamma^\vee) = 1$ and $(\gamma, \alpha^\vee) \geq 2$. Finally, since $s_\alpha(\gamma^\vee) = \gamma^\vee - \alpha^\vee = -\beta^\vee$, it follows that $\gamma \in I(s_\alpha) \smallsetminus \{\alpha\}$. Conversely, if $\gamma \in I(s_\alpha) \smallsetminus \{\alpha\}$ satisfies $(\gamma, \alpha^\vee) \geq 2$, then set $\beta = -s_\alpha(\gamma) \in R^+$. Then α is short, γ and β are long, and $\beta^\vee = -s_\alpha(\gamma^\vee) = \alpha^\vee - \gamma^\vee$. This shows that $\alpha^\vee$ is large. □

7. Gromov-Witten invariants

Given $w \in W$ and $d \in H_2(X)$, define the Gromov-Witten variety

$$\text{GW}_d(w) = \text{ev}_2^{-1}(X(w)) \ \subset \overline{\mathcal{M}}_{0,2}(X, d)\,.$$

We have $\dim \text{GW}_d(w) = \ell(wW_P) + (c_1(T_X), d) - 1$, and Theorem 1 implies that $\text{ev}_1(\text{GW}_d(w)) = \Gamma_d(X(w)) = X(w \cdot z^P_d)$. We need the following consequence of [5, Prop. 3.3].

Proposition 7.1 ([5])**.** *The variety $\text{GW}_d(w)$ is unirational, and the evaluation map $\text{ev}_1 : \text{GW}_d(w) \to \Gamma_d(X(w))$ is a locally trivial fibration over the open B-orbit in $\Gamma_d(X(w))$. In particular, the general fibers of ev_1 are unirational.*

Theorem 7.2. *Let $w \in W^P$ and $0 < d \in H_2(X)$. Then $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)]$ is non-zero in $H^*(X)$ if and only if we have $d = \alpha^\vee + \mathbb{Z}\Delta_P^\vee$ for some root $\alpha \in R^+$ such that $\ell(ws_\alpha W_P) = \ell(w) + (c_1(T_X), \alpha^\vee) - 1$. In this case α is P-cosmall and $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] = [X(ws_\alpha)]$.*

Proof. Assume that $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] \neq 0$. Then the inequalities $\dim X(w \cdot z_d^P) \leq \ell(w) + \ell(z_d^P) \leq \ell(w) + (c_1(T_X), d) - 1 = \dim \mathrm{GW}_d(w)$ must be equalities, and Theorem 6.2 implies that $d = \alpha^\vee + \mathbb{Z}\Delta_P^\vee \in H_2(X)$ for some P-cosmall root α. Since $\ell(w \cdot z_d^P W_P) = \ell(w) + \ell(z_d^P W_P)$ and $z_d^P W_P = s_\alpha W_P$, we obtain from Proposition 3.3 that $\Gamma_d(X(w)) = X(w \cdot s_\alpha) = X(ws_\alpha)$. Finally, Proposition 7.1 implies that $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] = [\Gamma_d(X(w))]$. On the other hand, if $\alpha \in R^+$ satisfies $d = \alpha^\vee \in H_2(X)$ and $\ell(ws_\alpha W_P) = \ell(w) + (c_1(T_X), d) - 1$, then the inequalities $\ell(ws_\alpha W_P) \leq \ell(w) + \ell(s_\alpha W_P) \leq \ell(w) + (c_1(T_X), d) - 1$ must be equalities, so Theorem 6.1 implies that α is P-cosmall. Similarly, the inequalities $\ell(ws_\alpha W_P) \leq \dim X(w \cdot z_d^P) \leq \dim \mathrm{GW}_d(w)$ are equalities, hence $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] \neq 0$. □

Let $H_T^*(X)$ denote the T-equivariant cohomology ring of X. Each T-stable closed subvariety $Z \subset X$ defines an equivariant class $[Z] \in H_T^*(X)$. Pullback along the structure morphism $X \to \{\text{point}\}$ gives $H_T^*(X)$ the structure of an algebra over the ring $\Lambda := H_T^*(\text{point})$, and $H_T^*(X)$ is a free Λ-module with basis $\{[Y(w)] : w \in W^P\}$. For any class $\Omega \in H_T^*(X)$ we let $\int_X \Omega \in \Lambda$ denote the proper pushforward of Ω along the structure morphism $X \to \{\text{point}\}$. The Kontsevich space $\overline{\mathcal{M}}_{0,n}(X, d)$ has a natural T-action given by $(t.f)(y) = t.f(y)$ for any stable map $f : C \to X$ and $t \in T$, and the evaluation maps $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X, d) \to X$ are T-equivariant. Given classes $\Omega_1, \ldots, \Omega_n \in H_T^*(X)$ and $d \in H_2(X)$, the associated *equivariant Gromov-Witten invariant* is defined by

$$I_d(\Omega_1, \ldots, \Omega_n) = \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \mathrm{ev}_1^*(\Omega_1) \cdot \mathrm{ev}_2^*(\Omega_2) \cdots \mathrm{ev}_n^*(\Omega_n) \ \in \Lambda\,.$$

Notice that Theorem 7.2 holds for the equivariant class $(\mathrm{ev}_1)_*[\mathrm{GW}_d(w)] \in H_T^*(X)$, with the same proof.

Corollary 7.3. *Let $w, u \in W^P$ and $0 < d \in H_2(X)$. The two-point Gromov-Witten invariant $I_d([Y(u)], [X(w)])$ is non-zero if and only if there exists a root $\alpha \in R^+ \smallsetminus R_P^+$ such that $d = \alpha^\vee + \mathbb{Z}\Delta_P^\vee$, $\ell(ws_\alpha W_P) = \ell(w) + (c_1(T_X), \alpha^\vee) - 1$, and $uW_P = ws_\alpha W_P$. In this case α is P-cosmall and $I_d([Y(u)], [X(w)]) = 1$.*

Proof. Since we have $I_d([Y(u)], [X(w)]) = \int_X [Y(u)] \cdot (\mathrm{ev}_1)_*[\mathrm{GW}_d(w)]$ by the projection formula, the corollary follows from Theorem 7.2. □

The divisor axiom [18] (see also [13, (40)]) is valid for equivariant Gromov-Witten invariants. Let $Z \subset X$ be any T-stable divisor and $0 < d \in H_2(X)$, and consider the variety $\mathrm{ev}_n^{-1}(Z) \subset \overline{\mathcal{M}}_{0,n}(X, d)$ and the morphism $\phi : \mathrm{ev}_n^{-1}(Z) \to \overline{\mathcal{M}}_{0,n-1}(X, d)$ that discards the n-th marked point in the domain of its argument. For a general stable map $f : C \to X$ in $\overline{\mathcal{M}}_{0,n-1}(X, d)$ we can identify the fiber $\phi^{-1}(f)$ with $f^{-1}(Z) \subset C$, so it follows from Kleiman's transversality theorem [17] that $\#\phi^{-1}(f) = ([Z], d) = \int_d [Z]$ for all points f in a dense open subset of $\overline{\mathcal{M}}_{0,n-1}(X, d)$. We deduce that $\phi_*[\mathrm{ev}_n^{-1}(Z)] = ([Z], d) \cdot [\overline{\mathcal{M}}_{0,n-1}(X, d)]$, so the projection formula implies that

$$I_d(\Omega_1, \ldots, \Omega_{n-1}, [Z]) = ([Z], d) \cdot I_d(\Omega_1, \ldots, \Omega_{n-1}) \ \in \Lambda \tag{9}$$

for all classes $\Omega_1, \dots, \Omega_{n-1} \in H_T^*(X)$. In particular, the equivariant Gromov-Witten invariant $I_d(\Omega_1, \dots, \Omega_{n-1}, [Z])$ depends only on the class of Z in the ordinary cohomology ring $H^*(X)$ and not on its equivariant class in $H_T^*(X)$.

Corollary 7.4. *Let $u, w \in W^P$, $\beta \in \Delta$, and $0 < d \in H_2(X)$. If the Gromov-Witten invariant $I_d([Y(u)], [Y(s_\beta)], [X(w)])$ is non-zero, then there exists a unique root $\alpha \in R^+ \smallsetminus R_P^+$ such that* (i) *$d = \alpha^\vee + \mathbb{Z}\Delta_P^\vee$,* (ii) *$wW_P = us_\alpha W_P$, and* (iii) *$\ell(wW_P) = \ell(uW_P) + 1 - (c_1(T_X), \alpha^\vee)$. If $\alpha \in R^+ \smallsetminus R_P^+$ is any root satisfying* (i), (ii), *and* (iii), *then we have $\langle [Y(u)], [Y(s_\beta)], [X(w)] \rangle_d = (\omega_\beta, \alpha^\vee) \in \mathbb{Z}$.*

Proof. If $I_d([Y(u)], [X(w)], [Y(s_\beta)]) \neq 0$ then it follows from (9) and Corollary 7.3 that there exists a root $\gamma \in R^+ \smallsetminus R_P^+$ such $d = \gamma^\vee + \mathbb{Z}\Delta_P^\vee$, $uW_P = ws_\gamma W_P$, and $\ell(wW_P) = \ell(uW_P) + 1 - (c_1(T_X), \gamma^\vee)$. Since $u^{-1}ws_\gamma \in W_P$ we deduce that $\alpha := u^{-1}w(-\gamma) \in R^+ \smallsetminus R_P^+$ satisfies (i), (ii), and (iii). The uniqueness of α follows from Lemma 2.2. □

Remark 7.5. The K-theoretic two-point invariants are easier to compute, since Proposition 7.1 together with the K-theoretic Gysin formula of [7, Thm. 3.1] imply that $(\mathrm{ev}_1)_*[\mathcal{O}_{\mathrm{GW}_d(w)}] = [\mathcal{O}_{X(w \cdot z_d^P)}] \in K_T(X)$ for any degree $d \geq 0$. It follows that any equivariant K-theoretic two-point Gromov-Witten invariant of X is given by

$$\begin{aligned}
\chi_{\overline{\mathcal{M}}_{0,2}(X,d)}(\mathrm{ev}_1^*[\mathcal{O}_{Y(u)}] \cdot \mathrm{ev}_2^*[\mathcal{O}_{X(w)}]) \;&=\; \chi_X([\mathcal{O}_{Y(u)}] \cdot [\mathcal{O}_{X(w \cdot z_d^P)}]) \\
=\; \chi_X(\mathcal{O}_{Y(u) \cap X(w \cdot z_d^P)}) \;&=\; \begin{cases} 1 & \text{if } uW_P \leq w \cdot z_d^P W_P\,; \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

We refer to [7, §4] for notation. Unfortunately, the K-theoretic invariants do not satisfy a divisor axiom, so this formula does not reveal any 3-point invariants.

8. The equivariant quantum Chevalley formula

The T*-equivariant quantum cohomology ring* $\mathrm{QH}_T(X)$ is an algebra over the polynomial ring $\Lambda[q] := \Lambda[q_\beta : \beta \in \Delta \smallsetminus \Delta_P]$, where $\Lambda = H_T^*(\text{point})$, which as a $\Lambda[q]$-module is defined by $\mathrm{QH}_T(X) = H^*(X) \otimes_{\mathbb{Z}} \Lambda[q]$. The multiplicative structure of $\mathrm{QH}_T(X)$ is given by

$$[Y(u)] \star [Y(v)] = \sum_{w,d} I_d([Y(u)], [Y(v)], [X(w)])\, q^d\, [Y(w)]\,,$$

where the sum is over $w \in W^P$ and $0 \leq d \in H_2(X)$, and we write $q^d = \prod_\beta q_\beta^{(\omega_\beta, d)}$. It was proved in [23, 24] that if $v = s_\beta$ is a simple reflection, then the product $[Y(u)] \star [Y(s_\beta)]$ contains no *mixed* terms, i.e. if $d \neq 0$ then the coefficient of $q^d[Y(w)]$ is always an integer. This fact is also a consequence of Corollary 7.4.

To state the equivariant quantum Chevalley formula, we need some notation. Since G is simply connected, each integral weight $\lambda \in \mathbb{Z}\{\omega_\beta \mid \beta \in \Delta\}$ can be identified with a character $\lambda : T \to \mathbb{C}^*$. Let $\mathbb{C}_\lambda$ be the corresponding one-dimensional representation of T, defined by $t.z = \lambda(t)z$. This representation can be viewed as a T-equivariant vector bundle over a point, so it defines the equivariant Chern class $c_T(\lambda) := c_1^T(\mathbb{C}_\lambda) \in \Lambda$. This class should *not* be confused with the class that λ might represent in $H^2(X) = \mathbb{Z}\{\omega_\beta \mid \beta \in \Delta \smallsetminus \Delta_P\}$ by the notation of section 2. The ring Λ is the polynomial ring over $\mathbb{Z}$ generated by the classes $c_T(\omega_\beta)$ for $\beta \in \Delta$. The equivariant quantum Chevalley formula is the following result [10, 14, 24].

Theorem 8.1. *Let $u \in W^P$ and $\beta \in \Delta \smallsetminus \Delta_P$. Then we have*

$$[Y(u)] \star [Y(s_\beta)] =$$
$$\sum_\alpha (\omega_\beta, \alpha^\vee)\,[Y(us_\alpha)] \;+\; c_T(\omega_\beta - u.\omega_\beta)\,[Y(u)] \;+\; \sum_\alpha (\omega_\beta, \alpha^\vee)\,q^{\alpha^\vee}\,[Y(us_\alpha)]\;;$$

the first sum is over $\alpha \in R^+ \smallsetminus R^+_P$ such that $\ell(us_\alpha W_P) = \ell(uW_P) + 1$, and the second sum is over $\alpha \in R^+ \smallsetminus R^+_P$ such that $\ell(us_\alpha W_P) = \ell(uW_P) + 1 - (c_1(T_X), \alpha^\vee)$.

Proof. It follows from Corollary 7.4 that the second sum accounts for all terms with non-zero q-degrees. The remaining terms come from the equivariant product $[Y(u)] \cdot [Y(s_\beta)] \in H^*_T(X)$, and the coefficient of $[Y(w)]$ in this product is

$$c^w_{u,s_\beta} = \int_X [Y(u)] \cdot [Y(s_\beta)] \cdot [X(w)] = \int_X [Y(u) \cap X(w)] \cdot [Y(s_\beta)] \;\in \Lambda\,.$$

This coefficient is non-zero only if $u \leq w$ and $\ell(wW_P) \leq \ell(uW_P) + 1$. If $\ell(wW_P) = \ell(uW_P) + 1$, then the intersection $Y(u) \cap X(w)$ is a one-dimensional closed T-stable subvariety of X whose T-fixed points consist of $u.P$ and $w.P$. It follows that $Y(u) \cap X(w) = u.C_\alpha$ and $w.P = us_\alpha.P$ for some root $\alpha \in R^+ \smallsetminus R^+_P$, and we have $c^w_{u,s_\beta} = ([Y(s_\beta)], [C_\alpha]) = (\omega_\beta, \alpha^\vee)$ as claimed. This argument can also be found in e.g. [14, Lemma 8.1] or [3, Prop. 1.4.3].

The last remaining term is $c^u_{u,s_\beta}[Y(u)]$. The projection formula implies that

$$c^u_{u,s_\beta} = \int_X [Y(s_\beta)] \cdot [u.P] = [Y(s_\beta)]_{u.P}$$

where $[Y(s_\beta)]_{u.P} \in \Lambda$ is the restriction of $[Y(s_\beta)]$ to the T-fixed point $u.P \in X$. Set $\lambda = \omega_\beta$ and notice that $L_\lambda = G \times^P \mathbb{C}_{-\lambda}$ is a G-equivariant line bundle with action defined by $g'.[g, z] = [g'g, z]$. According to the Borel-Weil theorem [4, p. 99] there exists a B^{op}-stable section $\sigma \in H^0(X, L_\lambda)$, unique up to scalar, and we have $\mathbb{C}\sigma \cong \mathbb{C}_{-\lambda}$ as a T-representation. This implies that $\sigma : X \times \mathbb{C}_{-\lambda} \to L_\lambda$ is a morphism of T-equivariant line bundles. Since the zero section $Z(\sigma)$ is B^{op}-stable, we deduce from (4) that $Z(\sigma) = Y(s_\beta)$. The T-equivariant class of $Y(s_\beta)$ is therefore given by

$$[Y(s_\beta)] = [Z(\sigma)] = c^T_1(L_\lambda) - c^T_1(X \times \mathbb{C}_{-\lambda}) \in H^2_T(X)\,.$$

Since the fiber of L_λ over $u.P$ is $L_\lambda(u.P) \cong \mathbb{C}_{-u.\lambda}$, we obtain

$$[Y(s_\beta)]_{u.P} = c^T_1(\mathbb{C}_{-u.\lambda}) - c^T_1(\mathbb{C}_{-\lambda}) = c_T(\omega_\beta - u.\omega_\beta)\,.$$

This finishes the proof. □

Remark 8.2. The full strength of Theorem 1 is not necessary to prove the quantum Chevalley formula. We here sketch a short alternative argument that bypasses the combinatorial construction of z^P_d. It follows from Proposition 7.1 that $\Gamma_d(X(w))$ is a Schubert variety in X for all $w \in W^P$ and $d \in H_2(X)$. Using this we can show that, if $d > 0$, then there *exists* a positive root $\alpha \in R^+ \smallsetminus R^+_P$ such that $\Gamma_d(X(w)) = \Gamma_{d-\alpha^\vee}(X(w \cdot s_\alpha))$. In fact, if we write $\Gamma_d(X(w)) = X(u)$, let C be a T-stable curve from $v.P \in X(w)$ to $u.P$, and let $v.C_\alpha \subset C$ a component such that $\overline{C \smallsetminus v.C_\alpha}$ connects $vs_\alpha.P$ to $u.P$, then we must have $X(u) \subset \Gamma_{d-\alpha^\vee}(X(vs_\alpha)) \subset \Gamma_{d-\alpha^\vee}(X(w \cdot s_\alpha)) \subset \Gamma_{d-\alpha^\vee}(\Gamma_{\alpha^\vee}(X(w))) \subset \Gamma_d(X(w))$. The arguments proving Theorems 6.2 and 7.2 now show that $\dim \Gamma_d(X(w)) \leq \ell(w) + (c_1(T_X), d) - 1$, with equality if and only if $d = \alpha^\vee + \mathbb{Z}\Delta^\vee_P$ for some root $\alpha \in R^+$ such that

$\ell(ws_\alpha W_P) = \ell(w) + (c_1(T_X), d) - 1$. Theorem 7.2 and the quantum Chevalley formula follow from this.

References

[1] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005. MR 2133266 (2006d:05001)

[2] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002, Translated from the 1968 French original by Andrew Pressley. MR 1890629 (2003a:17001)

[3] M. Brion, *Lectures on the geometry of flag varieties*, Topics in cohomological studies of algebraic varieties, Trends Math., Birkhäuser, Basel, 2005, pp. 33–85. MR 2143072 (2006f:14058)

[4] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, vol. 231, Birkhäuser Boston Inc., Boston, MA, 2005. MR 2107324 (2005k:14104)

[5] A. S. Buch, P.-E. Chaput, L. Mihalcea, and N. Perrin, *Finiteness of cominuscule quantum K-theory*, to appear in Ann. Sci. Ec. Norm. Super., arXiv:1011.6658.

[6] A. S. Buch, A. Kresch, and H. Tamvakis, *Gromov-Witten invariants on Grassmannians*, J. Amer. Math. Soc. **16** (2003), no. 4, 901–915 (electronic). MR 1992829 (2004h:14060)

[7] A. S. Buch and L. C. Mihalcea, *Quantum K-theory of Grassmannians*, Duke Math. J. **156** (2011), no. 3, 501–538. MR 2772069 (2011m:14092)

[8] A. S. Buch and M. Samuel, *K-theory of minuscule varieties*, preprint, 2013.

[9] P.-E. Chaput, L. Manivel, and N. Perrin, *Quantum cohomology of minuscule homogeneous spaces*, Transform. Groups **13** (2008), no. 1, 47–89. MR 2421317 (2009e:14095)

[10] C. Chevalley, *Sur les décompositions cellulaires des espaces G/B*, Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, With a foreword by Armand Borel, pp. 1–23. MR 1278698 (95e:14041)

[11] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Ann. Sci. École Norm. Sup. (4) **7** (1974), 53–88, Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I. MR 0354697 (50 #7174)

[12] S. Fomin and A. N. Kirillov, *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Proceedings of the 5th Conference on Formal Power Series and Algebraic Combinatorics (Florence, 1993), vol. 153, 1996, pp. 123–143. MR 1394950 (98b:05101)

[13] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., vol. 62, Amer. Math. Soc., Providence, RI, 1997, pp. 45–96. MR 1492534 (98m:14025)

[14] W. Fulton and C. Woodward, *On the quantum product of Schubert classes*, J. Algebraic Geom. **13** (2004), no. 4, 641–661. MR 2072765 (2005d:14078)

[15] N. Gonciulea and V. Lakshmibai, *Flag varieties*, Actualités Mathématiques, Hermann, Paris, 2001.

[16] J.-M. Hwang and S. Kebekus, *Geometry of chains of minimal rational curves*, J. Reine Angew. Math. **584** (2005), 173–194. MR 2155089 (2006e:14019)

[17] S. L. Kleiman, *The transversality of a general translate*, Compositio Math. **28** (1974), 287–297. MR 0360616 (50 #13063)

[18] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. **164** (1994), no. 3, 525–562. MR 1291244 (95i:14049)

[19] B. Kostant and S. Kumar, *T-equivariant K-theory of generalized flag varieties*, J. Differential Geom. **32** (1990), no. 2, 549–603. MR 1072919 (92c:19006)

[20] T. Lam and M. Shimozono, *Quantum cohomology of G/P and homology of affine Grassmannian*, Acta Math. **204** (2010), no. 1, 49–90. MR 2600433 (2011h:14082)

[21] J. M. Landsberg and L. Manivel, *On the projective geometry of rational homogeneous varieties*, Comment. Math. Helv. **78** (2003), no. 1, 65–100. MR 1966752 (2004a:14050)

[22] C. Li and L. Mihalcea, *K-theoretic Gromov-Witten invariants of lines in homogeneous spaces*, preprint arχiv:1206.3593.

[23] L. Mihalcea, *Equivariant quantum Schubert calculus*, Adv. Math. **203** (2006), no. 1, 1–33. MR 2231042 (2007c:14061)

[24] L. C. Mihalcea, *On equivariant quantum cohomology of homogeneous spaces: Chevalley formulae and algorithms*, Duke Math. J. **140** (2007), no. 2, 321–350. MR 2359822 (2008j:14106)

[25] J. R. Stembridge, *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), no. 4, 353–385. MR 1406459 (97g:20046)

[26] E. Strickland, *Lines in G/P*, Math. Z. **242** (2002), no. 2, 227–240. MR 1980621 (2004d:14073)

[27] F. L. Zak, *Tangents and secants of algebraic varieties*, Translations of Mathematical Monographs, vol. 127, American Mathematical Society, Providence, RI, 1993, Translated from the Russian manuscript by the author. MR 1234494 (94i:14053)

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