Last time: $f \in S(\mathbb{A})$ Schwartz,

finite linear combinations $f = \prod_{p \leq \infty} f_p$, with

- f_∞ Schwartz (all derivs decay faster than poly)

i.e. $\forall A, B$, $\exists C$ s.t. $|f_\infty^{(A)}(x_\infty)| \leq \frac{C}{|1+|x_\infty||^B}$. $\forall x_\infty \in \mathbb{R}$.

& $\cdot f_p$ loc const, cpt supp $\cdot$ a.a. p, $f_p = \mathbb{1}_{\mathbb{Z}_p}$.

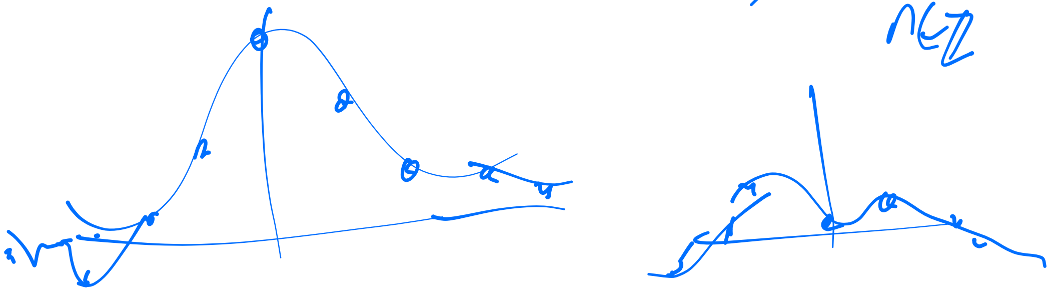
For $f \in S(\mathbb{A})$, $\hat{f}(\xi) = \int_{\mathbb{A}} f(x) e_{\mathbb{A}}(x)\, d\mu(x)$.

$\hat{f} \in S(\mathbb{A})$.

Adelic Poisson Summation formula: For $f \in S(\mathbb{A})$,

$$\sum_{q \in \mathbb{Q}} f(q) = \sum_{q \in \mathbb{Q}} \hat{f}(q).$$

Classical: for $f \in S(\mathbb{R})$, $\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m)$.



$G = \mathbb{R}$, $\Gamma = \mathbb{Z}$.

periodize
automorphize f.

pf: Let $F(x) := \sum_{n\in\mathbb{Z}} f(x+n)$ ($x \in \mathbb{R}$) on $\mathbb{R}/\mathbb{Z}$.

✓ Converges absolutely

$F(x+1) = F(x)$, Apply Fourier on S^1:

$$F(x) = \sum_{m\in\mathbb{Z}} \hat{F}(m)\, e(-mx)$$ Spectrum.

$$\hat{F}(m) = \int_{\mathbb{R}/\mathbb{Z}} F(x)\, e(mx)\, dx$$ $e^{2\pi i m x}$,

Fix D = fund dom for $\mathbb{R}/\mathbb{Z}$

("unfolding trick") $$= \int_D \sum_{n\in\mathbb{Z}} f(x+n)\, e(mx)\, dx.$$

abs conv

$$= \sum_{n\in\mathbb{Z}} \int_{D} f(x+n)\, e(mx)\, dx$$

$y = x+n, \quad dx = dy.$ Haar measure.

Haar

$$= \sum_{n\in\mathbb{Z}} \int_{D+n} f(y)\, e(m(y-n))\, dy$$

add char

$$= \sum_{n\in\mathbb{Z}} \int_{D+n} f(y)\, e(my)\, e(-mn)\, dy.$$

char trivial on $\mathbb{Z}$

$$= \sum_{n\in\mathbb{Z}} \int_{D+n} \underbrace{f(y)\, e(my)}_{\text{no } n\text{'s.}} \cdot dy.$$

$D \quad D+n$

"unfolding"

$$= \int_{\mathbb{R}} f(y)\, e(my)\, dy = \hat{f}(m).$$

<u>Lemma:</u> $\hat{F}(m) = \hat{f}(m)$

$$\sum_{n\in\mathbb{Z}} f(x+n) = F(x) = \sum_{m\in\mathbb{Z}} \hat{F}(m)\, e(-mx) = \sum_{m\in\mathbb{Z}} \hat{f}(m)\, e(-mx)$$

Set $x=0$. $$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in\mathbb{Z}} \hat{f}(m).$$

Aside: Recall: $$\widehat{\mathbb{1}_{\{a+p^n\mathbb{Z}_p\}}}(\xi) = e_p(a\cdot\xi)\cdot \mathbb{1}_{p^{-n}\mathbb{Z}_p}(\xi).$$

Fix $t>0$. Let $f_t(x) := f(x\cdot t)$.

Then $$\hat{f_t}(\xi) = \int_{\mathbb{R}} f(x\cdot t)\, e(\xi x)\, dx.$$

$y = x\cdot t$, $dy = dx\cdot t$

$$= \int_{\mathbb{R}} f(y)\, e\left(\xi \frac{y}{t}\right) \frac{dy}{t} = \frac{1}{t}\hat{f}\left(\frac{\xi}{t}\right).$$

$$\sum_{n\in\mathbb{Z}} f_t(n) = \boxed{\sum_{n\in\mathbb{Z}} f(n\cdot t) = \frac{1}{t}\sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{m}{t}\right)} = \sum_{m\in\mathbb{Z}} \hat{f_t}(m)$$

Try $f(x) = e^{-\pi x^2}$. $\hat{f}(\xi) = e^{-\pi\xi^2}$ (Ex).

$t = \frac{1}{1000}$. $\sum_{n \in \mathbb{Z}} e^{-\pi\left(\frac{n}{1000}\right)^2}$ ≈ 1000 terms until converged to a value.

$$= 1000 \cdot \sum_{m \in \mathbb{Z}} e^{-\pi(1000m)^2} \doteq 1000. + O(10^{-10^6})$$

$(m=0)$.

<u>Fourier transform:</u> $G = \mathbb{R}$, $d\mu = dx$, char $e^{2\pi i \xi x}$

Optimus Prime

$$\int_{\mathbb{R}} f(x) \cdot e(x \cdot \xi)\, dx = \hat{f}(\xi).$$

$(z = x+iy)$.

$(s \in \mathbb{C})$.

<u>Mellin transform:</u> $G = \mathbb{R}^{\times}_{>0}$, $d\mu = \frac{dx}{x}$, char $x \mapsto x^s$.

Bumblebee.

$$\tilde{f}(s) = \int_0^\infty f(x)\, x^s \frac{dx}{x}.$$

<u>Ex:</u> Change variables $x = e^y$ to see Mellin as version of Fourier. (& vice versa).

$t>0$.

$$\boxed{\sum_{n\in\mathbb{Z}} f(n\cdot t) = \frac{1}{t}\sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{m}{t}\right)}$$

Assume f is even. $\Rightarrow$ $\hat{f}$ is even.

$$\Rightarrow f(0) + 2\underbrace{\sum_{n\geq 1} f(nt)} = \underbrace{\frac{1}{t}\hat{f}(0)} + \underbrace{\frac{2}{t}\sum_{m\geq 1}\hat{f}\left(\frac{m}{t}\right)}_{\text{rapid}}.$$

ϑ-function, $\underbrace{\vartheta_f(t)}$ $:= \underbrace{\sum_{n\geq 1} f(nt)}$. ($t \in \mathbb{R}^\times_{>0}$)

$\vartheta_f(t)$ has rapid decay as $\underline{t\to\infty}$.

as $\underline{\underline{t\to 0}}$? Grows like $\boxed{\frac{1}{t}}$.

(Reminder) Mellin transform $\widetilde{\vartheta_f}(s) = \int_0^\infty \vartheta_f(t)\, t^s \frac{dt}{t}$. ($s \in \mathbb{C}$)

no issues

$\frac{1}{t} t^s \cdot \frac{1}{t}$

Abs conv? $\boxed{\operatorname{Re} s > 1}$ ✓.

$$\tilde{\vartheta}_f(s) = \int_0^\infty \sum_{n\geq 1} f(n\cdot t)\, t^s \,\frac{dt}{t}.$$

$$\overset{\text{abs conv}}{=} \sum_{n\geq 1} \int_0^\infty f(n\cdot t)\cdot t^s \,\frac{dt}{t}$$

let $y = n\cdot t$. , $\frac{dy}{y} = \frac{n\,dt}{n\cdot t} = \frac{dt}{t}$.

$$\overset{\text{Haar}}{=} \sum_{n\geq 1} \int_0^\infty f(y)\left(\frac{y}{n}\right)^s \frac{dy}{y}$$

$$\overset{\text{mult char}}{=} \underbrace{\sum_{n\geq 1} n^{-s}}_{\zeta(s)} \cdot \underbrace{\int_0^\infty f(y)\, y^s \,\frac{dy}{y}}_{\tilde{f}(s)}$$

$$= \zeta(s)\cdot \tilde{f}(s)$$

<u>Q</u>: $\tilde{f}(s)$ of $f(x) = e^{-\pi x^2}$?

$$\tilde{f}(s) = \int_0^\infty e^{-\pi x^2} x^s \,\frac{dx}{x}.$$

$$y = \pi x^2, \quad \frac{dy}{y} = \frac{\pi \cdot 2x \cdot dx}{\pi x^2} = 2 \cdot \frac{dx}{x}.$$

$$\tilde{f}(s) = \frac{1}{2}\int_0^\infty e^{-y}\left(\frac{y}{\pi}\right)^{\frac{s}{2}}\frac{dy}{y}$$

$$= \frac{1}{2}\pi^{-s/2}\underbrace{\int_0^\infty e^{-y}\, y^{s/2}\frac{dy}{y}}_{\Gamma\left(\frac{s}{2}\right)}$$

$$\boxed{\Gamma(s) := \cancel{\int_0^\infty e^{-y} y^{s-1} dy} = \int_0^\infty e^{-y} y^s \frac{dy}{y}.}$$

$$= \frac{1}{2}\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right) = \zeta_\infty(s).$$

Recall: Poisson summation: $\sum_{n\in\mathbb{Z}} f(n\cdot t) = \frac{1}{t}\sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{m}{t}\right).$

$$\rightarrow f(0) + 2\overbrace{\vartheta_f(t)}^{\sum_{n\geq 1} f(n\cdot t).} = \frac{\hat{f}(0)}{t} + \frac{2}{t}\underbrace{\sum_{m\geq 1}\hat{f}\left(\frac{m}{t}\right)}_{\vartheta_{\hat{f}}\left(\frac{1}{t}\right).}$$

Computed Result: $\tilde{\vartheta}_f(s) = \zeta(s)\,\tilde{f}(s)$

$$f(s) \cdot \ldots (s) \cdot f(s).$$

$$\tilde{\vartheta}_f(s) = \int_0^\infty \vartheta_f(t)\, t^s \frac{dt}{t} = \overbrace{\int_1^\infty \vartheta_f(t)\, t^s \frac{dt}{t}}^{\text{entire in } s.} + \int_0^1$$

$$\int_0^1 = \int_0^1 \Big[\frac{1}{t} \vartheta_{\hat f}\left(\frac{1}{t}\right) + \underbrace{\frac{\hat f(0)}{2t} - \frac{f(0)}{2}}\Big] t^s \frac{dt}{t}$$

$$\boxed{y = \frac{1}{t}. \quad \frac{dy}{y} = \frac{-t^{-2}\,dt}{t^{-1}} = -\frac{dt}{t}.}$$

$$= \int_1^\infty \vartheta_{\hat f}(y)\, y^{1-s} \cdot \frac{dy}{y} + \underbrace{\frac{\hat f(0)}{2} \frac{t^{s-1}}{s-1}\Big|_{t=0}^{1} - \frac{f(0)}{2}\frac{t^s}{s}\Big|_{t=0}^{1}}$$

$$\underbrace{\xi(s)\tilde f(s)}_{\text{Res}?} = \tilde\vartheta_f(s) = \underbrace{\int_1^\infty \Big[\vartheta_f(t)\, t^s + \vartheta_{\hat f}(t)\, t^{1-s}\Big]\frac{dt}{t}}_{\text{entire in } s.} \underbrace{- \frac{\hat f(0)}{2(1-s)} - \frac{f(0)}{2s}}_{s \neq 0, 1.}$$

Riemann Zeta function: For any nice f!

$$\xi(s) = \frac{1}{\tilde f(s)} \left[\int_1^\infty \Big[\vartheta_f(t)\, t^s + \vartheta_{\hat f}(t)\, t^{1-s}\Big]\frac{dt}{t} - \frac{\hat f(0)}{2(1-s)} - \frac{f(0)}{2s}\right].$$

Same as: $\sum_{n \in \mathbb{Z}} f(n) - \sum_{m \in \mathbb{Z}} \hat{f}(m) = 0$,

Given continuation of ζ from $\operatorname{Re} s > 1$ to $s \neq 0, 1$. If $f = \hat{f}$, then $\zeta(s) \cdot \tilde{f}(s)$ invariant under $s \mapsto 1-s$.
