Problem 1. Prove that the Segre map \( s : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N \) gives an isomorphism of \( \mathbb{P}^n \times \mathbb{P}^m \) with a closed subvariety of \( \mathbb{P}^N \), where \( N = nm + n + m \).

Problem 2. Assume that the characteristics of \( k \) is not 2. If \( C = V_f \subset \mathbb{P}^2 \) is any curve defined by an irreducible homogeneous polynomial \( f \in k[x, y, z] \) of degree 2, then \( C \cong \mathbb{P}^1 \).

Problem 3. (a) Any subspace of a separated space with functions is separated.
(b) A product of separated spaces with functions is separated.

Problem 4. Let \( X \) be a pre-variety such that for each pair of points \( x, y \in X \) there is an open affine subvariety \( U \subset X \) containing both \( x \) and \( y \).
(a) Show that \( X \) is separated.
(b) Show that \( \mathbb{P}^n \) has this property.

Problem 5. Let \( \varphi : X \to Y \) be a morphism of spaces with functions and suppose \( Y = \bigcup V_i \) is an open covering such that each restriction \( \varphi : \varphi^{-1}(V_i) \to V_i \) is an isomorphism. Then \( \varphi \) is an isomorphism.

Problem 6. [Hartshorne II.2.16 and II.2.17]
Let \( X \) be any variety and \( f \in k[X] \) a regular function.
(a) If \( h \) is a regular function on \( D(f) \subset X \) then \( f^nh \) can be extended to a regular function on all of \( X \) for some \( n > 0 \). [Hint: Let \( X = U_1 \cup \cdots \cup U_m \) be an open affine cover. Start by showing that some \( f^nh \) can be extended to \( U_i \) for each \( i \).]
(b) \( k[D(f)] = k[X]_f \).
(c) Suppose \( f_1, \ldots, f_r \in k[X] \) satisfy \( (f_1, \ldots, f_r) = k[X] \) and \( D(f_i) \) is affine for each \( i \). Then \( X \) is affine.
[Hint for (c): Use Problem 5.]

Problem 7. Let \( f : X \to Y \) be a continuous map of topological spaces, and let \( W \subset X \) be a subset.
(a) \( \overline{W} = X \) if and only if \( W \cap U \neq \emptyset \) for every non-empty open subset \( U \subset X \).
(b) If \( \overline{W} = X \) and \( f(X) = Y \), then \( f(W) = Y \).
(c) If \( X \) is irreducible and \( U \subset X \) is a non-empty open subset, then \( \overline{U} = X \).
(d) \( W \) is irreducible if and only if \( \overline{W} \) is irreducible. [By definition \( W \) is irreducible if, whenever \( W \subset F_1 \cup F_2 \) with \( F_i \subset X \) closed, we have \( W \subset F_i \) for some \( i \).]
(e) If \( W \) is irreducible then \( f(W) \) is irreducible.