Three combinatorial formulas for type A quiver polynomials and K-polynomials

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Type A quiver loci

- A **quiver** Q is a finite directed graph, and a **representation** of Q is an assignment of vector space to each vertex and linear map to each arrow.
- Q is of **type A** if its underlying graph is a type A Dynkin diagram.
- Once the vector spaces $K^{d_0}, \dots K^{d_n}$ at the vertices are fixed, the collection of representations is an algebraic variety, denoted by $\mathrm{rep}_Q(\mathbf{d})$. This variety carries the action of a **base change group**:

$$GL(\mathbf{d}) := GL(d_0) \times GL(d_1) \times \cdots \times GL(d_n).$$

- These orbit closures are called **quiver loci**.

Example

A representation of an equioriented type A quiver:

$$K^{d_0} \xrightarrow{V_1} K^{d_1} \xrightarrow{V_2} K^{d_2} \cdots \xrightarrow{V_n} K^{d_n}.$$

Here, V_i is a $d_{i-1} \times d_i$ matrix, and $\mathrm{rep}_Q(\mathbf{d})$ is the affine space of all sequences $(V_1, \dots, V_n)$. The base change group $GL(\mathbf{d})$ acts by:

$$(g_0, g_1, \dots, g_{n-1}, g_n) \cdot (V_1, \dots, V_n) = (g_0 V_1 g_1^{-1}, \dots, g_{n-1} V_n g_n^{-1}).$$

Equioriented type A quiver loci

The equioriented setting is well-understood. In particular:

- Orbits are determined by ranks of all products $V_i V_{i+1} \cdots V_j$, $i \leq j$.
- (Zelevinsky '85) The collection of these rank conditions is equivalent to certain Schubert-type rank conditions on an opposite Schubert cell in a partial flag variety. Eg. if Q has three arrows,

$$(V_1, V_2, V_3) \xrightarrow{\zeta} \begin{bmatrix} 0 & 0 & V_1 & I_{d_0} \\ 0 & V_2 & I_{d_1} & 0 \\ V_3 & I_{d_2} & 0 & 0 \\ I_{d_3} & 0 & 0 & 0 \end{bmatrix} \subseteq \begin{bmatrix} * & * & * & I_{d_0} \\ * & * & I_{d_1} & 0 \\ * & I_{d_2} & 0 & 0 \\ I_{d_3} & 0 & 0 & 0 \end{bmatrix} \cong P\backslash PwB_-.$$

  This map ζ is an equioriented **Zelevinsky map**.

- (Lakshmibai-Magyar '98) The Zelevinksy map is scheme-theoretic isomorphism which takes each orbit closure to a Schubert variety intersected with an opposite Schubert cell. Consequently, these quiver loci are normal and Cohen-Macaulay with rational singularities, F-split...
- The coordinate rings of equioriented type A quiver loci are naturally multigraded, and there exist multiple combinatorial formulas for their multidegrees and K-polynomials.

Goal: Generalize to all orientations.

Bipartite type A quiver loci

A type A quiver is **bipartite** if every vertex is a source or sink:

$$\begin{tikzcd}
K^{d_6} \arrow[dr, "V_6"'] & & K^{d_4} \arrow[dl, "V_5"] \arrow[dr, "V_4"'] & & K^{d_2} \arrow[dl, "V_3"] \arrow[dr, "V_2"'] & & K^{d_0} \arrow[dl, "V_1"] \\
& K^{d_5} & & K^{d_3} & & K^{d_1} &
\end{tikzcd}$$

$GL(\mathbf{d})$-orbits of bipartite type A quivers are completely determined by ranks of particular matrices: given an interval $[i,j] \subseteq Q$, define the matrix

$$Z_{[i,j]} = \begin{pmatrix} & & & V_i \\ & & V_{i+2} & V_{i+1} \\ & \iddots & \iddots & \\ V_{j-1} & V_{j-2} & & \\ V_j & & & \end{pmatrix}.$$

Let $\mathbf{r}_{[i,j]} := \mathrm{rank}\ Z_{[i,j]}$, and let $\mathbf{r}$ be the array of all $\mathbf{r}_{[i,j]}$. *Then, two representations in $rep_Q(\mathbf{d})$ lie in the same $\mathbf{GL}(\mathbf{d})$-orbit if and only if they have the same rank array $\mathbf{r}$.*

The bipartite Zelevinsky map

Theorem (Kinser-R)

- *There is a closed immersion from each representation space of a bipartite type A quiver to an opposite Schubert cell of a partial flag variety.*
- *This* **bipartite Zelevinsky map** *identifies each quiver locus with a Schubert variety intersected with the above opposite Schubert cell.*
- *Consequently, quiver loci are normal and C-M with rational singularities, F-split, orbit closure containment is determined by Bruhat order.*

Example

The image of $(V_1, V_2, V_3, V_4, V_5, V_6)$ under the bipartite Zelevinsky map is:

$$\left(\begin{array}{ccc|cccc} 0 & 0 & V_1 & I_{d_0} & 0 & 0 & 0 \\ 0 & V_3 & V_2 & 0 & I_{d_2} & 0 & 0 \\ V_5 & V_4 & 0 & 0 & 0 & I_{d_4} & 0 \\ V_6 & 0 & 0 & 0 & 0 & 0 & I_{d_6} \\ \hline I_{d_1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_{d_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{d_5} & 0 & 0 & 0 & 0 \end{array}\right) \subseteq \begin{pmatrix} * & I \\ I & 0 \end{pmatrix} \cong P\backslash P v_0 B^-.$$

Multigradings, quiver polynomials, and K-polynomials

The maximal torus $T \subseteq \mathbf{GL}(\mathbf{d})$ consisting of matrices which are diagonal in each factor induces a multigrading on $K[\mathrm{rep}_Q(\mathbf{d})]$ which makes the ideals of orbit closures homogeneous:

$$\begin{tikzcd}
y_3 \arrow[dr, "\beta_3"'] & & y_2 \arrow[dl, "\alpha_3"] \arrow[dr, "\beta_2"'] & & y_1 \arrow[dl, "\alpha_2"] \arrow[dr, "\beta_1"'] & & y_0 \arrow[dl, "\alpha_1"] \\
& x_3 & & x_2 & & x_1 &
\end{tikzcd}$$

Associate an alphabet $\mathbf{s}^j$ to the vertex x_j, and an alphabet $\mathbf{t}^i$ to the vertex y_i:

$$\mathbf{s}^j = s^j_1, s^j_2, \ldots, s^j_{\mathbf{d}(x_j)} \quad \text{and} \quad \mathbf{t}^i = t^i_1, t^i_2, \ldots, t^i_{\mathbf{d}(y_i)}.$$

The coordinate function $f^{\alpha_k}_{ij}$ (picking out (i,j)-entry of M_{α_k}) has degree $t^{k-1}_i - s^k_j$, and $f^{\beta_k}_{ij}$ has degree $t^k_i - s^k_j$.

With respect to the natural torus action on the opposite cell $\begin{bmatrix} * & I_{d_y} \\ I_{d_x} & 0 \end{bmatrix}$, the bipartite Zelevinsky map is T-equivariant.

Notation

- The K-**theoretic quiver polynomial** $K\mathcal{Q}_{\mathbf{r}}(\mathbf{t}/\mathbf{s})$ (resp., **quiver polynomial** $\mathcal{Q}_{\mathbf{r}}(\mathbf{t}-\mathbf{s})$) is the K-polynomial (resp., multidegree) of the quiver locus $\Omega_{\mathbf{r}}$ with respect to its embedding in $\mathrm{rep}_Q(\mathbf{d})$ and multigrading above.
- Let $\mathcal{A} = (a_1, a_2, \dots)$ and $\mathcal{B} = (b_1, b_2, \dots)$ be alphabets. Denote by $\mathfrak{G}_w(\mathcal{A};\mathcal{B})$ the **double Grothendieck polynomial** associated to w: if w_0 the longest element of the symmetric group S_m then

$$\mathfrak{G}_{w_0}(\mathcal{A};\mathcal{B}) = \prod_{i+j\leq m}\left(1-\frac{a_i}{b_j}\right),$$

  and $\mathfrak{G}_{s_i w}(\mathcal{A};\mathcal{B}) = \overline{\partial_i}\mathfrak{G}_w(\mathcal{A};\mathcal{B})$ whenever $\ell(s_i w) < \ell(w)$.
- The **double Schubert polynomial** $\mathfrak{S}_v(\mathcal{A};\mathcal{B})$ of a permutation v is obtained from $\mathfrak{G}_v(\mathcal{A};\mathcal{B})$ by substituting $1-\star$ for each variable $\star$, and then taking lowest degree terms.

The bipartite ratio formulas

- Let $\mathbf{r}$ be an array of ranks that determines a bipartite quiver orbit.
- Let $v(\mathbf{r})$ be the associated Zelevinsky permutation.
- Let v_* be the Zelevinsky permutation of the big $GL(\mathbf{d})$-orbit (which has closure $\mathrm{rep}_Q(\mathbf{d})$).

Theorem (Kinser-Knutson-R)

$$KQ_{\mathbf{r}}(\mathbf{t}/\mathbf{s}) = \frac{\mathfrak{G}_{v(\mathbf{r})}(\mathbf{t},\mathbf{s};\mathbf{s},\mathbf{t})}{\mathfrak{G}_{v_*}(\mathbf{t},\mathbf{s};\mathbf{s},\mathbf{t})} \qquad \text{and} \qquad Q_{\mathbf{r}}(\mathbf{t}-\mathbf{s}) = \frac{\mathfrak{S}_{v(\mathbf{r})}(\mathbf{t},\mathbf{s};\mathbf{s},\mathbf{t})}{\mathfrak{S}_{v_*}(\mathbf{t},\mathbf{s};\mathbf{s},\mathbf{t})}.$$

Main idea of proof.

Use the bipartite Zelevinsky map along with [Woo-Yong '12] on K-polynomials and multidegrees of Kazhdan-Lusztig varieties. $\square$

Pipe dreams and lacing diagrams

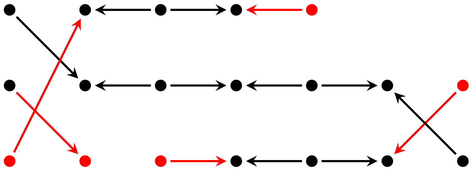
Consider the dimension vector $\mathbf{d} = (2, 2, 2, 3, 2, 2, 1)$, so that representations have the form:

$$\begin{tikzcd}
K^2 \arrow[dr, "V_{\beta_3}"'] & & K^2 \arrow[dl, "V_{\alpha_3}"] \arrow[dr, "V_{\beta_2}"'] & & K^2 \arrow[dl, "V_{\alpha_2}"] \arrow[dr, "V_{\beta_1}"'] & & K \arrow[dl, "V_{\alpha_1}"] \\
& K^2 & & K^3 & & K^2 &
\end{tikzcd}$$

Work with the orbit through:

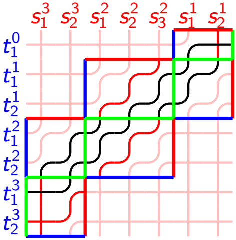
$$P = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \end{bmatrix} \right)$$

This sequence of partial permutations can be visualized with a **lacing diagram**:



Pipe dreams and lacing diagrams

The Zelevinsky image of the associated quiver locus is a Kazhdan-Lusztig variety which has **pipe dreams** supported inside the diagram of v_0 (i.e. the northwest quadrant of $\begin{bmatrix} * & I_{d_y} \\ I_{d_x} & 0 \end{bmatrix}$). For example:



Denote by $\mathrm{Pipes}(v_0, v(\mathbf{r}))$ all pipe dreams of $v(\mathbf{r})$ supported inside the Rothe diagram for v_0. Let P_* be the pipe dream which has a $+$ at position (i,j) if and only if (i,j) lies outside of the "zig-zag" region.

Lemma

Every element of $\mathit{Pipes}(v_0, v(\mathbf{r}))$ contains P_* as a subdiagram, and furthermore $\mathit{Pipes}(v_0, v_*) = \{P_*\}$.

Bipartite pipe formulas and component formulas

Theorem (Bipartite Pipe formula, Kinser-Knutson-R)

For any rank array $\mathbf{r}$, *we have*

$$K\mathcal{Q}_{\mathbf{r}}(\mathbf{t}/\mathbf{s}) = \sum_{P \in Pipes(v_0, v(\mathbf{r}))} (-1)^{|P| - l(v(\mathbf{r}))} (\mathbf{1} - \mathbf{t}/\mathbf{s})^{P \setminus P_*}$$

and

$$\mathcal{Q}_{\mathbf{r}}(\mathbf{t} - \mathbf{s}) = \sum_{P \in RedPipes(v_0, v(\mathbf{r}))} (\mathbf{t} - \mathbf{s})^{P \setminus P_*}.$$

Theorem (Bipartite component formula, Buch-Rimányi, Kinser-Knutson-R)

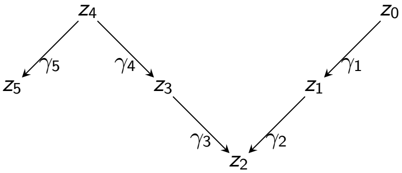
$$K\mathcal{Q}_{\mathbf{r}}(\mathbf{t}/\mathbf{s}) = \sum_{\mathbf{w} \in KW(\mathbf{r})} (-1)^{|\mathbf{w}| - \ell(v(\mathbf{r}))} \mathfrak{G}_{\mathbf{w}}(\mathbf{t}, \mathbf{s})$$

and

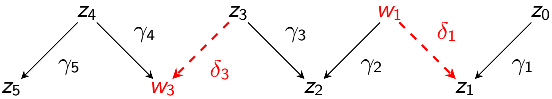
$$\mathcal{Q}_{\mathbf{r}}(\mathbf{t} - \mathbf{s}) = \sum_{\mathbf{w} \in W(\mathbf{r})} \mathfrak{S}_{\mathbf{w}}(\mathbf{t}, \mathbf{s})$$

From the bipartite orientation to arbitrary orientation

Associate a bipartite type A quiver to an arbitrarily oriented quiver by inserting vertices and arrows. Let Q be the quiver:



We construct an associated bipartite quiver $\widetilde{Q}$ by adding two new vertices w_1, w_3, and two new arrows δ_1, δ_3.



From bipartite to arbitrary orientation

Theorem (Kinser-R)

Let Q be a quiver of type A, and $\widetilde{Q}$ the associated bipartite quiver defined above. Let U be the open set in $\mathrm{rep}_{\widetilde{Q}}(\mathbf{d})$ where the maps over the added arrows are invertible. Then there is a morphism $\pi\colon U \to \mathrm{rep}_Q(\mathbf{d})$ which is equivariant with respect to the natural projection of base change groups $\mathbf{GL}(\widetilde{\mathbf{d}}) \to \mathbf{GL}(\mathbf{d})$. Each orbit closure $\overline{\mathcal{O}} \subseteq \mathrm{rep}_Q(\mathbf{d})$ for an arbitrary type A quiver is isomorphic to an open subset of an orbit closure of $\mathrm{rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}})$, up to a smooth factor. Namely, we have

$$\overline{\pi^{-1}(\mathcal{O})} \simeq G^* \times \overline{\mathcal{O}},$$

where the closure on the left hand side is taken in U.

Substitution to obtain formulas for arbitrary orientation

We can show that the K-polynomial of an orbit closure for Q is obtained from the K-polynomial of its corresponding orbit closure for $\widetilde{Q}$ by substitution of variables.

$$
\begin{array}{cc}
 & \begin{array}{cccc|cccc} \mathbf{s}^3 & \mathbf{t}^2 & \mathbf{s}^2 & \mathbf{s}^1 & \mathbf{t}^0 & \mathbf{s}^1 & \mathbf{t}^2 & \mathbf{t}^3 \end{array} \\
\begin{array}{c} \mathbf{t}^0 \\ \mathbf{s}^1 \\ \mathbf{t}^2 \\ \mathbf{t}^3 \\ \mathbf{s}^3 \\ \mathbf{t}^2 \\ \mathbf{s}^2 \\ \mathbf{s}^1 \end{array} &
\left(\begin{array}{cccc|cccc}
0 & 0 & 0 & V_{\gamma_1} & \mathbf{1}_{\mathbf{d}(z_0)} & & & \\
0 & 0 & V_{\gamma_2} & \mathbf{1}_{\mathbf{d}(z_1)} & & \mathbf{1}_{\mathbf{d}(z_1)} & & \\
0 & \mathbf{1}_{\mathbf{d}(z_3)} & V_{\gamma_3} & 0 & & & \mathbf{1}_{\mathbf{d}(z_3)} & \\
V_{\gamma_5} & V_{\gamma_4} & 0 & 0 & & & & \mathbf{1}_{\mathbf{d}(z_4)} \\
\hline
\mathbf{1}_{\mathbf{d}(z_5)} & & & & & & & \\
 & \mathbf{1}_{\mathbf{d}(z_3)} & & & & & 0 & \\
 & & \mathbf{1}_{\mathbf{d}(z_2)} & & & & & \\
 & & & \mathbf{1}_{\mathbf{d}(z_1)} & & & &
\end{array}\right)
\end{array}
$$

Thank you.