**k-MOMENTS OF DISTANCES BETWEEN CENTERS OF FORD CIRCLES**

SNEHA CHAUBEY, AMITA MALIK, AND ALEXANDRU ZAHARESCU

**Abstract.** In this paper, we investigate a problem on the distribution of Ford circles, which concerns moments of distances between centers of these circles that lie above a given horizontal line.

1. **Introduction**

Introduced in 1938 by Lester R. Ford [8], a Ford circle is a circle tangent to the $x$-axis at a given point with rational coordinates $(p/q, 0)$ in reduced form, centered at $(p/q, 1/(2q^2))$. Any two Ford circles are either disjoint or tangent to each other. In the present paper, we study a question concerning the distribution of Ford circles.

For a fixed interval $I := [\alpha, \beta] \subseteq [0, 1]$ with rational end points and for each large positive integer $Q$, we consider the set $\mathcal{F}_{I,Q}$ consisting of Ford circles with centers lying between the vertical lines $x = \alpha$ and $x = \beta$ or possibly on the line $x = \beta$ but not below the line $y = \frac{1}{2Q^2}$. Note that these are the Ford circles that are tangent to the real axis at the rational points $(a/q, 0)$ with $a/q$ in the interval $I = [\alpha, \beta]$ and $q \leq Q$. Let $N_I(Q)$ denote the number of elements in $\mathcal{F}_{I,Q}$. The circles $C_{Q,1}, C_{Q,2}, \ldots, C_{Q,N_I(Q)}$ in $\mathcal{F}_{I,Q}$ are arranged in such a way that any two consecutive circles are tangent to each other. For each $j$ in $\{1, 2, \ldots, N_I(Q)\}$, denote the center of $C_{Q,j}$ by $O_{Q,j}$ and the radius of $C_{Q,j}$ by $r_{Q,j}$.

For any positive integer $k$, consider the $k$-moment

$$\mathcal{M}_{k,I}(Q) := \frac{1}{|I|} \sum_{j=1}^{N_I(Q)-1} (D(O_{Q,j}, O_{Q,j+1}))^k,$$

where $D(O_{Q,j}, O_{Q,j+1})$ denotes the Euclidean distance between the centers $O_{Q,j}$ and $O_{Q,j+1}$. For all large $X$, we consider the average

$$\mathcal{A}_{k,I}(X) := \frac{1}{X} \int_X^{2X} \mathcal{M}_{k,I}(Q) \, dY,$$

where here and in what follows, $Y$ denotes a real variable and the positive integer $Q$ is a function of $Y$; more precisely, $Q$ is the integer part of $Y$. Although, as $Q$ increases, the sequence of individual distances $D(Q_{Q,j}, O_{Q,j+1})$ changes wildly as more and more circles of various sizes appear between any two given circles, the $k$-averages $\mathcal{A}_{k,I}(X)$ satisfy nice asymptotic formulas.

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Theorem 1.1. Fix an interval $I := [\alpha, \beta] \subseteq [0, 1]$ with $\alpha, \beta \in \mathbb{Q}$, and let $A_{k,I}(X)$ be defined as in (1.2). Then, for $k = 1$,

$$A_{1,I}(X) = \frac{6}{\pi^2} \log 4X + B_1(I) + O\left(\frac{\log^2 X}{X}\right),$$

(1.3)

where $B_1(I)$ is a constant depending only on the interval $I$.

We remark that when $I = [0, 1]$, the constant $B_1(I)$ is given by

$$B_1([0, 1]) = \frac{\gamma - 1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)},$$

where $\zeta(s)$ denotes the Riemann zeta function and $\gamma$ denotes Euler’s constant. In this case, we also obtain a better bound for the error term in (1.3), namely,

$$A_{1,[0,1]}(X) = \frac{6}{\pi^2} \log 4X + \frac{\gamma - 1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)} + O\left(\frac{1}{Xe^{c_0 (\log X)^{3/5}}(\log \log X)^{-1/5}}\right),$$

(1.4)

where $c_0 > 0$ is an absolute constant.

Theorem 1.2. For $I$ as in Theorem 1.1 and $k = 2$ in (1.2),

$$A_{2,I}(X) = B_2(I) + \frac{3 \log X}{\pi^2} + \frac{D_2(I)}{X^2} + O_c\left(X^{-21/10+\epsilon}\right),$$

where $B_2(I)$ and $D_2(I)$ are constants depending only on the interval $I$.

In particular, for $I = [0, 1]$, one has

$$B_2([0, 1]) = \frac{\zeta(3)}{2\zeta(4)} \text{ and } D_2([0, 1]) = \frac{3}{\pi^2} \left(\frac{\gamma - \zeta'(2)}{\zeta(2)} + \frac{5}{4} - \log 2\right).$$

In this case also, we obtain a better bound for the error term,

$$A_{2,[0,1]}(X) = \frac{\zeta(3)}{2\zeta(4)} + \frac{3 \log X}{\pi^2} + \frac{3}{\pi^2} \left(\frac{\gamma - \zeta'(2)}{\zeta(2)} + \frac{5}{4} - \log 2\right) \frac{1}{X^2}$$

$$+ O_c\left(\frac{\log^{5/3} X (\log \log X)^{1+\epsilon}}{X^3}\right).$$

(1.5)

Theorem 1.3. For $I$ as in Theorem 1.1 and $k \geq 3$ in (1.2),

$$A_{k,I}(X) = B_k(I) + O_k\left(\frac{1}{X^2}\right),$$

where $B_k(I)$ is a constant depending on $k$ and the interval $I$.

In particular, for the full interval $I = [0, 1]$, one has

$$B_k([0, 1]) = \frac{\zeta(2k-1)}{2^{k-1} \zeta(2k)}.$$

In this case, we obtain a second order term and a better bound for the error term,

$$A_{k,[0,1]}(X) = \frac{\zeta(2k-1)}{2^{k-1} \zeta(2k)} + k \frac{\zeta(2k-3)}{2^k \zeta(2k-2)} \frac{1}{X^2} + O_k\left(\frac{1}{X^3}\right).$$

(1.6)

It would be interesting to investigate similar questions for Apollonian circle packings.
2. General Setup

In this section, we fix a positive integer $k$ and express the $k$-th moment $M_{k,I}$ in terms of the Euler-phi function and the Mobius function. Next, we rewrite $A_{k,I}$ as an integral involving the Riemann zeta function, and then shift the path of integration based on the Vinogradov-Korobov zero free region. To proceed, we first review some facts about Farey fractions. Given a positive integer $Q$, by a Farey fraction of order $Q$, we mean a rational number in reduced form in the interval $[0, 1]$ with denominator at most $Q$. We denote by $F_Q$ the sequence of Farey fractions of order $Q$, arranged in order of increasing size. Two Farey fractions $a/b < c/d$ in $F_Q$ are neighbours if and only if $bc - ad = 1$ and $b + d > Q$. The Farey sequence $F_Q$ is in bijection with the set of Ford circles tangent to the real line at points in the interval $[0, 1]$ and radius at least $1/2Q^2$. Therefore, the distance between the centers $O_{Q,j}$ and $O_{Q,j+1}$ of two consecutive Ford circles $C_{Q,j}$ and $C_{Q,j+1}$ is given by

$$D(Q_{Q,j}, O_{Q,j+1}) = \frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2}, \quad (2.1)$$

where $a_j/q_j$ and $a_{j+1}/q_{j+1}$ are neighbours in $F_Q$ and correspond to a pair of Ford circles tangent to the $x$-axis at $x = a_j/q_j$ and $x = a_{j+1}/q_{j+1}$ respectively. Thus, $N_I(Q)$ is same as the number of Farey fractions of order $Q$ inside the interval $I = [\alpha, \beta]$. For two neighbouring Farey fractions $a'/q' < a/q$ of order $Q$ in the interval $I$, we note that $a' \equiv -\bar{q} \pmod{q'}$, since $q'a - qa' = 1$. The notation $\bar{x} \pmod{n}$ is used for the multiplicative inverse of $x \pmod{n}$ in the interval $[1, n]$ for positive integers $x$ and $n$ with $\gcd(x, n) = 1$. Thus, the conditions $a'/q' \in (\alpha, \beta]$ and $a/q \in (\alpha, \beta]$ are equivalent to

$$\bar{q} \in [q' - q'\beta, q' - q'\alpha], \quad \text{and} \quad \bar{q}' \in (q\alpha, q\beta], \quad (2.2)$$

respectively. Questions on the distribution of Farey fractions have been studied extensively, see for example [1], [2], [4], [9] and the references therein.

For a fixed $k$ and the interval $I$, from (1.1) and (2.1), one has

$$|I| M_{k,I}(Q) = \sum_{j=1}^{N_I(Q)-1} \left( \frac{1}{2q_j^2} + \frac{1}{2q_{j+1}^2} \right)^k$$

$$= \frac{1}{2^{k-1}} \sum_{j=2}^{N_I(Q)} \frac{1}{q_j^{2k}} + \frac{1}{2^{k}q_1^{2k}} - \frac{1}{2^{k}q_1^{2k}N_I(Q)} + \frac{1}{2^{k}} \sum_{j=1}^{N_I(Q)-1} \sum_{i=1}^{k-1} \binom{k}{i} \left( \frac{1}{q_j^{2k-i}q_{j+1}^{i-1}} \right)^2$$

$$= \frac{1}{2^{k-1}} \sum_{1 \leq q \leq Q} \frac{1}{q^{2k} \sum_{\alpha \leq \beta \leq q, (\alpha, \beta) = 1} 1} + \frac{1}{2^{k}} \sum_{i=1}^{k-1} \binom{k}{i} \sum_{j=1}^{N_I(Q)-1} \frac{1}{q_j^{2k-i}q_{j+1}^{i-1}} + \frac{1}{2^{k}} \frac{1}{q_1^{2k}} \left( \frac{1}{2^{k}q_1^{2k}} \right)^2$$

$$=: S_k + S'_k + R_k(I), \quad (2.3)$$

where in the last inequality, without loss of generality, we assume that for large $Q$, the endpoints of the interval $I = [\alpha, \beta]$ are Farey fractions of order $Q$ since $\alpha, \beta \in \mathbb{Q}$. This implies
$R_k(I)$ in (2.3) is a constant depending only on $k$ and end-points of the interval $I$. Therefore,

$$|I| A_{k,I} = \frac{1}{X} \int_X^{2X} (S_k + S_k' + R_k(I)) \, dY$$

$$= \frac{1}{X} \int_1^{2X} S_k \, dY - \frac{1}{X} \int_1^{X} S_k \, dY + \frac{1}{X} \int_X^{2X} S_k' \, dY + R_k(I).$$

(2.4)

Consider,

$$\frac{1}{X} \int_1^{X} S_k \, dY = \frac{1}{2k-1} \int_1^{X} \sum_{1 \leq q \leq Q} \frac{1}{q^{2k}} \sum_{\alpha q < a < \beta q \atop (a,q) = 1} \mu(d) \, dY$$

$$= \frac{1}{2k-1} \int_1^{X} \sum_{q \leq Q} \frac{1}{q^{2k}} \sum_{\alpha q < a < \beta q} \mu(d) \sum_{d | (a,q)} 1 \, dY$$

$$= \frac{1}{2k-1} \int_1^{X} \sum_{q \leq Q} \frac{1}{q^{2k}} \sum_{d | q} \mu(d) \sum_{\alpha q < a < \beta q \atop \frac{\alpha q}{d} < \frac{\beta q}{d}} 1 \, dY$$

$$= \frac{1}{2k-1} \int_1^{X} \left( \frac{|I|}{q^{2k}} \sum_{q \leq X} \frac{1}{q^{2k}} \sum_{\alpha q < a < \beta q \atop \frac{\alpha q}{d} < \frac{\beta q}{d}} \mu(d) \left( \frac{\beta q}{d} - \frac{\alpha q}{d} \right) \right) \, dY$$

$$= \frac{|I|}{2k-1} \sum_{1 \leq q \leq X} \frac{1}{q^{2k}} \left( \frac{1 - \frac{q}{X}}{X} \right) - \frac{1}{2k-1} \int_1^{X} \sum_{d,m \geq 1 \atop dm \leq Y} \frac{\mu(d)}{d^{2k} m^{2k}} \left( \frac{\beta m}{m^{2k}} - \frac{\alpha m}{m^{2k}} \right) \, dY$$

$$= \frac{|I|}{2k-1} \sum_{1 \leq q \leq X} \frac{1}{q^{2k}} \left( \frac{1 - \frac{q}{X}}{X} \right) - \frac{1}{2k-1} \sum_{1 \leq d \leq X} \frac{\mu(d)}{d^{2k}} \sum_{1 \leq m \leq \frac{X}{d}} \left( \frac{\beta m}{m^{2k}} - \frac{\alpha m}{m^{2k}} \right)$$

$$+ \frac{1}{2k-1} \sum_{1 \leq d \leq X} \sum_{1 \leq m \leq \frac{X}{d}} \frac{\mu(d)}{d^{2k-1}} \sum_{1 \leq m \leq \frac{X}{d}} \left( \frac{\beta m}{m^{2k-1}} - \frac{\alpha m}{m^{2k-1}} \right)$$

$$=: S_{k,1} - S_{k,2} + S_{k,3}. \quad (2.5)$$

We first estimate the sums $S_{k,2}$ and $S_{k,3}$. Since for $x \geq 1$, and $a \geq 2$,

$$\sum_{n \geq x} \frac{1}{n^a} = O \left( x^{1-a} \right),$$
one has

\[ S_{k,2} = \frac{1}{2^{k-1}} \sum_{1 \leq d \leq X} \frac{\mu(d)}{d^{2k}} \left( \sum_{m \geq 1} \frac{\beta m - \alpha m}{m^{2k}} \right) \right) - \sum_{m > X/d} \frac{\beta m - \alpha m}{m^{2k}} \right) \]

\[ = \frac{C_{2k,1}}{2^{k-1}} \sum_{1 \leq d \leq X} \frac{\mu(d)}{d^{2k}} + O \left( \sum_{1 \leq d \leq X} \frac{\mu(d)}{d^{2k}} \sum_{m > X/d} \frac{|\beta m - \alpha m|}{m^{2k}} \right) \]

\[ = \frac{C_{2k,1}}{2^{k-1} \zeta(2k)} + O \left( \frac{\log X}{X^{2k-1}} \right), \tag{2.6} \]

where for a natural number \( j \), we denote

\[ C_{j,1} := \sum_{m \geq 1} \frac{\beta m - \alpha m}{m^j}. \]

Similarly for \( k \geq 2 \),

\[ S_{k,3} = \frac{1}{2^{k-1}X} \sum_{1 \leq d \leq X} \frac{\mu(d)}{d^{2k-1}} \left( \sum_{m \geq 1} \frac{\beta m - \alpha m}{m^{2k-1}} \right) \right) - \sum_{m > X/d} \frac{\beta m - \alpha m}{m^{2k-1}} \right) \]

\[ = \frac{C_{2k-1,1}}{2^{k-1}} \frac{1}{X} \sum_{1 \leq d \leq X} \frac{\mu(d)}{d^{2k-1}} + O \left( \frac{1}{X} \sum_{1 \leq d \leq X} \frac{|\mu(d)|}{d^{2k-1}} \sum_{m > X/d} \frac{|\beta m - \alpha m|}{m^{2k-1}} \right) \]

\[ = \frac{C_{2k-1,1}}{2^{k-1} \zeta(2k-1)} \frac{1}{X} + O \left( \frac{\log X}{X^{2k-1}} \right). \tag{2.7} \]

In a similar fashion one estimates \( S_{k,3} \) for \( k = 1 \),

\[ |S_{1,3}| = \left| \frac{1}{X} \sum_{1 \leq d \leq X} \frac{\mu(d)}{d} \sum_{m < X/d} \frac{|\beta m - \alpha m|}{m} \right| \]

\[ \leq \frac{1}{X} \sum_{1 \leq d \leq X} \frac{|\mu(d)|}{d} \sum_{m < X/d} \frac{|\beta m - \alpha m|}{m} \right| = O \left( \frac{\log^2 X}{X} \right). \tag{2.8} \]

Next we consider the sum

\[ S_{k,1} = \frac{|I|}{2^{k-1}} \sum_{1 \leq q \leq X} \frac{\phi(q)}{q^{2k}} \left( 1 - \frac{q}{X} \right). \]

The arithmetic function \( \phi(n)n^{-2k} \) is multiplicative and its Dirichlet series is given by

\[ \sum_{n=1}^{\infty} \frac{\phi(n)n^{-2k}}{n^s} = \frac{\zeta(s + 2k - 1)}{\zeta(s + 2k)}, \]

which converges for \( \Re(s) > 2 - 2k \). For a complex number \( s \), we write \( s = \sigma + it \). By Perron’s formula ([12, page 130]), for \( c > 0 \),

\[ \frac{1}{X} \sum_{q \leq X} \frac{\phi(q)}{q^{2k}} (X - q) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{X^s \zeta(s + 2k - 1)}{s(s + 1)\zeta(s + 2k)} \, ds. \tag{2.9} \]
Fix \( T, U > 0 \) such that \( 2 \leq T \leq X, \ X^2 \leq U \leq X^{2k} \) and \( c = \frac{a}{\log X} \) for some absolute constant \( a > 0 \). Let
\[
d = -2k + 1 - \frac{A}{(\log 2T)^{2/3} (\log \log 2T)^{1/3}},
\]
where \( A \) will be a suitably chosen absolute constant. In order to evaluate the above integral, we modify the path of integration from \( c - iU \) to \( c + iU \) along the line segments \( l_j, 1 \leq j \leq 9 \) described below.

We let \( l_1 \) be the half line from \( c + iU \) to \( c + i\infty \), \( l_2 \) be the line segment from \(-2k + 1 + iU\) to \( c + iU \), \( l_3 \) be the line segment from \(-2k + 1 + iT\) to \(-2k + 1 + iU\), \( l_4 \) be the line segment from \( d + iT \) to \(-2k + 1 + iT\), \( l_5 \) be the line segment from \( d - iT \) to \( d + iT\), \( l_6 \) be the line segment from \(-2k + 1 - iT\) to \( d - iT\), \( l_7 \) be the line segment from \(-2k + 1 - iU\) to \(-2k + 1 - iT\), \( l_8 \) be the line segment with endpoints \(-2k + 1 - iU\) and \( c - iU\), and lastly let \( l_9 \) be the half line from \( c - i\infty \) to \( c - iU \). The main contribution on the right side of (2.9) comes from the residues at the poles of the function
\[
f_k(s) := \frac{X^s \zeta(s + 2k - 1)}{s(s + 1)\zeta(s + 2k)},
\]
encountered when we modified the path of integration. By the residue theorem,
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f_k(s) \ ds = \sum \text{Res}(f_k(s)) + \sum_{m=1}^{9} J_m,
\]
(2.10)
where \( J_i \) is the integral of \( f_k(s) \) along \( l_i \). Here the sum \( \sum \text{Res}(f_k(s)) \) is taken over all the poles of \( f_k(s) \) inside the region bounded by segments \( l_2, l_3, \ldots, l_8 \) and the vertical segment joining \( c - iU \) and \( c + iU \). To estimate the integrals \( J_1, \ldots, J_9 \), we use standard bounds for \( \zeta(s) \) and \( \frac{1}{\zeta(s)} \) ([13, page 47]),
\[
\zeta(\sigma + it) = \begin{cases} 
O \left(t^{\sigma-\frac{1}{2}} \log t\right), & -1 \leq \sigma \leq 0, \\
O \left(t^{\frac{1}{2} - \sigma} \log t\right), & 0 \leq \sigma \leq 1, \\
O (\log t), & 1 \leq \sigma \leq 2, \\
O (1), & \sigma \geq 2,
\end{cases}
\]
and
\[
\frac{1}{\zeta(\sigma + it)} = \begin{cases} 
O (\log t), & 1 \leq \sigma \leq 2, \\
O (1), & \sigma \geq 2.
\end{cases}
\]
We also use the Vinogradov-Koroborov zero free region ([14], [11]),
\[
\sigma \geq 1 - B (\log t)^{-\frac{2}{3}} (\log \log t)^{-\frac{1}{3}},
\]
where
\[
\frac{1}{\zeta(s)} = O \left((\log t)^{\frac{2}{3}} (\log \log t)^{\frac{1}{3}}\right),
\]
and \( B \) is an absolute constant.
3. First Moment

In this section we provide a proof of Theorem 1.1. For the first moment, we set $k = 1$ in Section 2.

Proof of Theorem 1.1. From (2.3), we note that

$$|I| = \sum_{q \leq Q} \frac{1}{q^2} \sum_{\substack{\alpha q < \alpha \leq \beta q \\ (\alpha, q) = 1}} 1 + \left( -\frac{1}{2q_1^2} + \frac{1}{2q_2^2} \right) = S_1 + R_1(I).$$

Therefore, as in (2.4) we have

$$|I| = \frac{1}{X} \int_1^{2X} S_1 \, dY - \frac{1}{X} \int_1^X S_1 \, dY + R_1(I). \quad (3.1)$$

Now from (2.5),

$$\frac{1}{X} \int_1^X S_1 \, dY = S_{1,1} - S_{1,2} + S_{1,3}. \quad (3.2)$$

The sums $S_{1,2}$ and $S_{1,3}$ have already been estimated in (2.6) and (2.8) respectively. In order to estimate $S_{1,1}$, we bound the integrals $J_m$ in (2.10) as follows. One has

$$|J_1|, |J_0| = O \left( \int_U^\infty \frac{|X^{c+it}| |\zeta(c+1+it)|}{c+it |c+1+it| |\zeta(c+2+it)|} \, dt \right)$$

$$= O \left( X^c \int_U^\infty \frac{\log t}{t^2} \, dt \right) = O \left( \frac{\log U}{U} \right).$$

And,

$$|J_2|, |J_8| = O \left( \int_{-1}^0 \frac{|X^{\sigma+iU}| |\zeta(1+\sigma+iU)|}{\sigma + iU |\sigma + 1+iU| |\zeta(2+\sigma+iU)|} \, d\sigma \right)$$

$$= O \left( \frac{(\log U)^2}{U^2} \int_{-1}^0 \left( \frac{X}{\sqrt{U}} \right)^\sigma \, d\sigma + \frac{\log U}{U^2} \int_0^c X^\sigma \, d\sigma \right) \approx O \left( \frac{\log^2 U}{U^2} \right).$$

Next,

$$|J_3|, |J_7| = O \left( \int_T^U \frac{|X^{-1+it}| |\zeta(it)|}{-1+it |it| |\zeta(1+it)|} \, dt \right) = O \left( X^{-1} \int_T^U \frac{\log t}{t^{3/2}} \, dt \right)$$

$$= O \left( \frac{\log^2 T}{X \sqrt{T}} \right).$$

Also,

$$|J_4|, |J_6| = O \left( \int_d^{-1} \frac{|X^{\sigma+iT}| |\zeta(1+\sigma+iT)|}{\sigma + 1+iT |\sigma+iT| |\zeta(\sigma+iT)|} \, d\sigma \right)$$

$$= O \left( \frac{(\log T)^{1/2} (\log \log T)^{1/2}}{T^2} \int_d^{-1} X^\sigma |\zeta(1+\sigma+iT)| \, d\sigma \right)$$

$$= O \left( \frac{(\log T)^{1/2} (\log \log T)^{1/2}}{T^2} \int_d^{-1} \left( \frac{X}{\sqrt{T}} \right)^\sigma \, d\sigma \right) \approx O \left( \frac{(\log T)^{1/2} (\log \log T)^{1/2}}{X T^{3/2}} \right).$$
Lastly,

\[ |J_5| = O \left( \int_{-T}^{T} \frac{|X^{d+it}| |\zeta(1+d+it)|}{|d+it||d+1+it||\zeta(d+2+it)|} \, dt \right) \]

\[ = O \left( X^d \int_{-T}^{T} \frac{t^{-1/2-d}(\log(2+|t|))^{5/3}(\log \log(3+|t|))^{1/3}}{1+t^2} \, dt \right) = O(X^d). \]

Collecting all the above estimates and setting \( U = X^2 \) and \( T = \exp \left( c_1 (\log X)^{3/5} \left( \log \log X \right)^{-1/5} \right) \), one obtains

\[ S_{1,1} = |I| \text{ Res}(f_1(s)) + O \left( \frac{1}{X^{c_0(\log X)^{3/5}(\log \log X)^{-1/5}}} \right), \quad (3.3) \]

where \( c_0 \) and \( c_1 \) are suitable positive absolute constants. Here, in the prescribed region, \( f_1(s) \) has only one pole at \( s = 0 \) of order two with residue

\[ \text{Res}(f_1(s)) = \frac{\log X}{\zeta(2)} + \frac{\gamma - 1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)}. \]

From (2.6), (2.8), (3.1), (3.2), (3.3) and above,

\[ A_{1,I}(X) = \frac{6}{\pi^2} \log 4X + \frac{\gamma - 1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)} + \frac{C_{2,I}}{|I|} + \frac{R_1(I)}{|I|} + O \left( \frac{\log^2 X}{X} \right). \]

This concludes the proof of Theorem 1.1.

**Remark.** In the case of the full interval \( I = [0,1] \), we observe that \( R_1([0,1]) = 0 \) in (2.3) and \( S_{1,2} = 0 = S_{1,3} \) in (2.5). Therefore, \( S_{1,1} \) in (3.2) is the only term which contributes to the average \( A_{1,[0,1]} \) in (3.1) and we obtain

\[ A_{1,[0,1]}(X) = \frac{6}{\pi^2} \log 4X + \frac{\gamma - 1}{\zeta(2)} - \frac{\zeta'(2)}{\zeta^2(2)} + O \left( \frac{1}{X^{c_0(\log X)^{3/5}(\log \log X)^{-1/5}}} \right), \]

as claimed in (1.4).

**4. Higher Moments**

In this section, we prove Theorem 1.2 and Theorem 1.3. We first estimate the integral

\[ \frac{1}{X} \int_{1}^{X} S_k \, dY \quad \text{for } k \geq 2 \text{ in (2.4).} \]

From (2.5),

\[ \frac{1}{X} \int_{1}^{X} S_k \, dY = S_{k,1} - S_{k,2} + S_{k,3}. \]

Estimates for \( S_{k,2} \) and \( S_{k,3} \) for \( k \geq 2 \) have already been obtained in (2.6) and (2.7). For \( k \geq 2 \), estimates for

\[ S_{k,1} = \frac{|I|}{2k-1X} \sum_{q \leq X} \phi(q) \frac{1}{q^{2k}} (X - q) \]
can be obtained as before where we set \( U = X^{2k} \). In this case, the corresponding function \( f_k(s) \) has poles at \( s = 0, s = -1 \) and \( s = 2 - 2k \) in the region described before. All these poles are simple and the sum of the residues of \( f_k(s) \) at these poles is given by

\[
\sum \text{Res}(f_k(s)) = \frac{\zeta(2k - 1)}{\zeta(2k)} - \frac{\zeta(2k - 2)}{\zeta(2k - 1)} - \frac{1}{(2k - 3)(2k - 2)\zeta(2) X^{2k-2}}.
\]

One can estimate the line integrals \( J_m \) of the function \( f_1(s) \) along \( l_i \) for \( 1 \leq i \leq 9 \) in (2.10) as before. In this case one has

\[
1 \int_{c-i\infty}^{c+i\infty} \frac{X^s \zeta(s + 2k - 1)}{s(s + 1)\zeta(s + 2k)} ds = \frac{\zeta(2k - 1)}{\zeta(2k)} - \frac{\zeta(2k - 2)}{\zeta(2k - 1)} X^{k - 1} + \frac{1}{(2k - 3)(2k - 2)\zeta(2) X^{2k - 2}}
\]

+ \( O\left(\frac{1}{X^{2k - 1} e^{c_0 (log X)^{3/5} (log log X)^{-1/5}}}\right) \).

Therefore, from (2.9) and the above equation, we obtain

\[
S_{k,1} = \frac{|I|\zeta(2k - 1)}{2^{k - 1}\zeta(2k)} - \frac{|I|\zeta(2k - 2)}{2^{k - 1}\zeta(2k - 1)} X^{k - 1} + \frac{|I|}{2^{k - 1}(2k - 3)(2k - 2)\zeta(2) X^{2k - 2}}
\]

+ \( O\left(\frac{1}{X^{2k - 1} e^{c_0 (log X)^{3/5} (log log X)^{-1/5}}}\right) \).

From (2.5), (2.6), (2.7) and above, we derive

\[
\frac{1}{X} \int_X^{2X} S_k dY = \frac{|I|\zeta(2k - 1)}{2^{k - 1}\zeta(2k)} - \frac{C_{2k,1}}{2^{k - 1}\zeta(2k)} + \frac{|I|(1 - 2^{2k - 3})}{2^{3k - 4}(2k - 3)(2k - 2)\zeta(2) X^{2k - 2}}
\]

+ \( O\left(\frac{\log X}{X^{2k - 1}}\right) \),

(4.1)

In order to prove Theorem 1.2 and Theorem 1.3, it remains to estimate the remaining integral

\[
\frac{1}{X} \int_X^{2X} S_k' dY
\]

for \( k \geq 2 \) in (2.4).

**Proof of Theorem 1.2.** For \( k = 2 \) in (2.3), we have

\[
|I|\mathcal{M}_{2,1}(Q) = \sum_{j=1}^{N_j(Q)-1} \left( \frac{1}{2q_j^2} + \frac{1}{2q_j^2} \right)^2
\]

\[
= \sum_{q \leq Q} \frac{1}{q^4} \sum_{\alpha q < \alpha \leq \beta q} 1 + \sum_{j=1}^{N_j(Q)-1} \left( \frac{1}{q_j^2} \right)^2 + \left( - \frac{1}{4q_1^4} + \frac{1}{4q_1^4} \right)
\]

\[
= S_2 + S_2' + R_2(I).
\]

Therefore,

\[
|I|\mathcal{A}_{2,1} = \frac{1}{X} \int_X^{2X} S_2 dY + \frac{1}{X} \int_X^{2X} S_2' dY + R_2(I).
\]

(4.2)

From [3, Theorem 2], we obtain

\[
S_2' = \frac{|I|}{2} S_0(Q) + \frac{C_{2,1}}{Q^2} + O_x \left( Q^{-21/10 + \epsilon} \right),
\]
where

\[ S_0(Q) = \frac{12}{\pi^2 Q^2} \left( \log Q + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{1}{2} \right) + O_\varepsilon \left( \frac{\log^{5/3} Q (\log \log Q)^{1+\varepsilon}}{Q^3} \right). \]

We remark in passing that the saving in the exponent above (from $-2$ to $-21/10$) was obtained by employing Weil type estimates ([7], [10], [15]) for Kloosterman sums.

Next, we have

\[ \frac{1}{|I| \int_X^{2X} S'_2 \, dY} = \frac{3}{\pi^2} \frac{\log X}{X^2} + \left( \frac{3}{\pi^2} \left( \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{3}{2} - \log 2 \right) + \frac{C_{2,l}}{2|I|} \right) \frac{1}{X^2} + O_\varepsilon \left( X^{-21/10+\varepsilon} \right). \]

Combining (4.1), (4.2) and above, we conclude that

\[ A_{2,l} = \frac{|I|}{2|I| \zeta(4)} - \frac{R_2(I)}{|I|} + \frac{3}{\pi^2} \frac{\log X}{X^2} + \left( \frac{3}{\pi^2} \left( \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{5}{4} - \log 2 \right) + \frac{C_{2,l}}{2|I|} \right) \frac{1}{X^2} \]

\[ + O_\varepsilon \left( X^{-21/10+\varepsilon} \right). \]

This completes the proof of Theorem 1.2.

\[ \square \]

Remark. For the full interval $I = [0, 1]$, observe that $R_2(I) = 0$, and

\[ S'_2 = \frac{6}{\pi^2 Q^2} \left( \log Q + \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{1}{2} \right) + O_\varepsilon \left( \frac{\log^{5/3} Q (\log \log Q)^{1+\varepsilon}}{Q^3} \right). \]

This along with (4.1) and (4.2) proves (1.5),

\[ A_{2,[0,1]}(X) = \frac{\zeta(3)}{2\zeta(4)} - \frac{1}{2} + \frac{3}{\pi^2} \frac{\log X}{X^2} + \frac{3}{\pi^2} \left( \gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{5}{4} - \log 2 \right) \frac{1}{X^2} \]

\[ + O_\varepsilon \left( \frac{\log^{5/3} X (\log \log X)^{1+\varepsilon}}{X^3} \right). \]

Proof of Theorem 1.3. For $k \geq 3$,

\[ |I| M_{k,l}(Q) = \sum_{j=1}^{N_l(Q)-1} \left( \frac{1}{2 q_j^2} + \frac{1}{2 q_{j+1}^2} \right)^k \]

\[ = \sum_{q \leq Q} \frac{1}{q^{2k}} \prod_{\substack{\alpha q < a \leq \beta q \\ (a,q) = 1}} \frac{|I|}{2^{k-1}} + \frac{1}{2^k} \sum_{i=1}^{k} \binom{k}{i} \sum_{j=1}^{N_l(Q)-1} \left( \frac{1}{q_j^{q_i} q_{j+i}^{q_i}} \right)^2 \left( \frac{1}{2^k q_{l1}^{2k}} + \frac{1}{2^k q_{N^l(Q)}^{2k}} \right) \]

\[ = S_k + S'_k + R_k(I). \]

Therefore,

\[ |I| A_{k,l} = \frac{1}{X} \int_X^{2X} S_k \, dY + \frac{1}{X} \int_X^{2X} S'_k \, dY + R_k(I). \]

(4.3)

For each $1 \leq i \leq k - 1$, consider the sum

\[ S_{k,i} := \sum_{j=1}^{N_l(Q)-1} \frac{1}{q_j^{2i} q_{j+1}^{2k-2i}}. \]
For any positive integer \(m\), let \(L_m\) denote the set
\[
L_m := \left\{ l \in \mathbb{N} : l > m, Q - m < l \leq Q, \gcd(m, l) = 1, \quad \bar{l} \pmod{m} \in (m\alpha, m\beta), \bar{m} \pmod{l} \in [l - l\beta, l - l\alpha) \right\}.
\]

Employing (2.2), we have
\[
S_{k,i} = \sum_{1 \leq r \leq Q} \sum_{q \in L_r} \frac{1}{q^{2k-2i}} \sum_{1 \leq q \leq Q} \sum_{r \in L_q} \frac{1}{q^{2k-2i}}.
\]

As noted earlier in Section 2, when \(q\) and \(r\) are denominators of neighbouring Farey fractions in \(F_Q\), then \(r + q > Q\). Therefore, \(q > r\) implies \(q > Q/2\) and for \(r > q\), we have \(r > Q/2\).

Also,
\[
\sum_{q \in L_r} 1 = O(\phi(r)) \quad \text{and} \quad \sum_{r \in L_q} 1 = O(\phi(q)).
\]

Using the above relations and the fact that for \(x \geq 2\) and \(a \geq 3\),
\[
\sum_{1 \leq n \leq x} \frac{\phi(n)}{n^a} = \frac{\zeta(a-1)}{\zeta(a)} + O(x^{2-a}), \quad (4.4)
\]
we obtain
\[
S_{k,i} \leq \left(\frac{2}{Q}\right)^{2i} \sum_{r \leq Q} \frac{1}{r^{2k-2i}} \sum_{q \in L_r} 1 + \left(\frac{2}{Q}\right)^{2k-2i} \sum_{q \leq Q} \frac{1}{q^{2i}} \sum_{r \in L_q} 1
\]
\[
= O\left(\frac{1}{Q^{2i}} \sum_{r \leq Q} \frac{1}{r^{2k-2i}} \phi(r)\right) + O\left(\frac{1}{Q^{2k-2i}} \sum_{q \leq Q} \frac{1}{q^{2i}} \phi(q)\right)
\]
\[
= O\left(\log\frac{Q}{Q^{2i}}\right) + O\left(\log\frac{Q}{Q^{2k-2i}}\right).
\]

Here on the far right side, the first \(\log Q\) may be replaced by 1 unless \(i = k - 1\), and the second \(\log Q\) may be replaced by 1 unless \(i = 1\). Hence,
\[
\frac{1}{X} \int_X^{2X} S'_k \, dY = \frac{1}{X} \int_X^{2X} \frac{1}{2^k} \sum_{i=1}^{k-1} \binom{k}{i} S_{k,i} \, dY = O\left(\frac{1}{X^2}\right).
\]

This combined with (4.1) and (4.3) yields
\[
\mathcal{A}_{k,I} = \frac{|I|\zeta(2k-1) - C_{2k,I}}{|I|^{2k-1}\zeta(2k)} + \frac{R_k(I)}{|I|} + O_k\left(\frac{1}{X^2}\right) \quad \text{for} \quad k \geq 3,
\]

which completes the proof of Theorem 1.3.
Remark. In the case of the full interval \( I = [0, 1] \), note that \( R_k([0, 1]) = 0 \), and

\[
S_{k,i} = \sum_{1 \leq q, r \leq Q} \frac{1}{q^{2i} r^{2k-2i}} \sum_{\substack{1 \leq q, r \leq Q \\gcd(q,r) = 1, \\ r < Q/2, \ \ q+r > Q}} \frac{1}{q^{2i} r^{2k-2i}} + \sum_{\substack{1 \leq q, r \leq Q \\gcd(q,r) = 1, \ \ q < Q/2, \ \ q+r > Q}} \frac{1}{q^{2i} r^{2k-2i}} + \sum_{\substack{1 \leq q, r \leq Q \\gcd(q,r) = 1, \ \ q, r \geq Q/2}} \frac{1}{q^{2i} r^{2k-2i}}
\]

\[=: \Sigma_{1,i} + \Sigma_{2,i} + \Sigma_{3,i}.\]

First we estimate the sum \( \Sigma_{1,i} \) for \( 1 \leq i \leq k - 2 \). Note that in this case \( r < Q/2 \), therefore \( q > Q/2 \) since \( r + q > Q \). Also, \( \frac{1}{q} = \frac{1}{Q} \left( 1 + O \left( \frac{Q - q}{Q} \right) \right) \) gives,

\[
\Sigma_{1,i} = \sum_{1 \leq r < Q/2} \frac{1}{r^{2k-2i}} \sum_{\substack{1 \leq q \leq Q \\gcd(q,r) = 1 \\ Q-r < q \leq Q}} \frac{1}{q^{2i}} + O \left( \frac{Q - q}{Q^{2i+1}} \right)
\]

\[
= \sum_{1 \leq r < Q/2} \frac{1}{r^{2k-2i}} \sum_{\substack{1 \leq q \leq Q \\gcd(q,r) = 1 \\ Q-r < q \leq Q}} \frac{1}{q^{2i}} + O \left( \frac{1}{Q^{2i+1}} \sum_{1 \leq r < Q/2} \sum_{\substack{1 \leq q \leq Q \\gcd(q,r) = 1 \\ Q-r < q \leq Q}} \frac{1}{q^{2k-2i-1}} \right)
\]

\[
= \frac{1}{Q^{2i}} \sum_{1 \leq r < Q/2} \phi(r) \frac{\phi(r)}{r^{2k-2i}} + O \left( \frac{1}{Q^{2i+1}} \sum_{1 \leq r < Q/2} \phi(r) \frac{\phi(r)}{r^{2k-2i-1}} \right).
\]

Using (4.4), for \( 1 \leq i \leq k - 2 \),

\[
\Sigma_{1,i} = \frac{\zeta(2k - 2i - 1)}{\zeta(2k - 2i)} \frac{1}{Q^{2i}} + O \left( \frac{1}{Q^{2i+1}} \right).
\]

Using

\[
\sum_{n \leq x} \frac{\phi(n)}{n^2} = \frac{\log x}{\zeta(2)} + O (1), \quad \text{and} \quad \sum_{n \leq x} \frac{\phi(n)}{n} = O (x),
\]

we have, for \( i = k - 1 \),

\[
\Sigma_{1,k-1} = \frac{1}{\zeta(2)} \frac{\log(Q/2)}{Q^{2k-2}} + O \left( \frac{1}{Q^{2k-2}} \right).
\]

Similarly, for the sum \( \Sigma_{2,i} \) for \( 2 \leq i \leq k - 1 \),

\[
\Sigma_{2,i} = \frac{\zeta(2i - 1)}{\zeta(2i)} \frac{1}{Q^{2k-2i}} + O \left( \frac{1}{Q^{2k-2i+1}} \right),
\]

and

\[
\Sigma_{2,1} = \frac{1}{\zeta(2)} \frac{\log(Q/2)}{Q^{2k-2}} + O \left( \frac{1}{Q^{2k-2}} \right).
\]
Lastly, for $1 \leq i \leq k - 1$,

$$\Sigma_{3,i} = O\left(\frac{1}{Q^{2k-2}}\right).$$

Therefore,

$$S'_k = \frac{k\zeta(2k-3)}{2^{k-1}\zeta(2k-2)} \frac{1}{Q^2} + O\left(\frac{1}{Q^3}\right).$$

This combined with (4.3) gives (1.6),

$$A_{k,[0,1]} = \frac{\zeta(2k-1)}{2^{k-1}\zeta(2k)} - \frac{1}{2^{k-1}} + \frac{k\zeta(2k-3)}{2^k\zeta(2k-2)} \frac{1}{X^2} + O_k\left(\frac{1}{X^3}\right).$$

REFERENCES