Last time: $\underbrace{\mathbb{A}}_{\text{ring of adeles}} = \{ x = (x_\infty, x_2, x_3, \dots) \mid$

$\forall p \le \infty,\ x_p \in \mathbb{Q}_p,\ \underbrace{\text{a.a. } p}_{},\ x_p \in \mathbb{Z}_p \} = \mathbb{R} \times \prod_p{}' \mathbb{Q}_p$

"almost all" = "all but finitely many".

Top generated by: $\prod_{p \in S} U_p \times \prod_{p \notin S} \mathbb{Z}_p$, for $|S| < \infty$, U_p open in $\mathbb{Q}_p$.

In this topology, $\mathbb{Q}$ (embedded diagonally) is discrete: set $(-1,1) \times \prod_p \mathbb{Z}_p$ contains only 0, because all others have $|q|_{\mathbb{A}} = \prod_{p \le \infty} |q|_p = 1$.

We wanted a fund domain for (+) action of $\mathbb{Q}$ on $\mathbb{A}$

E. Artin '46 Indiana Margaret Machett ideles ~~zeta~~ Hecke ~~zeta~~ functions

Tate '50 adelic Poisson summation + "Mellin-type transform"

becomes $GL(1, \mathbb{A})$ aut reps. Fourier analysis.

1947 Bargmann classifies irreps of $SL(2,\mathbb{R})$

Gelfand-Naimark. → '62 ICM Gelfand $GL(2,\mathbb{A})$, -Graev-PS.

I.e. Give me "simplest" representative, say in $\mathbb{R}$ of $\mathbb{Z}$-coset. $x = 3.14159\ldots$, $\{x\} = 0.14159\ldots$

shift by floor function into $[0,1)$.

<u>Recall:</u> D open fund domain for $\Gamma \circlearrowright X$ if:

① $\forall x \in X$, $\exists \gamma \in \Gamma$ s.t. $\gamma \cdot x \in \overline{D}$.

② $\forall x, y \in D$, $\gamma \in \Gamma$, $\gamma x = y \Rightarrow \gamma = 1$, $x = y$.

We want D for $\mathbb{Q} \ni q : x \mapsto q + x$. ($x \in \mathbb{A}$)

<u>Guess:</u> $D \overset{?}{=} (-1,1) \times \prod_p \mathbb{Z}_p$?. Given arbitrary $x = (x_\infty, x_2, x_3, x_5, \ldots)$ find $q \in \mathbb{Q}$ s.t. $x + q \in D$?. I.e. $|x_p + q|_p \leq 1$

If $x_p = \underbrace{a_{-N}p^{-N} + \cdots + a_{-1}p^{-1}} + a_0 + a_1 p + \cdots$

then need to kill off $\uparrow$, by adding q.

Need to do this simultaneously with a single q over finitely many primes. $p \in S = \{p : x_p \notin \mathbb{Z}_p\}$.

CRT: Given $p_1^{e_1}, \ldots, p_k^{e_k}$ powers of distinct primes, any $c_j \in \mathbb{Z}/p_j^{e_j}$, $\exists x \in \mathbb{Z}$ s.t. $x \equiv c_j \ (p_j^{e_j})$.

Thm: (Weaker approximation): Given S, $|S| < \infty$, $c_p \in \mathbb{Z}_p$ $(\forall p \in S)$, Given $\varepsilon > 0$, $\exists\, q \in \mathbb{Z}$, s.t.

$$|c_p - q|_p < \varepsilon.$$

pf: Want: $|c_p - q|_p < \varepsilon$. Choose m s.t. $p^{-m} \le \varepsilon$,

then $c_p = a_0 + a_1 p + a_2 p^2 + \cdots + a_m p^m + \underline{\qquad}$

$|c_p - q|_p \le p^{-m} \iff q \equiv a_0 + a_1 p + \dots + a_{m-1} p^{m-1} \pmod{p^m}$

CRT $\Rightarrow$ q exists in $\mathbb{Z}$!

<u>Thm (Weak Approx):</u> Given S, $c_p \in \mathbb{Q}_p$, $\varepsilon > 0$,

$\exists\, q \in \mathbb{Q}$ s.t. $|c_p - q|_p < \varepsilon$, $\forall p \in S$.

(And denominator $| \prod p^{N_p}$)

<u>Pf:</u> Now $c_p = a_{-N_p} p^{-N_p} + \dots + \dots$ with $\varepsilon = \underline{\min(\varepsilon \cdot p^{-N_p})}$,

Apply <u>Weaker Approx</u> to $\underbrace{c_p p^{N_p}}_{\in \mathbb{Z}_p}$ $\cap$ divide $q_* \in \mathbb{Z}$

by p^N's. $|q_* - c_p p^{N_p}|_p < \varepsilon \cdot p^{-N_p}$ $\Updownarrow$

Use $q = \dfrac{q_*}{\prod p^{N_p}}$ satisfies goal. $\left|\dfrac{q_*}{p^{N_p}} - c_p\right|_p < \varepsilon$.

<u>Thm</u> (Strong Approximation):

Given $x \in \mathbb{A}$, $\exists\, q \in \mathbb{Q}$ s.t.

① $x + q \in \overline{(0,1) \times \prod_p \mathbb{Z}_p} = D$ &

② If $x, y \in D$, $x = y + q$, $q \in \mathbb{Q} \Rightarrow q = 0$, $x = y$.

<u>Pf:</u> ② $x, y \in D$, $x - y \in (-1,1) \times \prod \mathbb{Z}_p$ only $\mathbb{Q}_p t$ is $q = 0$ ✓.

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① Fix $\underline{x} = (x_\infty, x_2, x_3, \dots) \in \mathbb{A}$. Let

$S(x) = S = \{ p : x_p \notin \mathbb{Z}_p \}$, $|S| < \infty$, These

$x_p \in \mathbb{Q}_p$. Apply Weak Approximation with S, x_p's, $\varepsilon = 1$, $\Rightarrow$ obtain $\boxed{q \in \mathbb{Q}}$ s.t. $|\underbrace{x_p + q}|_p \leq 1$ $(\forall p \in S)$.

This $q \in \mathbb{Z}_p$ for $p \notin S$ (by pf of WA). $\mathbb{Z}_p$.

& $\mathbb{Z}_p$ is a ring, so: $x_p + q \in \mathbb{Z}_p \ \forall p$.

Last step: find $n \in \mathbb{Z}$ s.t. $x_\infty + q + n \in [0,1)$.

$\uparrow \mathbb{Z}_p$.

$\Rightarrow x_p + (q+n) \in \mathbb{Z}_p \ \forall p$. Use $\underline{q+n} \in \mathbb{Q}$.

<u>Eg:</u> $x = \left(\pi, \left(\tfrac{1}{2} : \mathbb{Q}_2\right), \left(\tfrac{1}{3} : \mathbb{Q}_3\right), \text{all } \mathbb{Z}_p\text{s}\right)$.

Which (unique) $q \in \mathbb{Q}$ moves x into D?

$q = -\frac{5}{6}$, $\qquad \frac{1}{2} - \frac{5}{6} = -\frac{1}{3} \in \mathbb{Z}_2$.

$= \frac{1}{2} + \frac{1}{3}$

$\qquad \frac{1}{3} - \frac{5}{6} = -\frac{1}{2} \in \mathbb{Z}_3$.

$\lfloor x_\infty + q \rfloor = \lfloor 3.14 - .8 \rfloor = \lfloor 2.3 \rfloor = +2 \qquad n = -2$.

$q + n = -2 - \frac{5}{6} = \frac{-17}{6}$.

$x + q = \left(\pi - \frac{17}{6}, \frac{1}{2} - \frac{17}{6}, \frac{1}{3} - \frac{17}{6}, \mathbb{Z}_p \ldots\right)$

$$\{0,1\} \quad \uparrow \quad \mathbb{Z}_2 \quad \uparrow \quad \mathbb{Z}_3 \quad \uparrow \quad \dots$$

<u>Def:</u> ideles $\mathbb{A}^\times = \{x = (x_\infty, x_2, \dots),\ \forall p \le \infty,\ x_p \in \mathbb{Q}_p^\times$ & a.a. p, $x_p \in \mathbb{Z}_p^\times\}$

$\cap$ $\mathbb{A}$.

(X) <u>group</u> of ideles

$$x \cdot y = (x_\infty \cdot y_\infty,\ x_2 \cdot y_2,\ x_3 \cdot y_3,\ \dots\dots)$$

$$1/x = (1/x_\infty,\ 1/x_2,\ \underbrace{\dots\dots}_{\text{eventually all in } \mathbb{Z}_p^\times})$$

Topology? $\longrightarrow$ $\boxed{\text{Basis } X = \prod_{p\in S} U_p \times \prod_{p\notin S} \mathbb{Z}_p^\times}$, $p\in S$, U_p open in $\mathbb{Q}_p^\times$,

OR: ~~Subspace topology~~? Not the same.

$y\in Y$, $\left(\prod_{p\in S} V_p \times \prod_{p\notin S} \mathbb{Z}_p\right) \cap \mathbb{A}^\times$. $\boxed{V_p \text{ open in } \mathbb{Q}_p.}$

$$y = (\underbrace{y_\infty, y_2, \dots,}_{\neq 0 \quad \neq 0 \quad \dots} \overbrace{y_p}^{\mathbb{Z}_p^\times}, \dots)$$

Difference: Uniformity in S!!! Given X, $\exists S$, s.t. $\forall x\in X$, $\forall p\notin S$, $x_p \in \mathbb{Z}_p^\times$.

Given Y, $\forall y \in Y$, $\exists S = S(y)$ s.t. $\forall p\notin S(y)$, $y_p \in \mathbb{Z}_p^\times$.

Eg: Sequence that is in Y not X.

$y_7 = (1; 2, 3, 5, 7, 1, 1, \dots)$. $\nexists X$ containing $\{y_p\}$.

$\{y_p\} \subset Y \qquad \{y_p\} \not\subset X.$

Back $\mathbb{A}$ Character on $\mathbb{A}$?

On $\mathbb{Q}_p$ $(p \le \infty)$, $x \mapsto e^{2\pi i x}$ hom to $\mathbb{C}^\times$,

On $\mathbb{R}$, $e^{2\pi i x}$ invariant under $\mathbb{Z}$

i.e. char is trivial on $\mathbb{Z}$. Same for $\mathbb{Z}_p$.

$$x_p = a_{-N} p^{-N} + \cdots + a_{-1} p + \underbrace{a_0 + a_1 p + \cdots}_{} \mapsto e^{2\pi i (a_{-N} p^{-N} + \cdots + a_{-1} p^{-1})}$$

Want: Char on $\mathbb{A}$ trivial on $\mathbb{Q}$??.

$$T_n: (x_\infty, x_2, x_3, \dots) \mapsto \prod_{p \le \infty} e_p(x_p) = e_{\mathbb{A}}(x),$$

$e_p(x_p) = e^{2\pi i x_p}$ & $e_\infty(x_\infty) = e^{-2\pi i x_\infty}$.

Product has 1's almost everywhere.

<u>Claim</u>: $\forall q \in Q$, $e_A(q:A) = 1$.