

ALGEBRAIC GEOMETRY PROBLEMS, FEBRUARY 2011

Some are easy, others are trickier. You are encouraged to collaborate about the hard ones. Have fun!

Problem 1. Show that $I(\mathbb{A}^n) = (0)$.

Problem 2. If $I \subset R$ is any ideal, show that \sqrt{I} is a radical ideal.

Problem 3. [Hartshorne I.1.2 and I.1.11]

(a) Show that the set $X = \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in k\}$ is closed in \mathbb{A}^3 and find $I(X)$.

(b) Same for the subset $Y = \{(t^3, t^4, t^5) \in \mathbb{A}^3 \mid t \in k\}$ of \mathbb{A}^3 .

(c) Show that $I(Y)$ can't be generated by less than three polynomials.

Hint: Is $I(Y)$ a graded ideal? Are you sure??

Problem 4. Let R be a commutative ring. The following are equivalent:

(a) R is Noetherian.

(b) Every ascending chain of ideals in R stabilizes.

(c) Every non-empty collection of ideals of R has a maximal element.

Problem 5. Show that $W = \{(x, y, z) \in \mathbb{A}^3 \mid x^2 = y^3 \text{ and } y^2 = z^3\}$ is an irreducible closed subset of \mathbb{A}^3 and find $I(W)$.

Hint: Construct a homomorphism $k[x, y, z] \rightarrow k[T]$ with kernel $I(W)$.

Problem 6. Find $\sqrt{(y^2 + 2xy^2 + x^2 - x^4, x^2 - x^3)}$.

Problem 7. Let X be a Noetherian topological space.

(a) If an irreducible closed set Y is contained in a union $\cup X_i$ of finitely many closed sets X_i , then $Y \subset X_i$ for some i .

(b) X has finitely many components.

(c) X is the union of its components.

(d) X is not the union of any proper subset of its components.

Problem 8. Let X be any space with functions and $Y \subset \mathbb{A}^n$ an affine variety. Show that a function $f : X \rightarrow k$ is a morphism if and only if each coordinate function $f_i : X \rightarrow k$ is regular for $1 \leq i \leq n$.

Problem 9. Let $X = V(xy - zw) \subset \mathbb{A}^4$ and $U = D(y) \cup D(w) \subset X$. Define a regular function $f : U \rightarrow k$ by $f = x/w$ on $D(w)$ and $f = z/y$ on $D(y)$. Show that there are no polynomial functions $p, q \in A(X)$ such that $q(a) \neq 0$ and $f(a) = p(a)/q(a)$ for all $a \in U$.

Problem 10. Let X be an affine variety such that the affine coordinate ring $A(X)$ is a unique factorization domain. Let $U \subset X$ be an open subset. Show that if $f : U \rightarrow k$ is any regular function, then there exist $p, q \in A(X)$ such that $q(x) \neq 0$ and $f(x) = p(x)/q(x)$ for all $x \in U$.

- Problem 11.** (a) $k[\mathbb{A}^n \setminus \{0\}] = k[x_1, \dots, x_n]$ for $n \geq 2$.
 (b) $\mathbb{A}^n \setminus \{0\}$ is not an affine variety for $n \geq 2$.
 (c) Every global regular function on \mathbb{P}^n is constant, i.e. $k[\mathbb{P}^n] = k$.
 (d) \mathbb{P}^n is not quasi-affine for $n \geq 1$.

Problem 12. Let $\varphi : \mathbb{A}^1 \rightarrow V(y^2 - x^3) \subset \mathbb{A}^2$ be the morphism given by $\varphi(t) = (t^2, t^3)$. Show that φ is bijective, but not an isomorphism.

Problem 13. Let $X \subset \mathbb{P}^n$ be a projective variety with projective coordinate ring $R = k[x_0, \dots, x_n]/I(X)$. Let $f \in R$ be a non-constant homogeneous element. Show that $D_+(f) \subset X$ is an open affine subvariety with affine coordinate ring $k[D_+(f)] = R_{(f)}$.

Problem 14. Let X be any space with functions. A map $\varphi : \mathbb{P}^n \rightarrow X$ is a morphism if and only if $\varphi \circ \pi : \mathbb{A}^{n+1} \setminus \{0\} \rightarrow X$ is a morphism.

Problem 15. Prove that the Segre map $s : \mathbb{P}^n \times \mathbb{P}^m \rightarrow \mathbb{P}^N$ gives an isomorphism of $\mathbb{P}^n \times \mathbb{P}^m$ with a closed subvariety of \mathbb{P}^N , where $N = nm + n + m$.

Problem 16. Let $\varphi : X \rightarrow Y$ be a morphism of spaces with functions and suppose $Y = \bigcup V_i$ is an open covering such that each restriction $\varphi : \varphi^{-1}(V_i) \rightarrow V_i$ is an isomorphism. Then φ is an isomorphism.

Problem 17. Assume that the characteristics of k is not 2. If $C = V_+(f) \subset \mathbb{P}^2$ is any curve defined by an irreducible homogeneous polynomial $f \in k[x, y, z]$ of degree 2, then $C \cong \mathbb{P}^1$.

- Problem 18.** (a) Any subspace of a separated space with functions is separated.
 (b) A product of separated spaces with functions is separated.

Problem 19. Let X be a pre-variety such that for each pair of points $x, y \in X$ there is an open affine subvariety $U \subset X$ containing both x and y .

- (a) Show that X is separated.
 (b) Show that \mathbb{P}^n has this property.

Problem 20. [Hartshorne II.2.16 and II.2.17]

Let X be any variety and $f \in k[X]$ a regular function.

(a) If h is a regular function on $D(f) \subset X$ then $f^n h$ can be extended to a regular function on all of X for some $n > 0$. [Hint: Let $X = U_1 \cup \dots \cup U_m$ be an open affine cover. Start by showing that some $f^n h$ can be extended to U_i for each i .]

(b) $k[D(f)] = k[X]_f$.

(c) Let R be a k -algebra and let $f_1, \dots, f_r \in R$ be elements that generate the unit ideal, $(f_1, \dots, f_r) = R$. If R_{f_i} is a finitely generated k -algebra for each i , then R is a finitely generated k -algebra.

(d) Suppose $f_1, \dots, f_r \in k[X]$ satisfy $(f_1, \dots, f_r) = k[X]$ and $D(f_i)$ is affine for each i . Then X is affine.

Problem 21. Let E be the elliptic curve $V_+(y^2z - x^3 + xz^2) \subset \mathbb{P}^2$ and let $f, g : E \dashrightarrow \mathbb{P}^1$ be the rational maps defined by $f(x : y : z) = (x : z)$ and $g(x : y : z) = (y : z)$. (These are just projections to the x and y axis on the open subset $D_+(z)$.)

- Find the maximal open sets in E where f and g are defined as morphisms.
- Find the degrees of the field extensions $k(t) \subset k(E)$ induced by f and g .
- Find the cardinality of $f^{-1}(p)$ and $g^{-1}(p)$ when $p \in \mathbb{P}^1$ is a typical point. (Part of the exercise is to define what “typical” means.)

Problem 22. Let X be a projective variety and $\varphi : \mathbb{P}^1 \dashrightarrow X$ any rational map. Show that φ is defined as a morphism on all of \mathbb{P}^1 .

Problem 23. (a) If X has components X_1, \dots, X_m then $\dim(X) = \max \dim(X_i)$.
 (b) $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Problem 24. The commutative algebra result *lying over* states that if $R \subset S$ is an integral extension of commutative rings and $P \subset R$ is a prime ideal, then there is some prime $Q \subset S$ such that $Q \cap R = P$.

(a) Use lying over to show that if $\varphi : X \rightarrow Y$ is a dominant morphism of irreducible varieties, then $\varphi(X)$ contains a dense open subset of Y .

(b) If $\varphi : X \rightarrow Y$ is any morphism of varieties, then its image $\varphi(X)$ is *constructible*, i.e. a finite union of locally closed subsets of Y .

Problem 25. [Hartshorne I.5.2]

Assume $\text{char}(k) \neq 2$. Locate the singular points of the surfaces $X = V(xy^2 - z^2)$, $Y = V(x^2 + y^2 - z^2)$, and $Z = V(xy + x^3 + y^3)$ in \mathbb{A}^3 . (Take a look at the nice pictures in Hartshorne!)

Problem 26. Assume $\text{char}(k) = 0$. Let $X = V_+(f) \subset \mathbb{P}^n$ be a hypersurface given by a square-free homogeneous polynomial $f \in k[x_0, \dots, x_n]$.

- Show that $X_{\text{sing}} = V_+(\frac{\partial f}{\partial x_0}, \dots, \frac{\partial f}{\partial x_n})$.
- Show that $X_{\text{sing}} \neq X$.

Problem 27. [Shafarevich II.1.13]

(a) Show that an intersection of r hypersurfaces in \mathbb{P}^r is never empty.

(b) Let $X \subset \mathbb{P}^n$ be a hypersurface of degree at least two, such that X contains a linear subspace $L \subset \mathbb{P}^n$ of dimension $r \geq n/2$. Prove that X is singular. [Hint: Choose the coordinates on \mathbb{P}^n such that $L = V_+(x_{r+1}, x_{r+2}, \dots, x_n) \subset \mathbb{P}^n$.]

Problem 28. [Shafarevich II.1.10].

Let $X \subset \mathbb{P}^n$ be a hypersurface of degree three. If X has two different singular points, then X contains the line joining them.

Problem 29. If X is a variety and $x \in X$, we define the Zariski cotangent space to X at x to be $\mathfrak{m}_x/\mathfrak{m}_x^2$. The Zariski tangent space is the dual vector space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$. Show that if $f : X \rightarrow Y$ is a morphism of varieties with $f(x) = y$, then f induces linear maps $\mathfrak{m}_y/\mathfrak{m}_y^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$ and $(\mathfrak{m}_x/\mathfrak{m}_x^2)^* \rightarrow (\mathfrak{m}_y/\mathfrak{m}_y^2)^*$.

Problem 30. [Mostly Hartshorne I.6.3]

Give examples of varieties X and Y , a point $P \in X$, and a morphism $\varphi : X \setminus \{P\} \rightarrow Y$ such that φ can't be extended to a morphism on all of X in each of the cases:

- (a) X is a non-singular curve and Y is not projective.
- (b) X is a curve, P is a singular point on X , Y is projective.
- (c) X is non-singular of dimension at least two, Y is projective.

Problem 31. Let X and Y be curves and $\varphi : X \rightarrow Y$ a birational morphism.

- (a) X_{sing} is a proper closed subset of X .
- (b) $\varphi(X_{\text{sing}}) \subset Y_{\text{sing}}$.
- (c) If $y \in Y$ is a non-singular point, then $\varphi^{-1}(y)$ contains at most one point.

Problem 32. Two non-singular projective curves are isomorphic if and only if they have the same function field.

Problem 33. Resolution of singularities for curves.

Let X be a curve with smooth locus $U = X - X_{\text{sing}}$. Prove that there exists a non-singular curve \tilde{X} with a finite morphism $\varphi : \tilde{X} \rightarrow X$ such that the restriction $\varphi : \varphi^{-1}(U) \rightarrow U$ is an isomorphism. (For resolution of singularities in higher dimension, one can only hope for a “proper” morphism φ .)

Problem 34. Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$. Show that if $P \in E$ is any point then $E \setminus \{P\}$ is affine.

Problem 35. [Hartshorne I.6.2]

Let $E = V(y^2 - x^3 + x) \subset \mathbb{A}^2$, $\text{char}(k) \neq 2$.

- (a) E is a non-singular curve.
- (b) The units in $k[E]$ are the non-zero elements of k . [Hints: Define an automorphism $\sigma : k[E] \rightarrow k[E]$ fixing x and sending y to $-y$. Then define a norm $N : k[E] \rightarrow k[x]$ by $N(a) = a\sigma(a)$. Show that $N(1) = 1$ and $N(ab) = N(a)N(b)$.]
- (c) $k[E]$ is not a unique factorization domain.
- (d) Show that E is not rational.

Problem 36. Let $m_0, m_1, \dots, m_N \in k[x_0, \dots, x_n]$ be all the monomials of degree d . The *Veronese embedding* is the map $v_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$ defined by

$$v_d(x_0 : \dots : x_n) = (m_0(x_0, \dots, x_n) : \dots : m_N(x_0, \dots, x_n)).$$

- (a) Show that v_d is an isomorphism of \mathbb{P}^n with a closed subvariety in \mathbb{P}^N .
- (b) Let $S \subset \mathbb{P}^n$ be a hypersurface of degree d , i.e. $S = V_+(f)$ where $f \in k[x_0, \dots, x_n]$ is a form of degree d . Show that $S = v_d^{-1}(H)$ for a unique hyperplane $H \subset \mathbb{P}^N$.

Problem 37. Let L_1, L_2 , and L_3 be lines in \mathbb{P}^3 such that none of them meet.

- (a) There exists a unique quadric surface $S \subset \mathbb{P}^3$ containing L_1, L_2 , and L_3 . [Hint: Start by applying an automorphism of \mathbb{P}^3 to make the lines nice.]
- (b) S is the disjoint union of all lines $L \subset \mathbb{P}^3$ meeting L_1, L_2 , and L_3 .
- (c) Let $L_4 \subset \mathbb{P}^3$ be a fourth line which does not meet L_1, L_2 , or L_3 . Then the number of lines meeting L_1, L_2, L_3 , and L_4 is equal to the number of points in $L_4 \cap S$, which is one, two, or infinitely many.

Problem 38. An *algebraic group* is a pre-variety G together with morphisms $m : G \times G \rightarrow G$ and $i : G \rightarrow G$, and an identity element $e \in G$, such that G is a group in the usual sense when m is used to define multiplication and i maps any element to its inverse element.

- (a) Show that $\mathrm{GL}_n(k)$ is an algebraic group.
- (b) Show that any algebraic group is separated.
- (c) Show that \mathbb{P}^1 is not an algebraic group, i.e. it is not possible to find morphisms $m : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$ and $i : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ satisfying the group axioms.
- (d) Challenge: How about \mathbb{P}^n for $n \geq 2$?

Problem 39. Let G be an irreducible algebraic group acting on a variety X , i.e. we have a morphism $G \times X \rightarrow X$ such that the axioms for a group action are satisfied.

- (a) Show that each orbit in X is locally closed.
- (b) Each orbit is a non-singular variety.

Problem 40. Let $X \subset \mathbb{P}^5$ be the subset of points $(x_0 : \cdots : x_5)$ such that the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ x_3 & x_4 & x_5 \end{bmatrix}$$

has rank one. Show that X is a non-singular rational closed subvariety of \mathbb{P}^5 , and find its dimension and degree.

Problem 41. Let \mathcal{F} be a sheaf on X and $p \in X$ a point. Prove the following from the definition of the stalk \mathcal{F}_p :

- (a) Each element of \mathcal{F}_p has the form s_p for some section $s \in \mathcal{F}(U)$, $p \in U$.
- (b) Let $s \in \mathcal{F}(U)$, $p \in U$. Then $s_p = 0 \Leftrightarrow s|_V = 0$ for some $p \in V \subset U$.
- (c) Let $s \in \mathcal{F}(U)$. Prove that $s = 0$ if and only if $s_p = 0 \forall p \in U$.

Problem 42. [Hartshorne II.1.2]

Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on X . Show that φ is surjective if and only if the following condition holds: for every open set $U \subset X$, and for every $s \in \mathcal{G}(U)$, there is a covering $U = \bigcup V_i$ of U and sections $t_i \in \mathcal{F}(V_i)$ such that $\varphi_{V_i}(t_i) = s|_{V_i}$ for all i .

Problem 43. [Hartshorne II.1.14]

Let \mathcal{F} be a sheaf on X and $s \in \mathcal{F}(X)$ a global section. Show that the set $\{p \in X \mid s_p \neq 0\}$ is a closed subset of X .

Problem 44. Let $\varphi : \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of abelian sheaves on X . Show that $\ker(\varphi)_p = \ker(\varphi_p)$ and $\mathrm{Im}(\varphi)_p = \mathrm{Im}(\varphi_p)$ for all $p \in X$.

Problem 45. Let $f : X \rightarrow Y$ be a continuous map and \mathcal{G} a sheaf on Y . Show that $(f^{-1}\mathcal{G})_p = \mathcal{G}_{f(p)}$ for all $p \in X$.

Problem 46. Let $f : X \rightarrow Y$ be a continuous map, \mathcal{F} a sheaf on X , and \mathcal{G} a sheaf on Y . Show that the map $\mathrm{Hom}(\mathcal{G}, f_*\mathcal{F}) \rightarrow \mathrm{Hom}(f^{-1}\mathcal{G}, \mathcal{F})$ constructed in class is bijective.

Problem 47. (a) Let X be an affine variety, M a $k[X]$ -module, and \mathcal{F} an \mathcal{O}_X -module. Show that $\mathrm{Hom}_{k[X]}(M, \Gamma(X, \mathcal{F})) \cong \mathrm{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathcal{F})$.

(b) If X is affine and M and N are $k[X]$ -modules then $\tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} = (M \otimes_{k[X]} N)^\sim$.

(c) If $f : X \rightarrow Y$ is a morphism of varieties and \mathcal{G} is a (quasi-) coherent \mathcal{O}_Y -module, then $f^*\mathcal{G}$ is a (quasi-) coherent \mathcal{O}_X -module.

Problem 48. (a) X is a ringed space, \mathcal{F} and \mathcal{G} are \mathcal{O}_X -modules. Then the assignment $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ defines an \mathcal{O}_X -module. It is denoted $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

(b) Let \mathcal{L} be an invertible \mathcal{O}_X -module. Show that $\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ is also invertible and that $\mathcal{L}^{-1} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{O}_X$.

Problem 49. Let X be a scheme of characteristic $p > 0$, $F : X \rightarrow X$ the Frobenius morphism, and \mathcal{L} an invertible \mathcal{O}_X -module. Show that $F^*\mathcal{L} \cong \mathcal{L}^{\otimes p}$.

Problem 50. A morphism $f : X \rightarrow Y$ of varieties is called *affine* if for every open affine set $V \subset Y$ the inverse image $f^{-1}(V)$ is also affine. f is called *finite* if it is affine and $k[f^{-1}(V)]$ is a finitely generated $k[V]$ -module for all open affine $V \subset Y$.

Let $Y = \bigcup V_i$ be an open affine covering of Y such that $f^{-1}(V_i)$ is affine $\forall i$. Show that f is affine. If $k[f^{-1}(V_i)]$ is a finitely generated $k[V_i]$ -module for all i then f is finite.

Problem 51. (a) Let X be a complete variety and $f : X \rightarrow Y = \mathrm{Spec}\text{-}m(k)$ the unique morphism to a point. Show that $f^* : \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ is an isomorphism.

(b) Find a projective variety X and a birational morphism $f : X \rightarrow Y$ such that $f_*\mathcal{O}_X$ is not locally free on Y .

Problem 52. (a) $Y \subset \mathbb{P}^n$ is a hypersurface of degree d with ideal sheaf $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n}$. Show that $\mathcal{I}_Y \cong \mathcal{O}(-d)$.

(b) Let $v_d : \mathbb{P}^n \rightarrow \mathbb{P}^N$ be the Veronese embedding, $N = \binom{n+d}{n} - 1$. Show that $(v_d)^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \mathcal{O}_{\mathbb{P}^n}(d)$.

Problem 53. Let $\varphi : \mathbb{P}^n \rightarrow \mathbb{P}^m$ be any non-constant morphism. Then $\dim \varphi(\mathbb{P}^n) = n$. Furthermore, φ is the composition of a Veronese embedding $v_d : \mathbb{P}^n \rightarrow \mathbb{P}^{N-1}$, a projection $\mathbb{P}(k^N) - \mathbb{P}(L) \rightarrow \mathbb{P}(k^N/L)$ for some linear subspace $L \subset k^N$, and an inclusion of a linear subspace $\mathbb{P}(k^N/L) \subset \mathbb{P}^m$.

If X is any variety, let $\pi : \bar{X} \rightarrow X$ denote its normalization.

Problem 54. (a) If Y is a normal variety and $f : Y \rightarrow X$ a dominant morphism, then there exists a unique morphism $\bar{f} : Y \rightarrow \bar{X}$ such that $f = \pi \circ \bar{f}$.

(b) Give a counter example to (a) when f is not dominant.

Problem 55. $X = V(xy - z^2) \subset \mathbb{A}^3$ is normal. [Hint: $k[X] = k[x, xt, xt^2] \subset k(x, t)$ where $t = z/x$.]